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# The Distance-regular Antipodal Covers of Classical Distance-regular Graphs

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## 1. Introduction.

If $\Gamma$ is a graph and $\gamma$ is a vertex of $\Gamma$, then let us write $\Gamma_i(\gamma)$ for the set of all vertices of $\Gamma$ at distance $i$ from $\gamma$, and $\Gamma(\gamma) = \Gamma_1(\gamma)$for the set of all neighbours of $\gamma$ in $\Gamma$. We shall also write $\gamma \sim \delta$ to denote that $\gamma$ and $\delta$ are adjacent, and $\gamma^\perp$ for the set $\{\gamma\} \cup \Gamma(\gamma)$ of $\gamma$ and its all neighbours. $\Gamma_i$ will denote the graph with the same vertices as $\Gamma$, where two vertices are adjacent when they have distance $i$ in $\Gamma$.

The graph $\Gamma$ is called *distance-regular* with diameter $d$ and *intersection array* $\{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\}$ if for any two vertices $\gamma, \delta$ at distance $i$ we have $|\Gamma_{i+1}(\gamma) \cap \cap \Gamma(\delta)| = b_i$ and $|\Gamma_{i-1}(\gamma) \cap \Gamma(\delta)| = c_i (0 \leq i \leq d)$. Clearly $b_d = c_0 = 0$ (and $c_1 = 1$). Also, a distance-regular graph $\Gamma$ is regular of degree $k = b_0$, and if we put $a_i = k - b_i - c_i$ then $|\Gamma_i(\gamma) \cap \Gamma(\delta)| = a_i$ whenever $d(\gamma, \delta) = i$. We shall also use the notations $k_i = |\Gamma_i(\gamma)|$ (this is independent of the vertex $\gamma$), $\lambda = a_1, \mu = c_2$. For basic properties of distance-regular graphs, see Biggs [4].

The graph $\Gamma$ is called *imprimitive* when for some $I \subseteq \{0, 1, \ldots, d\}, I \neq \{0\}, I \neq \neq \{0, 1, \ldots, d\}$, having distance in $I$ is an equivalence relation, or, equivalently, when for some $i, 1 \leq i \leq d$, the graph $\Gamma_i$ is disconnected.

Two obvious types of imprimitive graphs are *bipartite* graphs of diameter $d \geq 2$ (here $\Gamma_2$ is disconnected) and the *antipodal* graphs (graphs such that $\Gamma_d$ is an equivalence relation) of diameter $d \geq 2$. Smith [31] showed that an imprimitive distance-regular graph of valency $k > 2$ is bipartite or antipodal (or both). (He stated his result for distance-transitive graphs, but his proof is easily extended to arbitrary distance-regular graphs. Of course the distance-regular graphs of valency $k = 2$ are just the polygons.)

This result may be rephrased by saying that if $k > 2$ and having distance in $I$ is an equivalence relation, then $I = \{0\}$, or $I = \{0, d\}$, or $I = \{0, 2, 4, \ldots\}$, or $I = \{0, 1, \ldots, d\}$.

When $\Gamma$ is connected and bipartite of diameter $d \geq 2$, then $\Gamma_2$ has two components, and the graphs induced on these components by $\Gamma_2$ are called the *halved* graphs of $\Gamma$. Clearly, the halved graphs of $\Gamma$ have diameter $[d/2]$

When $\Gamma$ is antipodal then we can define a new graph, the *folded* graph of $\Gamma$, which has the equivalence classes of $\Gamma_d$ as vertices, and where two equivalence classes are adjacent whenever they contain adjacent vertices. Clearly, the folded graph of $\Gamma$ has diameter $[d/2]$.

Given an arbitrary distance-regular graph (not a polygon), we may obtain a primitive distance-regular graph after halving at most once and folding at most once. Thus, the big project of classifying all distance-regular graphs naturally goes in two stages: first all primitive distance-regular graphs, and next, given a distance-regular graph $\Gamma$, find all the distance-regular graphs $\Delta$ such that $\Gamma$ is a halved graph or the folded graph of $\Delta$. For the current state of affairs concerning the first stage, cf. Bannai and Ito [2,3]. The question of what distance-regular $\Gamma$ are halved graphs has been attacked by Hemmeter [21,22]. In this note we address the question what distance-regular graphs have distance-regular antipodal covers.

## 2. Geometry of antipodal covering graphs.

A map $\pi : \Delta \rightarrow \Gamma$, where $\Gamma$ and $\Delta$ are graphs, is called a *covering* when $\pi$ is a graph morphism (i.e., $\pi$ maps points to points and edges to edges), and for each vertex $\delta$ of $\Delta$ the restriction of $\pi$ to the set $\delta^{\perp}$ is injective. If $\Delta$ is moreover distance-regular of diameter $D$, then $\pi$ is called an *antipodal covering* when two vertices of $\Delta$ have the same image under $\pi$ if and only if they have distance $D$. We call $\Delta$ a cover (antipodal cover) of $\Gamma$. Although every graph is a cover of itself (with $\pi = id.$), no graph on more than one point is an antipodal cover of itself. When $|\pi^{-1}(\gamma)| = r$ for all vertices $\gamma$ of $\Gamma$, then $\Delta$ is also called an $r$-cover of $\Gamma$.

A *geodesic* in a graph $\Gamma$ is a path $\gamma_0, \gamma_1, \ldots, \gamma_t$, where $\gamma_{i-1} \sim \gamma_i$ $(1 \leq i \leq t)$ and $d(\gamma_0, \gamma_t) = t$. For two vertices $\alpha, \beta$ of $\Gamma$ we let $C(\alpha, \beta)$ denote the union of all geodesics between $\alpha$ and $\beta$ in $\Gamma$.

The following two propositions give strong necessary conditions for the existence of antipodal covers of distance-regular graphs. Indeed, they will be the major tools in the rest of this paper. (Recall that if $\Gamma$ has diameter $d$, and $\Delta$ is a distance-regular antipodal cover of $\Gamma$, then $\Delta$ has diameter $D = 2d$ or $D = 2d + 1$.)

**Proposition 2.1.** *Suppose that $\Gamma$ is distance-regular of diameter $d \geq 2$ and has a distance-regular antipodal $r$-cover of diameter $2d$. Then for any two vertices $\gamma, \delta$*

*of* $\Gamma$ *with* $d(\gamma,\delta) = d$, *the subgraph* $C(\gamma,\delta)\setminus\{\gamma,\delta\} = \bigcup_{j=1}^{d-1}\Gamma_j(\gamma)\cap\Gamma_{d-j}(\delta)$ *of* $\Gamma$ *is the disjoint union of* $r$ *subgraphs of equal size.*

**Proof.** Let $\Delta$ be a distance-regular antipodal $r$–cover of diameter $2d$ of $\Gamma$, with covering map $\pi : \Delta \to \Gamma$. Let $\gamma_1 \in \pi^{-1}(\gamma)$, and let $\pi^{-1}(\delta) = \{\delta_1,\ldots,\delta_r\}$. Let $C_j$ be the union of all geodesics in $\Delta$ between $\gamma_1$ and $\delta_j$ $(1 \le j \le r)$ and $C = \cup_j C_j \setminus \{\gamma_1,\delta_j\}$. Then $\pi|_C : C \to C(\gamma,\delta)$ is an isomorphism. ∎

**Proposition 2.2.** *Suppose that* $\Gamma$ *is distance-regular of diameter* $d \ge 2$ *and has a distance-regular antipodal* $r$*–cover of diameter* $2d+1$. *Let* $\gamma,\delta$ *be vertices of* $\Gamma$ *with* $d(\gamma,\delta) = d$, *and put* $E = \{\delta\}\cup(\Gamma(\delta)\cap\Gamma_d(\gamma))$. *Then the collection of sets* $C(\gamma,\varepsilon)\setminus\{\gamma,\varepsilon\}(\varepsilon\in E)$ *can be partitioned into* $r$ *nonempty parts such that sets from different parts are disjoint, and all edges joining vertices from sets in different parts are contained in* $\Gamma(\gamma)$. ∎

An immediate corollary of this last proposition is that the dual polar graphs, and, more generally, the collinearity graphs of regular near $2d$–gons (for definitions, see Shult and Yanushka [30]), do not have antipodal distance-regular covers of odd diameter.

**Corollary 2.3.** *If* $d \ge 2$, *and any two adjacent vertices* $\delta,\varepsilon$ *in* $\Gamma_d(\gamma)$ *have a common neighbour in* $\Gamma_{d-1}(\gamma)$, *then* $\Gamma$ *does not have distance-regular antipodal covers of diameter* $2d+1$. *In particular, this holds for the collinearity graph of a regular near* $2d$*–gon.*

**Proof.** Clearly, if $\Gamma(\delta)\cap\Gamma(\varepsilon)\cap\Gamma_{d-1}(\gamma)\ne\emptyset$, then $C(\gamma,\delta)\setminus\{\gamma,\delta)\}$ and $C(\gamma,\varepsilon)\setminus\{\gamma,\varepsilon)\}$ are not disjoint. But if $\Gamma$ is a near $2d$–gon, and $\delta\varepsilon$ in an edge in $\Gamma_\delta(\gamma)$, then by definition of regular near $2d$–gon, the line containing $\delta\varepsilon$ has a (unique) point in $\Gamma_{d-1}(\gamma)$. ∎

**Corollary 2.4.** *If* $d \ge 2$ *and for any two adjacent vertices* $\delta,\varepsilon$ *in* $\Gamma_d(\gamma)$ *there is a point in* $\Gamma(\gamma)\cap\Gamma_{d-1}(\delta)\cap\Gamma_{d-1}(\varepsilon)$, *then* $\Gamma$ *does not have distance-regular antipodal covers of diameter* $2d+1$. *Similarly, if* $d \ge 3$ *and if* $d(\gamma,\delta) = d$ *then for any two vertices* $\alpha,\beta\in\Gamma(\delta)\cap\Gamma_{d-1}(\gamma)$ *there is a vertex* $\varepsilon\in\Gamma(\gamma)\cap\Gamma_{d-2}(\alpha)\cap\Gamma_{d-2}(\beta)$, *then* $\Gamma$ *does not have distance-regular antipodal covers of diameter* $2d$. ∎

Trivially, bipartite graphs do not have distance-regular antipodal covers of odd diameter. As was already remarked in the introduction, antipodal graphs of diameter $d \ge 3$ do not have any distance-regular antipodal covers.

