Input–Output Structure of Linear Differential/Algebraic Systems

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***Abstract*—Systems of linear differential and algebraic equations occur in various ways, for instance, as a result of automated modeling procedures and in problems involving algebraic constraints, such as zero dynamics and exact model matching. Differential/algebraic systems may represent an input–output relation in a highly redundant way; still, it is often of interest to determine the input–output structure in terms of the original parameters. This is the subject of the present paper. Specifically, explicit formulas in terms of original data will be given to answer the following questions: Do the given equations determine a transfer matrix, and if so, what is the pole/zero structure at infinity of that transfer matrix? The approach is based on two characteristic sequences of subspaces, one in the input space and one in the output space.**

I. Introduction

In recent years, automated modeling of large systems has become a feasible proposition. Software packages have become available that are able to produce a set of equations describing the dynamical behavior of a given system from its physical description—say, a circuit layout or, more generally, an interconnection of blocks, each with a known behavior. As a consequence of the automation of the modeling process, *redundancy* of system descriptions has increasingly become a factor that has to be taken into account explicitly. The human modeler who works with small systems will, in general, have no problem avoiding redundancy by a suitable incorporation of algebraic relations in differential equations. For large systems that require computer-aided modeling techniques, however, one is forced to recognize that any systematic modeling procedure leads, in the first instance, to a set of algebraic and differential equations that will often be highly redundant. This redundancy will have to be addressed, in one way or another. Actually, the importance of dealing with redundant descriptions has already been stressed in the 1970's by several authors, notably Rosenbrock [23], [24] and Luenberger [15], [16].

The present paper is concerned with the input–output structure of systems of linear differential and algebraic equations, such as may arise from computer-aided modeling procedures. More precisely, our aims are the following. Suppose that we have a set of linear differential and algebraic equations, and that some of the variables are grouped together as a vector y and some of the other variables are collected in a vector u. We want to answer the following questions.

1) Do the equations establish an input–output relation between u and y? (The exact formulation of this will be discussed below.)

2) If so, is the transfer matrix from u to y proper, and what is its rank?

3) In more detail, what is the pole/zero structure at infinity of the transfer matrix? (Again, the precise meaning of this question will be described below.)

It turns out that all of these questions can be handled in the same framework. This paper will present expressions which answer the above questions in terms of the given differential/algebraic system itself, without calling for a reduction procedure.

It should be emphasized that the results of this paper are not only relevant in the context of computer-aided modeling. Systems of algebraic and differential equations also arise when algebraic constraints are imposed on standard state-space systems; this is done, for instance, in the study of zero dynamics and in exact model-matching problems. In these cases, it is important to have a description of the input–output structure in terms of original data. For systems that depend on parameters, redundant descriptions are sometimes preferable to minimal descriptions because they allow more freedom to incorporate the dependence on the parameters in a nice way; a recent example of this occurs in H^∞ theory [25]. In order to fully use the advantages of the redundant form, one will again have to be able to express important system properties directly in terms of the redundant description.

In the absence of redundancy, the questions that we posed above are still partly nontrivial. For instance, the standard state-space representation makes evident that inputs and outputs are related by a proper transfer matrix, but the zero structure at infinity is not immediately transparent. There exists a considerable literature on state-space formulas for transfer zeros at infinity: see, for instance, [5], [20], [22]. The pole/zero structure at infinity of systems in descriptor form was studied in [14], [18], [30]. In these works, the poles and zeros at infinity of the transfer matrix are determined under various assumptions on the absence of redundancy (cf. also [28], [29]). Here, we shall consider general systems of linear algebraic and

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differential equations written in first-order form and derive formulas for the pole/zero structure at infinity of the transfer matrix (if the equations do determine a unique transfer matrix) under arbitrary redundancy. The results just mentioned then follow as corollaries, as will be shown below.

The key observation which makes it possible to treat the problem on such a general level is that the answers to all questions formulated above are provided by two particular sequences of subspaces, one in the input space and one in the output space. Having established this, one only needs to be able to determine the two sequences in terms of a general system of linear differential and algebraic equations, and this turns out to be not too difficult. The organization of the paper is as follows. Section II contains notation and basic definitions; in particular, we propose a unified notation for a variety of subspace recursions occurring in the geometric approach to linear systems. In Section III, we introduce the two characteristic sequences of subspaces, and prove that they indeed fully describe the input–output structure of the system (in the sense that they provide the answers to the questions 1)–3) formulated above). The main task is then to compute these sequences in terms of some general first-order representation, for which we take the so-called "pencil form." This computation is carried out in Section IV, although the actual work is deferred to the Appendix. The results can be reformulated in terms of any other first-order representation, as is shown for the descriptor form in Section V. The final Section VI contains conclusions.

II. Definitions and Notation

A. Input–Output Systems

First, we have to define more precisely the "input–output structure" of a linear system as indicated by questions 1)–3) in the Introduction. For this, we shall use the notion of "external behavior" as developed by Willems (see, for instance, [33]). Consider a system of linear ordinary differential and algebraic equations with constant coefficients, in which some of the variables are collected into a vector y and some of the other variables are collected into a vector u. The system might be written as

$$A(\sigma)\xi + B(\sigma)u + C(\sigma)y = 0 \tag{2.1}$$

where σ denotes differentiation, ξ contains all variables that have not been taken either into y or u, and $A(s)$, $B(s)$, and $C(s)$ are polynomial matrices. Although σ may also denote, for instance, the shift operator (in discrete-time systems) without any change to the theory below, for concreteness we shall keep using the term "differentiation."

The *external behavior* $\mathscr{B}$ of the system (2.1) is the set of all (y, u) trajectories for which there exists a ξ trajectory such that (2.1) holds. We use the word "trajectory" here as an abbreviation of "C^∞ vector-valued function defined on the real line," where the vector space to which the function maps is clear from the context. We use $C^\infty(\mathbb{R})$ for convenience; any other of the usual function classes may be substituted without consequences for the theory in the following sections, provided Proposition 2.1 below still holds (which may require a rewording of the definitions in the next paragraph).

We shall say that *y processes u* in the system (2.1) if the linear space of trajectories $\{y|(y,0) \in \mathscr{B}\}$ is finite-dimensional. In other words, y processes u if u determines y up to a finite number of constants ("initial conditions"). We shall say that *u is free* in the system (2.1) if for every trajectory u there exists a trajectory y such that $(y, u) \in \mathscr{B}$. For the class of behaviors described by equations of the form (2.1), the definitions of "processing" and "free" come down to the same as the definitions of the same terms given in a more general context by Willems [32], [33]; this is shown by the following proposition (cf. [32, sect. 4.5.1]), which follows immediately from standard theorems for differential equations. Note that, by the "elimination theorem" [33, theorem IV.3], any behavior given by equations of the form (2.1) can also be represented in the form (2.2) below.

