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Disjoint Cycles in Directed Graphs on the Torus and the Klein Bottle

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We give necessary and sufficient conditions for a directed graph embedded on the torus or the Klein bottle to contain pairwise disjoint circuits, each of a given orientation and homotopy, and in a given order. For the Klein bottle, the theorem is new. For the torus, the theorem was proved before by P. D. Seymour. This paper gives a shorter proof of that result. © 1993 Academic Press, Inc.

1. INTRODUCTION

Let S be the torus or the Klein bottle. We call a function $\phi: S \to S$ a *shift* if there exists a continuous function $\Phi: S \times [0, 1] \to S$ such that

$$\begin{aligned} &\text{(i)} \quad \Phi(x, 0) = x,\ \Phi(x, 1) = \phi(x), \text{ for all } x \in S, \\ &\text{(ii)} \quad \Phi(\cdot, t) \text{ is a homeomorphism on } S, \text{ for each } t \in [0, 1]. \end{aligned} \tag{1}$$

Let G be a directed graph embedded on S (without crossings). Let $C_1, ..., C_k$ be pairwise disjoint simple closed curves on S, each being nonnullhomotopic. We characterize when there exists a shift of S bringing each C_i to a directed cycle in G (with the same orientation as C_i), under the assumption that $S \backslash C_1$ is a cylinder. (This is automatically the case if S is the torus.)

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For the torus, this characterization was given in [2]. In this paper, we give a shorter proof, while for the Klein bottle the result is new. For compact surfaces of genus more than one, as well as for the Klein bottle in case $S\backslash C_1$ is *not* a cylinder, a characterization is given in [1]. So the present paper closes the gap. (Note that $S\backslash C_1$ being a cylinder implies that S is the torus or the Klein bottle.)

In studying this problem, we assume without loss of generality that $C_1, ..., C_k$ occur in this order around S. That is, we assume that there exists a closed curve D_0 crossing each of $C_1, ..., C_k$ exactly once, and in this order. If S is the torus, each curve D gives a natural interpretation of "left" and "right" with respect to D. If S is the Klein bottle, we choose for each curve D some interpretation of "left" and "right," arbitrarily but fixed when going along D from its begining point to its end point. Define a sequence

$$(\alpha_1, ..., \alpha_k) \tag{2}$$

by $\alpha_i = +1$ if C_i crossed D_0 from left to right, and $\alpha_i = -1$ if C_i crosses D_0 from right to left.

Let D be any curve on S, with end points in faces of G. We assume here and below that any such curve has only a finite number of intersections with G. Moreover, we assume that each intersection with G is in a vertex. (We can add a vertex at each intersection.)

We say that a crossing of D with any C_i is *positive* if it is a crossing in the same direction as D_0, and *negative* otherwise. If D has π positive crossings with C_1 and ν negative crossings with C_1, then the *winding number* $w(D)$ of D is equal to $\pi - \nu$.

Let D traverse vertices $v_1, ..., v_m$ of G, in this order (repetition allowed). We associate with D a sequence

$$i_G(D) = (X_1, ..., X_m), \tag{3}$$

where each X_j is a subset of $\{+1, -1\}$. The set X_j is defined as follows. Consider the segment of D when traversing v_j, going from face F to face F', say, of G. Let $e_1, ..., e_d$ be the edges incident with v_j, choosing indices in such a way that $F, e_1, ..., e_t, F', e_{t+1}, ..., e_d$ occur in this order clockwise at v_j, for some t. Then $+1 \in X_j$ if and only if at least one of $e_1, ..., e_t$ is directed towards v_j and at least one of $e_{t+1}, ..., e_d$ is directed away from v_j. Similarly, $-1 \in X_j$ if and only if at least one of $e_1, ..., e_t$ is directed away from v_j and at least one of $e_{t+1}, ..., e_d$ is directed towards v_j.

