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Generalized hexagons of even order

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Abstract

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An elaborate version is given of Kantor's construction of the known generalized hexagons of order (q, q^3) and of order (q, q) for q a power of 2.

1. Introduction

Nearly all of van Lint's lectures at the Combinatorial Theory Seminar on Wednesday afternoons in Eindhoven were extremely well presented and quite entertaining. The single exception the first author recalls has been van Lint's presentation of the generalized hexagon of order (2, 2). The purpose was an elementary construction, generalizing to the known generalized hexagons of higher order. The present paper outlines a way in which van Lint might have wanted to progress from that 'exceptional' lecture on, especially since ovoids and spreads are among the ingredients.

2. The method

We describe how to construct generalized polygons à la Kantor. In 1959, Tits introduced the notion of generalized polygons (cf. [4]). In [3], Kantor presented a construction of the generalized polygons which we treat here in greater detail (with proofs). We restrict to the case where q is even, because then there is an

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interesting correspondence with ovoids and spreads. It suffices for the purpose of this paper to define a generalized n-gon of order (s, t) as a bipartite graph Γ with parts P and L such that

- the vertices from P have valency $t+1$ while those from L have valency $s+1$;
- the diameter of Γ is n;
- Γ contains no cycles of length less than $2n$.

As usual the set of vertices at distance i from v in Γ is denoted by $\Gamma_i(v)$. Let I be an index set of size $t+1$. Given Q and a collection of subgroups $\{Q_j^{(i)}\}_{i\in I, 0\le j<n}$, we define the graph $\Gamma = \Gamma(Q, \{Q_j^{(i)}\}_{i,j})$ by the following rules.

Vertices: distinguished vertices ∞ and ℓ_i for $i \in I$; for each $i \in I$ and $1 \le j \le n-2$, all right cosets of Q over the subgroups $Q_j^{(i)}$; all elements of Q.

Adjacency: the vertex ∞ is adjacent to the vertices ℓ_i for $i \in I$; besides ∞ the vertex ℓ_i is adjacent to all cosets of Q over $Q_{n-2}^{(i)}$; for each $i \in I$, $j \in \{2, \ldots, n-2\}$ and $g, h \in Q$ the coset $gQ_j^{(i)}$ is adjacent to the coset $hQ_{j-1}^{(i)}$ if it contains the latter, that is, if $g^{-1}h \in Q_j^{(i)}Q_{j-1}^{(i)}$; for each i, the elements of Q are adjacent to the cosets of $Q_1^{(i)}$ they belong to.

We shall express the axioms of a generalized n-gon in terms of the system $\{Q_j^{(i)}\}_{i,j}$. First the orders of the subgroups involved and their inclusions are specified.

(O) For each $i \in I$ and $1 \le j \le n-1$, the group $Q_{j-1}^{(i)}$ is a subgroup of $Q_j^{(i)}$ with index $|Q_j^{(i)} : Q_{j-1}^{(i)}|$ equal to t or s depending on whether $n-j$ is even or odd; moreover, $Q_0^{(i)} = 1$ and $Q_{n-1}^{(i)} = Q$ for all i.

The second condition concerns the absence in Γ of cycles of length less than $2n$.

(C) For any natural h, for any indices $i_1, i_2, \ldots, i_h$ such that $i_m \ne i_{m+1}$ for $1 \le m \le h-1$ and $i_1 \ne i_h$, and for any $1 \le j_1, j_2, \ldots, j_h \le n-1$ such that $\sum_m j_m = n-1$, we have $1 \notin Q_{j_1}^{(i_1)\#} Q_{j_2}^{(i_2)\#} \cdots Q_{j_h}^{(i_h)\#}$.

Here, as usual, $H^\#$ for a group H stands for $H - \{1\}$. The proposition below generalizes Kantor's generalized quadrangle construction to arbitrary generalized polygons and is probably folklore.