## 3. Parameter conditions for distance-regular antipodal covers.

Let $\Gamma$ be distance-regular of diameter $d$ and with parameters $a_i, b_i, c_i$, and $\Delta$ a distance-regular antipodal $r$–cover of $\Gamma$ of diameter $D > 2$ and with parameters $A_i, B_i, C_i$. (We shall use these notations everywhere below, and also such notations as $K = B_0, \Lambda = A_1, M = C_2$.) Then $r = 1 + k_D = 1 + B_d/C_{D-d}$ and we have $b_i = B_i, c_i = C_i (0 \le i \le d-1), B_i = C_{D-i} (0 \le i \le D, i \neq d), c_d = rC_d$ if $D = 2d$, and $c_d = C_d$ if $D = 2d+1$. (Gardiner [19].)

In other words, suppose that $\Gamma$ has intersection array $\iota(\Gamma) = \{b_0, \dots, b_{d-1}; c_1, \dots, c_d\}$. If $\Delta$ has a diameter $2d > 2$, then

$$\iota(\Delta) = \{b_0, b_1, \dots, b_{d-1}, \frac{r-1}{r} c_d, c_{d-1}, \dots, c_1; c_1, \dots, c_{d-1}, \frac{1}{r} c_d, b_{d-1}, \dots, b_0\}$$

(so that $r | c_d$ and $r \le c_d / \max(c_{d-1}, c_d - b_{d-1})$) and if $\Delta$ has a diameter $2d+1$, then

$$\iota(\Delta) = \{b_0, b_1, \dots, b_{d-1}, t(r-1), c_d, c_{d-1}, \dots, c_1; c_1, \dots, c_{d-1}, c_d, t, b_{d-1}, \dots, b_0\}$$

for some integer $t$, where $t(r-1) \le \min(b_{d-1}, a_d)$ and $c_d \le t$. Clearly, given $\iota(\Gamma)$, there are only finitely many possibilities for $r$ and $t$, and if $c_d > \min(b_{d-1}, a_d)$, there are none. Sometimes one uses the notations $r.\Gamma$ and $(r.\Gamma)_t$ for such graphs (intersection arrays).

From the above it follows that if $c_d > 2b_{d-1}$ then the graph has no covers. We shall often want to apply this for the complement of a given strongly regular graph. (A connected graph is called strongly regular when it is distance regular of diameter 2. The complement of strongly regular graph is either disconnected or again strongly regular.) So, let us translate this condition for this situation.

**Lemma 3.1.** *Let $\Gamma$ be strongly regular with intersection array $\{k, b_1; 1, c_2\}$. Then its complementary graphs $\overline{\Gamma}$ does not have distance regular antipodal covers when $b_1 > 2c_2$.* ∎

Concerning the spectrum we can say the following. The eigenvectors of $\Gamma$ yield in the obvious way eigenvectors of $\Delta$ that are constant on the fibers $\pi^{-1}(\gamma)$, so that the eigenvalues of $\Gamma$ are also eigenvalues of $\Delta$, and it is easy to see that their multiplicity in $\Delta$ will be the same as in $\Gamma$. If $\Gamma$ has eigenvalues $\theta_0 \ge \theta_1 \ge \dots \ge \theta_d$ and $\Delta$ has eigenvalues $\Theta_0 \ge \Theta_1 \ge \dots \ge \Theta_D$, then $\Theta_0 = \theta_0, \Theta_2 = \theta_1, \dots, \Theta_{2d} = \theta_d$, i.e. the eigenvalues of $\Gamma$ interlace the "new" eigenvalues of $\Delta$.

**Proposition 3.2.** *If $D = 2d$, then the $d$ eigenvalues of $\Delta$ that are not eigenvalues of $\Gamma$ are the eingenvalues of the $d \times d$ matrix*

$$\begin{pmatrix} 0 & k & & & \\ 1 & \lambda & b_1 & & \bigcirc \\ & \mu & \ddots & \ddots & \\ & & \ddots & & \\ & \bigcirc & & \ddots & b_{d-2} \\ & & & c_{d-1} & a_{d-1} \end{pmatrix}.$$

*Thus, these eigenvalues do not depend on $r$, and their multiplicity is proportional to $r-1$. If $D = 2d+1$, then the $d+1$ eigenvalues of $\Delta$ (with parameters $(r.\Gamma)_t$) that are not eigenvalues of $\Gamma$ are the eigenvalues of the $(d+1) \times (d+1)$ matrix*

$$\begin{pmatrix} 0 & k & & & \\ 1 & \lambda & b_1 & & \bigcirc \\ & \mu & \ddots & \ddots & \\ & & \ddots & & \\ & \bigcirc & & a_{d-1} & b_{d-1} \\ & & & c_d & a_d - rt \end{pmatrix}.$$

*Thus, these eigenvalues depend only on $rt$.*

**Corollary 3.3.** *In particular, in case $d = 1, D = 3$, the two new eigenvalues $\theta$ are the two roots of $\theta^2 - (k-1-r\mu)\theta - k = 0$, (with $\mu = \mu(\Delta)$), and have multiplicity $m(\theta) = \dfrac{k(k+1)(r-1)}{k+\theta^2}$. Thus, either $k-1 = r\mu$ and $k\mu$ is even, or $\theta$ is an integer, and $(r-1)k\mu$ is even.*

**Proof.** The parity restrictions follow by counting edges in $\Delta(\infty)$ for some vertex $\infty$ of $\Delta$. ■

**Corollary 3.4.** *In particular, in case $d = 2, D = 4$, the two new eigenvalues $\theta$ are the two roots of $\theta^2 - \lambda\theta - k = 0$, and occur with multiplicity $m(\theta) = \dfrac{(r-1)v}{2+\lambda\theta/k}$. Consequently, either $\lambda = 0$, or $\lambda^2 + 4k$ is a square (and these eigenvalues are integral).* ■

All of the above is already contained in Biggs and Gardiner [6].

## Covers of the classical graphs.

In the sections below we determine almost all distance-regular antipodal covering graphs of classical distance-regular graphs of diameter $d \geq 2$, and of some related graphs. (Here "classical" is defined by what appears below.) Specifically, we treat a) Johnson graphs, b) Hamming graphs, c) Grassmann graphs, d) Dual polar graphs, e) Bilinear forms graphs f) Alternating forms graphs, g) Hermitean forms graphs (but only for $d \geq 3$), h) Quadratic forms graphs, i) $E_7$ graphs, j) Affine $E_6$ graphs, and k) Witt graphs. The result is that no such cover exist, except in those cases where covers were known already, and in these cases the covers turn out to be unique. (Only for the Witt graphs a few open cases remain; see §14.) For the definitions and properties of all graphs involved, see the (hopefully soon to be published) book Brouwer, Cohen and Neumaier [9]. Note that we shall write "cover" instead of "distance-regular antipodal cover".

## 4. Johnson graphs and related graphs.

The *Johnson graphs* $J(n,k)$ (also known as $\binom{n}{k}$) has as vertex set the set of $k$-subsets of a fixed $n$-set, where two $k$-sets are adjacent when they meet in a $(k-1)$-set (or, equivalently, when their union is a $(k+1)$-set). The diameter of this graph is $d = \min(k, n-k)$. For $d \geq 2$, no covers exist, as follows immediately from §2.

The *Odd graph* $O_{d+1}$ is the distance-$d$-graph of the Johnson graph $J(2d+1, d)$ (i.e., vertices are $d$-sets in a fixed $(2d+1)$-set, adjacent when they are disjoint). For $d = 2$, the Odd graph $O_3$ is better known as the Petersen graph.

Given a graph $\Gamma$ with vertex set $X$, its *bipartite double* is the graph with vertex set $X \times \{0,1\}$ where $(x,i) \sim (y,j)$ if and only if $x \sim y$ and $i \neq j$. The bipartite double of a graph is bipartite, and it is connected precisely when $\Gamma$ is connected and non-bipartite. The bipartite double of the Odd graph $O_{d+1}$ is again distance-regular; it may be described as the graph with as vertices the subsets of sizes $d$ and $d+1$ in a $(2d+1)$-set, with as adjacency (symmetrized) proper inclusion.

**Proposition 4.1.** (cf. Ivanov [26]) *The Odd graph $O_{d+1}(d \geq 3)$ has a unique distance-regular antipodal cover, namely its bipartite double. The Odd graph $O_3$ has two nonisomorphic distance-regular antipodal covers, namely its bipartite double (sometimes called the Desargues graph) and the 1-skeleton of the dodecahedron.*

**Proof.** The standard conditions on $t$ and $r$ for covers $r.O_m$ or $(r.O_m)$ (cf. §3), immediately yield that we have one of the possibilities mentioned, or $(3.O_3)_1$ or $2.O_4$ or $(2.O_{2t+1})_t$ with $t \geq 2$. The last possibility is ruled out by the inequality

$$\text{if} \quad a_i \neq 0 \quad \text{then} \quad b_i + c_i \leq a_i + \frac{a_{i+1}b_i}{a_i} + \frac{a_{i-1}c_i}{a_i}$$

(see Brouwer and Lambeck [10], or Faradjev, Ivanov and Ivanov [18], Prop. 3.3) applied for $i = t$, or by the geometric condition of Proposition 2.2; $(3.O_3)_1$ has nonintegral multiplicities, and $2.O_4$ is ruled out by detailed explicit investigation along the lines of Smith [32], who did the distance-transitive case. ∎

The Johnson graphs $J(2k,k)$ are antipodal, and their quotients are called the *folded Johnson graphs* $FJ(2k,k)$. For $k \geq 4$ the Johnson graphs $J(2k,k)$ are the only covers of the folded Johnson graphs $FJ(2k,k)$. (Indeed, this follows from the local characterization of the Johnson graphs, cf.Blokhuis and Brouwer [7].)

The Johnson graphs $J(2k,k)$ (also known as the *triangular graphs* $T(n)$) and the folded Johnson graphs $FJ(8,4)$ and $FJ(10,5)$ are strongly regular, and their complementary graphs are also strongly regular. The graph $\overline{T(5)}$, the complement of $T(5)$, is the Petersen graph, and was treated above.

**Proposition 4.2.** *The graphs $\overline{T(n)}$ have no distance-regular antipodal covers for $n > 7$. $\overline{T(6)}$ and $\overline{T(7)}$ have unique distance-regular antipodal covers; these are 3–covers of diameter 4 on 45 resp. 63 vertices.*

**Proof.** Concerning covers of diameter 4, for $\gamma \not\sim \delta$ we have $C(\gamma,\delta) \setminus \{\gamma,\delta\} \cong \overline{T(n-3)}$, which is connected for $n = 5$ and $n \geq 8$. The graphs $\overline{T(3)}$ and $\overline{T(4)}$ are isomorphic to $3K_1$ and $3K_2$, respectively, and indeed there exist unique 3–covers of $T(6)$ and $T(7)$, cf. Smith [33], Hall [20] and Ito [25].

