Perturbation Theory for Dual Semigroups V. Variation of Constants Formulas

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1. INTRODUCTION

If A_0 is the infinitesimal generator of a strongly continuous semigroup $T_0(t), t \geq 0$, on a Banach space X, the dual A_0^* of A_0 is the weak* generator of the dual semigroup $T_0^*(t) = (T_0(t))^*$, $t \geq 0$, on X^* in the following sense:

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$$x^* \in D(A_0^*) \quad \text{and} \quad A_0^* x^* = y^* \tag{1.1}$$

iff

$$\frac{d}{dt}\langle x, T_0^*(t)x^*\rangle = \langle x, T_0^*(t)y^*\rangle\,, \quad x \in X, t \geq 0\,.$$

For any $x^* \in X^*$, $t \geq 0$ we have $\int_0^t T_0^*(r)x^* dr \in D(A_0^*)$ (1.2)

and

$$A_0^*\Big(\int_0^t T_0^*(r)x^* dr\Big) = T_0^*(t)x^* - x^*\,.$$

Here $\int_0^t T_0^*(r)x^* dr$ has to be interpreted as the weak* integral

$$\langle x, \int_0^t T_0^*(r)x^* dr\rangle = \int_0^t \langle T_0(r)x, x^*\rangle dr\,. \tag{1.3}$$

In *Clément et al.* (1989b) it is shown that the perturbed operator $A^\times = A_0^* + C$, where $C : \overline{D(A_0^*)} \to X^*$ is bounded and linear, generates a weakly* continuous semigroup $T^\times$ on X^* in the sense of (1.1) and (1.2). Note that in general $X^\odot := \overline{D(A_0^*)} \neq X^*$. In *Clément et al.* (1989b) the restriction $T^\odot$ of $T^\times$ on $X^\odot$ is constructed first via a variation of constants formula and then extended to the space X^*. In *Clément et al.* (1989c) a general Hille-Yosida type characterization is derived for the weak* generators of weakly* continuous semigroups (in the sense of (1.1) and (1.2)).

In this paper we derive variation of constants formulas for the semigroup $T^\times$ (rather than for its restriction $T^\odot$ to $X^\odot$). The construction presented here — which is independent of the approach in *Clément et al.* (1987, 1989a,b,c) — relies on the observation that $A^\times$ generates an 'integrated semigroup' $S^\times$ on X^* such that $S^\times(t)$ is locally Lipschitz in the operator norm. $S^\times$ can also be described by a variation of constants formula. See *Arendt* (1987), *Kellermann* (thesis), *Kellermann&Hieber* (1989), *Neubrander* (1988), *Thieme* (to appear) for some background material concerning 'integrated semigroups.'

An alternative approach, which does not take the operator C as a starting point but considers 'multiplied integrals' of the dual semigroup T_0^* instead, is presented by *Diekmann, Gyllenberg&Thieme* (preprint).

The formulas derived in this paper will allow easy derivation of certain properties of $T^{\times}$. We expect that they will play a crucial role in extending the perturbation theory from dual semigroups to dual evolutionary systems and in handling quasilinear Cauchy problems on non-reflexive dual Banach spaces. Such problems arise from physiologically structured population models — see *Metz&Diekmann* (1986) for reference — in which population growth couples back to individual development.

2. BASIC IDEAS AND RESULTS

In *Clément et al.* (1989b) a strongly continuous semigroup $T^{\odot}$ is constructed on $X^{\odot} = \overline{D(A_0^*)}$ via the variation of constants formula

$$T^{\odot}(t)x^{\odot} = T_0^{\odot}(t)x^{\odot} + \int_0^t T_0^*(t-\tau)C\,T^{\odot}(\tau)x^{\odot}d\tau, \quad x^{\odot} \in X^{\odot} \tag{2.1}$$

with $T_0^{\odot}$ denoting the restriction of T_0^* to $X^{\odot}$. Then $T^{\odot}$ is extended to X^* by the so-called intertwining formula

$$T^{\times}(t) = (\lambda I - A^{\times})T^{\odot}(t)(\lambda I - A^{\times})^{-1}.$$