Proposition 2.1. *Suppose Q is a group admitting a system $\{Q_j^{(i)}\}_{i,j}$ of subgroups satisfying the conditions* (O) *and* (C). *Then $\Gamma(Q, \{Q_j^{(i)}\}_{i,j})$ is a generalized n-gon of order (s, t).*

Proof. Set $\Gamma = \Gamma(Q, \{Q_j^{(i)}\}_{i,j})$. It is readily checked that

$$\Gamma_j(\infty) = \begin{cases} \{\infty\} & \text{if } j = 0, \\ \{\ell_i \mid i \in I\} & \text{if } j = 1, \\ \bigcup_{i\in I} Q/Q_{n-j}^{(i)} & \text{if } 2 \le j \le n-1, \\ Q & \text{if } j = n, \end{cases}$$

and that Γ is bipartite with parts $P = \bigcup_i \Gamma_{2i}(\infty)$ and $L = \bigcup_i \Gamma_{2i+1}(\infty)$. Condition (O) implies that the number of vertices in Γ coincides with the number of vertices of a generalized n-gon of order (s, t) and that the valency of a vertex of Γ is equal to $t+1$ or $s+1$ depending on whether it lies in part P or L of Γ.

Thus Γ is a generalized n-gon precisely when it contains no cycles of length less than $2n$. It follows from the construction and the absence of inclusions other than those specified that Γ has no cycles of length less than $2n$ passing through ∞. It follows from (C) that Γ has no cycles of length less than $2n$ not passing through ∞. □

There is a converse to the proposition to the effect that any generalized n-gon Γ of order (s, t) affording a subgroup Q that, for some $x \in \Gamma$, stabilizes $\{x\} \cup \Gamma_1(x)$ vertex-wise, and is transitive on $\Gamma_n(x)$, is isomorphic to a $\Gamma(Q, \{Q_j^{(i)}\}_{i,j})$ for certain subgroups $Q_j^{(i)}$ of Q. But we do not need it here.

3. The construction

We set $k = \mathbb{F}_q$ and $\ell = \mathbb{F}_{q^3}$. For $x \in \ell$, we denote by $T(x)$ and $N(x)$ the trace and norm, respectively, of x over k. Viewing ℓ as a 3-dimensional vector space over k, the following set V has a natural structure of an 8-dimensional vector space over k:

$$V = k \times \ell \times \ell \times k = \{(\alpha, b, c, \delta) \mid \alpha, \delta \in k; b, c \in \ell\}.$$

We shall work with the bilinear form $g : V \times V \to \mathbb{F}_q$ given by

$$g((\alpha, b, c, \delta), (\alpha', b', c', \delta')) = \alpha\delta' + T(bc').$$

We use it to define the quadratic form $f : V \to \mathbb{F}_q$ by $f(x) = g(x, x)$ so that

$$f(\alpha, b, c, \delta) = \alpha\delta + T(bc).$$

Thus, the bilinear form associated with f is $(x, x') \mapsto g(x, x') + g(x', x)$.

From now on, q is even. Thus (V, f) is an $\Omega^+(8, k)$ space. Using the form f, we extend V by k to a group on the set

$$Q = V \times k,$$

the elements of which we denote by $(\alpha, b, c, \delta; \zeta)$ with $\alpha, \delta, \zeta \in k$ and $b, c \in \ell$, or just (x, ζ) with x representing (α, b, c, δ). It becomes a group of order q^9 by means of the following multiplication rule:

$$(x; \zeta) \cdot (x'; \zeta') = (x + x'; g(x, x') + \zeta + \zeta') \quad (x, x' \in V; \zeta, \zeta' \in k).$$

The center $Z = \{(0; \zeta) \mid \zeta \in k\}$ of Q coincides with its commutator subgroup, and $Q/Z \cong V$. Such a group is sometimes called a 'special' group.