Concerning covers of diameter 5, if $\delta\varepsilon$ is an edge in $\Gamma_2(\gamma)$ then $\gamma^\perp \cap \delta^\perp \cap \varepsilon^\perp \cong \overline{T(n-4)} \neq \emptyset$ for $n \geq 6$. Thus, such covers are possible only for the Petersen graph, and those we have seen already. ∎

The complement of the folded Johnson graph $FJ(8,4)$ is the Grassmann graph of the lines in $PG(3,2)$, and we shall see below that there are no covers. Finally, from the parameters of $\overline{FJ(10,5)}$ one immediately sees that it has no covers. (Indeed, it has $84 = c_d > 2b_{d-1} = 42$.)

## 5. The Hamming graphs.

The *Hamming graph* $H(n,q)$ has as vertex set the collection of all $n$–tuples with entries in a fixed $q$–set, where two $n$–tuples are adjacent when they differ in only one coordinate. This graph has diameter $d = n$ (for $q > 1$).

**Proposition 5.1.** *Let $n,q \geq 2$. Then $H(n,q)$ has no distance-regular antipodal covers, except for $H(2,2)$, the quadrangle, which is covered by the octagon.*

**Proof.** Terwilliger [35]'s bound $d \leq (k + c_d)/(\lambda + 2)$ on the diameter of the distance-regular graphs containing a quadrangle holds with equality for $H(n,q)$ (since $d = n$, $k = d(q-1)$, $c_d = d$, $\lambda = q-2$). For a cover we find $D \leq (K + C_D)/(\Lambda + 2) = 2d(q-1)/q < 2d$, so covers cannot contain quadrangles, and necessarily $D = 4, d = 2$. Concerning covers of diameter 4 of the "lattice graph" $H(2,q)$, we see that lines ($q$–cliques) lift to lines in the cover, and for $q > 2$ one quickly arrives at a contradiction. ∎

For $q = 2$, the Hamming graph $H(n,2)$ is antipodal (it is the $n$–cube); the folded graph is usually denoted by $\square_n$.

**Proposition 5.2.** *The only distance-regular antipodal cover of diameter $D \geq 3$ of the folded n–cube $\square_n$ is the n–cube, except for $n = 4, 5$ where unique covers with intersection arrays $\{4,3,3,1;1,1,3,4\}$ and $\{5,4,1,1;1,1,4,5\}$ exist.*

**Proof.** Neumaier [27] proves that a graph with $\lambda = 0$, $\mu = 2$ and valency $k$ has at most $2^k$ vertices, with equality only for the $k$–cube. This shows that for $D \geq 5$ the only covers are $D$–cubes. If $D = 4, d = 2$, then $n \in \{4,5\}$, and if $M \neq 2$, then $M = 1$ and we find the two arrays mentioned above. It is not difficult to show that there are unique graphs corresponding to these arrays. (They are known as $4.K_{4,4}$ and Wells' graph, respectively.) Finally, for $D = 3, d = 1$ we see that no feasible parameter sets exist different from that of the 3–cube. ■

For $q = 2$, the Hamming graph $H(n,2)$ is also bipartite; let us denote the halved graph by $\frac{1}{2}H(n,2)$. If $n$ is even, then $\frac{1}{2}H(n,2)$ is still antipodal; let us denote the folded halved (or halved folded) $n$–cube by $\frac{1}{2}\square_n$.

**Proposition 5.3.** *For $n \geq 4$, the halved n–cube $\frac{1}{2}H(n,2)$ does not have distance-regular antipodal covers. For (even) $n \geq 8$, the only distance-regular antipodal cover of $\frac{1}{2}\square_n$ is $\frac{1}{2}H(n,2)$.*

**Proof.** For even $n$, $\frac{1}{2}H(n,2)$ has diameter $d = n/2$ and $c_d = \binom{n}{2}, b_{d-1} = 1$, so no covers exist. For odd $n$, this graph has diameter $d = [n/2]$ and $c_d = \binom{n-1}{2}, b_{d-1} =$ $= 3$, so for $n > 5$ no covers exist. But covers of $\frac{1}{2}H(5,2)$ must have diameter 4, and are ruled out by Corollary 3.4.

Concerning the folded halved $n$–cubes $\frac{1}{2}\square_n (n \geq 8)$, these are locally $T(n)$. By Neumaier [28] (Propositions 2(ii) and 3) any connected locally triangular graph is the halved graph of a bipartite graph with $\mu = 2$ (indeed, for $n > 4$ the missing points are uniquely found as the maximal cliques). It follows that any cover of a folded halved $n$–cube gives rise to a cover of the folded $n$–cube, but since $n \geq 8$, such a cover must be the $n$–cube. ■

The Hamming graph $H(2,q)$ and the folded cubes $\square_4$ and $\square_5$, the halved Hamming graphs $\frac{1}{2}H(4,2)$ and $\frac{1}{2}H(5,2)$, and the halved folded cubes $\frac{1}{2}\square_8$ and $\frac{1}{2}\square_{10}$ are strongly regular, and we have to consider possible covers of their complements. Now $\overline{H(2,2)}$ and $\overline{\square}_4$ and $\overline{\frac{1}{2}H(4,2)}$ are disconnected, and fall off. Next, $\overline{H(2,3)} \cong H(2,3)$ and $\overline{\square}_5 \cong \frac{1}{2}H(5,2)$ and $\overline{\frac{1}{2}H(5,2)} \cong \square_5$, so these have been considered already. Next, $\overline{H(2,q)}$ with $q > 3$ has $c_d > 2b_{d-1}$ and hence has no covers. The complement of the folded halved 8–cube is the alternating forms graph on $\mathbf{F}_2^4$, and will be considered below (it has no covers). Finally, $\overline{\frac{1}{2}\square_{10}}$ has $c_2 > 2b_1$ and hence has no covers.

There are a few more graphs related to the Hamming graphs. First of all, there are the *Doob graphs* (cf. Doob [16]), direct products of a number of $K_4$'s and Shrikhande graphs. These have the same parameters as the Hamming graphs $H(n,4)$, and hence covers are ruled out by the same argument. (For the Shrikhande graph itself, and for its complement, covers of diameter 4 are ruled out by Corollary 3.4, and covers of diameter 5 are not feasible either.)

Lastly, the folded $n$-cubes are characterized by the parameters except when $n = 6$ (cf. Terwilliger [34]). For $n = 6$ there are precisely three nonisomorphic graphs with intersection array $\{6,5,4;1,2,6\}$(cf. Hussain [24]). The argument used above shows that none of these has distance-regular antipodal covers.

## 6. Grassmann graphs.

Let V be an $n$-dimensional vector space over the finite field $\mathbf{F}_q$ with $q$ elements, $q$ a prime power. The *Grassmann graph of the e-subspaces of V* has vertex set $\left[{V \atop e}\right]$, the collection of linear subspaces of $V$ of dimension $e$. Two subspaces $Y, Z$ are adjacent whenever $\dim Y \cap Z = e - 1$.

**Proposition 6.1.** *Grassmann graphs of diameter $d \geq 2$ do not admit distance-regular antipodal covers.*

**Proof.** The Grassmann graph of the $e$-subspaces of $V$ has diameter $d =$ $= \min(e, n-e)$; indeed, more generally we have $d(Y,Z) = e - \dim(Y \cap Z)$ for $e$-subspaces $Y, Z$ of $V$. Since the Grassmann graphs $\left[{V \atop e}\right]$ and $\left[{V \atop n-e}\right]$ are isomorphic (any nondegenerate symmetric bilinear form provides an isomorphism), we may assume $n \geq 2e$, so that $d = e$. Now if $d(Y,Z) = d$, then $\Gamma(Y) \cap \Gamma_{d-1}(Z)$ is the direct product $\left[{Y \atop e-1}\right] \times \left[{Z \atop 1}\right]$ (a grid) and hence is connected, so that by Proposition 2.1 there are no covers of even diameter. Similarly, if also $d(X,Z) = d$ and $d(X,Y) = 1$, then any $e$-subspace of $V$ containing $X \cap Y$ and meeting $Z$ in a nonzero vector is a common neighbour of $X$ and $Y$ in $\Gamma_{d-1}(Z)$, so that by Corollary 2.3 there are no covers of odd diameter either. ∎

Related to the Grassmann graphs are the so-called *double Grassmann graphs.* Suppose that $V$ is a vector space of dimension $2m+1$ over $\mathbf{F}_q$. Let $\Gamma$ be the graph whose vertex set is $\left[{V \atop m}\right] \cup \left[{V \atop m+1}\right]$, and such that $Y \sim Z$ for vertices $Y, Z$ of $\Gamma$ if and only if $Y \neq Z$ and either $Y \subset Z$ or $Z \subset Y$. It is isomorphic to the bipartite double of $\Delta_m$, where $\Delta$ is the Grassmann graph $\left[{V \atop m}\right]$. Note that $d(Y,Z) = \dim(Y/Y \cap Z) + \dim(Z/Y \cap Z)$, so that $\Gamma$ has diameter $2m+1$.

**Proposition 6.2.** *The double Grassmann graphs of diameter $d > 1$ do not have distance-regular antipodal covers.*

**Proof.** Since a double Grassmann graph $\Gamma$ is bipartite, it cannot have covers of odd diameter. Assume first $m > 1$, i.e., $d > 3$. Suppose $d(X,W) = d$, say $X$ is an $m$-space and $W$ an $(m+1)$-space with $X \cap W = 0$. If $Y, Z$ are vertices in $\Gamma_{d-1}(X) \cap \Gamma(W)$ then they are $m$-spaces contained in $W$. Now for any 1-space $P$ contained in $Y \cap Z$ the vertex $U = X + P$ of $\Gamma$ lies in $\Gamma(X) \cap \Gamma_{d-2}(Y) \cap \Gamma_{d-2}(Z)$, and hence by Corollary 2.4 there are no covers of even diameter either. Remains the case $m = 1$, where we have the incidence graph of the points and lines of a projective plane. Let $\Delta$ be a distance-regular bipartite antipodal $r$-cover of a bipartite graph $\Gamma$ of diameter 3. Since $\Delta$ is bipartite, its spectrum is symmetric around 0; since it has diameter 6, its spectrum contains 7 distinct eigenvalues. It follows that 0 is an eigenvalue of $\Delta$, and hence $-k/\mu$ is an eigenvalue of $\frac{1}{2}\Delta$. Now Biggs [5] teaches us that the multiplicity of an eigenvalue $x$ different from the valency and from $-1$ in an antipodal distance-regular graph of diameter 3 equals $m = (r-1)(k+1)k/(k+x^2)$. Applying this to $\frac{1}{2}\Delta$ with $x \to -k/\mu$ and $k \to k(k-1)/\mu$, we find $m = (r-1)(k-1)(k^2-k+\mu)/(k\mu+k-\mu)$. From this more general nonexistence results can be derived, but here we are only interested in the case $\mu = 1, k = q+1$, and integrality of $m$ implies $2q+1 = 3(r-1)$. But $r$ divides $c_3 = q+1$, and it follows that $q = 1$, contradiction. So, also for $m = 1$ there are no covers. ■

Another related graph is the incidence graph of points and hyperplanes in a projective space, the bipartite graph with point set $\begin{bmatrix} V \\ 1 \end{bmatrix} \cup \begin{bmatrix} V \\ n-1 \end{bmatrix}$ (where $\dim V = n$) and symmetrized inclusion as adjacency. For $n = 3$ this is the graph just studied. For $n > 3$, $-k/\mu$ is not an (algebraic) integer and hence this graph has no covers. (For $n = 4$ this graph is the dual polar graph $[D_3(q)]$, see below.)