Proposition 2.1: Let a behavior $\mathscr{B}$ be given by

$$\mathscr{B} = \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \middle| [R_1(\sigma) \quad R_2(\sigma)] \begin{bmatrix} y \\ u \end{bmatrix} = 0 \right\} \tag{2.2}$$

where $[R_1(s) \quad R_2(s)]$ is a polynomial matrix of full row rank. The following statements hold:

i) y processes u if and only if $R_1(s)$ has full column rank

ii) u is free if and only if $R_1(s)$ has full row rank.

Again following Willems, we shall say that the behavior $\mathscr{B}$ stems from an *input–output system* if both conditions of the above proposition hold. In this case, the proposition shows that $R_1(s)$ must be invertible, and then the *transfer matrix* of the system is defined by $T(s) = -R_1^{-1}(s)R_2(s)$. An input–output system determines its transfer matrix completely, but the converse is not quite true [33, proposition VIII.8].

B. Pole / Zero Structure at Infinity

The transfer matrix corresponding to an input–output system given by equations of the form (2.1) will be a matrix of rational functions. We recall some of the associated definitions. The *rank* of a rational matrix is its rank as a matrix over the field of rational functions. A rational matrix is said to be *proper* if it does not have poles at infinity, and *biproper* if it is proper and has a proper inverse. The detailed pole/zero structure at infinity is obtained from the following theorem describing the "local Smith–McMillan form at infinity" [11], [28].

Theorem 2.2: For every rational matrix $T(s)$, there exist biproper matrices $M(s)$ and $N(s)$ such that

$$M(s)T(s)N(s) = \begin{bmatrix} D(s) & 0 \\ 0 & 0 \end{bmatrix},$$

$$D(s) = \operatorname{diag}(s^{n_1}, \cdots, s^{n_r}). \tag{2.3}$$

The indexes $n_1, \cdots, n_r \in \mathbb{Z}$ are unique up to order; in particular, we may choose $M(s)$ and $N(s)$ such that $n_1 \geq \cdots \geq n_r$.

The *pole/zero structure at infinity* of $T(s)$ is defined by the set of indexes $n_1,\cdots,n_r$ given by the theorem above (cf. also [19], [23]).

Various alternative ways to describe the pole/zero structure are occasionally convenient. If the indexes $n_1,\cdots,n_r$ satisfy

$$n_1 \geq \cdots \geq n_k > 0$$
$$= n_{k+1} = \cdots = n_{r-l} > n_{r-l+1} \geq \cdots \geq n_r \quad (2.4)$$

one usually says that the rational matrix $T(s)$ has *k poles at infinity of orders* $n_1,\cdots,n_k$ and *l zeros at infinity of orders* $n_{r-l+1},\cdots,n_r$. It is often convenient, however, to count a zero of order n as a pole of order $-n$ (or vice versa, a pole of order n as a zero of order $-n$), so that one can say that the rational matrix $T(s)$ of Theorem 2.2 has r poles at infinity of orders $n_1,\cdots,n_r$ (or that it has r zeros at infinity of orders $-n_1,\cdots,-n_r$). With this convention, the pole/zero structure at infinity is also completely determined by each one of the following functions of $k \in \mathbb{Z}$:

- p_k = the number of poles at infinity of order $\geq k$
- s_k = the number of poles at infinity of order $\leq k$
- j_k = the number of zeros at infinity of order $\geq k$
- t_k = the number of zeros at infinity of order $\leq k$.

Although these functions are obviously related by simple rules ($p_k = r - s_{k-1} = t_{-k} = r - j_{-k+1}$), not one of them stands out as being most convenient for all cases, and so we shall use all four. Further obvious remarks are that r is equal to the rank of $T(s)$, and that $T(s)$ is proper rational if and only if $p_1 = 0$.

C. Subspace Recursions

Subspace recursions will be used extensively below as a tool of calculation. In an attempt at systemization, we propose here the following system of notation. Let X, Y, U, and Z be finite-dimensional linear spaces, and let K: $Z \to X$, L: $Z \to X$, M: $Z \to Y$, and N: $U \to X$ be linear mappings. Recall the notation $K^{-1}X_0 = \{z \in Z | Kz \in X_0\}$ where X_0 is a subspace of X. Define the following recursions:

i) $T^k(M, sK - L; T_0)(k \geq 0)$ is defined by

$$T^0 = T_0, \quad T^{k+1} = K^{-1}L[T^k \cap \ker M] \quad (2.5)$$

where $T_0 \subset Z$ satisfies $K^{-1}L[T_0 \cap \ker M] \supset T_0$.

ii) $T^k(sK - L, N; T_0)(k \geq 0)$ is defined by

$$T^0 = T_0, \quad T^{k+1} = K^{-1}[LT^k + \operatorname{Im} N] \quad (2.6)$$

where $T_0 \subset Z$ satisfies $K^{-1}[LT_0 + \operatorname{Im} N] \supset T_0$.

iii) $V^k(M, sK - L; V_0)(k \geq 0)$ is defined by

$$V^0 = V_0, \quad V^{k+1} = L^{-1}KV^k \cap \ker M \quad (2.7)$$

where $V_0 \subset Z$ satisfies $L^{-1}KV_0 \cap \ker M \subset V_0$.

iv) $V^k(sK - L, N; V_0)(k \geq 0)$ is defined by

$$V^0 = V_0, \quad V^{k+1} = L^{-1}[KV^k + \operatorname{Im} N] \quad (2.8)$$

where $V_0 \subset Z$ satisfies $L^{-1}[KV_0 + \operatorname{Im} N] \subset V_0$.

We also introduce the simplified notations

$$T^k(M, sK - L) = T^k(M, sK - L; \{0\}),$$
$$T^k(sK - L, N) = T^k(sK - L, N; \{0\}) \quad (2.9)$$
$$V^k(M, sK - L) = V^k(M, sK - L; Z),$$
$$V^k(sK - L, N) = V^k(sK - L, N; Z) \quad (2.10)$$

and

$$T^k(sK - L) = T^k(0, sK - L) = T^k(sK - L, 0) \quad (2.11)$$
$$V^k(sK - L) = V^k(0, sK - L) = V^k(sK - L, 0). \quad (2.12)$$

The notation has been chosen such that the nondecreasing sequences are denoted by T and the nonincreasing sequences by V. There is a certain mnemonic value to this: the letter V points downward, whereas with some imagination, the letter T is a version of an arrow pointing upward. The parameter s in the above notation formally only serves to keep the mappings K and L apart, but the usage has some advantages such as the abbreviations

$$\begin{bmatrix} sK - L \\ -M \end{bmatrix} = s\begin{bmatrix} K \\ 0 \end{bmatrix} - \begin{bmatrix} L \\ M \end{bmatrix},$$
$$[sK - L \quad -N] = s[K \quad 0] - [L \quad N]. \quad (2.13)$$

It should be noted that

$$T^k(M, sK - L) = T^k\left(\begin{bmatrix} sK - L \\ M \end{bmatrix}\right),$$
$$V^k(M, sK - L) = V^k\left(\begin{bmatrix} sK - L \\ M \end{bmatrix}\right) \quad (2.14)$$

so that the notation $T^k(M, sK - L)$ and $V^k(M, sK - L)$ may be seen as merely a typographical convenience. All sequences defined in the above way have limits as $k \to \infty$; these will be denoted by $T^*(M, sK - L; T_0)$ and so on. The subspace $T^*(M, sK - L; T_0)$ is the smallest subspace T such that $T \supset K^{-1}L[T \cap \ker M]$ and $T \supset T_0$, and similar characterizations hold for the other limits.