A variation of constants formula of type (2.1) is not possible for $T^{\times}$ because C is assumed to be defined on $X^{\odot}$ only. In order to overcome this difficulty we shall justify the following formula in section 6:

$$\begin{aligned} T^{\times}(t)x^* &= T_0^*(t)x^* + w^* - \lim_{\lambda\to\infty}\int_0^t T_0^*(t-\tau)C\lambda(\lambda - A_0^*)^{-1}T^{\times}(\tau)x^*\,d\tau \\ &= T_0^*(t)x^* + w^* - \lim_{\lambda\to\infty}\int_0^t T^{\times}(t-\tau)C\lambda(\lambda - A_0^*)^{-1}T_0^*(\tau)x^*\,d\tau. \end{aligned} \tag{2.2}$$

$T^{\times}(t)$ can be represented by a 'generation' expansion

$$T^{\times}(t) = \sum_{n=0}^{\infty} T_n^{\times}(t) \tag{2.3}$$

with $T_0^{\times} = T_0^*$ and

$$T_{n+1}^{\times}(t)x^* = w^* - \lim_{\lambda\to\infty}\int_0^t T_0^*(t-\tau)C\lambda(\lambda - A_0^*)^{-1}T_n^{\times}(t)x^*\,d\tau. \tag{2.4}$$

The series (2.3) converges in the operator norm. We shall see in section 6 that the $w^* - \lim_{\lambda\to\infty}$ in (2.2) and (2.4) holds uniformly for t in bounded intervals, $\|x^*\| \leq 1$ and that $T^{\times *}(t)x$ is a continuous X^{**} valued function of t for any $x \in X$.

Strangely enough we have not been able to prove these results directly. So we take a detour which is of its own interest. It is well known that $A^\times = A_0^* + C$ satisfies the resolvent estimates and therefore generates an 'integrated semigroup' $S^\times(t), t \geq 0$, on X^* which is locally Lipschitz in t with respect to the operator norm. See *Arendt* (1987), *Kellermann* (thesis), *Kellermann&Hieber* (1989). Actually it is possible to write down a variation of constants formula for $S^\times$, namely

$$\begin{aligned} S^\times(t) &= S_0^\times(t) + \int_0^t S_0^\times(t-\tau)d_\tau(CS^\times(\tau)) \\ &= S_0^\times(t) + \int_0^t S^\times(t-\tau)d_\tau(CS_0^\times(\tau)) \end{aligned} \tag{2.5}$$

with

$$S_0^\times(t) = \int_0^t T_0^*(r)dr$$

being the 'integrated semigroup' generated by A_0^*. The Stieltjes integrals in (2.5) hold in the operator norm. From the first formula in (2.5) we realize that $S^\times(t)x^*$ can be differentiated in the weak* sense yielding

$$\begin{aligned} T^\times(t)x^* &:= \frac{d^*}{dt}S^\times(t)x^* \\ &= T_0^*(t)x^* + \int_0^t T_0^*(t-\tau)d_\tau(CS^\times(\tau)x^*) \\ &= T_0^*(t)x^* + \int_0^t T^\times(t-\tau)d_\tau(CS_0^\times(\tau)x^*). \end{aligned} \tag{2.6}$$

The first integral in (2.6) is a weak* Stieltjes integral. The second equality in (2.6) will reveal that $X^{**} \ni T^{\times *}(t)x$ is a continuous function of t for $x \in X$. This will imply that the second integral in (2.6) makes sense as a weak* Stieltjes integral. As we will see in section 6 the second equality in (2.6) also shows that $(\lambda - A_0^*)^{-1}T^\times(t)$ is locally Lipschitz in t with respect to the operator norm because $(\lambda - A_0^*)^{-1}T_0^*(t)$ has this property. (2.6) will then imply (2.2).