We let $\mathrm{SL}(2, q^3)$ act on Q as determined by the action of its generators

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad [t] = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad (t \in \ell)$$

via

$$w(\alpha, b, c, \delta; \zeta) = (\delta, c, b, \alpha; f(\alpha, b, c, \delta) + \zeta),$$
$$[t](\alpha, b, c, \delta; \zeta)$$
$$= (\alpha, \alpha t + b, \alpha t^{q+q^2} + b^q t^{q^2} + b^{q^2} t^q + c,$$
$$\alpha N(t) + T(bt^{q+q^2}) + T(ct) + \delta; \zeta + \alpha T(bt^{q^2+q}) + T(b^{q+1}t^{q^2}) + \alpha^2 N(t)).$$

We are now ready to construct the groups $Q_j^{(i)}$. Since $Q_0^{(i)} = 1$ and $Q_5^{(i)} = Q$, we only need to work for $j = 1, 2, 3, 4$. In the respective cases, the orders are q, q^4, q^5, q^8. In [1] we find an ovoid in (V, f) left invariant by $\mathrm{SL}(2, \ell)$. We lift each of its elements to a subgroup $Q_1^{(i)}$ defined as follows:

$$Q_1^{(\infty)} = \{(0, 0, 0, \delta; 0) \mid \delta \in k\},$$
$$Q_1^{(t)} = [t] \cdot w \cdot Q_1^{(\infty)}$$
$$= [t]\{(\alpha, 0, 0, 0; 0) \mid \alpha \in k\}$$
$$= \{(\alpha, \alpha t, \alpha t^{q+q^2}, \alpha t^{1+q+q^2}; \alpha^2 N(t)) \mid \alpha \in k\}.$$

According to [2] a spread in (V, f) can be constructed as follows:

$$\Sigma(\infty) = \{(0, 0, c, \delta) \mid \delta \in k; c \in \ell\},$$
$$\Sigma(i) = [i] \cdot w \cdot \Sigma(\infty),$$

where $i \in \ell$. Then $\{\Sigma(i) \mid i \in \ell \cup \{\infty\}\}$ is a spread of (V, f) stabilized by $\mathrm{SL}(2, \ell)$. We lift the spread to Q in order to find the groups $Q_2^{(i)}$. We start with

$$Q_2^{(\infty)} = \{(0, 0, c, \delta; 0) \mid \delta \in k; c \in \ell\}.$$

Then, by $\mathrm{SL}(2, \ell)$-invariance, we must have, for $t \in \ell$,

$$Q_2^{(t)} = [t] \cdot w \cdot Q_2^{(\infty)}$$
$$= [t] \cdot \{(\alpha, b, 0, 0; 0) \mid \alpha \in k; b \in \ell\}$$
$$= \{(\alpha, \alpha t + b, \alpha t^{q+q^2} + b^q t^{q^2} + b^{q^2} t^q, T(bt^{q+q^2});$$
$$\alpha T(bt^{q^2+q}) + T(b^{q+1}t^{q^2}) + \alpha^2 N(t)) \mid \alpha \in k; b \in \ell\}.$$

We set $Q_3^{(i)} = ZQ_2^{(i)}$ and let $Q_4^{(i)}$ be the index q subgroup of Q containing Z whose preimage in V is the hyperplane perpendicular (with respect to f) to $Q_1^{(i)}Z/Z$.

Theorem. *For Q and $Q_j^{(i)}$ as above, the graph $\Gamma(Q, \{Q_j^{(i)}\}_{i,j})$ is a generalized hexagon of order (q, q^3).*

Proof. Condition (O) is readily verified. The subscript sequences $j_1, \ldots, j_h$ of condition (C) that need to be checked in order to verify that we have a generalized hexagon, are:

$$j_1, \ldots, j_h = \begin{cases} 11111 \\ 1112 \\ 122 \\ 113 \\ 23 \\ 14 \end{cases}$$

We check the condition (C) for each of these sequences individually, working from bottom to top.