The notation proposed here allows the formulation of many concepts in one framework. For instance, for a standard state-space system $\dot{x} = Ax + Bu$, $y = Cx$, the controllable subspace is $T^*(sI - A, B)$, the unobservable subspace is $V^*(C, sI - A)$, and the largest controlled invariant subspace contained in $\ker C$ is $V^*\left(\begin{bmatrix} sI - A \\ C \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}\right)$. Many more examples will be seen below.

The actual numerical computation of the recursions above is a subject that is closely related to the computation of the Kronecker canonical form. We will not go into this, but refer to [7] and [2].

D. Miscellaneous

The space of rational functions with values in a vector space W will be denoted by $W(s)$, and the subspace of proper rational W-valued functions will be written as $W_\infty(s)$. Any element of $W_\infty(s)$ has a Laurent series at infinity of the form

$$w(s) = w_0 + w_1 s^{-1} + w_2 s^{-2} + \cdots \tag{2.15}$$

and we shall write $w(\infty) = w_0$.

If a vector space W is formed as the direct sum of two spaces Y and U (so $W = Y \oplus U$), we shall use the symbols π_Y and π_U to denote the natural projections of W onto Y and U, respectively. That is to say,

$$\pi_Y : \begin{bmatrix} y \\ u \end{bmatrix} \mapsto y \quad \left(\begin{bmatrix} y \\ u \end{bmatrix} \in W \right) \tag{2.16}$$

and π_U is defined likewise.

If X_0 is a subspace of a linear space X, the *codimension* of X_0 is

$$\operatorname{codim} X_0 = \dim X/X_0 = \dim X - \dim X_0. \tag{2.17}$$

III. Characteristic Subspace Sequences

As already noted, each behavior that can be represented in the form (2.1) can also be represented in the form

$$R_1(\sigma)y + R_2(\sigma)u = 0 \tag{3.1}$$

where $R(s) = [R_1(s) \quad R_2(s)]$ is a polynomial matrix of full row rank. To the polynomial matrix $R(s)$ we can associate the rational vector space $\ker R(s) = \{w(s) \in W(s) | R(s)w(s) = 0\}$, where $W(\simeq \mathbb{R}^{p+m})$ is the direct sum of the output space $Y(\simeq \mathbb{R}^p)$ and the input space $U(\simeq \mathbb{R}^m)$. Although $R(s)$ is not uniquely determined by the behavior, the rational vector space $\ker R(s)$ is (cf., for instance, [26, corollary 2.5], [33, proposition III.3]). Therefore, everything that is defined in terms of $\ker R(s)$ is uniquely determined by the behavior as well.

We shall define two sequences of subspaces, one in the input space U and one in the output space Y, both of which we shall derive from a sequence of subspaces of $W = Y \oplus U$. This latter sequence is defined as follows.

Definition 3.1: Let a behavior $\mathscr{B}$ with external variables y and u be represented as in (3.1), and denote $W = Y \oplus U$. With $\mathscr{B}$, we associate the following sequence of subspaces of W, defined for $k \in \mathbb{Z}$:

$$W^k = \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \in W | \exists \begin{bmatrix} y(s) \\ u(s) \end{bmatrix} \in W_\infty(s) \quad \text{such that} \quad \begin{bmatrix} y(s) \\ s^{-k}u(s) \end{bmatrix} \in \ker [R_1(s) \quad R_2(s)] \quad \text{and} \quad \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} y(\infty) \\ u(\infty) \end{bmatrix} \right\}. \tag{3.2}$$

We also define $Y^k = \pi_Y W^k$ and $U^k = \pi_U W^k$.

Lemma 3.2: For all $k \in \mathbb{Z}$, we have

i) $\dim W^k = \dim W^0$

ii) $Y^k \subset Y^{k-1}$

iii) $U^k \subset U^{k+1}$

iv) $\dim Y^k + \dim U^{k-1} = \dim W^0$.

Proof: Equality i) follows from

$$\dim \ker [s^k R_1(s) \quad R_2(s)] = \dim \ker [R_1(s) \quad R_2(s)] = \dim W^0;$$

for the final equality, see the proof of Lemma 5.1 in [13]. Next, the inclusions ii) and iii) are immediate from the definition, while equality iv) follows from i) and

$$\begin{aligned} \dim W^k &= \dim Y^k + \dim \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \in W^k | y = 0 \right\} \\ &= \dim Y^k + \dim \Bigg\{ u \in U | \exists \begin{bmatrix} y(s) \\ u(s) \end{bmatrix} \in W_\infty(s) \quad \text{such that} \quad sy(s) \in W_\infty(s), \\ &\qquad \begin{bmatrix} s^k y(s) \\ u(s) \end{bmatrix} \in \ker [R_1(s) \quad R_2(s)], \quad \text{and} \quad u = u(\infty) \Bigg\} \\ &= \dim Y^k + \dim U^{k-1}. \end{aligned} \tag{3.3}$$

This completes the proof of the lemma.

The above lemma shows that the sequence $\{Y^k\}$ is nonincreasing and the sequence $\{U^k\}$ is nondecreasing. Since Y and U are finite-dimensional, both sequences must consequently have limits both as $k \to \infty$ and as $k \to -\infty$. In obvious notation, we shall denote the limit spaces by Y_+^*, Y_-^*, U_+^*, and U_-^*, respectively. By part ii) of Lemma 3.2, we have $Y_+^* \subset \cdots \subset Y^k \subset Y^{k-1} \subset \cdots \subset Y_-^*$, and by part iii), $U_-^* \subset \cdots \subset U^k \subset U^{k+1} \subset \cdots \subset U_+^*$. The next theorem shows the importance of the limit spaces.

Theorem 3.3: Let a behavior $\mathscr{B}$ with external variables y and u be given by

$$R_1(\sigma)y + R_2(\sigma)u = 0$$

where the polynomial matrix $[R_1(s) \quad R_2(s)]$ has full row rank. The following statements hold for $\mathscr{B}$:

i) y processes u if and only if $Y_+^* = \{0\}$

ii) u is free if and only if $U_+^* = U$.

If these conditions are both satisfied, then there exists a unique transfer matrix $T(s)$, and we have

iii) $\dim \ker T(s) = \dim U_-^*$

iv) $\dim \operatorname{im} T(s) = \dim Y_-^*$.