For any finite sequence x and any integer $w > 0$ we define x^w as the concatenation of w copies of x. If $x = (\xi_1, ..., \xi_s)$ and $y = (\eta_1, ..., \eta_t)$, then we let $x \prec y$ if there exist indices $1 \leqslant j_1 < j_2 < \cdots < j_s \leqslant t$ such that $\xi_i \in \eta_{j_i}$ for $i = 1, ..., s$. Moreover, $x \ll y$ if $x' \prec y$ for some cyclic permutation x' of x.

2. The Torus

We first consider the torus.

THEOREM 1. *Let S be the torus. Then there exists a shift of S bringing $C_1, ..., C_k$ to directed cycles in G, if and only if, for each closed curve D of positive winding number, one has*

$$(\alpha_1, ..., \alpha_k)^{w(D)} \leqslant i_G(D). \tag{4}$$

Proof. Necessity of the condition is trivial. Suppose now that the condition is satisfied. We may assume that each face of G is an open disk. (In any face F not being an open disk, we can put a new vertex v, with arcs from v to each vertex incident with F.)

We consider the torus as being the quotient space of $\mathbb{C}\setminus\{0\}$ by identifying any $y, z \in \mathbb{C}$ if $z = 2^u y$ for some integer u. Let $\pi\colon \mathbb{C}\setminus\{0\} \to S$ be the quotient map. We make this construction in such a way that each lifting of each C_i to $\mathbb{C}\setminus\{0\}$ is a closed curve enclosing 0. More precisely, there exist closed curves Γ_i ($i \in \mathbb{Z}$) so that $\pi \circ \Gamma_i = C_i$ for each $i \in \mathbb{Z}$, taking indices of $C_i \bmod k$. We can take the indices in such a way that Γ_{i+1} encloses Γ_i, and such that $\Gamma_{i+k} = 2\Gamma_i$ for each integer i. Moreover, we assume that Γ_i has clockwise orientation if $\alpha_i = +1$ and anti-clockwise orientation if $\alpha_i = -1$ (taking indices of $\alpha_i \bmod k$).

The inverse image $H := \pi^{-1}[G]$ of G is an infinite graph embedded in $\mathbb{C}\setminus\{0\}$. For any curve P on $\mathbb{C}\setminus\{0\}$ we denote $i_H(P) := i_G(\pi \circ P)$. (So $i_H(P)$ can be defined similarly as we defined $i_G(P)$ above.)

Now for each integer i, let $\mathscr{R}_i$ be the set of faces F of H so that there exists an integer $t \leqslant i$ and a curve P starting in a face enclosed by Γ_t and ending in F, such that

$$(\alpha_t, \alpha_{t+1}, ..., \alpha_i) \not\leqslant i_H(P). \tag{5}$$

We show

CLAIM. *$\bigcup \mathscr{R}_i$ is bounded, for each integer i.*

Proof. We show that in the definition of $\mathscr{R}_i$ we can restrict P to curves traversing at most kf faces of H, where f denotes the number of faces of G, from which the claim follows (as it implies that $\bigcup \mathscr{R}_i$ is at most kf faces "away from" the bounded set enclosed by Γ_i).

Let P be a curve starting in a face enclosed by Γ_t and ending in F, satisfying (5) and traversing a minimum number of faces of H. Suppose P

traverses more than kf faces. We show that there exists a $t' \leqslant i$ and a curve P' starting in a face enclosed by $\Gamma_{t'}$ and ending in F such that

$$(\alpha_{t'}, \alpha_{t'+1}, ..., \alpha_i) \not\prec i_H(P'), \tag{6}$$

and such that P' traverses fewer faces of H than P does, which means a contradiction.

Since P traverses more than kf faces of H, there exists a face F' of G so that $\pi \circ P$ traverses F' more than k times. So P can be decomposed as $P = P_0 \cdot P_1 \cdot P_2 \cdot \cdots \cdot P_k \cdot P_{k+1}$, where for each $j = 1, ..., k$, $\pi \circ P_j$ is a curve with end points in F', intersecting G at least once. Without loss of generality, each such $\pi \circ P_j$ is a closed curve.