14. The membership $0 \in Q_1^{(a)\#} Q_4^{(b)\#}$ for $a \neq b$ contradicts the geometrically evident fact that any two points of the ovoid in (V, f) are nonperpendicular.

23. If $0 \in Q_2^{(a)\#} Q_3^{(b)\#}$ then, without loss of generality, $a = \infty$, so, computing $\bmod Z$, we have, for $b \neq a$:

$$(\alpha,\ \alpha b + x,\ \alpha b^{q+q^2} + x^q b^{q^2} + x^{q^2} b^q,\ T(xb^{q+q^2})) \in \{(0, 0, c, \delta) \mid \delta \in k; c \in \ell\},$$

leading to the contradiction $\alpha = x = 0$. A geometric way of expressing the relation $\bmod Z$ is that the incidence relation between the ovoid and the spread is a bijective correspondence.

113. Suppose $0 \in Q_1^{(a)\#} Q_1^{(b)\#} Q_3^{(c)\#}$ with a, b, c distinct (so that the points $Q_1^{(a)}Z/Z$ and $Q_1^{(b)}Z/Z$ of the ovoid do not lie in the spread element $Q_3^{(c)}/Z$). Looking $\bmod Z$, this implies the existence of a 2-space π in (V, f) having two points in the ovoid and a third point in a member of the spread. Thus π must be a singular space, contradicting the fact that the two points of π from the ovoid are nonperpendicular.

122. Suppose a, b, c are distinct and $0 \in Q_1^{(a)\#} Q_2^{(b)\#} Q_2^{(c)\#}$. By 3-transitivity of S, we may take $a = 1$, $b = 0$, and $c = \infty$. Thus there are $\delta_3 \in k$, $c_3 \in \ell$, with $\delta_3 \neq 0$ or $c_3 \neq 0$, and $\alpha_3 \in k - \{0\}$ and $\alpha_2 \in k$ and $b_2 \in \ell$, not both 0, such that

$$0 = (\alpha_1, \alpha_1, \alpha_1, \alpha_1; \alpha_1^2)(\alpha_2, b_2, 0, 0; 0)(0, 0, c_3, \delta_3; 0).$$

Computing $\bmod Z$, we see that $\alpha_1 = \alpha_2 = b_2 = \delta_3 = c_3$. This yields

$$\begin{aligned} 0 &= (\alpha_1, \alpha_1, \alpha_1, \alpha_1; \alpha_1^2)(\alpha_1, \alpha_1, 0, 0; 0)(0, 0, \alpha_1, \alpha_1; 0) \\ &= (0, 0, 0, 0; \alpha_1^2), \end{aligned}$$

which contradicts $c_3 \neq 0$.

1112. Suppose again $0 \in Q_1^{(a)\#} Q_1^{(b)\#} Q_1^{(c)\#} Q_2^{(d)\#}$. Without loss of generality we take $d = \infty$ and $a = 0$. Now there are four cases to distinguish: according to $bc = \infty 0$, ∞c (with $c \neq 0$), 10, $1c$ (with $c \neq 0, 1, \infty$). We treat them separately.

$abcd = 0\infty 0\infty$.

The above condition leads to the equation

$$0 = (\alpha_1, 0, 0, 0; 0)(0, 0, 0, \alpha_2; 0)(\alpha_3, 0, 0, 0; 0)(0, 0, c_4, \delta_4; 0).$$

Now, mod Z considerations give $c=0$, $\alpha_1=\alpha_3$, and $\alpha_2=\delta_4$. Thus, the right-hand side becomes $(\alpha_1, 0, 0, \alpha_2; \alpha_1\alpha_2)^2=(0, 0, 0, 0; \alpha_1\alpha_2)$. Equating this to the left-hand side, we see that at least one of α_1, α_2 must be zero, and we are done.

$abcd=0\infty1\infty$.