Proof: From Definition 3.1, it follows that

$$\dim Y_+^* = \dim \ker R_1(s) \tag{3.4}$$

which proves i). Denote the number of rows of $[R_1(s) \quad R_2(s)]$ by r. From Lemma 3.2-iv), we have

$$\begin{aligned} \dim U_+^* &= \dim W^0 - \dim Y_+^* \\ &= \dim \ker [R_1(s) \quad R_2(s)] - \dim \ker R_1(s) \\ &= \dim Y + \dim U - r - \dim \ker R_1(s) \\ &= \dim U - r + \dim \operatorname{im} R_1(s) \end{aligned} \tag{3.5}$$

and this yields ii). Parts iii) and iv) follow in a completely analogous way.

The above theorem shows that the limits of the sequences we have defined determine the existence of the transfer matrix and its rank. In the next theorem, we consider the sequences themselves, and show that they determine the pole/zero structure at infinity of the transfer matrix.

Theorem 3.4: Let a behavior be given as in Theorem 3.3, *and assume that the transfer function* $T(s)$ *exists. Denote the number of poles at infinity of* $T(s)$ *of order* $\geq k$ *by* p_k *and the number of poles at infinity of order* $\leq k$ *by* s_k $(k \in \mathbb{Z})$. *Then*

$$p_k = \dim Y^k \tag{3.6}$$

and

$$s_k = \dim U^k - \dim U_-^*. \tag{3.7}$$

Proof: We can write W^k in the following way:

$$W^k = \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \in W | \exists \begin{bmatrix} y(s) \\ u(s) \end{bmatrix} \in W_\infty(s) \quad \text{such that} \right.$$

$$\left. y(s) = s^{-k}T(s)u(s) \quad \text{and} \quad \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} y(\infty) \\ u(\infty) \end{bmatrix} \right\}. \tag{3.8}$$

It is clear from this formulation that $\dim Y^k$ and $\dim U^k$ are invariants under left and right multiplication of $T(s)$ by biproper matrices. Therefore, we may assume that $T(s)$ is in the Smith–McMillan form at infinity (2.3). One then verifies directly that (3.6) holds. For (3.7), note that $s_k + p_{k+1} = \operatorname{rank} T(s) = \dim Y_-^*$ by Theorem 3.3. Therefore, we have $s_k = \dim Y_-^* - \dim Y^{k+1}$, which can be rewritten as (3.7) by Lemma 3.2-iv).

Remark 3.5: Note that the orders of the poles and zeros at infinity of $T(s)$ can be derived from either the p_k's or the s_k's. One therefore has a choice to consider either subspaces of Y or subspaces of U.

Remark 3.6: Of course, the integers $\dim Y^k$ and $\dim U^k$ are still determined if the transfer matrix does not exist, but in that case, their interpretation is unclear.

Remark 3.7: For discrete-time systems, it is possible to express the subspaces W^k directly in terms of the behavior. The subspace W^0 is obtained as the set of all values at time 0 of trajectories that are zero for $t < 0$ (cf. [31, sect. 5], and [13, sect. 2]). The subspaces W^k are obtained by the same construction, carried out with respect to the behavior $\mathscr{B}_k = \{(y,u) | (\sigma^k y, u) \in \mathscr{B}\}$.

IV. Characterization in Terms of a Pencil Representation

In the preceding section, we established that the input–output structure of a behavior described by a system of linear differential and algebraic equations is completely determined by two subspace sequences. Our next goal is to give expressions for these sequences in terms of a general first-order representation. There are several ways to write such a representation for a behavior with external variables $w = (y, u)$; these include the *pencil form*

$$\begin{aligned} \sigma G z &= F z \\ w &= H z, \end{aligned} \tag{4.1}$$

the *dual pencil form*

$$\sigma K x + L x + M w = 0, \tag{4.2}$$

and the *descriptor form*

$$\begin{aligned} \sigma E x &= A x + B u \\ y &= C x + D u. \end{aligned} \tag{4.3}$$

In the authors' experience, the pencil form stands out as the most convenient for the problem we want to discuss. It is possible, however, to reformulate the results for systems in dual pencil form and for systems in descriptor form, and we shall do this for the descriptor form in the next section.

Since the two characteristic subspace sequences are defined in terms of the rational vector space $\ker R(s)$, we first need an expression for this space in terms of the representation (4.1). The following result is a special case of formula (4.4) in [13].

Lemma 4.1: Let the behavior $\mathscr{B}$ *given by* (4.1) *be represented by*

$$R(\sigma) w = 0 \tag{4.4}$$

where $R(s)$ *is a polynomial matrix of full row rank. We then have*

$$\ker R(s) = H[\ker(sG - F)]. \tag{4.5}$$

Using this, we obtain the following characterization of the subspaces W^k in trms of a pencil representation.

Lemma 4.2: Let a behavior be given by

$$\begin{aligned} \sigma G z &= F z \\ y &= H_y z \\ u &= H_u z, \end{aligned} \tag{4.6}$$

and let the sequence of subspaces W^k *be defined as in Definition* 3.1. *Let* $k \in \mathbb{Z}$, $y \in Y$, *and* $u \in U$. *Then* $[y^T\ u^T]^T \in W^k$ *if and only if there exists a rational vector* $z(s)$ *with Laurent expansion*

$$z(s) = z_{-l}s^l + z_{-l+1}s^{l-1} + \cdots + z_0 + z_1 s^{-1} + \cdots + z_k s^{-k} + \cdots \tag{4.7}$$

such that the following conditions hold:

i) $(sG - F)z(s) = 0$

ii) (*for* $k \geq 0$) $H_y z(s)$ *and* $s^k H_u z(s)$ *are proper*, $y = H_y z_0$, $u = H_u z_k$

iii) (*for* $k \leq 0$) $s^{-k} H_y z(s)$ *and* $H_u z(s)$ *are proper*, $y = H_y z_{-k}$, $u = H_u z_0$.

In the next step, we translate the above characterizations in terms of subspace recursions. We use the notation introduced in Section II-C.

Lemma 4.3: Let a behavior be given by a pencil representation (4.6). *For all $k \geq 0$, we have*

$$Y^{-k} = H_y\big[V^*(sG - F) \cap T^k\big(H_y, sG - F; T^*(H, sG - F)\big)\big] \quad (4.8)$$

$$Y^{k} = H_y\big[T^*(H, sG - F) \cap V^k\big(H_u, sG - F; V^*(sG - F)\big)\big] \quad (4.9)$$

$$U^{-k} = H_u\big[T^*(H, sG - F) \cap V^k\big(H_y, sG - F; V^*(sG - F)\big)\big] \quad (4.10)$$

$$U^{k} = H_u\big[V^*(sG - F) \cap T^k\big(H_u, sG - F; T^*(H, sG - F)\big)\big]. \quad (4.11)$$

Proof: See Appendix.

