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PERTURBING SEMIGROUPS BY SOLVING STIELTJES RENEWAL EQUATIONS

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Dedicated to George Maltese on the occasion of his 60th birthday

(Submitted by: Glenn Webb)

Abstract. We develop a perturbation theory for strongly continuous semigroups and dual semigroups not based on perturbation of infinitesimal generators but on certain families of bounded linear operators describing the cumulative effect of the feedback. The theory extends the theory of perturbation of generators by bounded or relatively bounded linear operators. The theory is applied to problems of structured population dynamics which cannot, to the best of our knowledge, be treated using a more conventional perturbation theory.

1. Introduction. Consider a linear dynamical system whose state space X is a Banach space. Assume that the *output* of the system can be described by a (bounded) linear map B from X into another Banach space Y. So, when $t \mapsto u(t) \in X$ describes the time evolution of the state, the map $t \mapsto Bu(t) \in Y$ gives the output as a function of time.

The *cumulative output* up to time t is, by definition, $\int_0^t Bu(s)ds$. For autonomous systems there exists a semigroup of operators $T_0(t)$ on X such that $u(t) = T_0(t)x$, where x denotes the initial state at time $t = 0$ and we can define cumulative output maps, parameterized by t, by

$$V_0(t)x = \int_0^t BT_0(s)xds.$$

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The semigroup property of T_0, i.e., $T_0(t+s) = T_0(t)T_0(s)$, implies the algebraic relation

$$V_0(t+s) - V_0(t) = V_0(s)T_0(t). \tag{1.1}$$

The relation (1.1) has its own meaningful interpretation: The cumulative output between t and $t+s$, given by the difference on the left hand side, is obtained by letting the state of the system evolve for t time units and then calculating the cumulative output over a time interval of length s. In the stochastic literature the relation (1.1) is used to characterize additive functionals, see for instance Fukushima (1980).

So, from the very beginning, one may alternatively describe output in terms of a family of operators $V_0(t)$ satisfying relation (1.1), rather than in terms of B. The advantage can be seen in retrospect. If $V_0(t)$ satisfies (1.1), one can show that $V_0(t)$ can be differentiated on $\mathcal{D}(A_0)$, the domain of the generator A_0 of $T_0(t)$, and that $B = V_0'(0)$ is an A_0-bounded operator. But, in many applications, B can neither be extended from $\mathcal{D}(A_0)$ to the whole space X nor be given a concrete characterization on $\mathcal{D}(A_0)$, often for the simple reason that $\mathcal{D}(A_0)$ cannot be described in concrete terms either.

Suppose now that v is an *input* (or a *control*), that is, a mapping from $[0,\infty)$ to Y, which is fed into the system via the linear operator $C : Y \to X$. The variation-of-constants formula

$$u(t) = T_0(t)x + \int_0^t T_0(t-s)Cv(s)ds \tag{1.2}$$

specifies how to obtain the state $u(t)$ at time t, given the initial state x and the input v. We emphasize that this formula has a clear interpretation. The question under which conditions for x and v the function u is differentiable (in whatever sense) and satisfies the abstract differential equation

$$\frac{du}{dt} = A_0 u + Cv$$

is certainly of mathematical interest, but does not necessarily add to the understanding of the underlying 'real-world' problem. As a mathematical model, the integral expression is just as good as the differential equation.

The variation-of-constants formula (1.2) can be rewritten as a Stieltjes integral in two different ways, either as

$$u(t) = T_0(t)x + \int_0^t T_0(t-s)F(ds), \tag{1.3}$$

where

$$F(t) = \int_0^t Cv(s)ds$$

is the *cumulative input*, or as

$$u(t) = T_0(t)x + \int_0^t U_0(ds)v(t-s), \tag{1.4}$$

where the operator family U_0 is defined by

$$U_0(t) = \int_0^t T_0(s)Cds. \tag{1.5}$$

If the state of the system is initially at the origin, i.e., $x = 0$, and if the constant input v is the step function

$$v(t) = \begin{cases} 0, & \text{if } t < 0, \\ y, & \text{if } 0 \leq t, \end{cases}$$

then $U_0(t)y$ describes the time evolution of the state. We therefore call the operator family U_0 a *step response* corresponding to the semigroup T_0. Obviously U_0 satisfies the relation

$$U_0(t+s) - U_0(t) = T_0(t)U_0(s). \tag{1.6}$$

The interpretation of this relation is as follows. If the system evolves in response to the step function input $v(t)$, then the state at time s is $U_0(s)y$. In the time interval from s to $t+s$ this state evolves to $T_0(t)U_0(s)y$ and in the same time interval the input will give a contribution $U_0(t)y$. The sum of these contributions must equal the state at time $t+s$ obtained in response to the input in the time interval $[0, t+s)$.

There are important applications where the variation-of-constants formula (1.2) cannot be used because the instantaneous input $Cv(t)$ is not a well-defined element of the state space X, but where the response to the input makes perfect sense. As an example we mention the flow of a continuous fluid with a point source. The source itself corresponds to a Dirac measure whereas the flow streaming out of the source is a continuous function. The variation-of-constants formula (1.4) is therefore more general than (1.2) and even if U_0 cannot be defined by (1.5) using an operator $C : Y \to X$ it is still clear from the interpretation that U_0 should satisfy (1.6) and we can actually define a step response of a semigroup to be an operator family satisfying that relation.

Finally we consider *feedback* — or re-investment — by equating input to output. This can be done in two alternative ways depending on whether we focus on cumulative output or on step response. In the first case we choose the variation-of-constants formula (1.3) as starting point and restore the autonomous character of the system by showing that $u(t) = T(t)x$ for a (perturbed) semigroup $T(t)$ and that $F(t) = V(t)x$ with a cumulative output family V for T. Note that (1.3) is an explicit formula once V is known:

$$T(t) = T_0(t) + \int_0^t T_0(t-s)V(ds). \tag{1.7}$$

In this approach the essence of the perturbation can be expressed as a relation between the original and the perturbed output families as can be seen by formally applying B to this identity and integrating with respect to time:

$$V(t) = V_0(t) + \int_0^t V_0(t-s)V(ds). \tag{1.8}$$

The other approach is based on (1.4). By putting $u(t) = v(t) = T(t)x$ in this formula we obtain an integral equation for u (or T). A formal solution, as we shall show below, is given by the explicit formula

$$T(t) = T_0(t) + \int_0^t U(ds)T_0(t-s), \tag{1.9}$$

where U is the solution of the equation

$$U(t) = U_0(t) + \int_0^t U(t-s)U_0(ds). \tag{1.10}$$

Notice that multiplying (1.9) by C from the right and integrating with respect to time one obtains (1.10).

Let us recapitulate the situation. The "data" are a semigroup $T_0(t)$ and either a family of cumulative output operators $V_0(t)$ or a family $U_0(t)$ of step response operators — rather than the generator A_0 of T_0 and the maps B or C. In the first case we find the solution V of the *renewal equation* (1.8) and define $T(t)$ by the now explicit formula (1.7). In the second case we solve U from the renewal equation (1.10) and define $T(t)$ by (1.9). In both cases the claim is that T is a semigroup. The properties of being a cumulative output or a step response, respectively, are inherited by V and U from V_0 and U_0 with these properties now holding with respect to T instead of T_0; i.e.,

$$V(t+s) - V(t) = V(s)T(t), \tag{1.11}$$

$$U(t+s) - U(t) = T(t)U(s). \tag{1.12}$$

To substantiate this claim we have to prove existence and uniqueness of a solution to (1.8) and to (1.10) and to show that the Stieltjes convolutions in (1.7) and in (1.9) can be given appropriate meanings. It turns out that the approach based on cumulative output is very well suited for studying perturbation of dual semigroups, whereas step response operators are the right approach to perturbation of strongly continuous semigroups. In fact we shall show that the two approaches are dual to each other.

For the study of the asymptotic behavior of the semigroup T it is useful to know the spectral properties of its generator A. To indicate that these can also be obtained in terms of the cumulative output family V_0 and the resolvent $R(\lambda, A_0)$ of the generator A_0 of the unperturbed semigroup T_0 we formally take Laplace transforms. Define

$$\hat{V}(\lambda) = \int_0^\infty e^{-\lambda s}V(ds). \tag{1.13}$$

Then (1.8) transforms into

$$\hat{V}(\lambda) = \hat{V}_0(\lambda) + \hat{V}_0(\lambda)\hat{V}(\lambda) \tag{1.14}$$

and (1.7) transforms into

$$R(\lambda, A) = R(\lambda, A_0) + R(\lambda, A_0)\hat{V}(\lambda). \tag{1.15}$$

Equations (1.14) and (1.15) imply that

$$R(\lambda, A) = R(\lambda, A_0)(I - \hat{V}_0(\lambda))^{-1}. \tag{1.16}$$

In a similar manner we obtain

$$R(\lambda, A) = (I - \hat{U}_0(\lambda))^{-1} R(\lambda, A_0). \tag{1.17}$$

We conclude that we can analyze the spectrum of A by analyzing the spectrum of A_0 and the singularities of $\lambda \mapsto (I - \hat{V}_0(\lambda))^{-1}$ (or $\lambda \mapsto (I - \hat{U}_0(\lambda))^{-1}$ if we use the approach based on step response). Stated differently, the resolvent of the generator of the perturbed semigroup can be expressed explicitly in terms of the "data".

The idea of modeling renewal processes (in the scalar case) as *Stieltjes* convolution equations goes back at least as far as Feller (1941) and has now become standard in probability theory (cf. Feller, 1966). Prüß (1984) considered Stieltjes convolutions and resolvent families in the context of abstract Volterra integrodifferential equations.