The Grassmann graph $\begin{bmatrix} V \\ 2 \end{bmatrix}$ of the lines in a projective space is strongly regular, but its complement does not have covers, except in the case of the lines in $PG(3,2)$, where this complementary graph is isomorphic to the folded Johnson graph $FJ(8,4)$ and has the unique cover $J(8,4)$. (Indeed, this follows immediately from Proposition 2.1 and Corollary 2.3.)

## 7. Dual polar graphs.

Let $q$ be a prime power. Let $V$ be one of the following spaces equipped with a specified form:

$[C_d(q)] = \mathbb{F}_q^{2d}$ with a nondegenerate symplectic form;

$[B_d(q)] = \mathbb{F}_q^{2d+1}$ with a nondegenerate quadratic form;

$[D_d(q)] = \mathbb{F}_q^{2d}$ with a nondegenerate quadratic form of (maximal) Witt index $d$;

$[^2D_{d+1}(q)] = \mathbb{F}_q^{2d+2}$ with a nondegenerate quadratic form of (non-maximal) Witt index $d$;

$[^2A_{2d}(r)] = \mathbb{F}_q^{2d+1}$ with a nondegenerate Hermitean form ($q = r^2$);

$[^2A_{2d-1}(r)] = \mathbb{F}_q^{2d}$ with a nondegenerate Hermitean form ($q = r^2$).

Background on the spaces and their forms can be found in Artin [1] and Dieudonné [15]. The spaces $[C_d(q)], [B_d(q)], [D_d(q)], [^2D_{d+1}(q)], [^2A_{2d}(r)], [^2A_{2d-1}(r)]$ are often named $Sp(2d,q), \Omega(2d+1,q), \Omega^+(2d,q), \Omega^-(2d+2,q), U(2d+1,r)$ and $U(2d,r)$, respectively.

A subspace of $V$ is here called *isotropic* whenever the form vanishes completely on this subspace. (In standard terminology this is called totally isotropic, or, in certain cases, totally singular.) Maximal isotropic subspaces have dimension $d$ (in other words, are $d$–spaces of $V$). The *dual polar graph (on $V$)* has as vertices the maximal isotropic subspaces; two points $\gamma, \delta$ are adjacent whenever $\dim \gamma \cap \delta = d - 1$.

**Proposition 7.1.** *Dual polar spaces of diameter $d \geq 3$ do not admit distance-regular antipodal covers. For $d = 2$ the generalized quadrangle of order $(2,2)$ has a unique distance-regular antipodal 3–cover, and many of the (thin) generalized quadrangles of order $(1,q)$ (complete bipartite graphs $K_{q+1,q+1}$) have covers, but no other covers occur.*

**Proof.** (We shall use the terminology and results of the theory of near polygons. See Shult and Yanushka [30], Cameron [12,13], Brouwer and Wilbrink [11] and Shult [29].) Since dual polar graphs are regular near polygons, it follows from Corollary 2.3 that there are no covers of odd diameter. If $\Gamma$ is a dual polar graph of diameter $d \geq 3$ and $d(\gamma, \delta) = d, \alpha, \beta \in \Gamma(\delta) \cap \Gamma_{d-1}(\gamma)$, then there is a unique quad $Q$ containing $\{\alpha, \beta, \delta\}$. Since all point-quad relations are classical, there is a unique point $\varepsilon \in \Gamma_{d-2}(\gamma) \cap Q$, and $\varepsilon \sim \alpha, \beta$. It follows from Proposition 2.1 that no covers of diameter $2d$ exist.

Dual polar graphs of diameter $d = 2$ are generalized quadrangles of order $(s,t)$ with $t = q$ and $s = q, q, 1, q^2, q^{3/2}, q^{1/2}$ in the six respective cases, By Corollary 3.4 we must have either $s = 1$, or $(s-1)^2 + 4s(t+1)$ is a square. This latter condition is of the form

$$(p^a + 1) + 4p^b = c^2$$

for certain integers $a, b, c$, when $q$ is a power of the prime $p$. It follows that $(c - p^a - 1)(c + p^a + 1) = 4p^b$ and hence $\frac{1}{2}(c \pm (p^a + 1))$ are powers of $p$. Hence $c - p^a - 1 = 2$ and $c + p^a + 1 = 2p^b, p^a + 1 = p^b - 1, p = 2, a = 1, b = 2, c = 5$. Thus our generalized quadrangle was either GQ$(1,q)$ or GQ$(2,2)$. The former is the complete bipartite graph $K_{q+1,q+1}$; the latter is $\overline{T(6)}$. We know already that the latter has a unique 3–cover. Concerning the former, its covers are precisely the incidence graphs of symmetric $(r,\mu)$–nets, with $\mu r = q + 1$, and many such graph are known. [For example, when $r = 2$ these correspond precisely to Hadamard matrices of order $q + 1$, and these exists for each prime power $q \equiv 3 (\text{mod} 4)$. For $q = 5$ a unique graph $3.K_{6,6}$ exists; for $q = 9$ there are no examples.] ∎

The dual polar graphs on $[D_n(q)]$ are bipartite, but the halved graphs of diameter $d = [n/2] \geq 2$ do not have covers. (This follows again by looking at the geometry of the near polygon.)

The six dual polar spaces with $d = 2$ and the halved dual polar spaces $\frac{1}{2}[D_4(q)]$ and $\frac{1}{2}[D_5(q)]$ are strongly regular, and we may consider the complementary graphs.

Moreover, the dual polar spaces with $d = 2$ are generalized quadrangles, and we may consider their duals. However, no covers occur.

[The complement of $GQ(1,q)$ is disconnected, and that of $GQ(q,1)$ was treated earlier. The complement of $GQ(s,t)$ with $s,t > 1$ has $c_2 > 2b_1$ except for $s = 2$ and $(s,t) = (3,2)$. But no $GQ(3,2)$ exists, $\overline{GQ(2,2)} \cong T(6)$ was treated earlier, and covers of $\overline{GQ(2,4)}$ (that is, of the Schläfli graph $E_6(1)$) are ruled out by Corollary 3.4. The dual generalized quadrangles have order $(p^a, p^b)$ with $a, b \geq 0$. If $a, b > 0$ then the above argument shows that no covers exists. Concerning $GQ(q,1)$, this is $H(2, q+1)$, and hence has no covers either. Finally, for $\frac{1}{2}[D_5(q)]$ and all $q$ and for $\frac{1}{2}[D_4(q)]$ and $q > 3$ one finds that $c_2 > 2b_1$; in the remaining cases Corollary 3.4 is violated.]

## 8. Bilinear forms graphs.

Set $V = \mathbf{F}_q^d$ and $W = \mathbf{F}_q^e$. Let $B$ be the vector space (of dimension $de$ over $\mathbf{F}_q$) of bilinear maps from $V \times W$ to $\mathbf{F}_q$. (Then $B$ is canonically isomorphic to $(V \otimes W)^*$, the vector space of linear maps from $V \otimes W$ to $\mathbf{F}_q$.) The null space of $f$ in $V$ is defined as $\{v \in V | f(v, W) = 0\}$. Defining $f \sim g$ if and only if $\mathrm{rk}(f - g) = 1$ (where the rank $\mathrm{rk}(f)$ of a bilinear map $f$ is the codimension of each of its null spaces (in $V$ and $W$)), we get a graph $\Gamma$ called the *bilinear forms graph (over $\mathbf{F}_q$ with dimension $d$ and $e$)*. In a bilinear forms graph, two vertices $\gamma, \delta$ have distance $i$ if and only if $\mathrm{rk}(\gamma - \delta) = i$. Note that we might also consider $B$ (and $\Gamma$) as the set of $d \times e$ matrices over $\mathbf{F}_q$ (where the distance of two matrices is the rank of their difference).

A useful geometric description of the bilinear forms graph is that of collinearity graph of so-called *attenuated spaces*: Let $V$ be a vector space of dimension $d + e$ over $\mathbf{F}_q$ and $W$ a subspace of $V$ of dimension $e$. Then the corresponding attenuated space is the collection of subspaces $U$ of $V$ with $U \cap W = 0$, where subspaces $U$ of dimension $d$ are called "points", and those of dimension $d - 1$ "lines", and incidence is symmetrized inclusion. The identification with the above description of the bilinear forms graph follows by choosing a fixed "point" $U_0$; now each "point" $U$ defines a map $f_U : U_0 \to W$ by $F_U(u_0) + u_0 \in U$ for $u_0 \in U_0$ (i.e., $\{f_U(u_0)\} = W \cap (U - u_0)$), and one easily verifies that $\dim(U \cap U') = e - r$ if and only if $\mathrm{rk}(f_U - f_{U'}) = r$. But clearly the space $\mathrm{Hom}(U_0, W)$ of maps from $U_0$ into $W$ is isomorphic with $(U_0 \otimes W)^*$. Note that $\Gamma$ is a subgraph of the Grassmann graph of $d$-spaces in $V$.

Here, and in the sequel we shall use geometric language whole talking about vector spaces; for example, we call subspaces $X, Y$ disjoint when they have no projective point in common, i.e., when $X \cap Y = 0$. This cannot cause confusion.