For $j = 0, ..., k$, let h_j be the smallest integer h for which

$$(\alpha_t, \alpha_{t+1}, ..., \alpha_h) \not\prec i_H(P_0 \cdot P_1 \cdot \cdots \cdot P_j). \tag{7}$$

Then there exist j', j'' so that $0 \leqslant j' < j'' \leqslant k$ and so that $h_{j'} \equiv h_{j''} \pmod k$. Let $h' := h_{j'}$ and $h'' := h_{j''}$.

Since $\pi \circ (P_{j'+1} \cdot \cdots \cdot P_{j''})$ is a closed curve on S, there exists a $z \in \mathbb{C} \setminus \{0\}$ and a $u \in \mathbb{Z}$ so that $P_{j'+1} \cdot \cdots \cdot P_{j''}$ goes from z to $2^u z$.

Since the closed curve $\pi \circ (P_{j'+1} \cdot \cdots \cdot P_{j''})$ has winding number u, we know

$$(\alpha_1, ..., \alpha_{ku}) \leqslant i_H(P_{j'+1} \cdot \cdots \cdot P_{j''}). \tag{8}$$

Suppose $ku > h'' - h'$. Then

$$(\alpha_{h'}, \alpha_{h'+1}, ..., \alpha_{h''}) \prec i_H(P_{j'+1} \cdot \cdots \cdot P_{j''}), \tag{9}$$

since $h' \equiv h'' \pmod k$. Since $(\alpha_t, ..., \alpha_{h'-1}) \prec i_H(P_0 \cdot \cdots \cdot P_{j'})$ (by definition of $h' = h_{j'}$), (9) implies $(\alpha_t, ..., \alpha_{h''}) \prec i_H(P_0 \cdot \cdots \cdot P_{j''})$. This contradicts the definition of $h'' = h_{j''}$.

So $ku \leqslant h'' - h'$. Consider the curve

$$P' := (2^u(P_0 \cdot \cdots \cdot P_{j'})) \cdot P_{j''+1} \cdot \cdots \cdot P_{k+1}. \tag{10}$$

Let $t' := t + ku$. Then $t' = t + ku \leqslant t + h'' - h' \leqslant i$ (since $t \leqslant h'$ and $h'' \leqslant i$). Now

$$(\alpha_{t'}, \alpha_{t'+1}, ..., \alpha_{h''}) \not\prec i_H(2^u(P_0 \cdot \cdots \cdot P_{j'})) \tag{11}$$

(as $(\alpha_{t'}, \alpha_{t'+1}, ..., \alpha_{h'+ku}) = (\alpha_t, \alpha_{t+1}, ..., \alpha_{h'}) \not\prec i_H(P_0 \cdot \cdots \cdot P_{j'}) = i_H(2^u(P_0 \cdot \cdots \cdot P_{j'}))$, by definition of $h' = h_{j'}$, and as $h' + ku \leqslant h''$). Moreover,

$$(\alpha_{h''}, \alpha_{h''+1}, ..., \alpha_i) \not\prec i_H(P_{j''+1} \cdot \cdots \cdot P_{k+1}) \tag{12}$$

(since otherwise $(\alpha_t, ..., \alpha_i) \prec i_H(P)$, as $(\alpha_t, ..., \alpha_{h''-1}) \prec i_H(P_0 \cdot \cdots \cdot P_{j''})$, by definition of $h'' = h_{j''}$).

Relations (11) and (12) directly imply (6).

This ends the proof of the Claim. ∎

Clearly, each face F enclosed by Γ_i belongs to $\mathscr{R}_i$ (since we can take $t = i$ and for P any curve remaining in F). Moreover, $\mathscr{R}_{i+k}$ can be obtained from $\mathscr{R}_i$ by multiplying the faces in $\mathscr{R}_i$ by 2.

The faces in $\mathscr{R}_i$ induce a connected subgraph of the dual graph of H, as one easily checks. (If P is the arc connected to $F \in \mathscr{R}_i$ then every face traversed by P belongs to $\mathscr{R}_i$.) Hence the arcs on the boundary of the unbounded connected component of $\mathbb{C} \setminus \overline{\bigcup \mathscr{R}_i}$ form a simple closed curve; call it Δ_i. (Here $\bar{X}$ denotes the topological closure of X.)