The paper is organized as follows. In Section 2 we recall some basic facts on abstract Stieltjes integrals and convolutions and we establish the existence of a resolvent family of certain types of kernels. In Section 3 we define the step response and cumulative output of a semigroup and formulate the main perturbation results (Theorem 3.4 and Corollary 3.5) in terms of these concepts and in Section 4 we interpret the perturbation results in terms of the more familiar notion of infinitesimal generators. Finally the theory is illustrated by two examples from structured population dynamics in Section 5.

2. Stieltjes convolutions and the abstract renewal equation. Let $S(t)$, $t \geq 0$, be a strongly continuous family of linear bounded operators on a Banach space X and $f : [0, \infty) \to X$ a continuous function. Usually the convolution of S with f is defined by

$$(S * f)(t) = \int_0^t S(s)f(t-s)ds = \int_0^t S(t-s)f(s)ds.$$

Defining

$$U(t)x = \int_0^t S(r)x dr,$$

this convolution can be rewritten as

$$(S * f)(t) = \int_0^t U(ds)f(t-s),$$

with the latter integral being interpreted in the sense of Stieltjes. Following Hönig (1975) we define a more general Stieltjes integral and later a more general convolution.

By a *bilinear triple* $\mathcal{B} = (X, Y, Z)$ we mean a triple of Banach spaces X, Y and Z, with a bilinear mapping $X \times Y \to Z$. If $x \in X$ and $y \in Y$, then $xy \in Z$ denotes the value of the bilinear mapping. We shall assume that

$$\|xy\| \leq \|x\| \|y\|. \tag{2.1}$$

Assumption (2.1) is not essential for the existence of Stieltjes convolutions, but it is of importance for proving continuity properties of the convolutions.

If x and y are functions defined on $[0,t]$ with values in X and Y, respectively, then the Stieltjes integral

$$\int_0^t x(ds)y(s) \tag{2.2}$$

is defined as the limit of sums

$$\sum_{j=0}^{n}(x(r_{j+1})-x(r_j))y(s_j)\,, \quad s_j \in [r_j, r_{j+1}], \tag{2.3}$$

with $0 = r_0 < \ldots < r_{n+1} = t$, when the partition $r_0, \ldots, r_{n+1}$ gets finer, whenever this limit exists in the norm-topology of Z. The integral

$$\int_0^t x(s)y(ds) \tag{2.4}$$

is of course defined analogously as the limit of sums

$$\sum_{j=0}^{n} x(s_j)\,(y(r_{j+1})-y(r_j)), \quad s_j \in [r_j, r_{j+1}]. \tag{2.5}$$

As a matter of fact the integral (2.4) is simply a special case of the integral (2.2) with $\mathcal{B} = (Y, X, Z)$ and bilinear mapping $(y, x) \to xy$. Integration by parts can be performed in the usual way: If x and y are functions from $[0,t]$ to X and Y respectively, then $\int_0^t x(ds)y(s)$ exists if and only if $\int_0^t x(s)y(ds)$ exists and

$$\int_0^t x(ds)y(s) = x(t)y(t) - x(0)y(0) - \int_0^t x(s)y(ds).$$

This can easily be seen by reordering the terms in the approximating sums (2.3) and (2.5).

We shall mainly be concerned with two special cases, namely $\mathcal{B} = (\mathcal{L}(X), X, X)$ and $\mathcal{B} = (\mathcal{L}(X), \mathcal{L}(X), \mathcal{L}(X))$, with the natural bilinear mappings. Thus we are interested in convolutions of the types

$$(U \star x)(t) = \int_0^t U(ds)x(t-s), \tag{2.6}$$

$$(U \star V)(t) = \int_0^t U(ds)V(t-s), \tag{2.7}$$

and

$$\int_0^t U(t-s)V(ds), \tag{2.8}$$

where U and V are families of bounded linear operators on a Banach space X and x is a function from the positive real axis to X.

Let $\mathcal{B} = (X, Y, Z)$ be a bilinear triple. A function $x : [0, t] \to X$ is said to be of *bounded $\mathcal{B}$-variation* on $[0, t]$ if

$$V_{\mathcal{B},t}(x) = \sup\{\| \sum_{i=1}^{n} (x(t_i) - x(t_{i-1}))y_i\|\} < \infty, \tag{2.9}$$

where the supremum is taken over all partitions $0 = t_0 < \ldots < t_n = t$ of $[0, t]$ and all $y_i \in Y$ with $\|y_i\| \le 1$. The number $V_{\mathcal{B},t}$ is called the $\mathcal{B}$-variation of x on $[0, t]$. If a function $x : \mathbb{R}_+ \to X$ is of bounded $\mathcal{B}$-variation on every interval $[0, t]$, $t > 0$, then x is said to be locally of bounded $\mathcal{B}$-variation. If $\mathcal{B} = (X, X^*, \mathbb{C})$, then we obtain the usual notion of bounded variation. To see this, notice that for any partition of $[0, t]$ we trivially have

$$\sup\Big\{\Big|\sum_{i=1}^{n}\langle x(t_i) - x(t_{i-1}), x_i^*\rangle\Big| : x_i^* \in X^*, \|x_i^*\| \le 1\Big\} \le \sum_{i=1}^{n} \|x(t_i) - x(t_{i-1})\|.$$

The reverse inequality holds since by the Hahn-Banach Theorem there exist $x_i^* \in X^*$ of norm 1 such that $\langle x(t_i) - x(t_{i-1}), x_i^*\rangle = \|x(t_i) - x(t_{i-1})\|$.

The following proposition shows that for our two main examples, the concepts of bounded $\mathcal{B}$-variation are actually the same. Compare Hönig (1975), Theorem 4.4.

Proposition 2.1. *Let U be a function from $[0, t]$ to $\mathcal{L}(X)$ and let $\mathcal{B}_1 = (\mathcal{L}(X), X, X)$ and $\mathcal{B}_2 = (\mathcal{L}(X), \mathcal{L}(X), \mathcal{L}(X))$. Then U is of bounded $\mathcal{B}_1$-variation if and only if it is of bounded $\mathcal{B}_2$-variation and $V_{\mathcal{B}_1,t}(U) = V_{\mathcal{B}_2,t}(U)$.*

Proof. Let $0 = t_0 < \ldots < t_n = t$ be a partition of $[0, t]$. It suffices to show that

$$\begin{aligned}
&\sup\Big\{\Big\|\sum_{i=1}^{n}[U(t_i) - U(t_{i-1})]V_i\Big\| : V_i \in \mathcal{L}(X), \|V_i\| \le 1\Big\}\\
&= \sup\Big\{\Big\|\sum_{i=1}^{n}[U(t_i) - U(t_{i-1})]x_i\Big\| : x_i \in X, \|x_i\| \le 1\Big\}.
\end{aligned}$$

On the one hand we have

$$\begin{aligned}
&\sup\Big\{\Big\|\sum_{i=1}^{n}[U(t_i) - U(t_{i-1})]V_i\Big\| : V_i \in \mathcal{L}(X), \|V_i\| \le 1\Big\}\\
&= \sup\Big\{\Big\|\sum_{i=1}^{n}[U(t_i) - U(t_{i-1})]V_i x\Big\| : V_i \in \mathcal{L}(X), \|V_i\| \le 1, x \in X, \|x\| \le 1\Big\}\\
&\le \sup\Big\{\Big\|\sum_{i=1}^{n}[U(t_i) - U(t_{i-1})]x_i\Big\| : x_i \in X, \|x_i\| \le 1\Big\}.
\end{aligned}$$

On the other hand, given $x_i \in X$, $i = 1, \ldots, n$, choose $y \in X$ of norm one and let f be a continuous linear functional on X of norm one satisfying $f(y) = 1$ (Such an f exists by the Hahn-Banach Theorem). Define $V_i \in \mathcal{L}(X)$ by $V_i x = f(x)x_i$. Then $V_i(y) = x_i$ and $\|V_i\| = \|x_i\|$ for all $i = 1, \ldots, n$. Since $\|y\| = 1$ this yields the reverse inequality.

A function $U : [0,t] \to \mathcal{L}(X)$ which is of bounded $\mathcal{B}$-variation with respect to either (and hence both) of the bilinear triples $\mathcal{B}_1, \mathcal{B}_2$ of the preceding proposition is said to be of *bounded semi-variation* on $[0,t]$. If $U : \mathbb{R}_+ \to X$ is of bounded semi-variation on every interval $[0,t]$, $t > 0$, then U is said to be locally of bounded semi-variation.

The following fundamental existence result explains the importance of the notion of bounded $\mathcal{B}$-variation.

Theorem 2.2 (cf. Hönig, 1975, p. 24). *Let $\mathcal{B} = (X,Y,Z)$ be a bilinear triple and let $x : [0,t] \to X$ be of bounded $\mathcal{B}$-variation. If $y : [0,t] \to Y$ is continuous, then the Stieltjes integral*

$$\int_0^t x(ds)y(s)$$

exists and the estimate

$$\left\| \int_0^t x(ds)y(s) \right\| \leq V_{\mathcal{B},t}(x) \sup_{0\leq s\leq t} \|y(s)\| \tag{2.10}$$

holds.

Theorem 2.2 guarantees the existence of all the kinds of convolutions considered in this paper. For instance the convolution (2.6) exists in the norm topology of X whenever the family U of bounded linear operators on X is locally of bounded semi-variation and x is a continuous function from $\mathbb{R}_+$ to X. If U is locally of bounded semi-variation and V is a uniformly continuous operator family (i.e., a continuous mapping from $\mathbb{R}_+$ to $\mathcal{L}(X)$ equipped with the operator norm topology), then the convolution (2.7) makes sense in the operator norm topology. If V is only strongly continuous, then the convolution (2.7) can still be defined pointwise:

$$(U \star V)(t)x = \int_0^t U(ds)V(t-s)xds. \tag{2.11}$$

The estimate (2.10) shows that $(U \star V)(t)$ is a bounded linear operator on X. If U and V are both uniformly continuous, if $U(0) = V(0) = 0$ and if either U or V is locally of bounded semi-variation, then $(U \star V)(0) = 0$ and

$$(U \star V)(t) = \int_0^t U(ds)V(t-s) = \int_0^t U(t-s)V(ds). \tag{2.12}$$

This follows immediately by integration by parts.