The generation expansion (2.3), (2.4) is derived similarly using a generation expansion for $S^\times$. The following formula is particularly helpful in studying the dependence of $T^\times$ on C and T_0^*. Set $V_\infty(t) = CS^\times(t)$,

$V_0(t) = CS_0^\times(t)$ and consider the second equality in (2.6) and (2.5):

$$\begin{cases} T^\times(t)x^* = T_0^*(t)x^* + \int_0^t T_0^*(t-\tau)d_\tau(V_\infty(\tau)x^*) \\ V_\infty(t) = V_0(t) + \int_0^t V_0(t-\tau)d_\tau V_\infty(\tau) \\ \qquad = V_0(t) + \int_0^t V_\infty(t-\tau)d_\tau V_0(\tau) \end{cases} \tag{2.7}$$

In the next section a convolution calculus for locally Lipschitz operator kernels will be developed in which V_∞ plays the role of a resolvent kernel for V_0.

3. A CONVOLUTION CALCULUS FOR LOCALLY LIPSCHITZ CONTINUOUS OPERATOR KERNELS

3.1. *LIPSCHITZ KERNELS AND THEIR CONVOLUTION*

By a *kernel* (of operators) we mean a family $U(t), t \geq 0$, of linear bounded operators on a Banach space Y which satisfies

$$U(0) = 0 \tag{3.1}$$

and is locally Lipschitz in t (with respect to the operator norm), i.e. for any $t > 0$ there exists a $\Lambda_t > 0$ such that

$$\|U(r) - U(s)\| \leq \Lambda_t|r-s|, \quad 0 \leq r, s \leq t. \tag{3.2}$$

The kernels form a vector space in an obvious way. We define seminorms $\|\cdot\|_t$ by

$$\|U\|_t := \sup_{0\leq r\neq s\leq t} \frac{\|U(r)-U(s)\|}{|r-s|}, \quad t > 0. \tag{3.3}$$

By (3.1), $U(0) = 0$, we have

$$\sup_{0\leq r\leq t} \|U(r)\| \leq t\|U\|_t. \tag{3.4}$$

With these seminorms the kernels form a Fréchet space which becomes an algebra in the following way: For two kernels U, V we define the convolution $\star$ by

$$(U \star V)(t) = \int_0^t U(t-r)d_rV(r). \tag{3.5}$$

The integral in (3.5) is a Stieltjes integral in the operator norm, i.e. it is the limit of sums

$$\sum_{j=0}^{n} U(t-s_j)(V(r_{j+1})-V(r_j)), \quad s_j \in [r_j, r_{j+1}]$$

with $0 = r_0 < \ldots < r_{n+1} = t$, when the partition $r_0, \ldots, r_{n+1}$ gets finer. By reordering the sums one easily checks that

$$(U \star V)(t) = \int_0^t d_r U(r) V(t-r) \tag{3.6}$$

with the integral being the limit of sums

$$\sum_{j=0}^{n} (U(r_{j+1} - U(r_j))V(t-s_j), \quad s_j \in [r_j, r_{j+1}],$$

$0 = r_0 < \ldots < r_{n+1} = t$. It is convenient to extend the kernels to $\mathbf{R}$ by setting

$$U(t) = 0, \quad t \leq 0. \tag{3.7}$$

Then they are locally Lipschitz on $\mathbf{R}$ and

$$(U \star V)(s) = \int_0^t U(s-r) d_r V(r), s \leq t. \tag{3.8}$$

$U \star V$ is a kernel again; actually we have the following inequalities in terms of the seminorms $\|\cdot\|_t$.