**Proposition 8.1.** *The bilinear forms graph of diameter $d \geq 2$ do not have distance-regular antipodal covers.*

**Proof.** (i) We show that $C(\gamma,\delta)\setminus\{\gamma,\delta\}$ is connected when $d(\gamma,\delta)=d$.

The graph $\Gamma$ is isomorphic to the collinearity graph of the attenuated space of the $d$–spaces in a $(d+e)$–space $V$ disjoint from a given $e$–space $W$. Let $\gamma,\delta$ correspond to $d$-spaces $X,Y$, respectively, then $X\cap Y=0$. The vertices of $\Gamma_{d-i}(\gamma)\cap\Gamma_i(\delta)$ correspond to maximal totally isotrophic subspaces $Z$ with $W\cap Z=0, \dim(X\cap Z)= = i, \dim(Y\cap Z)=d-i$. But the pair of disjoint spaces $W,X$ spans $V$, and so we have a bijective projection $\pi: Y\to X$ (defined by: the line $\langle y,\pi(y)\rangle$ meets $W$), and if we put $E=X\cap Z$, $F=(\pi(Y\cap Z)$ then $E\cap F=0$ (since $Z\cap W=0$). Thus, there is a 1—1 correspondence between pairs $(E,F)$ of disjoint subspaces of $X$ with $\dim E+\dim F=d$ and subspaces $Z$ on a geodesic from $X$ to $Y$. We find the following description of $C(\gamma,\delta)$:

*Let $\gamma,\delta$ be two vertices at distance $j$ in a bilinear forms graph $\Gamma$. Then $C(\gamma,\delta)$ is isomorphic to the graph with as vertices the pairs $(E,F)$ of subspaces of a $j$–space $X$ such that $E\cap F=0, E+F=X$, where $(E,F)\sim(E',F')$ if and only if $\dim(E+E')/(E\cap E')+\dim(F+F')/(F\cap F')=2$. The vertices of $\Gamma_{j-i}(\gamma)\cap\Gamma_i(\delta)$ correspond to pairs $(E,F)$ with $\dim E=i, \dim F=j-i$.*

It follows immediately that $C(\gamma,\delta)\setminus\{\gamma,\delta\}$ is connected; even $\Gamma_{d-1}(\gamma)\cap\Gamma(\delta)$ is connected. Thus, by Proposition 2.1, $\Gamma$ does not have covers of even diameter.

(ii) If also $Z\cap X=0, Y\sim Z$, then there is a 1–space $P\subseteq X$ with $(P+(Y\cap \cap Z))\cap W=0$, so that $P+(Y\cap Z)$ is a common neighbour of $Y$ and $Z$ in $\Gamma_{d-1}(X)$. Thus, by Corollary 2.3, $\Gamma$ does not have covers of odd diameter. ∎

**Remark.** The special case $q\geq 4, d\geq 3, e\geq 2d$ of this proposition was already proved in Huang [23].

## 9. Alternating forms graphs.

Set $V=\mathbf{F}_q^n$ and let $A$ be the $n(n-1)/2$-dimensional vector space of (bilinear) alternating forms on $V$. Thus $f\in A$ if and only if $f$ is a bilinear form on $V$ and $f(x,x)=0$ for all $x\in V$. We define $\mathrm{rk}(f)=\dim(V/\mathrm{Rad} f)$, where $\mathrm{Rad} f=\{x\in V|f(x,y)$ for all $y\in V\}$. Note that $\mathrm{rk}(f)$ takes even values only for $f\in A$.

The *alternating forms graph* $\Gamma$ *on* $V$ is defined on the points of $A$ by $\gamma\in\Gamma(\delta)$ for $\gamma,\delta\in A$ whenever $\mathrm{rk}(\gamma-\delta)=2$; thus two vertices $\gamma,\delta$ have distance $i$ if and only if $\mathrm{rk}(\gamma-\delta)=2i$.

We could also have defined $A$ (and $\Gamma$) as the set of all skew symmetric $n\times n$ matrices over $\mathbf{F}_q$ with zero diagonal (where two matrices are adjacent whenever their difference has rank 2).

The alternating forms graph can be described inside the dual polar graph $[D_q(q)]$. Indeed, let $\Delta$ be the collinearity graph of the dual polar space $[D_q(q)]$, let $\infty$ be a vertex of $\Delta$, and put $A = \Delta_d(\infty)$. Then $A$ is a coclique in $\Delta$, but the graph included on $A$ by $\Delta_2$ is isomorphic to the alternating forms graph on $\mathbf{F}_q^d$.

This is easy to see directly, but we shall need the geometric interpretation of this fact later. What it means is that if $\gamma, \delta \in \Delta_d(\infty)$ and $\infty, \gamma, \delta$ correspond to subspaces $U, V, W$ (with $U \cap V = U \cap W = 0$), then $W$ determines a symplectic polarity $\perp_W$ on $V$ defined by

$$Z^{\perp_W} = ((Z^{\perp} \cap W) + U) \cap V.$$

In this way each maximal totally isotropic subspace $W$ disjoint from $U$ gives rise to a polarity $\perp_W$ of $V$, and $\perp_W$ will be nondegenerate precisely when $V \cap W = 0$. (In fact, $\mathrm{Rad}(\perp_W) = V \cap W$.) Each symplectic polarity $\perp_W$ distinct from the totally degenerate $\perp_V$ is determined by precisely $q-1$ subspaces $W$ (and these do not have common points outside $V$).

**Proposition 9.1.** *The alternating forms graphs $\Gamma$ on $\mathbf{F}_q^n$ with $n \geq 4$ do not have distance-regular antipodal covers of even diameter.*

**Proof.** Note that $\Gamma$ has diameter $d = [n/2]$. We show that $C(\gamma, \delta) \setminus \{\gamma, \delta\}$ is connected when $d_\Gamma(\gamma, \delta) = d$. Since for arbitrary $\alpha, \beta$ the graph $C(\alpha, \beta)$ is determined up to isomorphism by $d_\Gamma(\alpha, \beta)$, independent of $n$, we may suppose $n = 2d$. Let $\Delta$ be the collinearity graph of the dual polar space $[D_n(q)]$ , then $\Gamma$ is isomorphic to the graph induced by $\Delta_2$ on $\Delta_n(\infty)$ for some fixed vertex $\infty$ of $\Delta$. Let $\infty, \gamma, \delta$ correspond to maximal totally isotropic subspaces $U, V, W$, respectively, then $U, V, W$ are pairwise disjoint. The vertices of $\Gamma_{d-i}(\gamma) \cap \Gamma_i(\delta)$ correspond to maximal totally isotropic subspaces $Z$ with $U \cap Z = 0$, $\dim V \cap Z = 2i$, $\dim W \cap Z = n - 2i$. But $Y = V \cap Z$ determines $W \cap Z = W \cap Y^{\perp}$ and $Z = Y + (W \cap Z)$, and $Z \cap U = 0$ precisely when $Y \cap Y^{\perp_W} = 0$. Thus we find the following description of $C(\gamma, \delta)$:

*Let $\gamma, \delta$ be two vertices at distance $j$ in an alternating forms graph $\Gamma$. Then $C(\gamma, \delta)$ is isomorphic to the graph with as vertices the nondegenerate subspaces $Y$ of a $2j$-space $V$ provided with a nondegenerate symplectic form, where two subspaces $Y_1, Y_2$ are adjacent when $\dim(Y_1 + Y_2)/(Y_1 \cap Y_2) = 2$. The vertices of $\Gamma_{j-i}(\gamma) \cap \gamma_i(\delta)$ correspond to the subspaces $Y$ with $\dim Y = 2i$.*

It follows immediately that $C(\gamma, \delta) \setminus \{\gamma, \delta\}$ is connected; even $\Gamma_{d-1}(\gamma) \cap \Gamma(\delta)$ is connected. ∎

**Proposition 9.2.** *The alternating forms graph $\Gamma$ on $\mathbf{F}_q^n$ with $n \geq 4$ do not have distance-regular antipodal covers of odd diameter.*

**Proof.** We want to show that if $\delta, \varepsilon \in \Gamma_d(\gamma)$, $\delta \sim \varepsilon$, then $\Gamma(\gamma) \cap \Gamma_{d-1}(\delta) \cap \Gamma_{d-1}(\varepsilon) \neq \emptyset$. Translating this to polar space situation, we see that if $n$ is even, then the size of this set is the number of hyperbolic lines meeting an $(n-2)$-subspace of an $n$-space with a nondegenerate symplectic form in at least one point. But this is always nonzero. If $n$ is odd, then we may suppose that $\gamma \cap \delta \cap \varepsilon = \emptyset$ — otherwise we can divide out this intersection and reduce to the case $n$ even. But now $|\Gamma(\gamma) \cap \Gamma_{d-1}(\delta) \cap \Gamma_{d-1}(\varepsilon)|$ is the number of hyperbolic lines in an $(n-2)$-subspace of an $n$-space with a symplectic form with radical of dimension 1. Also this is nonzero. ∎

For $n = 4, 5$ these graphs are strongly regular, and we may consider the complementary graphs. For $n = 4$ and $q \geq 4$ and for $n = 5$ these complementary graphs have $c_2 > 2b_1$ and hence have no covers. For $n = 4, q = 3$ covers with odd diameter fail on $c_2 = 306 > b_1 = 170$, and with even diameter on Corollary 3.4. For $n = 4, q = 2$, the complementary graph is the folded halved 8–cube, and was treated earlier.

## 10. Hermitean forms graphs.

Set $V = \mathbf{F}_q^n$, where $q = r^2$ with $r$ a prime power, and let $H$ stand for the $n^2$–dimensional vectorspace over $\mathbf{F}_r$ of the Hermitean forms on $V$. Thus $f \in H$ if and only if $f(x, y)$ is linear in $y$, and $f(y, x) = \overline{f(x, y)}$ for all $x, y \in V$.

The *Hermitean forms graph on* $V$ is the graph whose vertices are the members of $H$ and in which $\gamma, \delta \in H$ are adjacent whenever $\mathrm{rk}(\gamma - \delta) = 1$. Here, $\mathrm{rk}(\gamma)$ and $\mathrm{Rad}\gamma$ have the same meaning as in the previous section. As before we have $d(\gamma, \delta) = \mathrm{rk}(\gamma - \delta)$. Note that we could also have defined $H$ as the set of Hermitean $n \times n$ matrices over $\mathbf{F}_q$.

The Hermitean forms graph can be described inside the dual polar graph $[^2A_{2d-1}(r)]$. Indeed, let $\Delta$ be the collinearity graph of the dual polar graph $[^2A_{2d-1}(r)]$ (with $r^2 = q$), and let $\infty$ be a vertex of $\Delta$. Then the subgraph $\Delta_d(\infty)$ of $\Delta$ is isomorphic to the Hermitean forms graph on $\mathbf{F}_q^d$. (And just as in the previous section we can define polarities $\perp_W$ – this time these are Hermitean.)

**Proposition 10.1.** *The Hermitean forms graph $\Gamma$ of diameter $d \geq 3$ do not have distance-regular antipodal covers of diameter $2d$, except in case $d = 3, q = 4$, where unique antipodal 2- and 4-covers of diameter 6 exist.*

**Proof.** Similarly to the proof of Proposition 9.1 we find here:

*Let $\gamma, \delta$ be two vertices at distance $j$ in a Hermitean forms graph $\Gamma$. Then $C(\gamma, \delta)$ is isomorphic to the graph with as vertices the nondegenerate subspaces $Y$ of a $j$-space $V$ provided with a nondegenerate Hermitean form, where two vertices $Y_1, Y_2$ are adjacent when* $\dim(Y_1 + Y_2)/(Y_1 \cap Y_2) = 1$. *The vertices of $\Gamma_{j-i}(\gamma) \cap \Gamma_i(\delta)$ correspond to the subspaces $Y$ with* $\dim Y = i$.