If U is only strongly continuous the convolution $\int_0^t U(t-s)V(ds)$ does not necessarily make sense. But for families of adjoint operators such a convolution can be defined in a weak* sense.

Lemma 2.3. *Let U be a family of bounded linear operators on a Banach space X which is locally of bounded semi-variation; that is, locally of bounded $\mathcal{B}$-variation with respect to either of the bilinear triples in Proposition* 2.1. *Then for any $x^* \in X^*$ the function $t \mapsto U^*(t)x^*$ is locally of bounded $\mathcal{B}'$-variation with respect to the bilinear triple $\mathcal{B}' = (X^*, X, \mathbb{C})$ and*

$$V_{\mathcal{B}',t}(U^*(\cdot)x^*) \leq V_{\mathcal{B},t}(U)\|x^*\|. \tag{2.13}$$

Proof. This is obvious, since

$$\begin{aligned}
&\sup\Big\{\Big|\sum_{i=1}^{n}\langle x_i,(U^*(t_i)-U^*(t_{i-1}))x^*\rangle\Big| : x_i\in X, \|x_i\|\le 1\Big\}\\
&=\sup\Big\{\Big|\sum_{i=1}^{n}\langle (U(t_i)-U(t_{i-1}))x_i,x^*\rangle\Big| : x_i\in X, \|x_i\|\le 1\Big\}\\
&\le\sup\Big\{\Big\|\sum_{i=1}^{n}(U(t_i)-U(t_{i-1}))x_i\Big\| : x_i\in X, \|x_i\|\le 1\Big\}\|x^*\|.
\end{aligned}$$

Let U be as in Lemma 2.3 and let T be a strongly continuous family of bounded linear operators on X. By Theorem 2.2 and Lemma 2.3 the integral $\int_0^t\langle T(t-s)x, U^*(ds)x^*\rangle$ exists and

$$\begin{aligned}
\Big|\int_0^t\langle T(t-s)x,U^*(ds)x^*\rangle\Big| &\le V_{\mathcal{B}',t}(U^*(\cdot)x^*)\sup_{0\le s\le t}\|T(t-s)x\|\\
&\le MV_{\mathcal{B},t}(U)\|x^*\|\|x\|,
\end{aligned}\tag{2.14}$$

where we have used (2.10), (2.13) and the Banach-Steinhaus Theorem. Thus we can define the convolution $(T^*\star U^*)(t)$ through the formula

$$\langle x,(T^*\star U^*)(t)x^*\rangle=\int_0^t\langle T(t-s)x,U^*(ds)x^*\rangle\tag{2.15}$$

and (2.14) shows that $(T^*\star U^*)(t)$ so defined is a bounded linear operator on X^*. It is clear from (2.15) that $(T^*\star U^*)(t)=(U\star T)(t)^*$, with $(U\star T)(t)$ defined as in (2.11).

Next we consider continuity properties of the Stieltjes convolution and its relation to the ordinary convolution.

Proposition 2.4. *Let $\mathcal{B}=(X,Y,Z)$ be a bilinear triple and let $x:\mathbb{R}_+\to X$, $y:\mathbb{R}_+\to Y$ be continuous functions. Assume that $x(0)=0$ and that x is locally of bounded $\mathcal{B}$-variation. Then the following assertions hold:*

(a) *$x\star y$ is a continuous function from $\mathbb{R}_+$ to Z and $(x\star y)(0)=0$.*
(b) *$\frac{d}{dt}(x*y)(t)=(x\star y)(t)$.*
(c) *If $X=Y=Z$, $y(0)=0$ and y is locally of bounded $\mathcal{B}$-variation, then so is $x\star y$ and*

$$V_{\mathcal{B},t}(x\star y)\le V_{\mathcal{B},t}(x)V_{\mathcal{B},t}(y).$$

Proof. (a) Notice that

$$(x\star y)(t)=\int_0^t x(ds)(y(t-s)-y(t))+x(t)y(t).\tag{2.16}$$

The continuity of $x\star y$ follows from (2.16) and the continuity of x and y together with the estimates (2.1) and (2.10). In particular $(x\star y)(0)=0$.

(b) First assume that y is continuously differentiable. The assertion for continuous y is then easily checked by a standard approximation argument. By integration by parts,

$$\frac{d}{dt}(x * y)(t) = x(t)y(0) + \int_0^t x(s)y'(t-s)ds = (x \star y)(t). \tag{2.17}$$

Notice that we have used $x(0) = 0$.

(c) Consider a partition $0 = t_0 < \cdots < t_{n+1} = t$ and elements $x_1, \ldots, x_{n+1}$ with $\|x_j\| \leq 1$. Extend y to the negative numbers by $y(r) = 0, r < 0$, and notice that with this convention

$$(x \star y)(t_j) = \int_0^t x(ds)y(t_j - s).$$

Then, by (2.10),

$$\Big\| \sum_{j=0}^n \Big((x \star y)(t_{j+1}) - (x \star y)(t_j)\Big)x_j \Big\| = \Big\| \int_0^t x(ds) \sum_{j=0}^n \big(y(t_{j+1}-s) - y(t_j - s)\big)x_j \Big\|$$

$$\leq V_{\mathcal{B},t}(x) \sup_{0 \leq s \leq t} \Big\| \sum_{j=0}^n \big(y(t_{j+1}-s) - y(t_j-s)\big)x_j \Big\| \leq V_{\mathcal{B},t}(x) V_{\mathcal{B},t}(y),$$

which implies the assertion.

The next proposition on the associativity of the Stieltjes convolutions could be formulated in greater generality, but we prefer to state it only in terms of operator families.

Proposition 2.5. *Let U and V be uniformly continuous families of bounded linear operators on X, $f : [0, \infty) \to X$ continuous. Then the following statements hold:*

(a) $U * (V * f) = (U * V) * f$.

(b) *If U, V are locally of bounded semi-variation and $U(0) = V(0) = 0$, then $U \star (V \star f) = (U \star V) \star f$.*

(c) *If U, V are as in* (b) *and $T(t)$, $t \geq 0$, is a strongly continuous family of bounded linear operators, then $U \star (V \star T) = (U \star V) \star T$.*

Proof. Assertion (a) is proved in a standard fashion.

Differentiation of the identity in (a) yields, according to (2.17),

$$\frac{d}{dt} U * (V * f) = U * \frac{d}{dt}(V * f) = U * (V \star f).$$

On the other hand,

$$\frac{d}{dt} U * (V * f) = U \star (V * f)$$

by Proposition 2.4 (b). We thus have

$$U * (V \star f) = U \star (V * f)$$

and repeating the above argument to this identity we obtain assertion (b).

The assertion in (c) follows directly from (b).

Theorem 2.6. *Let X be a Banach space, $\mathcal{B} = (\mathcal{L}(X), \mathcal{L}(X), \mathcal{L}(X))$ and let $\tau > 0$. The set $\mathcal{A}_\tau$ of all continuous functions $U : [0,\tau] \to \mathcal{L}(X)$ of bounded semi-variation satisfying $U(0) = 0$ is a Banach algebra with Stieltjes convolution as multiplication and $V_{\mathcal{B},\tau}$ as norm.*

Proof. It is clear that $\mathcal{A}_\tau$ is a vector space and that $V_{\mathcal{B},\tau}$ is a semi-norm. The condition $U(0) = 0$ implies that $V_{\mathcal{B},\tau}$ is in fact a norm. Proposition 2.4 shows that the Stieltjes convolution of two elements in $\mathcal{A}_\tau$ remains in $\mathcal{A}_\tau$ and that the crucial norm inequality holds. By Proposition 2.5, Stieltjes convolution is associative. It remains to be shown that $\mathcal{A}_\tau$ is complete.

Taking $x(s) = U(s)$, $y(s) \equiv I$ in (2.10) one obtains

$$\|U(t)\| \leq V_{\mathcal{B},t}(U).$$

As $V_{\mathcal{B},t}(U)$ is obviously increasing in t, we obtain

$$\sup_{0 \leq t \leq \tau} \|U(t)\| \leq V_{\mathcal{B},\tau}(U). \tag{2.18}$$

Let U_N be a Cauchy sequence in $\mathcal{A}_\tau$. By (2.18) it is also a Cauchy sequence with respect to the supremum norm and hence there exists a continuous $U : [0,\tau] \to \mathcal{L}(X)$ such that U_N converges uniformly to U. Obviously $U(0) = 0$. In order to see that $U_N \to U$ in the topology of $\mathcal{A}_\tau$, let $\epsilon > 0$. As U_N is a Cauchy sequence, there exists some N_ϵ such that $V_{\mathcal{B},\tau}(U_N - U_M) \leq \epsilon$ for $N, M \geq N_\epsilon$. Choose a fixed, but arbitrary, partition $0 = t_0 < \ldots < t_n = \tau$ and fixed, but arbitrary, elements $x_i \in X, |x_i| \leq 1$. Then

$$\left\| \sum_{i=1}^{n} [U_N(t_i) - U_M(t_i) - U_N(t_{i-1}) + U_M(t_{i-1})] x_i \right\| \leq \epsilon$$

for $N, M \geq N_\epsilon$. As $U_M(s) \to U(s)$ in the operator norm, we can take the limit $M \to \infty$ in this inequality and obtain

$$\left\| \sum_{i=1}^{n} [U_N(t_i) - U(t_i) - U_N(t_{i-1}) + U(t_{i-1})] x_i \right\| \leq \epsilon$$

for $N \geq N_\epsilon$. As our partition and our choice of elements x_i was arbitrary, we can take the supremum and obtain that U is of bounded semi-variation and satisfies $V_{\mathcal{B},\tau}(U_N - U) \leq \epsilon$ for $N \geq N_\epsilon$.