Lemma 3.1.

$$\|U \star V\|_t \leq \int_0^t \|U\|_{t-r} \, \|V\|_r dr \leq t \|U\|_t \|V\|_t .$$

Proof. Let $0 \leq r, s \leq t$. Then $(U \star V)(s) - (U \star V)(r)$ is approximated by sums

$$\sum_{j=0}^{n} (U(s-\sigma_{j+1}) - U(r-\sigma_{j+1}))(V(\sigma_{j+1}) - V(\sigma_j)) \tag{3.9}$$

with

$$0 = \sigma_0 < \cdots < \sigma_{n+1} = t .$$

The norm of the sum (3.9) can be estimated by

$$\sum_{j=0}^{n} \|U\|_{t-\sigma_{j+1}} \, |s-r| \, \|V\|_{\sigma_{j+1}} (\sigma_{j+1} - \sigma_j).$$

Taking the limit by refining the partitions we obtain

$$\|(U \star V)(s) - (U \star V)(r)\| \leq |s - r| \left(\int_0^t \|U\|_{t-\sigma} \|V\|_\sigma \, d\sigma \right).$$

This implies the first estimate. The second is trivial.

We can integrate $(U \star V)(t)$ and obtain a more familiar convolution.

Lemma 3.2. *$\int_0^t (U \star V)(r)dr = \int_0^t U(t-r)V(r)dr =: (U * V)(t)$. In other words,*

$$(U \star V)(t) = \frac{d}{dt}(U * V)(t)$$

with the differentiation holding in the operator norm.

Proof.

$$\int_0^t (U \star V)(r)dr = \int_0^t \int_0^t U(r-s)d_sV(s)\,dr = \int_0^t \left(\int_0^t U(r-s)dr \right) d_sV(s)$$

$$= \int_0^t \left(\int_0^{t-s} U(r)dr \right) d_sV(s) = -\int_0^t d_s \left(\int_0^{t-s} U(r)dr \right) V(s) = \int_0^t U(t-s)V(s)\,ds.$$

The second, fourth and fifth equality follow by approximating the integrals by sums and rearranging these, the first equality holds by definition, the third by standard integral calculus. Remember (3.7): $U(0) = V(0) = 0$. Noting that

$$\frac{d}{dt}(U * V) = U' * V = U * V',$$

provided the respective derivatives exist, we find that

$$U \star (V \star W) = \frac{d^2}{dt^2}(U * (V * W))$$

$$(U \star V) \star W = \frac{d^2}{dt^2}((U * V) * W).$$

As the associativity of $*$ is well-known and easily checked by standard integration theory, we have

Lemma 3.3. *$\star$ is associative, i.e. the Fréchet space of kernels is an algebra.*

In view of Lemma 3.1, the Fréchet space of kernels deserves the name Frechet algebra.

3.2 *RESOLVENT KERNELS*

The resolvent kernel V_∞ of a kernel V_0 is determined by the relation

$$V_\infty = V_0 + V_0 \star V_\infty = V_0 + V_\infty \star V_0 \,. \qquad (3.10)$$

If it exists the resolvent kernel is unique by its algebraic properties. See *Gripenberg et al.* (1990), Section 9.3, Lemma 3.3.

Remark. Often the resolvent kernel of a kernel W_0 is defined by

$$W_\infty = W_0 - W_0 \star W_\infty = W_0 - W_\infty \star W_0. \qquad (3.11)$$

See *Gripenberg et al.* (1990), Section 9.3. Note that (3.10) translates into (3.11) by setting $W_\infty = -V_\infty$, $W_0 = -V_0$. The concept of (3.11) seems to be more natural when 'frequency domain methods' are used whereas the concept of (3.10) is more convenient when exploiting order relations in case that Y is an ordered Banach space.

The standard construction of the resolvent kernel is the series of multiple convolutions:

$$V_\infty = \sum_{n=1}^{\infty} V^{\star n} \qquad (3.12)$$

with

$$V^{\star 1} = V_0,\; V^{\star(n+1)} = V^{\star n} \star V_0. \qquad (3.13)$$

The main point is showing the convergence of the series. (3.10) then follows from

Lemma 3.4. $V^{\star n} \star V_0 = V_0 \star V^{\star n}$

which is immediate by induction. From Lemma 3.1 we obtain by induction

Lemma 3.5. $\|V^{\star(n+1)}\|_t \le \frac{t^n}{n!}\|V_0\|_t^{n+1}, \quad n \ge 1\,.$

So $\sum_{n=1}^{\infty} \|V^{\star n}\|_t \le \|V_0\|_t \exp(t\|V_0\|_t)$ and the series (3.12) converges in the seminorms $\|\cdot\|_t$. By (3.4), $\sum_{n=1}^{\infty} V^{\star n}(t)$ converges in the operator norm uniformly for t in bounded intervals.

