LETTER TO THE EDITOR

Dear Editor,

Busy-period analysis of a correlated queue

Langaris (1987) presents a busy period analysis of a queue with positively correlated arrival and service processes. We would like to point out that his results are valid for the first busy period only, and not, as is claimed by Langaris, for busy periods in steady state.

Langaris considers a single-server queue in which the interarrival time and service time of a customer have a bivariate density function $g(t, s)$, with negative-exponential marginal density functions.

A considerable part of his analysis is devoted to $f_n(t, s)$, the probability density function of the following event: the busy period length is t and the number of customers served during this busy period is n, given that the first service of that busy period has length $s < t$. After obtaining the Laplace transform of $f_n(t, s)$, Langaris derives the Laplace transform of $b_n'(t)$, the probability density function of the following event: the busy period has length t and exactly n customers are served during this busy period. For this derivation we use the relation

$$(1) \qquad b_n'(t) = \int_{s=0}^{t} f_{n-1}(t, s) dA(s),$$

in which $A(\cdot)$ is the marginal distribution function of the service time of an arbitrary customer.

Langaris conditions the length of a new busy period on the length of the service time of the first customer in that busy period, assuming that this customer's service time has an ordinary distribution. This is where the problem arises: the assumption holds for the first busy period, but not for a busy period in steady state. Just like in the uncorrelated $M/G/1$ queue, the interarrival time of a customer starting a busy period is larger than the sojourn time of his predecessor, and hence is likely to be relatively long. Due to the assumed correlation between arrival and service processes (the essential feature of this model), the service time of this first customer is likely to be relatively long too. This effect has also been mentioned by Conolly and Hadidi (1969) in a queueing model in which the service time and the interarrival time of a customer are linearly related.

As a consequence, in Langaris (1987), Tables 1 and 2 are valid only for the first busy period. Furthermore, since the service time of the first customer in a busy period is understimated, ET, the mean length of a busy period in steady state, as given by Langaris, is smaller than the real mean busy period length in steady state. To obtain the correct value ET, one can use the balancing argument $ET/(ET + EI) = \rho$, in which

$\rho < 1$ is the workload, and EI is the mean idle period length. This balancing argument holds for any single-server delay queueing system. If the interarrival time of a customer is negative exponentially distributed, and the service time distribution $A(\cdot)$ has the property $A(0^+) = 0$, then EI is equal to the mean interarrival time. Hence, $ET = EA/(1-\rho)$. The mean number of customers in a busy period when the system is in steady state is given by ET/EA.

To obtain $b_n'(t)$, $A(\cdot)$ in (1) should be replaced by $A^1(\cdot)$, the service time distribution of the first customer in a busy period. To determine $A^1(\cdot)$, one can condition on the interarrival time of this customer. The interarrival time of the first customer in a busy period consists of two independent parts:

(i) The sojourn time of his predecessor, which has Laplace–Stieltjes transform $\alpha(\lambda+\omega)/\alpha(\lambda)$, with λ the parameter of the negative exponentially distributed interarrival times, and $\alpha(\cdot)$ the Laplace–Stieltjes transform of the sojourn time, as given in Conolly and Choo (1979);

(ii) The remaining interarrival time, which is again negative exponentially distributed, and has Laplace–Stieltjes transform $\lambda/(\lambda+\omega)$. Unfortunately, using these results, (1) becomes numerically intractable.

References

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Yours sincerely,

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