Let $\mathcal{A}$ denote the algebra of all uniformly continuous families U of bounded linear operators which are locally of bounded semi-variation and satisfy $U(0) = 0$. In other words, $U \in \mathcal{A}$ if and only if the restriction of U to $[0,\tau]$ belongs to $\mathcal{A}_\tau$ for all $\tau > 0$. Consider the convolution equation

$$V = V_0 + V_0 \star V = V_0 + V \star V_0 \tag{2.19}$$

with a given $V_0 \in \mathcal{A}$. A solution $V \in \mathcal{A}$ to (2.19) is called a *resolvent family* for V_0. Since $\mathcal{A}$ is an algebra the resolvent family is uniquely determined once its existence is guaranteed (cf. Gripenberg et. al (1990) Section 9.3, Lemma 3.3).

The standard way to prove existence of a resolvent family is to actually construct it through the series expansion

$$V = \sum_{j=0}^{\infty} V_n, \qquad V_{n+1} = V_0 \star V_n. \tag{2.20}$$

The main problem consists in finding a reasonable topology such that the series in (2.20) converges and that the limit is uniformly continuous and locally of bounded semi-variation. If

$$\lim_{t \downarrow 0} V_{\mathcal{B},t}(V_0) < 1, \tag{2.21}$$

then the norm of V_0 in $\mathcal{A}_\tau$ is less than one for τ small enough and hence standard Banach algebra arguments show that the series in (2.20) converges in the topology of $\mathcal{A}_\tau$ to an element of $\mathcal{A}_\tau$; in other words, the series expansion yields a local solution to the convolution equation (2.19). Note that (2.21) holds in particular if

$$\lim_{t \downarrow 0} V_{\mathcal{B},t}(V_0) = 0. \tag{2.22}$$

In order to obtain existence of a global solution we introduce equivalent weighted norms on $\mathcal{A}_\tau$.

Define, for all $U \in \mathcal{A}$ and all $\lambda \geq 0$,

$$\|U\|_{\lambda,t} = \sup \left\| \sum_{j=0}^{n} (U(t_{j+1}) - U(t_j)) e^{-\lambda t_j} x_j \right\|, \tag{2.23}$$

where the supremum is taken over all partitions $0 = t_0 < \cdots < t_{n+1} = t$ and all elements $x_1, \ldots, x_{n+1} \in X$ with $\|x_j\| \leq 1$. For each $t \geq 0, \lambda \geq 0$, the mapping $U \mapsto \|U\|_{\lambda,t}$ is a norm on $\mathcal{A}_t$ and a semi-norm on $\mathcal{A}$. The mapping $t \mapsto \|U\|_{\lambda,t}$ is monotone, non-decreasing, non-negative, and bounded on bounded intervals. For each t, the norms $\|\cdot\|_{\lambda,t}$ are equivalent to $V_{\mathcal{B},t}(\cdot) = \|\cdot\|_{0,t}$. If U and V belong to $\mathcal{A}$ then

$$\|(U \star V)\|_{\lambda,t} \leq \|U\|_{\lambda,t} \, \|V\|_{\lambda,t} \tag{2.24}$$

(this is proved exactly as Lemma 2.4 (c)).

Theorem 2.7. *Let $V_0 \in \mathcal{A}$ satisfy condition* (2.21). *Then there exists a unique resolvent family $V \in \mathcal{A}$ for V_0. V is given by the series* (2.20) *which converges in $\mathcal{A}_\tau$ for all $\tau > 0$. If* (2.22) *holds, then* $\lim_{t \downarrow 0} V_{\mathcal{B},t}(V) = 0$.

Proof. For any $\tau > 0$, we can choose $\lambda > 0$ such that $\|V_0\|_{\lambda,\tau} < 1$. It follows from (2.24) that

$$\|V_n\|_{\lambda,\tau} \leq (\|V_0\|_{\lambda,\tau})^n.$$

Hence the series (2.20) converges with respect to the norm $\|\cdot\|_{\lambda,\tau}$ which is equivalent to the norm of $\mathcal{A}_\tau$. Thus the series (2.20) converges in $\mathcal{A}_\tau$ for all $\tau > 0$ and hence the limit V belongs to $\mathcal{A}$. By induction we obtain from Proposition 2.5 (c) that $V_0 \star V_n = V_{n+1} = V_n \star V_0$. Summing up this relation over n and taking limits one concludes that V is a solution to equation (2.19). Finally, if (2.22) holds, then Proposition 2.4 (c) shows that $\lim_{t \downarrow 0} V_{\mathcal{B},t}(V_n) = 0$ for all n and hence the triangle inequality $V_{\mathcal{B},t}(V) \leq V_{\mathcal{B},t}(V - V_n) + V_{\mathcal{B},t}(V_n)$ implies $\lim_{t \downarrow 0} V_{\mathcal{B},t}(V) = 0$.

Remark 2.8. It is well known (Riesz and Sz.-Nagy, 1955, p. 17) that a scalar valued function and its variation have the same points of continuity and discontinuity. So one may conjecture that condition (2.22) holds for all $V_0 \in \mathcal{A}$. However, we will present an example elsewhere that (2.22) or the weaker condition (2.21) do not hold for $V_0 \in \mathcal{A}$ in general, even if X is an abstract M-space. It should be noted that the property (2.21) is *not* preserved by the resolvent family whereas the condition (2.22) is. For these reasons we shall in the next sections, when we develop the theory of step response, assume that the step response satisfies condition (2.22).

The importance of resolvent kernels consists in the fact that they provide solutions to convolution equations.

Proposition 2.9 (cf. Gripenberg et. al. (1990), Section 9.3, Lemma 3.5). *Let V_0 be a family of bounded linear operators on the Banach space X which is continuous with respect to the uniform operator topology, locally of bounded semi-variation and which satisfies $V_0(0) = 0$ and $\lim_{t\downarrow 0} V_{\mathcal{B},t}(V_0) < 1$. Let V be the resolvent family of V_0; that is, the unique solution of equation (2.19). Let U_0 be a strongly continuous family and W_0 a uniformly continuous family of bounded linear operators on X, $W_0(0) = 0$. Then the convolution equation*

$$U = U_0 + V_0 \star U$$

is uniquely solved by

$$U = U_0 + V \star U_0\,,$$

whereas

$$W = W_0 + W \star V_0$$

is uniquely solved by

$$W = W_0 + W_0 \star V\,.$$

3. Cumulative output, step response and perturbation of semigroups. Let T be a semigroup on the Banach space X; i.e., a family $\{T(t)\}_{t\geq 0}$ of bounded linear operators satisfying $T(t+s) = T(t)T(s)$, $T(0) = I$. At this point we do not specify any continuity properties of T, but in the sequel we shall only be interested in the cases of either a strongly continuous semigroup or the adjoint of a strongly continuous semigroup.

Definition 3.1. A family $V = \{V(t)\}_{t\geq 0}$ of bounded linear operators from X to a Banach space Y is called a *cumulative output* family for the semigroup T if the following properties hold:

(i) $V(0) = 0$.

(ii) $V(t+s) - V(t) = V(s)T(t), \quad t, s \geq 0$.

A family $V = \{V(t)\}_{t\geq 0}$ of bounded linear operators from Y to X is called a *step response* for T if it satisfies (i) and

(ii)′ $V(t+s) - V(t) = T(t)V(s), \quad t, s \geq 0$.

If $Y = X$ we say that V is a cumulative output (step response) *on* X.

The next elementary proposition states that cumulative output and step response are dual concepts. Moreover we establish that uniformly continuous step responses (cumulative outputs) of exponentially bounded semigroups are exponentially bounded as well. This statement is important because we will take Laplace transforms of V in Section 4.

Proposition 3.2. *Let T be a semigroup on a Banach space X and let U be a step response from Y to X (cumulative output from X to Y) for T. Then the following properties hold.*

(a) *$U^* = \{U(t)^*\}_{t\geq 0}$ is a cumulative output from X^* to Y^* (step response from Y^* to X^*) for the adjoint semigroup $T^* = \{T(t)^*\}_{t\geq 0}$.*

(b) *If U is uniformly continuous, so is U^*.*

(c) *If $Y = X$ and if U is locally of bounded semi-variation, then for all $x^* \in X^*$, $t \mapsto U^*(t)x^*$ is locally of bounded $\mathcal{B}'$-variation with respect to the bilinear triple $\mathcal{B}' = (X^*, X, \mathbb{C})$*

(d) *Let U be uniformly continuous. If the semigroup T is exponentially bounded, so is U.*

Proof. (a) The properties (i) , (ii) and (ii)$'$ follow immediately by taking adjoints.

(b) This is obvious since the operator norm is preserved when taking adjoints.

(c) This was already proved in Lemma 2.3.