As a corollary we have the estimate

Lemma 3.6. $\|V_\infty\|_t \le \|V_0\|_t \exp(t\|V_0\|_t)\,.$

The importance of resolvent kernels consists in solving convolution equations.

Lemma 3.7. (*Gripenberg et al.* (1990), Section 9.3, Lemma 3.5)

The convolution equation

$$U = U_0 + V_0 \star U$$

is uniquely solved by

$$U = U_0 + V_\infty \star U_0 ,$$

whereas

$$W = W_0 + W \star V_0$$

is uniquely solved by

$$W = W_0 + W_0 \star V_\infty .$$

Before we estimate the solutions of convolution equations we make the following simple observation which follows from Lemma 3.6.

Lemma 3.8. $1 + \int_0^t \|V_\infty\|_r dr \le \exp(t\|V_0\|_t)$.

Note that $\|V_0\|_t$ is a monotone non-decreasing function of t. The following is now easily derived from Lemma 3.7 and Lemma 3.1.

Lemma 3.9. *Let* W *solve* $W = W_0 + W \star V_0$ *or* $W = W_0 + V_0 \star W$. *Then*

$$\|W\|_t \le \|W_0\|_t \exp(t\|V_0\|_t) .$$

We use this lemma for studying the dependence of the resolvent kernel V_∞ on V_0. Let

$$U_\infty = U_0 + U_0 \star U_\infty$$
$$V_\infty = V_0 + V_0 \star V_\infty .$$

Then

$$U_\infty - V_\infty = (U_0 - V_0) + (U_0 - V_0) \star U_\infty + V_0 \star (U_\infty - V_\infty).$$

By Lemma 3.9

$$\|U_\infty - V_\infty\|_t \le \|U_0 - V_0 + (U_0 - V_0) \star U_\infty\|_t \exp(t\|V_0\|_t).$$

By Lemma 3.1

$$\|U_\infty - V_\infty\|_t \le \|U_0 - V_0\|_t\Big(1 + \int_0^t \|U_\infty\|_r dr\Big)\exp(t\|V_0\|_t).$$

By Lemma 3.8,

$$\|U_\infty - V_\infty\|_t \le \|U_0 - V_0\|_t \exp(t\|U_0\|_t)\exp(t\|V_0\|_t)\,.$$

So we have

Lemma 3.10. *Let U_∞, V_∞ be the resolvent kernels of U_0, V_0 respectively. Then*

$$\|U_\infty - V_\infty\|_t \le \|U_0 - V_0\|_t \exp(t[\|U_0\|_t + \|V_0\|_t])\,.$$

4. PERTURBATION OF LOCALLY LIPSCHITZ CONTINUOUS INTEGRATED SEMIGROUPS

It is well-known that an operator $A_0^\times$ on a Banach space Y generates an 'integrated semigroup' $S_0^\times(t)$, $t \ge 0$, on Y which is locally Lipschitz (with respect to the operator norm) iff $\lambda - A_0^\times$ can be continuously inverted for $\lambda > w$ and the resolvent estimates

$$\|(\lambda - A_0^\times)^{-n}\| \le \frac{M}{(\lambda - w)^n}, \quad \lambda > w,\, n \in \mathrm{N} \tag{4.1}$$

are satisfied. Actually

$$\|S_0^\times(t) - S_0^\times(r)\| \le M\, e^{wt}(t-r),\, t \ge r \ge 0\,. \tag{4.2}$$