It follows that $C(\gamma, \delta)$ is bipartite. For $d \geq 3$, the distance two graph of $C(\gamma, \delta) \setminus \{\gamma, \delta\}$ (with $d(\gamma, \delta) = d$) induced on $\Gamma_{d-1}(\gamma) \cap \Gamma(\delta)$ is isomorphic to the graph on the nonisotropic points of a $d$–space $V$ provided with a nondegenerate

Hermitean form, adjacent when the line joining them is hyperbolic. It is easy to see that this graph is connected precisely when $r > 2$. For $d \geq 4, C(\gamma,\delta) \setminus \{\gamma,\delta\}$ is connected also when $r = 2$. Thus, by Proposition 2.1 it follows that $\Gamma$ does not have antipodal covers of diameter $2d$, except possibly in case $d = 2$, or $(d,r) = (3,2)$. When $(d,r) = (3,2)$, then $C(\gamma,\delta) \setminus \{\gamma,\delta\} \cong 4C_6$ for $d(\gamma,\delta) = 3$, and such covers do exist, see §14. ∎

When $d = 2$, then by Corollary 3.4 we find that either $r \in \{2,3\}$, or $4r-3$ is a square, and if we have an antipodal $t$–cover, then $\sqrt{4r-3}|5(t-1)$. If $r = 2$, then the Hermitean forms graph is the folded 5–cube, and its unique antipodal double cover of diameter 4 is the Wells graph (see §5). If $r = 3$ then the Hermitean forms graph is the coset graph of the truncated ternary Golay code, and has an antipodal 3-cover (see §14). Concerning larger $r$ for which $4r-3$ is a square $(7, 13, 31, \ldots)$, nothing is known.

**Proposition 10.2.** *The Hermitean forms graphs $\Gamma$ of diameter $d \geq 2$ do not have distance-regular antipodal covers of diameter $2d+1$, except in case $d = 2, q = 4$ where we find the 5–cube.*

**Proof.** We show that if $\delta, \varepsilon \in \Gamma_d(\gamma), \delta \sim \varepsilon$, then $\Gamma(\gamma) \cap \Gamma_{d-1}(\delta) \cap \Gamma_{d-1}(\varepsilon) \neq \emptyset$. Translating to the polar space situation, we see that the size of this set is the number of nonisotropic points in a hyperplane of a $d$–space with nondegenerate Hermitean form. For $d \geq 3$ this is nonzero and we are done. For $d = 2, q = 4$, our graph is isomorphic to the folded 5–cube, and has a unique double cover of diameter 5, the 5–cube. For $d = 2, q > 4$ we use the full strength of Proposition 2.2. For $\delta \in \Gamma_2(\gamma)$, put $E = \{\delta\} \cup (\Gamma(\delta) \cap \Gamma_2(\gamma))$. We have to show that there is no nontrivial partition of $E$ with the property that if $\varepsilon, \varepsilon' \in E$ have a common neighbour in $\Gamma(\gamma)$ then they belong to the same part. Embed $\Gamma$ as $\Delta_2(\infty)$ in the collinearity graph $\Delta$ of a generalized quadrangle $GQ(r, r^2)$ with the property that any three pairwise noncollinear points have $r+1$ common neighbours. Put $A = \{\infty, \gamma, \delta\}^{\perp}$. Then all points of $E$ on some lines on $\delta$ not meeting $A$ belong to the same part $E_0$ of the partition. If $\varepsilon \in E \setminus E_0, \varepsilon$ on the line $\delta\alpha, \alpha \in A$, then for $\beta \in A \setminus \{\alpha\}$ let $\varsigma$ be the neighbour of $\varepsilon$ on $\gamma\beta$. Put $B = \{\infty, \delta, \gamma\}^{\perp}$. Since $r^2 + 1 > 2r + 1$, there are lines on $\delta$ not meeting $A$ or $B$, and a neighbour $\eta$ of $\varsigma$ on such a line would belong to $E_0$, but also to the same part as $\varepsilon$, contradiction. Thus $E = E_0$. ∎

For $n = 2$, the Hermitean forms graphs are strongly regular. We have $b_1 = q(r-1)$ and $c_2 = r(r-1)$, so for $r > 2, q > 4$ the complementary graph does not have covers. But when $r = 2$, $q = 4$ we are looking for covers of the complement of the folded 5–cube, that is, of the halved 5–cube, and by Proposition 5.3 there are none.

## 11. The quadratic forms graphs.

The graphs discussed in this section were constructed in Egawa [17].

Set $V = \mathbf{F}_q^n$. A *quadratic form on* $V$ *(over* $\mathbf{F}_q$*)* is a map $\gamma : V \to \mathbf{F}_q$ such that

$$\gamma(\lambda x) = \lambda^2\gamma(x) \quad \text{for all} \quad \lambda \in \mathbf{F}_q \quad \text{and} \quad x \in V$$

and such that $B_\gamma : V \times V \to \mathbf{F}_q$ defined by

$$B_\gamma(x, y) = \gamma(x + y) - \gamma(x) - \gamma(y) \qquad (x, y \in V)$$

is a symmetric bilinear form (the *symmetric bilinear form associated with* $\gamma$). Let $Q$ denote the $n(n+1)/2$–dimensional vector space of all quadratic forms on $V$. The *radical* of $\gamma$, denoted by Rad$\gamma$, is defined by

$$\mathrm{Rad}\gamma = \{x \in \mathrm{Rad}B_\gamma | \gamma(x) = 0\} = \{x \in V | \gamma(y) = \gamma(x + y) \quad \text{for all} \quad y \in V\},$$

where, of course, Rad$B_\gamma$ is defined as in §9. We observe that Rad$B_\gamma$ =Rad$\gamma$ if $q$ is odd and that $\dim(\mathrm{Rad}B_\gamma) \leq \dim(\mathrm{Rad}\gamma) + 1$ in general. The *rank* of $\gamma \in Q$, denoted rk$(\gamma)$, is the number $\mathrm{rk}(\gamma) = \dim(V/\mathrm{Rad}\gamma)$. Put $R_i = \{(\gamma, \delta) \in Q^2 | \mathrm{rk}(\gamma - \delta) = i\}$ and $R_i(\delta) = \{\gamma \in Q | (\gamma, \delta) \in R_i\}$.

The *quadratic forms graph on* $V$ has a vertex set $Q$ and $\gamma, \delta \in Q$ are called adjacent if $\mathrm{rk}(\gamma - \delta) \in \{1, 2\}$. In this section, we shall denote this graph by $\Gamma$.

For the study of $\Gamma$ we need an auxiliary graph $\Sigma$, the symmetric bilinear forms graph.

Let $S$ be the $n(n+1)/2$–dimensional vector space of symmetric bilinear forms on $V$. The *symmetric bilinear forms graph* $\Sigma$*on* $V$ is defined on the elements of $S$ by $\gamma \sim \delta$ whenever $\mathrm{rk}(\gamma - \delta) = 1$.

We could also have defined $S$ (and $\Sigma$) as the set of all symmetric $n \times n$ matrices over $\mathbf{F}_q$ (where two matrices are adjacent whenever their difference has rank 1).

The graph $\Sigma$ can be described inside the dual polar graph $[C_n(q)]$. Indeed, let $\Delta$ be the collinearity graph of the dual polar graph $[C_n(q)]$, and let $\infty$ be a vertex of $\Delta$. Then the subgraph $\Delta_n(\infty)$ of $\Delta$ is isomorphic to the symmetric bilinear forms graph on $\mathbf{F}_q^n$. (And just as in the previous sections we can define polarities $\perp_W$ - this time these are orthogonal.)

When $q$ is odd, then we may canonically identify the vertex sets of $\Gamma$ and $\Sigma$, and then the quadratic forms graph $\Sigma$ is the distance-1-or-2-graph of $\Sigma$, i.e., $\Gamma(\gamma) = \Sigma(\gamma) \cup \Sigma_2(\gamma)$ for all $\gamma$.

**Proposition 11.1.** *Let $q$ be odd. Then the quadratic forms graphs of diameter $d \geq 2$ do not have distance regular antipodal covers of diameter $2d$.*

**Proof.** Similarly to what we found in the proofs of Proposition 8.1 and 9.1, we find here:

*Let $q$ be odd, and let $\gamma, \delta$ be two vertices at distance $j$ in a symmetric bilinear forms graph $\Sigma$. Then $C_\Sigma(\gamma, \delta)$ is isomorphic to the graph with as vertices the nondegenerate subspaces $Y$ of a $j$-space $V$ provided with a nondegenerate quadratic form, where two subspaces $Y_1, Y_2$ are adjacent when $\dim(Y_1 + Y_2)/(Y_1 \cap Y_2) = 1$. The vertices of $\Gamma_{j-i}(\gamma) \cap \Gamma_i(\delta)$ correspond to the subspaces $Y$ with $\dim Y = i$.*

It follows that $C(\gamma, \delta)$ is bipartite. Concerning the quadratic forms graph $\Gamma$, if $j = d_\Sigma(\gamma, \delta)$ is even, then

$$C_\Gamma(\gamma, \delta) = \{\varepsilon \in C_\Sigma(\gamma, \delta) | d(\gamma, \varepsilon) \quad \text{even}\},$$

while if $j$ is odd, then

$$C_\Gamma(\gamma, \delta) = C_\Sigma(\gamma, \delta) \cup \{\varepsilon | d_\Sigma(\varepsilon, \gamma) + d_\Sigma(\varepsilon, \delta) = j + 1\}.$$