(d) It follows from (ii)$'$ that

$$U(n+1) = U(1) + T(1)U(n).$$

Hence

$$\|U(n+1)\| \leq \|U(1)\| + c\|U(n)\|$$

for some positive constant c independent of n. By induction

$$\|U(n+1)\| \leq (c+1)^n\|U(1)\|.$$

Now split $t = n + s$ with $0 \leq s < 1$. Then

$$\|U(t)\| \leq \|U(s)\| + \|U(n)\|\|T(s)\| \leq c_0 + (c+1)^{(n-1)}c_0 \leq 2c_0(c+1)^{(n-1)}$$

where $c_0 = \max\{\sup_{0\leq s<1}\|U(s)\|, \sup_{0\leq s<1}\|T(s)\|\}$. c_0 is finite since U is uniformly continuous and T is exponentially bounded. Choosing $\lambda = \log(c+1)$ we have

$$\|U(t)\| \leq 2c_0(c+1)^{(n-1)} = 2c_0e^{\lambda(n-1)} \leq 2c_0e^{\lambda t},$$

which shows that U is exponentially bounded. Taking adjoints we infer that a uniformly continuous cumulative output of an exponentially bounded semigroup is exponentially bounded, too.

We now turn to the perturbation problem outlined in the introduction. Motivated by Remark 2.8 we first introduce the notion of *regular* step response.

Definition 3.3. Let T be a semigroup on a Banach space X and let U be a step response for T on X. U is called a *regular step response* for T if it satisfies the following three conditions.

(i) The mapping $t \mapsto U(t)$ is continuous from $\mathbb{R}$ to $\mathcal{L}(X)$ equipped with the uniform operator topology.

(ii) U is locally of bounded semi-variation.

(iii) $\lim_{t\downarrow 0} V_{\mathcal{B},t}(U) = 0$.

Theorem 3.4. *Let T_0 be a strongly continuous semigroup on a Banach space X and let U_0 be a regular step response for T_0 on X. Let U be the unique solution of the renewal equation*

$$U(t) = U_0(t) + \int_0^t U(d\tau)U_0(t-\tau) \tag{3.1}$$

and let T be defined by

$$T(t) = T_0(t) + \int_0^t U(d\tau)T_0(t-\tau). \tag{3.2}$$

Then T is a strongly continuous semigroup and U is a regular step response for T on X. Moreover,

$$T(t) = T_0(t) + \int_0^t U_0(d\tau)T(t-\tau). \tag{3.3}$$

Proof. By Theorem 2.7, the renewal equation (3.1) has a unique solution U which satisfies condition (i) of Definition 3.1 and all three conditions of Definition 3.3. We have to show that T is a strongly continuous semigroup and that U and T are related by condition (ii)$'$ of Definition 3.1.

By Theorem 2.2 and Proposition 2.4 (a) the convolution integral in (3.2) makes sense and is continuous in the strong operator topology. By (3.1), (3.2) and the fact that U_0 is a step response for T_0 we have

$$\begin{aligned} U(t+s) &= U_0(t+s) + \int_0^{t+s} U(d\tau)U_0(t+s-\tau) \\ &= U_0(t) + T_0(t)U_0(s) + \int_0^t U(d\tau)[U_0(t-\tau) + T_0(t-\tau)U_0(s)] \\ &\quad + \int_0^s d_\tau[U(t+\tau) - U(t)]U_0(s-\tau) \\ &= U_0(t) + \int_0^t U(d\tau)U_0(t-\tau) + [T_0(t) + \int_0^t U(d\tau)T_0(t-\tau)]U_0(s) \\ &\quad + \int_0^s d_\tau[U(t+\tau) - U(t)]U_0(s-\tau), \end{aligned}$$

from which it follows that

$$U(t+s) - U(t) = T(t)U_0(s) + \int_0^s d_\tau[U(t+\tau) - U(t)]U_0(s-\tau).$$

In order to apply Proposition 2.9 to this equation we fix t and set

$$W(s) = U(t+s) - U(t), \quad W_0(s) = T(t)U_0(s).$$

This yields $W = W_0 + W \star U_0$ which is solved by $W = W_0 + W_0 \star U$ according to the second statement in Proposition 2.9. Translating back we obtain

$$U(t+s) - U(t) = T(t)U_0(s) + \int_0^s T(t)U_0(d\sigma)U(s-\sigma) = T(t)U(s),$$

which shows that U and T are related by condition (ii)′ of Definition 3.1.

We have already observed that the operator family is strongly continuous. Obviously $T(0) = I$. To complete the proof we check that T has the semigroup property:

$$\begin{aligned} T(t+s) &= T_0(t+s) + \int_0^{t+s} U(d\tau)T_0(t+s-\tau) \\ &= T_0(t)T_0(s) + \int_0^t U(d\tau)T_0(t-\tau)T_0(s) \\ &\quad + \int_0^s d_\tau[U(\tau+t) - U(t)]T_0(s-\tau) \\ &= \Big[T_0(t) + \int_0^t U(d\tau)T_0(t-\tau)\Big]T_0(s) + \int_0^s T(t)U(d\tau)T_0(s-\tau) \\ &= T(t)\Big[T_0(s) + \int_0^s U(d\tau)T_0(s-\tau)\Big] = T(t)T(s). \end{aligned}$$

Finally (3.3) follows from the first statement in Proposition 2.9.

Corollary 3.5. *Let T_0 be a strongly continuous semigroup on a Banach space X and let U_0 be a regular step response for T_0. Let U be the unique solution of* (3.1) *and define T by* (3.2). *Then the adjoint U^* of U satisfies the Stieltjes renewal equation*

$$U^*(t) = U_0^*(t) + \int_0^t U_0^*(t-\tau)U^*(d\tau) \tag{3.4}$$

and the adjoint T^* of T is given by

$$T^*(t)x^* = T_0^*(t)x^* + \int_0^t T_0^*(t-\tau)U^*(d\tau)x^*, \quad x^* \in X^*. \tag{3.5}$$

Furthermore

$$T^*(t)x^* = T_0^*(t)x^* + \int_0^t T^*(t-\tau)U_0^*(d\tau)x^*, \quad x^* \in X^*. \tag{3.6}$$

Proof. Let U be the unique solution of (3.1) and define T by (3.2). By Theorem 3.4, T is a strongly continuous semigroup with regular step response U on X. Taking adjoints of both sides of equation (3.2) one finds that U^* satisfies (3.4). In the same way (3.2) and (3.3) yield (3.5) and (3.6), respectively.

4. Infinitesimal properties of step responses and cumulative outputs. In this section we investigate in which sense cumulative outputs and step responses can be differentiated. We show that, if a strongly continuous semigroup T_0 with infinitesimal generator A_0 is perturbed by a step response as described in section 3, then the generator A of the perturbed semigroup T has the form

$$\begin{aligned} \mathcal{D}(A) &= \{x \in X : (I-P)x \in \mathcal{D}(A_0)\} \\ A &= \lambda P + A_0(I-P) \end{aligned}$$

for some linear operator P on X and some fixed $\lambda > 0$. We further show that perturbation of a dual semigroup T_0^* by a cumulative output family corresponds to perturbation of A_0^* by an A_0^*-bounded operator. The perturbation theory developed in this paper therefore generalizes both the theory of Desch and Schappacher (1987) and that of Clément, Diekmann, Gyllenberg, Heijmans and Thieme (1987, 1988, 1989a, 1989b). If the step response U_0 is Lipschitz continuous, then the perturbed semigroup is generated by the part of $A_0^{\odot *} + C$ in X for a bounded linear operator C from X to $X^{\odot *}$ and its adjoint by $A_0^* + C^\times$ with $C^\times$ being the restriction of C^* to $X^\odot$.

Let X be a Banach space and let T_0 be a strongly continuous semigroup with generator A_0 on X. Recall that the maximal space of strong continuity of the dual semigroup T_0^* on X^* is $X^\odot := \overline{\mathcal{D}(A_0^*)}$, and that the generator $A_0^\odot$ of $T_0^\odot := T_0^*|_{X^\odot}$ is the part of A_0^* in $X^\odot$ (cf. Butzer and Berens 1967). The Favard class of T_0 is by definition

$$\mathrm{Fav}(T_0) = \{x \in X : \|T_0(t)x - x\| = \mathcal{O}(t) \quad \text{as} \quad t \downarrow 0\}.$$

Equipped with the norm

$$\|x\|_{\mathrm{Fav}(T_0)} = \|x\| + \sup_{0<t\leq 1} \frac{1}{t}\|T_0(t)x - x\|,$$

$\mathrm{Fav}(T_0)$ is a Banach space.

Proposition 4.1. *Let U_0 be a regular step response for the strongly continuous semigroup T_0 on X. Then the following holds:*

(a) *$s \mapsto U_0^*(s)\int_0^t T_0^*(\tau)d\tau$ is continuously differentiable in the uniform operator topology and*

$$\frac{d}{ds}U_0^*(s)\int_0^t T_0^*(\tau)d\tau = U_0^*(t+s) - U_0^*(s). \tag{4.1}$$

(b) *$s \mapsto U_0^*(s)R(\lambda, A_0^*)$ is continuously differentiable in the uniform operator topology and*

$$\frac{d}{ds}U_0^*(s)R(\lambda, A_0^*)\Big|_{s=0} = \hat{U}_0^*(\lambda) \tag{4.2}$$

for all λ sufficiently large.

(c) *$s \mapsto U_0^*(s)x^\odot$ is continuously differentiable in the norm topology of X^* for all $x^\odot \in \mathcal{D}(A_0^*)$. Let*

$$C^\times x^\odot := \frac{d}{ds}U_0^*(s)x^\odot\Big|_{s=0} = \lim_{s\downarrow 0}\frac{1}{s}U_0^*(s)x^\odot,\ x^\odot \in \mathcal{D}(A_0^*). \tag{4.3}$$

Then

$$C^\times x^\odot = \hat{U}_0^*(\lambda)(\lambda I - A_0^*)x^\odot\,,\ x^\odot \in \mathcal{D}(A_0^*) \tag{4.4}$$

for all sufficiently large λ. Moreover,

$$C^\times x^\odot = \lim_{\lambda\to\infty}\lambda\hat{U}_0^*(\lambda)x^\odot\,,\ x^\odot \in \mathcal{D}(A_0^*). \tag{4.5}$$

(d) *$C^\times$ is an A_0^*-bounded operator with A_0^*-bound zero.*

Proof. Integrating relation (ii) in Definition 3.1 with respect to t one obtains

$$U_0^*(s)\int_0^t T_0^*(\tau)d\tau = \int_0^t [U_0^*(\tau+s) - U_0^*(\tau)]d\tau = \int_s^{t+s} U_0^*(\tau)d\tau - \int_0^t U_0^*(\tau)d\tau. \quad (4.6)$$

Differentiation with respect to s provides (a).