A simplification in the description of the limit subspaces is obtained from the following lemma.

Lemma 4.4: In the situation of the preceding lemma, we have

$$T^*\big(H_y, sG - F; T^*(H, sG - F)\big) = T^*(H_y, sG - F) \quad (4.12)$$

$$V^*(H_u, sG - F; V^*(sG - F)) = V^*(H_u, sG - F). \quad (4.13)$$

Proof: It follows immediately from the definitions that

$$T^k\big(H_y, sG - F; T^*(H, sG - F)\big) \supset T^k(H_y, sG - F) \supset T^k(H, sG - F) \quad (4.14)$$

for all $k \geq 0$. Consequently, we have

$$T^*\big(H_y, sG - F; T^*(H, sG - F)\big) \supset T^*(H_y, sG - F) \quad (4.15)$$

and

$$T^*(H_y, sG - F) \supset T^*(H, sG - F) = T^0\big(H_y, sG - F; T^*(H, sG - F)\big). \quad (4.16)$$

By induction, we get from the last line

$$T^*(H_y, sG - F) \supset T^*\big(H_y, sG - F; T^*(H, sG - F)\big), \quad (4.17)$$

and together with (4.15), this completes the proof of the first claim. The second claim is proved in a similar way.

It would, in fact, not be difficult to formulate the above lemma as a consequence of a general principle for iterations in lattices, but we prefer to keep the formulation concrete. We now come to the main result of this paper.

Theorem 4.5: Let a behavior with external variables y and u be given by (4.6). *The following statements hold:*

i) y processes u if and only if

$$H_y[T^*(H, sG - F) \cap V^*(H_u, sG - F)] = \{0\}. \quad (4.18)$$

ii) u is free if and only if

$$H_u[V^*(sG - F) \cap T^*(H_u, sG - F)] = U. \quad (4.19)$$

iii) Assume that both conditions above are satisfied, so that a unique transfer function $T(s)$ exists. Then we have

$$\dim \ker T(s) = \dim H_u\big[T^*(H, sG - F) \cap V^*(H_y, sG - F)\big] \quad (4.20)$$

$$\dim \operatorname{Im} T(s) = \dim H_y\big[V^*(sG - F) \cap T^*(H_y, sG - F)\big]. \quad (4.21)$$

iv) In the notation of Section II, we have for all $k \geq 0$,

$$p_k = \dim H_y\big[T^*(H, sG - F) \cap V^k\big(H_u, sG - F; V^*(sG - F)\big)\big] \quad (4.22)$$

$$t_k = \dim H_y\big[V^*(sG - F) \cap T^k\big(H_y, sG - F; T^*(H, sG - F)\big)\big] \quad (4.23)$$

$$s_k = \dim H_u\big[V^*(sG - F) \cap T^k(H_u, sG - F; T^*(H, sG - F))\big] - \dim H_u\big[T^*(H, sG - F) \cap V^*(H_y, sG - F)\big] \quad (4.24)$$

$$j_k = \dim\big[H_u\big[T^*(H, sG - F) \cap V^k\big(H_y, sG - F; V^*(sG - F)\big)\big] - \dim H_u\big[T^*(H, sG - F) \cap V^*(H_y, sG - F)\big]. \quad (4.25)$$

Proof: The statements follow by combining Theorems 3.3 and 3.4 with Lemmas 4.3 and 4.4.

The strength of the theorem lies in the fact that it allows an arbitrary amount of redundancy in the differential/algebraic system (4.6). Of course, the statements in the theorem will simplify if it is known that the system (4.6) satisfies certain nonredundancy conditions. Some simple conditions of this sort are shown in the lemma below. For an interpretation of (4.26) and (4.27) as nonredundancy conditions, cf. [4], [26], [27]. For purposes of comparison, recall that the following are necessary conditions for a representation of the form (4.1) to be minimal (see, for instance, [13, proposition 1.1]):

i) G is surjective

ii) $\ker G \cap \ker H = \{0\}$.

Lemma 4.6: Let $F, G: Z \to X$ and $H: Z \to W$ be linear mappings. We have

i) $T^*(H, sG - F) = \ker G$ if

$$F^{-1}[\operatorname{Im} G] \cap \ker G \cap \ker H = \{0\} \quad (4.26)$$

ii) $V^*(sG - F) = F^{-1}[\operatorname{Im} G]$ if

$$F[\ker G] + \operatorname{Im} G = X, \quad (4.27)$$

and in this case, $GV^*(sG - F) = \operatorname{Im} G$.

Proof: The first claim is immediate from the definitions. If (4.27) holds, we have $\ker G + V^1(sG - F) = \ker G + F^{-1}[\operatorname{Im} G] = Z$ so that $GV^1(sG - F) = \operatorname{Im} G$. This implies that $V^k(sG - F) = V^1(sG - F) = F^{-1}[\operatorname{Im} G]$ for all $k \geq 1$, so that also $V^*(sG - F) = F^{-1}[\operatorname{Im} G]$. Moreover, we have $GV^*(sG - F) = GV^1(sG - F) = \operatorname{Im} G$.

Below, we want to compare our geometric formulation with formulations in terms of matrix pencils. For this, we will need the following result which describes the rank and the pole/zero structure at infinity of an arbitrary matrix pencil in geometric terms. The facts described below are known ("essentially" since Kronecker, but see also [1], [3], [6], [18]); nevertheless, we give a proof in order to demonstrate that the result can be obtained from our main theorem by a straightforward calculation.

Corollary 4.7: Let X_1 and X_2 be finite-dimensional linear spaces, and let K and L be linear mappings from X_1 to X_2. Denote the number of zeros at infinity of $sK - L$ of order $\leq k (\geq k)$ by $t_k(j_k)(k \in \mathbb{Z})$. The following statements hold:

i) $\dim\ker(SK - L) = \dim(V^*(sK - L) \cap \ker K)$

ii) $\dim\operatorname{Im}(sK - L) = \dim(LT^*(sK - L) + \operatorname{Im} K)$

iii) $j_k = \dim(V^k(sK - L) \cap \ker K) - \dim(V^*(sK - L) \cap \ker K) \quad (k \geq 0)$

iv) $t_k = \dim(LT^{k+1}(sK - L) + \operatorname{Im} K)$ *for* $k \geq -1$, *and* $t_k = 0$ *for* $k \leq -2$.