Now it is easy to see that $C_\Gamma(\gamma, \delta) \setminus \{\gamma, \delta\}$ is connected in $\Gamma$ - even $\Gamma_{d-1}(\gamma) \cap \Gamma(\delta)$ is connected. (Indeed, if $d_\Sigma(\gamma, \delta) = 2d$, then $\Gamma_{d-1}(\gamma) \cap \Gamma(\delta) = \Sigma_{2d-2}(\gamma) \cap \Sigma_2(\delta)$ and this set induces in $\Gamma$ the graph on the nondegenerate lines in a $2d$–space with nondegenerate quadratic form, where two lines are adjacent when they meet, and for $d \geq 1$ this graph is connected. On the other hand, if $d_\Sigma(\gamma, \delta) = 2d - 1$, then $\Gamma_{d-1}(\gamma) \cap \Gamma(\delta) = (\Sigma_{2d-2}(\gamma) \cap \Sigma(\delta)) \cup (\Sigma_{2d-3}(\gamma) \cap \Sigma_2(\delta)) \cup (\Sigma_{2d-2}(\gamma) \cap \Sigma_2(\delta))$ and each vertex in the last summand determines a point-line flag $(P, L)$ with $P$ not contained in $L^\perp$ (i.e., at least one of $P$ and $L$ is nondegenerate). But $\Sigma_{2d-2}(\gamma) \cap \Sigma(\delta)$ is a nonempty clique in $\Gamma$, each point of $\Sigma_{2d-3}(\gamma) \cap \Sigma_2(\delta)$ has a neighbour (for $\Sigma$ and hence also for $\Gamma$) in this clique, and finally a vertex that determines the flag $(P, L)$ has a neighbour in $\Sigma_{2d-2}(\gamma) \cap \Sigma(\delta)$ when $P$ is nondegenerate (i.e., nonisotropic) and in $\Sigma_{2d-3}(\gamma) \cap \Sigma_2(\delta)$ when $L$ is nondegenerate.) ■

**Remark.** Let us see how one may compute $c_j$ from the description of $C(\gamma, \delta)$ above. Let $d(\gamma, \delta) = j$, and compute $c_j = |\Gamma_{j-1}(\gamma) \cap \Gamma(\delta)|$. If $d_\Sigma(\gamma, \delta) = 2j$ is even, then $c_j$ equals the number of nondegenerate lines in a $2j$–space with nondegenerate quadratic form. Let $t_i$ be the number of totally isotropic $i$–subspaces. Then

$$c_j = \begin{bmatrix} 2j \\ 2 \end{bmatrix} - \begin{bmatrix} 2j-2 \\ 1 \end{bmatrix} t_1 + q t_2.$$

If the form is hyperbolic (has Witt index $j$), we find

$$t_1 = \begin{bmatrix} j \\ 1 \end{bmatrix} (q^{j-1} + 1),$$
$$t_2 = \begin{bmatrix} j \\ 2 \end{bmatrix} (q^{j-1} + 1)(q^{j-2} + 1),$$

so that

$$c_j = q^{2j-2}(q^{2j} - 1)/(q^2 - 1).$$

If the form is elliptic (has Witt index $j-1$), we find

$$t_1 = \begin{bmatrix} j-1 \\ 1 \end{bmatrix}(q^j+1),$$

$$t_2 = \begin{bmatrix} j-1 \\ 2 \end{bmatrix}(q^j+1)(q^{j-1}+1),$$

and again the same value for $c_j$.

Finally, suppose $d_\Sigma(\gamma,\delta) = 2j-1$ is odd. Then

$$t_1 = \begin{bmatrix} j-1 \\ 1 \end{bmatrix}(q^{j-1}+1) = (q^{2j-2}-1)/(q-1),$$

and

$$t_2 = \begin{bmatrix} j-1 \\ 2 \end{bmatrix}(q^{j-1}+1)(q^{j-2}+1) = (q^{2j-2}-1)(q^{2j-4}-1)/(q^2-1)(q-1).$$

Thus,

$$|\Sigma_{2j-2}(\gamma)\cap\Sigma(\delta)| = \begin{bmatrix} 2j-1 \\ 1 \end{bmatrix} - t_1 = q^{2j-2}$$

and

$$|\Sigma_{2j-3}(\gamma)\cap\Sigma_2(\delta)| = \begin{bmatrix} 2j-1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2j-3 \\ 1 \end{bmatrix} t_1 + qt_2 = q^{2j-2}(q^{2j-2}-1)/(q^2-1).$$

Finally, let us investigate $\Sigma_{2j-2}(\gamma)\cap\Sigma_2(\delta)$ - this corresponds to the set of totally isotropic $n$-spaces $Z$ such that $U\cap Z = 0, \dim V\cap Z = n-2j+2$, and $\dim W\cap Z = = n-2$, where $U,V,W$ are totally isotropic $n$-spaces in $[C_n(q)]$ corresponding to $\infty,\gamma,\delta$, respectively. We must have $V\cap W \subseteq Z$, for otherwise $\langle V\cap Z, W\cap Z, V\cap W\rangle$ would be totally isotropic and properly contain $Z$. Thus, we may divide out $V\cap W$ and suppose that $n = 2j-1$ and $U,V,W$ are pairwise disjoint. Now we find

$$|\Sigma_{2j-2}(\gamma)\cap\Sigma_2(\delta)| = \frac{q^{2j-2}-1}{q-1}\cdot q^{2j-3}\cdot(q-1)+$$

$$\sum_{\varepsilon=\pm1}\frac{1}{2}q^{j-1}(q^{j-1}+\varepsilon)\cdot\Big(\frac{(q^{j-1}-\varepsilon)(q^{j-2}+\varepsilon)}{(q-1)}\cdot(q-1)+q^{j-2}(q^{j-1}-\varepsilon)\cdot(q-2)\Big)$$

$$= q^{2j-2}(q^{2j-2}-1).$$

[The three terms correspond to the $Z$ with $P = V\cap Z$ an isotropic point, a hyperbolic point ($\varepsilon = +1$) or an elliptic point ($\varepsilon = -1$), where $P$ is called hyperbolic (elliptic) when $P^{\perp_W}$ is. For each terms, the first factor is the number of choices for $P$, the second is the number of choices of $H = W\cap Z$ (in at least two terms differentiated between tangent hyperplane and nondegenerate hyperplane in $(W,\perp_V)$), and the last factor is the number of ways of extending $\langle P,H\rangle$ to $Z$.] Altogether, we find

$$c_j = q^{2j-2}\Big(1+\frac{q^{2j-2}-1}{q^2-1}+q^{2j-2}-1\Big) = q^{2j-2}\cdot\frac{q^{2j}-1}{q^2-1},$$

the same value as before.

**Proposition 11.2.** *Let $q$ be odd. Then the quadratic forms graphs of diameter $d \geq 2$ do not have distance-regular antipodal covers of diameter $2d+1$.*

**Proof.** Let $d(\gamma,\delta) = d(\gamma,\varepsilon) = d, d(\delta,\varepsilon) = 1$. We want to find a vertex in $\Gamma(\gamma) \cap \Gamma_{d-1}(\delta) \cap \Gamma_{d-1}(\varepsilon)$. Let $\infty, \gamma, \delta, \varepsilon$ correspond to totally isotropic $n$–spaces $U, V, W, X$. (Then $U \cap V = U \cap W = U \cap X = 0, \dim V \cap W$ and $\dim V \cap X$ are either 0 or 1, and $\dim W \cap X$ is either $n-2$ or $n-1$.) If $\dim W \cap X = n-2$, then we can find a totally isotropic $n$–space $Y$ with $\dim W \cap Y = \dim X \cap Y = n-1$. Now either the corresponding vertex $\eta$ in $\Gamma$ is in $\Gamma_{d-1}(\gamma)$, and we see that $C(\gamma,\delta) \cap C(\gamma,\varepsilon) \setminus \{\gamma\} \neq \emptyset$, or $\eta \in \Gamma_d(\gamma)$. Thus (using the full strength of Proposition 2.2 instead of Corollary 2.4) it suffices to consider the case $\dim W \cap X = n-1$. If $W \cap X$ contains a line $L$ that is nondegenerate in $(W, \perp_V)$, then $\langle L, L^{\perp} \cap V \rangle$ is the required vertex. Since $\dim W \cap X = n-1 \geq 3$, this is certainly the case when $n > 4$ or when $W \cap V = 0$ or $X \cap V = 0$. But when $n = 2d = 4, \dim W \cap V = \dim X \cap V = 1$, then we can choose a point $P$ in $W \cap X$ that is nonisotropic for $\perp_V$, and $\langle P, P^{\perp} \cap V \rangle$ is the required vertex. ■

Now let us turn the case when $q$ is even.

**Proposition 11.3.** *Let $q$ be even. Then the quadratic forms graphs of diameter $d \geq 2$ do not have distance-regular antipodal covers.*

**Proof.** If $\gamma$ is a quadratic form on $V$, then $B_\gamma$ is an alternating form, and to any alternating form $B$ there correspond $q^n$ quadratic forms $\gamma$ such that $B = B_\gamma$. The condition $\mathrm{rk}(\gamma - \delta) \in \{1,2\}$ for adjacency of the quadratic forms $\gamma$ and $\delta$ is equivalent to either $B_\gamma = B_\delta$ (and $\gamma \neq \delta$), or $\mathrm{rk}(B_\gamma - B_\delta) = 2$ and $\gamma$ and $\delta$ coincide on $\mathrm{Rad}(B_\gamma - B_\delta)$. Thus, the fibers $C_B = \{\gamma | B_\gamma = B\}$ for alternating forms $B$ are $q^n$-cliques, and if $\gamma \notin C_B$ then $\gamma$ is adjacent to either 0 or $q^2$ elements of $C_B$. As Egawa [17] shows, it is possible by looking at $\Gamma$ to distinguish the cases $\mathrm{rk}(\gamma-\delta) = 1$ and $\mathrm{rk}(\gamma - \delta) = 2$. (For $q > 2$ this is trivial, since the singular line $\{\gamma,\delta\}^{\perp\perp}$ has size $q$ in the former case, and 2 in the latter case.) This means that the fibers $C_B$ can be recognized, and that $\Gamma$ determines the alternating forms graph. Now if $\Delta$ is a distance-regular antipodal cover of $\Gamma$ (of diameter $\geq 4$), then cliques in $\Gamma$ lift to cliques in $\Delta$, and if $\Delta$ has diameter at least 5 then "adjacent" cliques lift to adjacent cliques. It follows that if $\Delta$ has diameter at least 5, then the quotient of $\Delta$ with respect to this partition into cliques covering fibers $C_B$ is a distance-regular antipodal cover of the alternating forms graph, but by Propositions 9.1 and 9.2 no such covers exist. Remains the case where $\Delta$ has diameter $2d = 4$ and $n \in \{3,4\}$, but this is excluded by Corollary 3.4. ■

For $n = 3,4$, the quadratic forms graphs are strongly regular, and we may consider the complementary graphs. But the quadratic forms graph on $\mathbf{F}_q^n$ has the same parameters as (and in case $(n,q) = (3,2)$ is isomorphic to) the alternating forms graph on $\mathbf{F}_q^{n+1}$, and the arguments that disposed of covers in the case of the alternating forms graphs also work here.