To prove (b), note that

$$\frac{1}{s}U_0^*(s)R(\lambda, A_0^*) = \frac{1}{s}\int_0^\infty e^{-\lambda\tau}U_0^*(s)T_0^*(\tau)d\tau = \int_0^\infty e^{-\lambda\tau}\frac{1}{s}[U_0^*(\tau+s) - U_0^*(\tau)]d\tau$$
$$\rightarrow \int_0^\infty e^{-\lambda\tau}U_0^*(d\tau) = \hat{U}_0^*(\lambda),$$

as $s \downarrow 0$ in the uniform operator topology.

For every $x^\odot \in \mathcal{D}(A_0^*)$ there is a unique x^* such that $x^\odot = R(\lambda, A_0^*)x^*$. By (b) we therefore have

$$C^\times x^\odot = \lim_{s\downarrow 0}\frac{1}{s}U_0^*(s)R(\lambda, A_0^*)x^* = \hat{U}_0^*(\lambda)x^* = \hat{U}_0^*(\lambda)\,(\lambda I - A_0^*)x^\odot$$

in the norm-topology of X^*. This proves the first part of (c) and that (4.4) holds. By (4.4),

$$\| C^\times x^\odot \| \leq \lambda \| \hat{U}_0^*(\lambda) \| \| x^\odot \| + \| \hat{U}_0^*(\lambda) \| \| A_0^* x^\odot \|, \quad (4.7)$$

which shows that $C^\times$ is A_0^*-bounded. Since $\lim_{\lambda\to\infty} \| \hat{U}_0^*(\lambda) \| \leq \lim_{t\downarrow 0} V_{\mathcal{B},t}(U_0) = 0$ by the regularity of U_0, the A_0^*-bound must be zero. Taking the limit as $\lambda \to \infty$ in (4.4) one obtains (4.5).

Part of the conclusion of Proposition 4.1 can be reformulated as follows. Every cumulative output family $\{V_0(t)\}_{t\geq 0}$ which is the adjoint of a regular step response $\{U_0(t)\}_{t\geq 0}$ for a strongly continuous semigroup T_0 is of the form

$$V_0(t) = C^\times \int_0^t T_0^*(\tau)d\tau \quad (4.8)$$

for some A_0^*-bounded operator $C^\times$ with A_0^*-bound zero defined by (4.3). One may ask whether the converse holds true: Is V_0, defined by (4.8) for an A_0^*-bounded operator on X^* with A_0^*-bound zero, a cumulative output family for T_0^*? It is clear that V satisfies conditions (i) and (ii) of Definition 3.2 so V_0 is indeed a cumulative output for T_0^*. Moreover, V is continuous in the uniform operator topology. It should be noted, however, that V_0 need not be the adjoint of a regular step response.

Theorem 4.2. *Let T_0 be a strongly continuous semigroup on the Banach space X with infinitesimal generator A_0. Let U_0 be a regular step response for T_0. Let T be the corresponding semigroup in accordance with Theorem 3.4. Then we have the following characterization of infinitesimal generators.*

(a) *The infinitesimal generator A of T is given by*

$$\begin{aligned}\mathcal{D}(A) &= \{x \in X : (I - \hat{U}_0(\lambda))x \in \mathcal{D}(A_0)\} \\ Ax &= \lambda\hat{U}_0(\lambda)x + A_0(I - \hat{U}_0(\lambda))x, \quad x \in \mathcal{D}(A),\end{aligned} \quad (4.9)$$

for every fixed sufficiently large $\lambda > 0$.

(b) *The weak*-generator* A^* *of* T^* *equals* $A_0^* + C^\times$, *where* $\mathcal{D}(A^*) = \mathcal{D}(A_0^*)$ *and*

$$\begin{aligned} \mathcal{D}(C^\times) := \{x^\odot \in X^\odot : \lim_{s\downarrow 0} \frac{1}{s} U_0^*(s)x^\odot \text{ exists } \} \supseteq \mathcal{D}(A_0^*) \\ C^\times x^\odot = \lim_{s\downarrow 0} \frac{1}{s} U_0^*(s)x^\odot , \quad x^\odot \in \mathcal{D}(C^\times). \end{aligned} \tag{4.10}$$

In particular, $X^\odot = \overline{\mathcal{D}(A_0^*)}$ *is invariant under the perturbed semigroup* T^* *and is the maximal subspace of* X^* *on which* $T^*(t)$ *is strongly continuous.*

(c) *The infinitesimal generator* A *of* T *can alternatively be described as*

$$\begin{aligned} \mathcal{D}(A) = \{x \in X : \lim_{s\downarrow 0} \frac{1}{s}\Big(T_0(s)x + U_0(s)x - x\Big) \text{ exists } \} \\ Ax = \lim_{s\downarrow 0} \frac{1}{s}\Big(T_0(s)x + U_0(s)x - x\Big), \quad x \in \mathcal{D}(A). \end{aligned} \tag{4.11}$$

Proof. (a) By Theorem 3.4, the formula (3.2) defines a strongly continuous semigroup T. By Proposition 3.2 (d), the formal Laplace-transforms of the introduction are valid, in particular we have the expression (1.17) for the resolvent of A. It follows from (1.17) that

$$\mathcal{D}(A) = (I - \hat{U}_0(\lambda))^{-1}(\mathcal{D}(A_0)) = \{x \in X : (I - \hat{U}_0(\lambda))x \in \mathcal{D}(A_0)\}.$$

Inverting both sides of (1.17) one obtains (4.9).

(b) Taking Laplace transforms of (3.4) and (3.5) (using weak* Riemann Stieltjes integrals in (3.5)) one obtains

$$R(\lambda, A^*) = R(\lambda, A_0^*)[I - \hat{U}_0^*(\lambda)]^{-1} \tag{4.12}$$

which shows that $\mathcal{D}(A^*) = \mathcal{D}(A_0^*)$. Inversion of both sides of (4.12) yields

$$A^* = A_0^* + \hat{U}_0^*(\lambda)(\lambda I - A_0^*).$$

Proposition 4.1 (c) now implies (4.10).

(c) Let $x \in \mathcal{D}(A)$. Then, by (3.3),

$$T(s)x = T_0(s)x + \int_0^s U_0(dr)\Big(x + \int_0^{s-r} T(\tau)Axd\tau\Big).$$

We integrate by parts:

$$T(s)x = T_0(s)x + U_0(s)x + \int_0^s U_0(r)T(s-r)Axdr.$$

This shows that $\lim_{s\downarrow 0} \frac{1}{s}(T_0(s)x + U_0(s)x - x)$ exists and equals Ax. To prove the reverse direction, let $x \in X$ such that $y = \lim_{s\downarrow 0} \frac{1}{s}(T_0(s)x + U_0(s)x - x)$ exists. Then, by part (b),

$$\langle y, x^\odot \rangle = \langle x, A^* x^\odot \rangle \quad \forall x^\odot \in \mathcal{D}(A^*).$$

Hence $x \in \mathcal{D}(A), Ax = y$ (cf. Ball, 1977).

Remark. a) It is tempting to write the expression (4.9) for A as

$$A = A_0 + (\lambda I - A_0)\hat{U}_0(\lambda). \tag{4.13}$$

However, the range of $\hat{U}_0(\lambda)$ need not lie in $\mathcal{D}(A_0)$ and therefore (4.13) does not necessarily make sense. A is a perturbation of A_0 only in the generalized sense of (4.9). Perturbations of the kind $A = \lambda P + A_0(I - P)$, with P a bounded operator from X to $\mathrm{Fav}(T_0)$, have first been considered by Desch and Schappacher (1987).

c) The characterization (4.11) is particularly satisfying because it uses only the data T_0 and V_0.

The generation result for T^* shows that perturbation of dual semigroups by cumulative output families is a generalization of the theory of bounded perturbations of dual generators developed by Clément, Diekmann, Gyllenberg, Heijmans and Thieme (1987, 1988, 1989a, 1989b). The next theorem shows that if the step response is Lipschitz continuous then the two perturbation theories are essentially equivalent.