The equation mod Z now reads:

$$0=(\alpha_1, 0, 0, 0)(0, 0, 0, \alpha_2)(\alpha_3, \alpha_3, \alpha_3, \alpha_3)(0, 0, c_4, \delta_4).$$

This gives mod Z that $\alpha_3=0$, done.

$abcd=010\infty$.

The equation mod Z reads

$$0=(\alpha_1, 0, 0, 0)(\alpha_2, \alpha_2, \alpha_2, \alpha_2)(\alpha_3, 0, 0, 0)(0, 0, c_4, \delta_4),$$

giving $\alpha_2=0$ from the third coordinate, done.

$abcd=01t\infty$.

Now

$$0=(\alpha_1, 0, 0, 0; 0)(\alpha_2, \alpha_2, \alpha_2, \alpha_2; \alpha_2^2)$$
$$(\alpha_3, \alpha_3 t, \alpha_3 t^{q+q^2}, \alpha_3 t^{1+q+q^2}; \alpha_3^2 N(t))(0, 0, c_4, \delta_4; 0).$$

The first four coordinates give

$$\alpha_3=\alpha_1+\alpha_2,$$
$$t=\alpha_2/\alpha_3,$$
$$c_4=\alpha_2+\alpha_3 t^{q+q^2}=\alpha_2+\alpha_2^2/\alpha_3,$$
$$\delta_4=\alpha_2+\alpha_3 N(t)=\alpha_2+\alpha_2^3/\alpha_3^2.$$

Now the Z coordinate of the above product can be computed to be:

$$\alpha_2^2+\alpha_3^2N(t)+\alpha_1\alpha_2+\alpha_3\delta_4+T(\alpha_3 t c_4)+\alpha_1\alpha_2+\alpha_2^2+\alpha_2^2$$
$$=\alpha_3\alpha_2+\alpha_2^3/\alpha_3=\alpha_2\frac{\alpha_3^2+\alpha_2^2}{\alpha_3}=\frac{\alpha_2\alpha_1^2}{\alpha_3}.$$

11111. Finally, suppose, $0\in Q_1^{(a)\#}Q_1^{(b)\#}Q_1^{(c)\#}Q_1^{(s)\#}Q_1^{(t)\#}$. Since there must be at least three different values of $a, b, c, \ldots$ involved, we may assume, without loss of generality, $a=0$, $b=1$, and $c=\infty$. Of course, $s\neq t\neq 0$. Writing out the membership assumption on 0, we have

$$0=(\alpha_1, 0, 0, 0; 0)(\alpha_2, \alpha_2, \alpha_2, \alpha_2; \alpha_2^2)(0, 0, 0, \alpha_3; 0)$$
$$(\alpha_4, \alpha_4 s, \alpha_4 s^{q+q^2}, \alpha_4 N(s); \alpha_4^2 N(s))(\alpha_5, \alpha_5 t, \alpha_5 t^{q+q^2}, \alpha_5 N(t); \alpha_5^2 N(t))$$

for certain $\alpha_1, \ldots, \alpha_5 \in k$. Now $(\alpha_1, 0, 0, 0; 0)(\alpha_2, \alpha_2, \alpha_2, \alpha_2; \alpha_2^2)=(\alpha_1+\alpha_2, \alpha_2, \alpha_2, \alpha_2; \alpha_1\alpha_2+\alpha_2^2)$, so the product of the first three is

$$(\alpha_1+\alpha_2, \alpha_2, \alpha_2, \alpha_2+\alpha_3; \alpha_1\alpha_3+\alpha_2\alpha_3+\alpha_1\alpha_2+\alpha_2^2).$$