Moreover we recall that by definition

$$S_0^\times(t)S_0^\times(r) = \int_0^t (S_0^\times(r+\tau) - S_0^\times(\tau))d\tau, \quad S_0^\times(0) = 0\,. \tag{4.3}$$

See *Arendt* (1987), *Kellermann* (thesis), *Kellermann&Hieber* (1989). The following relations hold between $A^\times$ and $S^\times$:

Lemma 4.1 *a) Let $x, y \in Y$. Then $x \in D(A_0^\times)$ and $A_0^\times x = y$ iff $\frac{d}{dt}S_0^\times(t)x = x + S_0^\times(t)y$ for all $t \ge 0$.*

b) $(\lambda - A_0^\times)^{-1} = \lambda\int_0^\infty e^{-\lambda t}S_0^\times(t)dt = \int_0^\infty e^{-\lambda t}d_tS_0^\times(t)$.

c) For any $y \in Y$, $t \ge 0$, $\int_0^t S_0^\times(r)ydr \in D(A_0^\times)$ and $A_0^\times \int_0^t S_0^\times(r)ydr = S_0^\times(t)y - ty$.

See, e.g., *Thieme* (to appear).

If $C : \overline{D(A_0^\times)} \to Y$ is a bounded linear operator, the operator $A^\times = A_0^\times + C$ also satisfies the estimates (4.1) (with different w). See the proof of Theorem 1.1 in *Pazy* (1983), Section 3.1. So $A^\times$ generates a locally Lipschitz continuous integrated semigroup $S^\times$. Compare Proposition 3.3 in *Kellermann&Hieber* (1989). Actually it is possible to find $S^\times$ as the solution of the variation of constants formula

$$\begin{aligned} S^\times(t) &= S_0^\times(t) + \int_0^t S^\times(t-\tau)d_\tau(CS_0^\times(\tau)) \\ &= S_0^\times(t) + \int_0^t S_0^\times(t-\tau)d_\tau(CS^\times(\tau)). \end{aligned} \tag{4.4}$$

Taking Laplace transforms one realizes that

$$\lambda \int_0^\infty e^{-\lambda t} S^\times(t)dt = (\lambda - A^\times)^{-1}.$$

Applying C to (4.4) we realize that $CS^\times(t)$ coincides with the resolvent kernel V_∞ of V_0, $V_0(t) = CS_0^\times(t)$. Hence

$$S^\times(t) = S_0^\times(t) + \int_0^t S_0^\times(t-\tau)d_\tau V_\infty(\tau). \tag{4.5}$$

In other words,

$$S^\times = S_0^\times + S_0^\times \star V_\infty. \tag{4.6}$$

In turn, we can first construct V_∞ as the resolvent kernel of V_0 and define $S^\times$ by (4.6). If we multiply (4.6) by C and compare with (3.10) we find that $V_\infty = CS^\times$. Using the expansion (3.12), (3.13) we obtain the generation expansion

$$\begin{aligned} S^\times(t) &= \sum_{n=0}^\infty S_n^\times(t) \\ S_{n+1}^\times(t) &= \int_0^t S_n^\times(t-\tau)d_\tau(CS_0^\times(\tau)) = \int_0^t S_0^\times(t-\tau)d_\tau(CS_n^\times(\tau)). \end{aligned} \tag{4.7}$$

In fact the definition

$$S_n^\times = S_0^\times \star V^{\star n}, \quad n \geq 1,$$

yields $CS_{n+1}^\times = V^{\star(n+1)}$ — see (4.7) and Lemma 3.4 — and so

$$\begin{aligned} S_{n+1}^\times &= S_0^\times \star (CS_n^\times), \\ S_{n+1}^\times &= S_0^\times \star (V^{\star n} \star V_0) = (S_0^\times \star V^{\star n}) \star V_0 = S_n^\times \star (CS_0^\times). \end{aligned}$$