Theorem 4.3. *Let $U_0, T_0, A_0, T, A, C^\times$ be as in Theorem 4.2 and assume in addition that U_0 is Lipschitz continuous with respect to the uniform operator topology. Then there exists a bounded operator C from X to $X^{\odot *}$ such that*

(a) *A is the part of $A_0^{\odot *} + C$ in X.*
(b) *$\mathcal{D}(C^\times) \supset X^\odot$ and $C^\times x^\odot = C^* x^\odot$, $\forall x^\odot \in X^\odot$.*

Proof. By proposition 4.1 (c),

$$C^\times x^\odot = \lim_{s \downarrow 0} \frac{1}{s} U_0^*(s) x^\odot \tag{4.14}$$

for all $x^\odot$ in $\mathcal{D}(A_0^*)$. Since U_0 is Lipschitz we have

$$\| \frac{1}{s} U_0^*(s) \| \leq L < \infty, \quad 0 < s \leq 1. \tag{4.15}$$

Therefore, the limit in (4.14) exists for all $x^\odot \in X^\odot = \overline{\mathcal{D}(A_0^*)}$ and defines a bounded linear operator $C^\times$ from $X^\odot$ to X^*. By Theorem 4.2, $\mathcal{D}(A^*) = \mathcal{D}(A_0^*)$ and $A^* - A_0^* = C^\times$. Theorem 3.1 of Diekmann, Gyllenberg and Heijmans (1989) (see also Robinson (1977)) now implies the existence of a bounded linear operator C from X to $X^{\odot *}$ such that

$$T(t)x = T_0(t)x + \int_0^t T_0^{\odot *}(t-\tau) C T_0(\tau) x d\tau, \quad x \in X.$$

It follows by the results of Clément, Diekmann, Gyllenberg, Heijmans and Thieme (1987) that (a) holds and that $A^* = A_0^* + C^*$. On the other hand $A^* = A_0^* + C^\times$ by Theorem 4.2. Therefore, C^* and $C^\times$ coincide on $\mathcal{D}(A_0^*)$ and hence (since both are bounded) on $X^\odot = \overline{\mathcal{D}(A_0^*)}$.

5. Two examples from structured population dynamics.

Example 5.1: *Age-structured population growth revisited.* Consider a population whose individuals are distinguished from one another by their age a and assume that this population is living under constant environmental conditions (in other words, we ignore time dependence and density dependence in order to arrive at a linear autonomous problem). A deterministic model for the growth of such a population requires the specification of two functions:

(i) the survival probability $\mathcal{F}(a)$,

(ii) the expected cumulative number of offspring $L(a)$.

So an individual of age a_0 survives till at least age $a_1 \geq a_0$ with probability $\mathcal{F}(a_1)/\mathcal{F}(a_0)$ and an individual of age a_0 will produce in the age bracket $[a_0, a_1]$ an expected number of offspring $\frac{L(a_1)-L(a_0)}{\mathcal{F}(a_0)}$ (note that, when taking the expectation of the number of offspring, we do not condition on survival till at least age a_1). Concerning these two model ingredients we assume throughout this section:

$(H_\mathcal{F})$ $\mathcal{F}(0) = 1$, $\mathcal{F}$ is positive, continuous and non-increasing on $\mathbb{R}_+ = [0, \infty)$.

(H_L) $L(0) = 0$, L is continuous and non-decreasing on $\mathbb{R}_+$.

In order to show how the description of the dynamics of the population fits into the framework introduced in this paper, we take $X = C_0(\mathbb{R}_+)$, provided with the supremum norm. Hence we can represent elements of X^* by (Borel) measures m defined on $\mathbb{R}_+$. The space X^* will be the population state space: at any given time instant the population size and composition is described by a (finite) non-negative measure m such that for any measurable subset ω of $\mathbb{R}_+$ the number of individuals with age $a \in \omega$ equals $m(\omega) = \int_\omega m(da)$. Now, if we disregard newborn for a moment, the individuals which are in ω (a short way of saying that their age belongs to ω) at time t, were in ω_t at time zero, where by definition

$$\omega_t = \{a_0 \in \mathbb{R}_+ : a_0 + t \in \omega\} = (\omega - t) \cap \mathbb{R}_+. \tag{5.1}$$

Therefore, we define

$$(T_0^*(t)m)(\omega) = \int_{\omega_t} \frac{\mathcal{F}(a_0+t)}{\mathcal{F}(a_0)} m(da_0) = \int_{[0,\infty)} \chi_\omega(a_0+t)\frac{\mathcal{F}(a_0+t)}{\mathcal{F}(a_0)} m(da_0). \tag{5.2}$$

One easily verifies that T_0^* is indeed the adjoint of the strongly continuous semigroup T_0 defined on $X = C_0(\mathbb{R}_+)$ by the explicit formula

$$(T_0(t)x)(a) = x(a+t)\frac{\mathcal{F}(a+t)}{\mathcal{F}(a)}. \tag{5.3}$$

As a next step we turn our attention towards reproduction. Let the measure m describe the population size and composition at time $t = 0$. Then

$$f(t,m) := \int_0^\infty \frac{L(a_0+t) - L(a_0)}{\mathcal{F}(a_0)} m(da_0) \tag{5.4}$$

is the expected cumulative number of direct offspring produced in the time interval $[0, t]$ ("direct" here means children, but not grandchildren etc.). This number is finite for whatever $m \in X^*$ if and only if

$$\frac{L(a_0+t) - L(a_0)}{\mathcal{F}(a_0)}$$

is bounded for $a_0 \to \infty$. Since all offspring has age zero at birth (by the very definition of age) we introduce

$$U_0^*(t)m = \delta f(t,m) \tag{5.5}$$

where δ is the Dirac measure at $a = 0$. Formally at least, $U_0^*(t)$ is the adjoint of $U_0(t)$ defined by

$$(U_0(t)x)(a) = x(0)\frac{L(a+t)-L(a)}{\mathcal{F}(a)}. \tag{5.6}$$

The assumption

$(H_{\mathcal{F},L})$ $\quad \forall t \geq 0 \quad \frac{L(a+t)-L(a)}{\mathcal{F}(a)} \to 0$ as $a \to \infty$

guarantees that $U_0(t)$ maps X into X and that $U_0^*(t)$ is indeed the adjoint. A trivial calculation reveals that

$$U_0(t+s) - U_0(t) = T_0(t)U_0(s). \tag{5.7}$$

Thus U_0 is a step response for T_0. Next we verify the regularity conditions. For $t_1 \geq t_2$,

$$\|U_0(t_1) - U_0(t_2)\| = \sup_{a\geq 0} \frac{L(a+t_1)-L(a+t_2)}{\mathcal{F}(a)},$$

which goes to zero for $t_1 - t_2 \downarrow 0$ (note that the condition $H_{\mathcal{F},L}$ for the behaviour at infinity guarantees uniform continuity). Next consider a partition $0 \leq t_0 < \ldots < t_{n+1} = s$ and elements $x_j \in X$ with $\|x_j\| \leq 1$. The estimate

$$\Big\|\sum_{j=0}^{n}(U_0(t_{j+1}) - U_0(t_j))x_j\Big\| = \sup_{a\geq 0}\Big|\sum_{j=0}^{n} x_j(0)\frac{L(a+t_{j+1})-L(a+t_j)}{\mathcal{F}(a)}\Big|$$
$$\leq \sup_{a\geq 0}\sum_{j=0}^{n}|x_j(0)|\frac{L(a+t_{j+1})-L(a+t_j)}{\mathcal{F}(a)} \leq \sup_{a\geq 0}\frac{L(a+s)-L(a)}{\mathcal{F}(a)} < \infty$$

shows that U_0 is locally of bounded semi-variation with $\|U_0\|(0+) = 0$. Here we have used that L is nondecreasing. We have thus shown that U_0 is a regular step response for T_0.

Remark: Under our conditions U_0 need not be of bounded variation with respect to the operator norm. For example, when $\mathcal{F}(a) \equiv 1$ and

$$L(a) = \begin{cases} 0, & \text{if } 0 \leq a < 1, \\ \sqrt{a-1}, & \text{if } 1 \leq a < 2, \\ 1, & \text{if } 2 \leq a, \end{cases}$$

this is not the case, as one can demonstrate as follows. Let ϵ_j be a sequence o positive real numbers such that

(i) $\sum_{j=0}^{\infty}\epsilon_j = 1$

(ii) $\sum_{j=0}^{\infty}\sqrt{\epsilon_j}$ diverges.

Define $t_0 = 0$ and $t_{j+1} = t_j + \epsilon_j$. Then t_j is an increasing sequence with limit one. Now observe that

$$\sum_{j=0}^{n} \|U_0(t_{j+1}) - U_0(t_j)\| = \sum_{j=0}^{n} \sup_{a\geq 0}(L(a+t_{j+1}) - L(a+t_j))$$
$$\geq \sum_{j=0}^{n} (L(1 - t_j + t_{j+1}) - L(1)) = \sum_{j=0}^{n} \sqrt{\epsilon_j} \to \infty$$

for $n \to \infty$.

Whenever L is globally Lipschitz continuous, on the other hand, it easily follows that U_0 is of bounded variation with respect to the operator norm.

The above example also shows that the property of being of bounded semi-variation is *not* inherited by the adjoint family. To see this, let m_j be the Dirac measure concentrated at the point t_j. Then using (5.4) and (5.5) one finds that

$$\left\| \sum_{j=0}^{n} (U_0^*(t_{j+1}) - U_0^*(t_j))m_j \right\| = \sum_{j=0}^{n} (L(1 - t_j + t_{j+1}) - L(1))$$

which tends to infinity as $n \to \infty$ exactly as above.

Calculating infinitesimal generators of the unperturbed and the perturbed semigroups is easy when $\mathcal{F}$ and L are smooth, but rather troublesome in general. For instance, the infinitesimal generator A_0 of T_0 is given by

$$D(A_0) = \{x \in C_0(\mathbb{R}_+) : a \mapsto x(a)\mathcal{F}(a) \text{ is } C^1 \text{ and } \frac{(x\mathcal{F})'}{\mathcal{F}} \in C_0(\mathbb{R}_+)\} \tag{5.8}$$

with

$$A_0 x = \frac{(x\mathcal{F})'}{\mathcal{F}},$$

but, depending on the assumptions we are willing to make concerning $\mathcal{F}$, a further elaboration can be given. Taking Laplace transforms, on the other hand, is easy irrespective of the smoothness of $\mathcal{F}$ and L: for $\operatorname{Re}\lambda$ sufficiently large we find

$$\begin{aligned}((\lambda I - A_0)^{-1}x)(a) &= \int_0^\infty e^{-\lambda\tau}(T_0(\tau)x)(a)d\tau = \int_0^\infty e^{-\lambda\tau}x(a+\tau)\frac{\mathcal{F}(a+\tau)}{\mathcal{F}(a)}d\tau \\ &= \int_a^\infty e^{-\lambda(s-a)}x(s)\frac{\mathcal{F}(s)}{\mathcal{F}(a)}ds.\end{aligned} \tag{5.9}$$

It follows that $\lambda \in \rho(A_0)$ for $\operatorname{Re}\lambda > -\mu_\infty$ whenever μ_∞ is such that there exists an $s_0 \geq 0$ such that $\frac{\mathcal{F}(a+s)}{\mathcal{F}(a)} \leq e^{-\mu_\infty(s-s_0)}$ for $s \geq s_0$ and $a \geq 0$. Moreover, the explicit expression shows at once that $(\lambda I - A_0)^{-1}$ is compact.