Then Δ_i is oriented clockwise if $\alpha_i = +1$, and anti-clockwise if $\alpha_i = -1$. This follows from the fact that any arc a of H on the boundary of $\overline{\bigcup \mathscr{R}_i}$ is oriented clockwise if $\alpha_i = +1$, and anti-clockwise if $\alpha_i = -1$ (clockwise and anti-clockwise with respect to $\overline{\bigcup \mathscr{R}_i}$). To see this, let a be incident with faces $F \in \mathscr{R}_i$ and $F' \notin \mathscr{R}_i$. By definition of $\mathscr{R}_i$, there exists a $t \leqslant i$ and a curve P starting in a face enclosed by Γ_t and ending in F, satisfying (5). We can extend P to a curve P' ending in F', by crossing a. Since $F' \notin \mathscr{R}_i$, $(\alpha_t, ..., \alpha_i) \prec i_H(P')$. Hence α_i must belong to the last set occurring in $i_H(P')$, giving the required statement.

Moreover, for each integer i, Δ_i is enclosed by Δ_{i+1}, without intersections. This follows from the fact that if F belongs to $\mathscr{R}_i$, then each face F' having a vertex in common with F belongs to $\mathscr{R}_{i+1}$. Indeed, by definition of $\mathscr{R}_i$, there exists a $t \leqslant i$ and a curve P starting in a face enclosed by Γ_t and ending in F, satisfying (5). We can extend P to a curve P' ending in F', by traversing a vertex incident with both F and F'. From (5) one derives $(\alpha_t, ..., \alpha_{i+1}) \not\prec i_H(P')$. Hence $F' \in \mathscr{R}_{i+1}$.

Since also $\Delta_{i+k} = 2\Delta_i$ for each i, it follows that $\pi \circ \Delta_1, ..., \pi \circ \Delta_k$ give disjoint closed curves on the torus S, of the same orientations as $C_1, ..., C_k$, respectively, and in the same order as $C_1, ..., C_k$. Shifting $C_1, ..., C_k$ to $\pi \circ \Delta_1, ..., \pi \circ \Delta_k$ gives the required shift. ∎

3. The Klein Bottle

We next consider the Klein bottle. Define $\alpha_i := -\alpha_{i-k}$ for $i = k+1, ..., 2k$.

Theorem 2. *Let S be the Klein bottle, such that $S \setminus C_1$ is a cylinder. Then there exists a shift of S bringing $C_1, ..., C_k$ to directed cycles in G, if and only* *if, for each orientation-preserving closed curve D of positive winding number, one has*

$$(\alpha_1, ..., \alpha_k, \alpha_{k+1}, ..., \alpha_{2k})^{w(D)/2} \preccurlyeq i_G(D). \tag{13}$$

Proof. The proof is similar to that of Theorem 1. We now consider the Klein bottle as being the quotient space of $\mathbb{C}\setminus\{0\}$ by identifying any $y, z \in \mathbb{C}$ if $z = 2^u y$ for some even integer u or $z = 2^u \bar{y}$ for some odd integer u. Again, let $\pi: \mathbb{C}\setminus\{0\} \to S$ be the quotient map, in such a way that there exist closed curves Γ_i ($i \in \mathbb{Z}$) so that $\pi \circ \Gamma_i = C_i$ for each $i \in \mathbb{Z}$, taking indices of C_i mod k. We can take the indices in such a way that Γ_{i+1} encloses Γ_i, and such that $\Gamma_{i+2k} = 2\Gamma_i$ for each integer i. Moreover, we assume that Γ_i has clockwise orientation if $\alpha_i = +1$ and anti-clockwise orientation if $\alpha = -1$, now taking indices of α_i mod $2k$.

Also the remainder of the proof is similar to that of Theorem 1. ∎

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