Moreover, the product of the fourth and the fifth is:

$$(\alpha_4 + \alpha_5,\ \alpha_4 s + \alpha_5 t,\ \alpha_4 N(s) + \alpha_5 N(t),\ \alpha_4 s^{1+q+q^2} + \alpha_5 t^{1+q+q^2};$$
$$\alpha_4\alpha_5 N(t) + T(\alpha_4\alpha_5 s t^{q+q^2}) + \alpha_4^2 N(s) + \alpha_5^2 N(t)).$$

Thus the product of all five is

$$(\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5,\ \alpha_2 + \alpha_4 s + \alpha_5 t,$$
$$\alpha_2 + \alpha_4 s^{q+q^2} + \alpha_5 t^{q+q^2},\ \alpha_2 + \alpha_3 + \alpha_4 N(s) + \alpha_5 N(t);$$
$$(\alpha_1 + \alpha_2)(\alpha_4 N(s) + \alpha_5 N(t)) + T(\alpha_2(\alpha_4 s^{q+q^2} + \alpha_5 t^{q+q^2}))$$
$$+ \alpha_1\alpha_3 + \alpha_2\alpha_3 + \alpha_1\alpha_2 + \alpha_2^2 + \alpha_4\alpha_5 N(t) + T(\alpha_4\alpha_5 s t^{q+q^2})$$
$$+ \alpha_4^2 N(s) + \alpha_5^2 N(t)).$$

Substituting the first four into the Z component we obtain:

$$(\alpha_1 + \alpha_2)(\alpha_2 + \alpha_3) + T(\alpha_2^2) + \alpha_1\alpha_3 + \alpha_2\alpha_3 + \alpha_1\alpha_2 + \alpha_2^2$$
$$+ \alpha_4\alpha_5 N(t) + T(\alpha_4\alpha_5 s t^{q+q^2}) + \alpha_4^2 N(s) + \alpha_5^2 N(t))$$
$$= \alpha_2^2 + \alpha_4\alpha_5 N(t) + T(\alpha_4\alpha_5 s t^{q+q^2}) + \alpha_4^2 N(s) + \alpha_5^2 N(t).$$

Multiplying the second and third component of the product element and subtracting the result of multiplying their sum by α_2, we obtain

$$\alpha_2^2 + \alpha_4^2 N(s) + \alpha_5^2 N(t) + \alpha_4\alpha_5(s t^{q+q^2} + s^{q+q^2} t).$$

This should be zero, so it may be added to the Z component of the product, yielding:

$$\alpha_4\alpha_5 N(t) + T(\alpha_4\alpha_5 s t^{q+q^2}) + \alpha_4\alpha_5(s t^{q+q^2} + s^{q+q^2} t).$$

Since $\alpha_4\alpha_5 \neq 0$, equating the Z component to zero, gives

$$N(t) + T(s t^{q+q^2}) + s t^{q+q^2} + s^{q+q^2} t = 0,$$

which, upon dividing by $N(t)$ (recall that $t \neq 0$ by assumption), can be rewritten as

$$1 + T(s/t) + s/t + (s/t)^{q+q^2} = 0.$$

Writing out the trace, we obtain

$$1 + (s/t)^q + (s/t)^{q^2} + (s/t)^{q+q^2} = 0,$$

which factors as

$$(1 + (s/t)^q)(1 + (s/t)^{q^2}) = 0.$$

All solutions of this equation have $s = t$, contradicting the above assumptions.

Thus, conditions (O) and (C) hold for $\{Q_j^{(i)}\}_{i,j}$, so the theorem follows from the proposition. □

Remarks. The generalized hexagons of the theorem are the well-known ones related to ${}^3D_4(q)$ (q a power of 2).

The $G_2(q)$ generalized hexagons are obtained by replacing ℓ with k throughout.

The first three cases in the proof of the theorem all follow from the orthogonal geometry (V, f) and properties of spreads and ovoids. Thus the construction of a generalized hexagon from the same group Q with possibly distinct ovoids and spreads impinges on the existence of a lift from V to Q such that condition (C) holds for the subscript sequences 122, 1112, 11111.

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