Proof: The transfer matrix of the pencil representation (F, G, H_y, H_u) with

$$G = [K \quad 0], \qquad F = [L \quad I],$$
$$H_y = [0 \quad I], \qquad H_u = [I \quad 0] \tag{4.28}$$

is given by $T(s) = sK - L$. All mappings have $X_1 \oplus X_2$ as their domain; F, G, and H_y map into X_2, whereas H_u maps into X_1. To obtain formulas for the rank and the pole/zero structure at infinity of $sK - L$, we have to compute the sequences occurring in Theorem 4.5. By Lemma 4.6, we have $T^*(H, sG - F) = \ker G = \ker K \oplus X_2$ and $GV^*(sG - F) = \operatorname{Im} G = \operatorname{Im} K$. Calculation yields further

$$V^1(H_u, sG - F; V^*(sG - F)) = \{0\} \oplus \operatorname{Im} K \tag{4.29}$$

$$V^k(H_u, sG - F; V^*(sG - F)) = \{0\} \oplus \{0\} \qquad (k \geq 2) \tag{4.30}$$

$$T^k\big(H_y, sG - F; T^*(H, sG - F)\big) = T^{k+1}(sK - L) \oplus X_2 \qquad (k \geq 0) \tag{4.31}$$

$$T^k(H_u, sG - F; T^*(H, sG - F)) = X_1 \oplus X_2 \qquad (k \geq 1) \tag{4.32}$$

$$V^k\big(H_y, sG - F; V^*(sG - F)\big) = V^k(sK - L) \oplus \{0\} \qquad (k \geq 1). \tag{4.33}$$

An application of Theorem 4.5 now produces the results above.

The corollary allows us to give an alternative formulation of the nonredundancy conditions of Lemma 4.6. Conversely, the proposition below can also be seen as providing a geometric characterization of matrix pencil properties; in this sense, it generalizes [1, theorems 8 and 10] to the not necessarily regular case.

Proposition 4.8: Let F, G: $Z \to X$ and H: $Z \to W$ be linear mappings. The condition (4.26) holds if and only if the matrix pencil $[sG^T - F^T \quad H^T]^T$ has full column rank and has no zeros at infinity.[1] *The condition (4.27) holds if and only if the matrix pencil $sG - F$ has full row rank and has no zeros at infinity.*

Proof: Using i) and iii) of Corollary 4.7, we see that the matrix pencil $[sG^T - F^T \quad H^T]^T$ has full column rank and no zeros at infinity if and only if $V^1(H, sG - F) \cap \ker G = \{0\}$. Because $V^1(H, sG - F)$ equals $F^{-1}[\operatorname{Im} G] \cap \ker H$ by definition (2.7), this proves the first claim. Next, note that for any matrix pencil $sG - F$, we have the inequalities

$$t_0 \leq r \leq \dim X \tag{4.34}$$

where t_0 denotes the number of zeros at infinity of order ≤ 0 and r is the rank of $sG - F$. By iv) of Corollary 4.7, the condition (4.27) is equivalent to $t_0 = \dim X$. From this, the statement in the proposition is immediate.

Corollary 4.9: Let a behavior be given by a system (4.6) which is such that the matrix pencil $[sG^T - F^T \; H_y^T \; H_u^T]^T$ has full column rank and no zeros at infinity. Under this condition, y processes u if and only if the matrix $[sG^T - F^T \; H_u^T]^T$ has full column rank.

Proof: Because $V^*(H_u, sG - F) \subset F^{-1}[\operatorname{Im} G] \cap \ker H_u$, the condition (4.18) for y to process u is equivalent, under (4.26), to

$$T^*(H, sG - F) \cap V^*(H_u, sG - F) = \{0\}. \tag{4.35}$$

Also, it follows from Lemma 4.6 that $T^*(H, sG - F) = \ker G$ if (4.26) holds. So, under this condition, (4.35) is equivalent to $V^*(H_u, sG - F) \cap \ker G = \{0\}$ which, by Corollary 4.7, is the condition for $[sG^T - FT \quad H_u^T]^T$ to have full column rank.

Corollary 4.10: Let a behavior be given by a system (4.6) which is such that the matrix pencil $sG - F$ has full row rank and has no zeros at infinity. Under this condition, u is free if and only if the matrix $[sG^T - F^T \quad H_u^T]^T$ has full row rank.

Proof: By Corollary 4.7, $[sG^T - F^T \quad H_u^T]^T$ has full row rank if and only if

$$\begin{bmatrix} F \\ H_u \end{bmatrix} T^*(H_u, sG - F) + \operatorname{Im} \begin{bmatrix} G \\ 0 \end{bmatrix} = X \oplus U. \tag{4.36}$$

Assume now that (4.27) holds. Because $\ker G \subset T^*(H_u, sG - F)$, we then have $FT^*(H_u, sG - F) + \operatorname{Im} G = X$. This means that the projection π_X of $X \oplus U$ onto

[1] In line with standard usage, "no zeros at infinity" is understood here and below as "no zeros at infinity of positive order."

X is surjective even if constrained to the space on the left-hand side of (4.36); since the kernel of the constrained projection is $H_u[T^*(H_u, sG - F) \cap F^{-1}[\operatorname{Im} G]]$, it follows that under condition (4.27), one has

$$\dim\left(\begin{bmatrix} F \\ H_u \end{bmatrix} T^*(H_u, sG - F) + \operatorname{Im}\begin{bmatrix} G \\ 0 \end{bmatrix}\right) = \dim X + \dim H_u\left[T^*(H_u, sG - F) \cap F^{-1}[\operatorname{Im} G]\right]. \quad (4.37)$$

Comparing this and (4.36) with the condition (4.19), and using the fact that $V^*(sG - F) = F^{-1}[\operatorname{Im} G]$ if (4.27) is satisfied (Lemma 4.6), we see that the corollary holds.

Corollary 4.11: Let a behavior be given by a system of the form (4.6). *Assume that the matrix pencils* $sG - F$ *and* $[sG^T - F^T \quad H_y^T \quad H_u^T]^T$ *have full row rank and full column rank, respectively, and that both have no zeros at infinity. Under these conditions, the system* (4.6) *has a unique transfer matrix* $T(s)$ *if and only if the matrix* $[sG^T - F^T \quad H_u^T]^T$ *is nonsingular. Moreover, if this holds, we have:*

i) for each $k \geq 1$, *the number of poles at infinity of order* k *of* $T(s)$ *is equal to the number of zeros at infinity of order* k *of* $[sG^T - F^T \quad H_u^T]^T$

ii) for each $k \geq 1$, *the number of zeros at infinity of order* k *of* $T(s)$ *is equal to the number of zeros at infinity of order* k *of* $[sG^T - F^T \quad H_y^T]^T$

iii) $\dim \ker T(s) = \dim \ker [sG^T - F^T \quad H_y^T]^T$

iv) $\dim \operatorname{Im} T(s) = \dim \operatorname{Im} [sG^T - F^T \quad H_y^T]^T - \dim X$, *where* X *denotes the codomain of* F *and* G.