As a byproduct, we obtain the estimate

$$\sum_{n=0}^{\infty} \|S_{n+1}^{\times}\|_t < \infty, \quad \text{for any } t > 0\,. \tag{4.8}$$

5. PERTURBATION OF DUAL SEMIGROUPS

If $T_0^*(t)$ is the dual semigroup on X^* associated with a strongly continuous semigroup T_0 on X — the infinitesimal generator of which is A_0 —, then

$$S_0^{\times}(t)x^* = \int_0^t T_0^*(r)x^* dr \tag{5.1}$$

defines the locally Lipschitz continuous 'integrated semigroup' $S_0^{\times}(t)$ on X^* which is generated by A_0^*. Let $C : \overline{D(A_0^*)} \to X^*$ be a bounded linear operator. Then the perturbed operator $A^{\times} = A_0^* + C$ with $D(A^{\times}) = D(A_0^*)$ generates the integrated semigroup given by (4.4). From the second equation in (4.4) we realize that $S^{\times}(t)x^*$ can be differentiated in the weak* sense and that

$$\begin{aligned} T^{\times}(t)x^* &:= \frac{d^*}{dt} S^{\times}(t)x^* \\ &= T_0^*(t)x^* + \int_0^t T_0^*(t-\tau) d_\tau (CS^{\times}(\tau)x^*)\,. \end{aligned} \tag{5.2}$$

The integral on the right hand side has to be interpreted in the weak* sense. We note that $X^{**} \ni T^{\times *}(t)x$ is a continuous function of t for $x \in X$. Taking this into account we obtain from the first equation in (4.4) that

$$T^{\times}(t)x^* = T_0^*(t)x^* + \int_0^t T^{\times}(t-\tau) d_\tau (CS_0^{\times}(\tau)x^*), \tag{5.3}$$

where the integral on the right hand side has to be interpreted in a weak* sense:

$$\langle x, \int_0^t T^{\times}(t-\tau) d_\tau (CS_0^{\times}(\tau)x^*)\rangle$$

is the limit of the sums

$$\sum_{j=0}^{n} \langle \Big(CS_0^{\times}(\tau_{j+1}) - CS_0^{\times}(\tau_j)\Big)x^*, T^{\times *}(t-\sigma_j)x\rangle,$$

$0 = \tau_0 < \cdots < \tau_{n+1} = t, \sigma_j \in [t_j, t_{j+1}]$, when the partition $\tau_0, \cdots, \tau_{n+1}, n \in$ N, gets finer.

From (1.2) and the second equality in (4.4) we realize that $S^\times(t)x^*$ $\in D(A_0^*)$ and

$$A_0^* S^\times(t)x^* = -x^* + T^\times(t)x^* - CS^\times(t)x^* .$$

In other words

$$T^\times(t)x^* = x^* + A^\times S^\times(t)x^* . \tag{5.4}$$

This is property (1.2) for $T^\times, A^\times$. Using (4.3), (5.4) and Lemma 4.1c) we can verify that $T^\times(t)$ is a semigroup. From Lemma 4.1a) we obtain that

$$x^* \in D(A^\times), A^\times x^* = y^* \quad \text{iff} \quad T^\times(t)x^* - x^* = S^\times(t)y^* , t \geq 0 .$$

This is equivalent to (1.1) for $T^\times, A^\times$. Hence we have shown that the weakly* continuous semigroup $T^\times$ generated by $A^\times$ in the sense of (1.1), (1.2) is obtained by the formulas (2.6).