## 12. $E_7$ graphs.

Let $\Gamma$ be the collinearity graph of the points in a building of type $E_7$ defined over $\mathbf{F}_q$, where the points are those objects whose residue is of type $E_6$. Then $\Gamma$ is distance-regular with intersection array

$$\{q(q^8+q^4+1)\frac{q^9-1}{q-1}, q^9(q^4+1)\frac{q^5-1}{q-1}, q^{17}; 1, (q^4+1)\frac{q^5-1}{q-1}, (q^8+q^4+1)\frac{q^9-1}{q-1}\}.$$

The corresponding Coxeter graph (the "$q=1$" case) is the Gosset graph on 56 points and with intersection array $\{27, 10, 1; 1, 10, 27\}$.

**Proposition 12.1.** *The collinearity graph of the points in a finite building of type $E_7$ (either thin or thick) does not admit distance-regular antipodal covers.*

**Proof.** In case the building is thin, we have the Gosset graph on 56 points. But since in this graph $k_3 = 1$, it follows that for $\gamma \in \Gamma$ no line is contained in $\Gamma_3(\gamma)$, and by Corollary 2.3 there are no covers of odd diameter. If $d(\gamma,\delta) = 3$, then $\Gamma_2(\gamma) \cap \Gamma(\delta)$ is the collinearity graph of a building of type $E_6$, and in particular is connected. Thus, by Proposition 2.1 there are no covers of even diameter. ■

## 13. The affine $E_6$ graph.

Let $\Delta$ be the collinearity graph of a finite thick building of type $E_7$, and $\infty$ a vertex of $\Delta$. Then the subgraph induced on $\Delta_3(\infty)$ is called the *affine $E_6$ graph.* A direct description may be given as follows.

Let $F$ be a field and denote by $\mathbf{K}_F$, or just $\mathbf{K}$ if no confusion arises, the 27-dimensional vector space over $F$ consisting of ordered triples $x = (x^{(1)}, x^{(2)}, x^{(3)})$ of $3\times 3$ matrices $x^{(i)}$ $(i = 1,2,3)$, supplied with the following symmetric cubic form.

$$\mathrm{Dt}(x) = \det x^{(1)} + \det x^{(2)} + \det x^{(3)} - \mathrm{tr} x^{(1)}x^{(2)}x^{(3)} \qquad (x = (x^{(1)}, x^{(2)}, x^{(3)}) \in \mathbf{K}).$$

Denote by $(\cdot,\cdot,\cdot)$ the linearization of Dt, i.e.,

$$(x,y,z) = \mathrm{Dt}(x+y+z) - \mathrm{Dt}(x+y) - \mathrm{Dt}(y+z) - \mathrm{Dt}(y+z) + \\ + \mathrm{Dt}(x) + \mathrm{Dt}(y) + \mathrm{Dt}(z) \qquad (x,y,z \in \mathbf{K}).$$

Let $\Gamma$ be the graph whose vertex set is $\mathbf{K}$ and in which $x, y$ are adjacent if and only if $\mathrm{Dt}(x-y, x-y, \mathbf{K}) = 0$. Then $\Gamma$ is said to be the *affine $E_6$-graph over $F$.* The appearance of $E_6$ in this name is due to the fact that the automorphism group of this graph is a semidirect product of the additive group of $\mathbf{K}$ and (an extension of a central cover of) the group of lie type $E_6$ defined over $F$.

**Proposition 13.1.** *The affine $E_6$ graph over $\mathbf{F}_q$ does not have distance-regular antipodal covers.*

**Proof.** This can easily be read off from the explicit description of $E_6$ in Cohen and Cooperstein [14]. One can give a geometric argument as follows. Consider $\Gamma$ as $\Delta_3(\infty)$ embedded in the $E_7$ graph $\Delta$. Note that for vertices $\gamma, \delta$ of $\Gamma$ the distances $d_\Gamma(\gamma,\delta)$ and $d_\Delta(\gamma,\delta)$ coincide. If $\gamma$ and $\delta$ are vertices of $\Gamma$ at distance 3, and $\alpha,\beta \in \Gamma_2(\gamma) \cap \Gamma(\delta), \alpha \not\sim \beta$, then inside the $E_7$ graph $\Delta$ the points $\alpha$ and $\beta$ determine a symplecton $S$. Let $\gamma^\perp \cap S = \{\varepsilon\}$, and $\infty^\perp \cap S = \{\varsigma\}$. Now $\delta^\perp \cap \varepsilon^\perp \cap S \setminus \varsigma^\perp$ is connected (indeed, $\delta^\perp \cap \varepsilon^\perp \cap S$ is the collinearity graph of a polar space of type $D_5$, and removing a hyperplane leaves this graph connected), it contains $\alpha$ and $\beta$, and is connected in $\Gamma_2(\gamma) \cap \Gamma(\delta)$, so that this latter set is connected, and by Proposition 2.1 $\Gamma$ does not have covers of even diameter.

If $\delta, \varepsilon \in \Gamma_3(\gamma)$, then let $L$ be the line $\delta\varepsilon$. If $\Delta_2(\infty) \cap L \neq \Delta_2(\gamma) \cap L$, then $\delta$ and $\varepsilon$ have a common neighbour in $\Gamma_2(\gamma)$. Otherwise, let $\Delta_2(\infty) \cap L = \Delta_2(\gamma) \cap L = \{\varsigma\}$, and let $S$ be the symplecton on $\gamma$ and $\varsigma$. Now $\gamma^\perp \cap \varsigma^\perp \cap S \cap \Delta_3(\infty) \neq \emptyset$, and it follows that $\Gamma(\gamma) \cap \Gamma_2(\delta) \cap \Gamma_2(\varepsilon) \neq \emptyset$, and hence $\Gamma$ does not have covers of odd diameter. ∎

## 14. The Witt graphs and related graphs.

The *large Witt graph* is the graph with as vertices the 759 blocks of a Steiner system $S(5,8,24)$, where two blocks are adjacent when they are disjoint.

**Proposition 14.1.** *The large Witt graph does not have distance-regular antipodal covers.*

**Proof.** This graph is near a hexagon, so covers of odd diameter are excluded by Corollary 2.3. When $\gamma$ and $\delta$ are vertices at distance 3, then $C(\gamma,\delta) \setminus \{\gamma,\delta\}$ is isomorphic to the incidence graph of the generalized quadrangle $GQ(2,2)$, and, in particular, is connected. Thus, by Proposition 2.1 there are no covers of even diameter either. ∎

The subgraph of the large Witt graph induced by the 506 blocks of $S(5,8,24)$ that miss a fixed symbol, is itself distance-regular (of diameter 3). Consideration of its parameters shows that it has no distance-regular antipodal covers of diameter 7, but 3–covers and 9–covers of diameter 6 have feasible intersection arrays.

**Problem.** Do there exist distance-regular graphs with intersection arrays $\{15,14,12,6,1,1;1,1,3,12,14,15\}$ or $\{15,14,12,8,1,1;1,1,1,12,14,15\}$?

The subgraph of the large Witt graph induced by the 330 blocks of $S(5,8,24)$ that miss two fixed symbols, is again distance-regular (of diameter 4). The parameters allow no covers of diameter 9, and 2– and 3–covers of diameter 8. Ivanov,

Ivanov and Faradjev [18] constructed an antipodal 3–cover, and Brouwer [8] showed that there is no antipodal 2–cover and that the 3–cover is unique.

If $C$ is the extended ternary Golay code (of word length 12 and dimension 6 over $\mathbb{F}_3$), then the graph with as vertices its cosets, where two cosets are adjacent when they contain vectors that differ by a vector of Hamming weight one, is distance-regular of diameter 3. This graph is a near hexagon, so there are no covers of odd diameter. There are no feasible parameter sets for covers of even diameter.

If $C$ is the truncated ternary Golay code (of word length 10 and dimension 6 over $\mathbb{F}_3$), then its coset graph is isomorphic to the Hermitean forms graph on $\mathbb{F}_9^2$. This graph has a triple cover, namely the coset graph of the shortened ternary Golay code (of word length 10 and dimension 5 over $\mathbb{F}_3$). No other covers are known; the parameters of the 2– and 6–covers also satisfy all known conditions.

If $C$ is the doubly truncated binary Golay code (of word length 21 and dimension 12 over $\mathbb{F}_2$), then its coset graph $\Gamma$ is isomorphic to the Hermitean forms graph on $\mathbb{F}_4^3$. This graph has a unique double cover, namely the coset graph of the code (of word length 21 and dimension 11 over $\mathbb{F}_2$) obtained by taking all code words in the binary Golay code that start with 00 or 11 and deleting these two coordinate positions. This graph also has a unique antipodal 4-cover, namely the coset graph of the code (of word length 21 and dimension 10 over $\mathbb{F}_2$) obtained by taking all code words in the extended binary Golay code that start with 000 or 111 and deleting these three coordinate positions. The graph $\Gamma$ has no other antipodal covers. (In particular, no antipodal 3–covers exist, although the corresponding intersection array is feasible.) The uniqueness proofs for these covers will be given elsewhere.

## 15. Conclusion.

In these investigations we encountered the following antipodal distance-regular graphs. First the infinite families with unbounded diameter.

0. the $2m$–gons;
1. the Johnson graphs $J(2k, k)$;
2. the doubled Odd graphs;
3. the Hamming cubes $H(n, 2)$;
4. the folded cubes $\Box_{2m}$.

Next, in order of decreasing diameter:

5. The Ivanov-Ivanov-Faradjev graph on 990 vertices $(d = 8)$;
6. some coset graphs related to the binary Golay code $(d = 6)$;
7. the dodecahedron $(v = 20, d = 5)$;

8. some coset graphs related to the ternary Golay code ($d = 4$);
9. the triple covers $3.\overline{T(6)}$ and $3.\overline{T(7)}$ ($v = 45, 63, d = 4$);
10. Wells graph ($v = 32, d = 4$);
11. various covers $r.K_{k,k}$ ($d = 4$).

We observed the following "sporadic" isomorphisms:

a) $H(2,3) \cong \overline{H(2,3)}$,

b) $\square_5 \cong \overline{\tfrac{1}{2}H(5,2)} \cong \mathrm{Herm}(\mathbf{F}_4^2)$,

c) $\overline{\tfrac{1}{2}\square_8} \cong \mathrm{Alt}(\mathbf{F}_2^4) \cong \mathrm{Quad}(\mathbf{F}_2^3)$,

d) $\overline{FJ(8,4)} \cong \begin{bmatrix}4\\2\end{bmatrix}_2$,

e) $\mathrm{Herm}(\mathbf{F}_4^3)$ is the coset graph of the doubly truncated binary Golay code,

f) $\mathrm{Herm}(\mathbf{F}_9^2)$ is the coset graph of the truncated ternary Golay code.

The following questions remain:

- determine the distance-regular antipodal covers of the Hermitean forms graphs of diameter 2;
- determine the distance-regular antipodal covers of the Witt graph on 506 vertices;
- determine the distance-regular antipodal covers of the classical generalized hexagons.

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