Proof: The statement about the existence of the transfer matrix is immediate from Corollaries 4.9 and 4.10. To prove claim i), we note that the number of poles at infinity of $T(s)$ of order $\geq k$ is, according to our main result, Theorem 4.5, given in general by $\dim H_y[T^*(H, sG - F) \cap V^k(H_u, sG - F;\ V^*(sG - F))]$. If we now assume that the two conditions of Lemma 4.6 hold, then we have, first of all, that $T^*(H, sG - F)$ equals $\ker G$. Moreover, we have $GV^*(sG - F) = \operatorname{Im} G$, which implies that $V^k(H_u, sG - F;\ V^*(sG - F)) = V^k(H_u, sG - F)$ for $k \geq 1$. Also note that $V^k(H_u, sG - F;\ V^*(sG - F)) \subset \ker H_u \cap F^{-1}[\operatorname{Im} G]$ because $V^*(sG - F) = F^{-1}[\operatorname{Im} G]$. Therefore, we get

$$\dim H_y\left[T^*(H, sG - F) \cap V^k(H_u, sG - F; V^*(sG - F))\right] = \dim\left(\ker G \cap V^k(H_u, sG - F)\right). \quad (4.38)$$

This proves claim i) because on the right-hand side, we have exactly the expression for the number of zeros at infinity of order $\geq k$ for the pencil $[sG^T - F^T \quad H_u^T]^T$. Claim ii) is proved analogously. The proofs for claims iii) and iv) are obtained from the proofs of Corollaries 4.9 and 4.10, respectively, by interchanging the roles of u and y.

The above results will look more familiar when stated in terms of a descriptor representation. We shall come to this now.

V. The Descriptor Representation

In this section, we consider the descriptor representation

$$\begin{aligned} \sigma Ex &= Ax + Bu \\ y &= Cx + Du. \end{aligned} \quad (5.1)$$

Here, the matrices E and A are not necessarily square; the domain of the mappings E and A will be denoted by X_d (descriptor space), while the codomain will be denoted by X_e (equation space). In order to get expressions in terms of the matrices E, A, B, C, and D for the subspaces that are of interest in this paper, we merely need to rewrite the descriptor representation (E, A, B, C, D) as a pencil representation (F, G, H_y, H_u) by defining

$$\begin{aligned} G &= [E \quad 0], & F &= [A \quad B], \\ H_y &= [C \quad D], & H_u &= [0 \quad I]. \end{aligned} \quad (5.2)$$

Because of the partitioning of these matrices and the particular form of G and H_u, the expressions for the two characteristic subspace sequences take on a new form. Straightforward computation leads to the following result.

Lemma 5.1: Let a behavior be given by a descriptor representation (E, A, B, C, D). *Then we have*

$$Y^{-k} = [C \quad D]\left([A \quad B]^{-1} EV^*(sE - A, B) \cap \left(T^k\left(\begin{bmatrix} sE - A \\ C \end{bmatrix}, \begin{bmatrix} B \\ D \end{bmatrix}; T^*(C, sE - A)\right) \oplus U\right)\right) \quad (k \geq 0) \quad (5.3)$$

$$Y^k = C\left[T^*(C, sE - A) \cap V^k(sE - A; V^*(sE - A, B))\right] \quad (k \geq 1) \quad (5.4)$$

$$U^{-k} = \begin{bmatrix} B \\ D \end{bmatrix}^{-1}\left[\begin{bmatrix} A \\ C \end{bmatrix} T^*(C, sE - A) + \begin{bmatrix} E \\ 0 \end{bmatrix} V^{k-1}\left(\begin{bmatrix} sE - A \\ C \end{bmatrix}, \begin{bmatrix} B \\ D \end{bmatrix}; V^*(sE - A, B)\right)\right] \quad (k \geq 1) \quad (5.5)$$

$$U^k = B^{-1}\left[AT^k(sE - A; T^*(C, sE - A)) + EV^*(sE - A, B)\right] \quad (k \geq 0). \quad (5.6)$$

Combining Lemma 5.1 with Theorems 3.3 and 3.4 allows one to describe the input–output structure in terms of descriptor parameters just as in Theorem 4.5. However, as already suggested by the looks of the formulas in the above lemma, this does not lead to very attractive results. Let us just formulate one proposition for descriptor systems at the most general level.

Proposition 5.2: Let a behavior be given by a descriptor representation (E, A, B, C, D). *There exists a unique transfer matrix* $T(s)$ *if and only if*

$$T^*(C, sE - A) \cap V^*(sE - A) \subset \ker C \quad (5.7)$$

and

$$AT^*(sE - A) + EV^*(sE - A, B) \supset \operatorname{Im} B. \quad (5.8)$$

The transfer matrix $T(s)$ is proper if and only if

$$T^*(C, sE - A) \cap A^{-1}EV^*(sE - A, B) \subset \ker C. \quad (5.9)$$

Proof: The conditions for the existence of the transfer matrix follow from Lemma 5.1 and Theorem 3.3 (also use the obvious analog of Lemma 4.4). It follows from Theorem 3.4 that $T(s)$ is proper if and only if $Y^1 = 0$. By Lemma 5.1, this condition translates into (5.9).

When we introduce the nonredundancy conditons of Lemma 4.6, we recover (with some extras) the results of [28], [29], which were obtained in these references in a completely different way. In our context, the proof is straightforward by use of the connection (5.2), together with, for instance, the fact that the pole/zero structure at infinity of the pencil

$$\begin{bmatrix} sG - F \\ H_u \end{bmatrix} = \begin{bmatrix} sE - A & B \\ 0 & I \end{bmatrix} \quad (5.10)$$

is the same as the pole/zero structure at infinity of $sE - A$, except for poles/zeros of order 0.

Corollary 5.3: Let a system be given in descriptor form (5.1). *Assume that the matrices* $[sE - A \quad B]$ *and* $[sE^T - A^T \quad C^T]^T$ *have full row rank and full column rank, respectively, and that both matrices have no zeros at infinity. The system then determines a unique transfer matrix $T(s)$ if and only if the matrix $sE - A$ is nonsingular. Moreover, in this case, we have*

i)

$$\dim \ker T(s) = \dim \ker \begin{bmatrix} sE - A & -B \\ C & D \end{bmatrix} \quad (5.11)$$

ii)

$$\dim \operatorname{Im} T(s) = \dim \operatorname{Im} \begin{bmatrix} sE - A & -B \\ C & D \end{bmatrix} - \dim X_e \quad (5.12)$$

iii) for all $k \geq 1$, the number of poles at infinity of order k of $T(s)$ equals the number of zeros at infinity of order k of $sE - A$

iv) for all $k \geq 1$, the number of zeros at infinity of order k of $T(s)$ equals the number of zeros at infinity of order k of the matrix

$$\begin{bmatrix} sE - A & -B \\ C & D \end{bmatrix}$$

v) the transfer matrix $T(s)$ is proper if and only if $A^{-1}[\operatorname{Im} E] \cap \ker E = \{0\}$.

Remark 5.4: For a standard state-space representation of the form $\sigma x = Ax + Bu$, $y = Cx + Du$, the nonredundancy conditions of the above corollary are automatically satisfied. One can therefore combine the above corollary with Corollary 4.7 and immediately obtain the standard geometric characterizations of the rank of the transfer matrix [21] and of its zero structure at infinity [17], [20], [22].