It is now easy to obtain a generation expansion for $T^\times$. Proceeding as before we can differentiate (4.7) in the weak* sense obtaining

$$\begin{aligned} T^\times_{n+1}(t)x^* \, :&= \frac{d^*}{dt} S^\times_{n+1}(t)x^* \\ &= \int_0^t T_0^*(t-\tau) d_\tau (CS^\times_n(\tau)) = \int_0^t T^\times_n(t-\tau) d_\tau (CS^\times_0(\tau)). \end{aligned} \tag{5.5}$$

It follows from (4.8) that

$$\sum_{n=0}^{\infty} T^\times_n(t)$$

converges in the operator norm uniformly on bounded intervals, hence the series in (4.7) can be differentiated in the weak* sense such that

$$T^\times(t)x^* = \frac{d^*}{dt} S^\times(t)x^* = \sum_{n=0}^{\infty} \frac{d^*}{dt} S^\times_n(t)x^* = \sum_{n=0}^{\infty} T^\times_n(t)x^* . \tag{5.6}$$

6. THE VARIATION OF CONSTANTS FORMULA (2.2)

In order to give a meaning to the integrals in (2.2) we prove

Lemma 6.1. *a) $(\lambda - A_0^*)^{-1} T_0^*(t)$ is locally Lipschitz in t with respect to the operator norm.*

b) The same holds for $(\lambda - A_0^*)^{-1} T^\times(t)$.

Proof: a)

$$(\lambda - A_0^*)^{-1}T_0^*(t) = T_0^*(t)(\lambda - A_0^*)^{-1}$$
$$= \int_0^t T_0^*(s)A_0^*(\lambda - A_0^*)^{-1}ds + (\lambda - A_0^*)^{-1}$$
$$= -\int_0^t T_0^*(s)ds + \int_0^t T_0^*(s)\lambda(\lambda - A_0^*)^{-1}ds + (\lambda - A_0^*)^{-1}.$$

b) By (5.2)

$$(\lambda - A_0^*)^{-1}T^\times(t) = (\lambda - A_0^*)^{-1}T_0^*(t) + \int_0^t (\lambda - A_0^*)^{-1}T_0^*(t-\tau)d_\tau(CS^\times(\tau)).$$

Part a) and Lemma 3.1 now imply the assertion.

In order to show the first equality in formula (2.2) we use formula (5.2) and prove that

$$w^* - \lim_{\lambda\to\infty} \int_0^t T_0^*(t-\tau)C\lambda(\lambda - A_0^*)^{-1}T^\times(\tau)x^*ds = \int_0^t T_0^*(t-\tau)d_\tau(CS^\times(\tau)x^*).$$

Note that the integrals on the left hand side can be approximated in the weak* sense by sums

$$\sum_j T_0^*(t-\tau_j)C\lambda(\lambda - A_0^*)^{-1}(S^\times(\tau_{j+1}) - S^\times(\tau_j))x^*$$

uniformly for large λ and uniformly for $\|x^*\| \leq 1$, t in bounded intervals.

The integral on the right hand side can be approximated in the weak* sense by sums

$$\sum_j T_0^*(t-\tau_j)C(S^\times(\tau_{j+1}) - S^\times(\tau_j))x^*$$

uniformly for large λ and uniformly for $\|x^*\| \leq 1$, t in bounded intervals.

So we only need to show that

$$\lambda(\lambda - A_0^*)^{-1}S^\times(r) \to S^\times(r),\ \lambda \to \infty$$

uniformly for r in bounded intervals. But

$$\lambda(\lambda - A_0^*)^{-1}S^\times(r) - S^\times(r) = (\lambda - A_0^*)^{-1}A_0^*S^\times(r)$$
$$= (\lambda - A_0^*)^{-1}(T^\times(r) - I - CS^\times(r))$$

— see (5.4) — and

$$\|(\lambda - A_0^*)^{-1}\| \to 0 \quad \text{for } \lambda \to \infty.$$

The second equality in (2.2) is shown similarly using (5.3). Note that $T^{\times *}(t)x, x \in X$, is a continuous X^{**} valued function of $t \geq 0$. (2.4) is derived from (5.5) in the same way. Note from (4.7) and (1.2) that $S_{n+1}^{\times}(t)x^* \in D(A_0^*)$ and

$$A_0^* S_{n+1}^{\times}(t)x^* = T_{n+1}^{\times}(t)x^* - x^* - CS_n^{\times}(t)x^* .$$

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