The half plane for which

$$(\hat{U}_0(\lambda)x)(a) = \int_0^\infty e^{-\lambda\tau}(U_0(d\tau)x)(a) = \frac{x(0)}{\mathcal{F}(a)}e^{\lambda a}\int_a^\infty e^{-\lambda s}L(ds) \tag{5.10}$$

is well-defined depends on the behaviour of L at infinity (when L approaches a finite limit exponentially with rate constant β_∞ it includes $\{\lambda : \operatorname{Re}\lambda > -\beta_\infty\}$). A straightforward calculation reveals that $\hat{U}_0(\lambda)$ has eigenvalue one if and only if λ is a root of the characteristic equation

$$\int_0^\infty e^{-\lambda s} L(ds) = 1. \tag{5.11}$$

The compactness of $\hat{U}_0(\lambda)$ is an immediate consequence of (5.10).

We conclude that, under the conditions $H_{\mathcal{F}}, H_L$ and $H_{\mathcal{F},L}$, our approach yields the existence of a dual semigroup on the space of measures for which the asymptotic behaviour for $t \to \infty$ can be deduced from an analysis of the characteristic equation (5.11), together with estimates for the radius of the essential spectrum in terms of the behaviour of L and $\mathcal{F}$ at infinity. The approach completely avoids the technicalities involved in a precise characterization of the domain of the generator.

Example 5.2: *Additional structure incorporated.* Under constant environmental conditions we can always parameterize the state of an individual by its age and its state at birth. As a consequence the following formulation is rather general, though the further specification of the model ingredients on the basis of (mechanistic) submodels for "growth" (movement through the individual state space), reproduction and survival may be a far from trivial task.

We assume that the birth-state takes values in a closed bounded subset D of $\mathbb{R}^n$. We shall usually denote the birth-state by z or ζ.

There are now three model ingredients:

(i) the survival probability $\mathcal{F}(a, z)$

(ii) the expected cumulative number of offspring $L(a, z)$

(iii) given that the mother had birth-state z and was of age a at the moment of giving birth, the distribution with respect to state at birth of the offspring is described by the probability measures $p(a, z)$ on D (i.e., $p(a, z)(\omega)$ is the probability that the offspring has state at birth belonging to the measurable subset ω of D).

Concerning these ingredients we assume:

($H_{\mathcal{F}}$) $\mathcal{F}$ is positive, continuous, $\mathcal{F}(0, z) = 1$ and $\mathcal{F}$ is non-increasing with respect to a,

(H_L) L is continuous, $L(0, z) = 0$ and L is non-decreasing with respect to a,

($H_{\mathcal{F},L}$) $\forall t \geq 0$, $\frac{L(a+t,z)-L(a,z)}{\mathcal{F}(a,z)} \to 0$ uniformly in z, as $a \to \infty$,

(H_p) $(a, z) \to p(a, z)$ is continuous with respect to the weak* topology of $M(D)$.

The operators $T_0(t)$ and $U_0(t)$ are defined on $C_0(\mathbb{R}_+ \times D)$ as follows:

$$(T_0(t)x)(a, z) = x(a + t, z)\frac{\mathcal{F}(a + t, z)}{\mathcal{F}(a, z)} \tag{5.12}$$

$$(U_0(t)x)(a, z) = \frac{1}{\mathcal{F}(a, z)} \int_0^t \Big(\int_D x(0, \zeta)p(a + \tau, z)(d\zeta)\Big)L(a + d\tau, z). \tag{5.13}$$

The definition of $T_0(t)$ is motivated exactly as in the foregoing section. But the idea behind $U_0(t)$ is slightly more complicated than before, so we shall spell out some of the steps in detail.

Consider an individual with age a_0 and birth-state z. During the time interval of length t the expected number of its offspring is

$$\frac{L(a_0+t,z)-L(a_0,z)}{\mathcal{F}(a_0,z)}$$

and their distribution with respect to birth-state is given by

$$\int_0^t p(a_0+\tau,z)\frac{1}{\mathcal{F}(a_0,z)}L(a_0+d\tau,z).$$

Note that this is a non-normalized distribution. Indeed, since $p(a,z)$ is a probability measure we find by evaluating this expression for the set D, $\int_0^t \frac{1}{\mathcal{F}(a_0,z)}L(a_0+d\tau,z) = \frac{L(a_0+t,z)-L(a_0,z)}{\mathcal{F}(a_0,z)}$ as required. So for a population of size and composition described by the measure m on $\mathbb{R}_+ \times D$ the birth-state distribution of the next generation is given by

$$\int_{\mathbb{R}_+\times D}\Big(\int_0^t p(a_0+\tau,z)\frac{1}{\mathcal{F}(a_0,z)}L(a_0+d\tau,z)\Big)dm(a_0,z)$$

and this formal calculation serves as the motivation to define

$$U_0^*(t)m = \delta_{a=0}\times\int_{\mathbb{R}_+\times D}\Big(\int_0^t p(a_0+\tau,z)\frac{1}{\mathcal{F}(a_0,z)}L(a_0+d\tau,z)\Big)dm(a_0,z), \quad (5.14)$$

where we use the notation that for any measure ν on D we define $\delta_{a=0}\times\nu$ as a measure on $\mathbb{R}_+\times D$ by

$$(\delta_{a=0}\times\nu)(\omega) = \nu(\{z\in D : (0,z)\in\omega\}).$$

Note that in (5.14), $p(a_0+\tau,z)$ is a measure on D.

By requiring that $U_0(t)$ is the pre-adjoint, we arrive at (5.13).

The estimate

$$\Big|\int_0^t\Big(\int_D x(0,\zeta)p(a+\tau,z)(d\zeta)\Big)L(a+d\tau,z)\Big| \le \sup_{\zeta\in D}|x(0,\zeta)|(L(a+t,z)-L(a,z)) \quad (5.15)$$

is the main tool to verify the appropriate conditions on $U_0(t)$. First of all it guarantees, because of $H_{\mathcal{F},L}$, that $U_0(t)$ maps indeed $C_0(\mathbb{R}_+\times D)$ into itself. Second, we find that, for $t_1 > t_2$,

$$\|U_0(t_1)-U_0(t_2)\| \le \sup_{a\ge 0, z\in D}\frac{L(a+t_1,z)-L(a+t_2,z)}{\mathcal{F}(a,z)} \to 0$$

as $t_1 - t_2 \downarrow 0$. And finally,

$$\Big\|\sum_{j=0}^n (U_0(t_{j+1})-U_0(t_j))x_j\Big\| \le \sup_{a\ge 0, z\in D}\sum_{j=0}^n\frac{L(a+t_{j+1},z)-L(a+t_j,z)}{\mathcal{F}(a,z)}$$

$$= \sup_{a\ge 0, z\in D}\frac{L(a+s,z)-L(a,z)}{\mathcal{F}(a,z)} < \infty$$

(once more because of $H_{\mathcal{F},L}$). It remains to verify the algebraic relation $U_0(t+s)$ $U_0(t) = T_0(t)U_0(s)$. Since

$$(U_0(t+s)x)(a,z) - (U_0(t)x)(a,z)$$
$$= \frac{1}{\mathcal{F}(a,z)} \int_t^{t+s} \Big(\int_D x(0,\zeta)p(a+\tau,z)(d\zeta) \Big) L(a+d\tau,z)$$

and

$$(T_0(t)U_0(s)x)(a,z) =$$
$$\frac{1}{\mathcal{F}(a+t,z)} \int_0^s \Big(\int_D x(0,\zeta)p(a+t+\tau,z)(d\zeta) \Big) L(a+t+d\tau,z) \frac{\mathcal{F}(a+t,z)}{\mathcal{F}(a,z)}$$

we find the required identity by a simple "shift" of the τ-integration variable.

To conclude we look at the Laplace transforms and derive the "characterist equation" which now takes the form

$$\begin{aligned}(\hat{U}_0(\lambda)x)(a,z) &= \int_0^\infty e^{-\lambda\tau} U_0(d\tau)(a,z) \\ &= \frac{1}{\mathcal{F}(a,z)} \int_0^\infty e^{-\lambda\tau} \Big(\int_D x(0,\zeta)p(a+\tau,z)(d\zeta) \Big) L(a+d\tau,z) \\ &= \frac{e^{\lambda a}}{\mathcal{F}(a,z)} \int_a^\infty e^{-\lambda\sigma} \Big(\int_D x(0,\zeta)p(\sigma,z)(d\zeta) \Big) L(d\sigma,z).\end{aligned}$$

Define the operator $K : C(D) \to C(D)$ by

$$(K\phi)(z) = \int_0^\infty e^{-\lambda\sigma} \Big(\int_D \phi(\zeta)p(\sigma,z)(d\zeta) \Big) L(d\sigma,z).$$

Clearly $\hat{U}_0(\lambda)$ has eigenvalue one if and only if K has eigenvalue one.

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