VI. Conclusions

The need for describing the input–output structure of general linear differential/algebraic systems arises in computer-aided modeling and various other applications. In this paper, we have given explicit formulas in terms of the original parameters for systems with an arbitrary amount of redundancy. These formulas allow one to establish whether the system determines an input–output relation at all, and if so, they describe the rank of the transfer matrix and its pole/zero structure at infinity. The formulas may be seen as generalizations of a number of classical results on the input–output structure of standard state-space systems and descriptor systems satisfying certain constraints; indeed, these results are recovered as corollaries. For the main line of our derivation, we have preferred the pencil representation over the often-used descriptor representaiton since the more symmetric treatment of "u" and "y" variables in the pencil representation is, at the level of generality of this paper, an advantage.

We have discussed only linear systems in this paper. To obtain explicit formulas as in the present paper for systems of nonlinear differential and algebraic equations would appear to be a challenging task, which is probably impossible to carry out in general. It should be noted, however, that promising advances have been made with methods from the mathematical discipline called (confusingly, in this context) differential algebra; see, for instance, [8]–[10].

Appendix

For the proof of Lemma 4.3, we need the following lemma, which is a reformulation of [12, theorem 2.8].

Lemma A.1: Let F, G: $Z \to X$ be linear mappings. A vector $z \in Z$ belongs to $V^*(sG - F)$ if and only if there exists a strictly proper rational function $\xi(s)$ such that $(sG - F)\xi(s) = Gz$.

Proof of Lemma 4.3: We begin with formula (4.8). First, let $y \in Y^{-k}$. It follows from Lemma 4.2 that there exists a rational vector $z(s)$ with Laurent expansion (4.7) such that

i) $(sG - F)z(s) = 0$

ii) $s^k H_y z(s)$ and $H_u z(s)$ are proper

iii) $y = H_y z_k$.

From i), it follows that $z_i \in V^*(sG - F)$ for all i. In particular, we have $z_k \in V^*(sG - F)$. From i) and ii), it follows that $z_{-l} \in T^1(H, sG - F), \cdots, z_0 \in T^{l+1}(H, sG - F)$, so we certainly have $z_0 \in T^*(H, sG - F) = T^0(H_y, sG - F;\ T^*(H, sG - F))$. Now, ii) implies that $z_1 \in T^1(H_y, sG - F;\ T^*(H, sG - F)), \cdots, z_k \in T^k(H_y, sG - F;\ T^*(H, sG - F))$. Finally, it follows from iii) that $y \in H_y[V^*(sG - F) \cap T^k(H_y, sG - F;\ T^*(H, sG - F))]$.

Conversely, assume now that $y \in H_y[V^*(sG - F) \cap T^k(H_y, sG - F;\ T^*(H, sG - F))]$. Then there exists $z_k \in V^*(sG - F) \cap T^k(H_y, sG - F;\ T^*(H, sG - F))$ such that $y = H_y z_k$. Since $z_k \in T^k(H_y, sG - F;\ T^*(H, sG - F))$, we can find $z_0, \cdots, z_{k-1}$ such that $Fz_{k-1} =$

$Gz_k, \cdots, Fz_0 = Gz_1$, while $H_y z_{k-1} = 0, \cdots, H_y z_0 = 0$ and $z_0 \in T^*(H, sG - F)$. Since $z_0 \in T^*(H, sG - F)$, there exist $z_{-1}, \cdots, z_{-l}$ such that $Gz_0 = Fz_{-1}, \cdots, Gz_{-l+1} = Fz_{-l}, Gz_{-l} = 0$, while $Hz_{-1} = \cdots = Hz_{-l} = 0$. Because $z_k \in V^*(sG - F)$, we can find $z_{k+1} \in V^*(sG - F)$ such that $Fz_k = Gz_{k+1}$. According to Lemma A.1, there exists a strictly proper rational function $\xi(s)$ such that $(sG - F)\xi(s) = Gz_{k+1}$. Now define

$$z(s) = z_{-l}s^l + z_{-l+1}s^{l-1} + \cdots + z_0 + z_1 s^{-1} + \cdots + z_k s^{-k} + \xi(s)s^{-k}. \quad \text{(A.1)}$$

Then it is easily seen that conditions i)–iii) hold. From Lemma 4.2, it now follows that $y \in Y^{-k}$.

To show (4.9), we first prove that $Y^k \subset H_y[T^*(H, sG - F) \cap V^k(H_u, sG - F; V^*(sG - F))]$. Let $y \in Y^k$. It follows from Lemma 4.2 that there exists a rational vector $z(s)$ with Laurent expansion (4.7) such that

i) $(sG - F)z(s) = 0$

ii) $H_y z(s)$ and $s^k H_u z(s)$ are proper

iii) $y = H_y z_0$.

As above, it follows that $z_0 \in T^*(H, sG - F)$ and $z_k \in V^*(sG - F)$. So, by definition, $z_k \in V^0(H_u, sG - F; V^*(sG - F))$. It follows from i) and ii) that $z_{k-1} \in V^1(H_u, sG - F; V^*(sG - F)), \cdots, z_0 \in V^k(H_u, sG - F; V^*(sG - F))$. Condition iii) now produces the desired result.

Next, we prove the reverse inclusion. Let $y \in H_y[T^*(H, sG - F) \cap V^k(H_u, sG - F; V^*(sG - F))]$. Then there exists $z_0 \in T^*(H, sG - F) \cap V^k(H_u, sG - F; V^*(sG - F))$ such that $y = H_y z_0$. Since $z_0 \in T^*(H, sG - F)$, there exist $z_{-1}, \cdots, z_{-l}$ such that $Gz_0 = Fz_{-1}, \cdots, Gz_{-l+1} = Fz_{-l}$, $Gz_{-l} = 0$, while $Hz_{-1} = 0, \cdots, Hz_{-l} = 0$. Furthermore, because $z_0 \in V^k(H_u, sG - F; V^*(sG - F))$, we can find $z_1, \cdots, z_k$ such that $Fz_0 = Gz_1, \cdots, Fz_{k-1} = Gz_k$, while $H_u z_0 = 0, \cdots, H_u z_{k-1} = 0$ and $z_k \in V^*(sG - F)$. The condition $z_k \in V^*(sG - F)$ implies, as above, that there exists a strictly proper rational function $\xi(s)$ for which $(sG - F)\xi(s) = Fz_k$. Define $z(s)$ as in (A.1); then it is easily seen that conditions i)–iii) hold. From Lemma 4.2, it now follows that $y \in Y^k$.

Formulas (4.10) and (4.11) are obtained from (4.8) and (4.9) by interchanging y and u (which also implies changing the sign of k).

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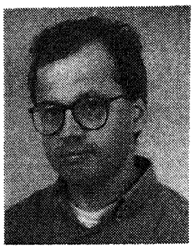

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