Wavelet Bases
Adapted to Inhomogeneous Cases

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Abstract. In this chapter we describe a general strategy for the construction of wavelet bases when the loss of invariance under translation prevents the use of Fourier techniques.

§1 Introduction

The construction of wavelet bases in the usual case of $L^2(\mathbb{R})$ is based on the use of Fourier techniques, *i.e.*, on the invariance under translation. However, in many practical cases, this invariance does not hold and one has to look for an other strategy. Such a new approach has been performed and used for the following examples:

- Wavelet basis of $L^2(\Omega)$, where Ω is an arbitrary open set of $\mathbb{R}^n$, with Dirichlet boundary conditions [8];
- Wavelet basis of $L^2(I)$, where I is an interval of $\mathbb{R}$, without any boundary condition [5,10], or with arbitrary boundary conditions [1,2];
- Wavelets adapted to the differential operator $T = D^\star aD$, where a is the operator of pointwise multiplication by the complex-valued function $a(x)$ which is not regular and which satisfies

$$| a(x) | \leq M = \| a \|_\infty \quad \text{and} \quad \mathrm{Re}\, a(x) \geq 1 \quad a.e.$$

(the last property means that $a(x)$ is accretive);
D is the operator $-i\frac{d}{dx}$ on the domain $H^1(\mathbb{R})$ [3].

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• Wavelets adapted to the "same" operator T where $\mathbb{R}$ is replaced by an interval $I \subset \mathbb{R}$ for which arbitrary boundary conditions are added. The domain V of D is now a closed subspace of $H^1(I)$ which "contains" the boundary conditions, and the domain $V^\star$ of its adjoint $D^\star$ is an other closed subspace of $H^1(I)$ which "contains" the complementary boundary conditions, [4];
• Wavelets associated with a family of irregular meshes [13].

Our aim is to present the generic construction of wavelet bases adapted to inhomogeneous cases, that is common to the treatment of most of the above examples. As an introduction, we will recall a number of basic ideas and definitions used in the homogeneous case, *i.e.,* in the case of $L^2(\mathbb{R})$. This will allow us to show where the differences start.

The multiresolution analysis, as it was defined by Meyer and Mallat [11] for $L^2(\mathbb{R})$, provides the general algorithm for the construction of wavelet bases for $L^2(\mathbb{R})$.

We recall that a multiresolution analysis of $L^2(\mathbb{R})$ consists of a sequence of nested closed subspaces $(V_j)_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ which allow to approximate any square-integrable function with an increasing accuracy :

$$V_j \subset V_{j+1}, \quad \bigcap_{j\in\mathbb{Z}} V_j = \{0\} \quad \text{and} \quad \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}).$$

Each V_j is invariant under dyadic translation:

$$\forall k \in \mathbb{Z} \quad f(x) \in V_j \iff f(x - k2^{-j}) \in V_j.$$

The spaces V_j are derived from each other by scaling:

$$f(x) \in V_j \iff f(2x) \in V_{j+1}.$$

And finally, there exists a regular and well localised function $g(x)$ in V_0 such that the set $(g(x-k))_{k\in\mathbb{Z}}$ forms a Riesz basis of V_0. The regularity r, $r \in \mathbb{N}$, and the localisation of the function $g(x)$ are expressed by

$$\begin{cases} (i) \ \ g \in C^{r-1}; \\ (ii) \ \ g^{(r)} \ \ \text{exists almost everywhere;} \\ (iii) \ \ \forall\, m,\ 0 \le m \le r,\ \forall\, p\ \exists\, C_{m,p} \quad |\, g^{(m)}(x) \,| \le C_{m,p}\,(1+ \mid x \mid)^{-p}. \end{cases} \tag{1}$$

We say that the so-defined multiresolution analysis is r-regular.

We see that this definition requires, among other properties, invariance under translation of the approximation spaces V_j. If we have functions restricted to a finite interval I, then the end points of the interval create an inhomogeneity which breaks the invariance under translation: when we want to construct a wavelet basis for $L^2(I)$, the multiresolution analysis must first be redefined. It is by far not the only example where the usual definition of multiresolution analysis is not suitable. The next section is devoted to the description of three of the above mentioned inhomogeneous cases.

Let us first return to the construction of wavelet bases in the classical case of $L^2(\mathbb{R})$. As we know, it starts by considering the subspace W_j, orthogonal to V_j in V_{j+1}. The fundamental theorem proves the existence of a function $\psi \in W_0$ which has the same decay and regularity properties as g and of which all translated copies (translated by integers) form an orthonormal basis of W_0. By scaling, one derives immediately that the functions $\psi_{jk}(x) = 2^{\frac{j}{2}}\psi(2^j x - k)$, with $k \in \mathbb{Z}$, form an orthonormal basis of W_j. Further, by construction we have

$$L^2 = \bigoplus_{j\in\mathbb{Z}} W_j . \tag{2}$$

Since the sum is orthogonal, the family $(\psi_{jk})_{j,k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$ and for any function $f \in L^2(\mathbb{R})$ we have

$$f = \sum_{j,k\in\mathbb{Z}} d_{jk}\psi_{jk} , \quad \text{where} \quad d_{jk} = \langle f, \psi_{jk}\rangle ,$$

$$\text{and} \quad \int |f|^2 = \sum_{j,k\in\mathbb{Z}} |d_{jk}|^2 .$$

In practical situations, one starts at a coarsest level j_0 and one considers only $j \geq j_0$. Thus, one uses the equality

$$L^2 = V_{j_0} \oplus \left(\bigoplus_{j=j_0}^{\infty} W_j \right) , \tag{3}$$

which gives

$$f(x) = \sum_{k\in\mathbb{Z}} c_{j_0k}g_{j_0k}(x) + \sum_{k\in\mathbb{Z}, j\geq j_0} d_{jk}\psi_{jk}(x).$$

The first sum describes the projection of f on V_{j_0}, *i.e.*, the approximation of f on the coarsest level, and the decomposition of f on each W_j, the missing details between two successive levels of approximation. Here we do not have equality of the norms anymore but only equivalence, *i.e.*, there exists two positive constants A and B, not depending on j_0, such that

$$A \int |f|^2 \leq \sum_{k\in\mathbb{Z}} |c_{j_0k}|^2 + \sum_{k\in\mathbb{Z}, j\geq j_0} |d_{jk}|^2 \leq B \int |f|^2,$$

which is also written as

$$\int |f|^2 \sim \sum_{k\in\mathbb{Z}} | c_{j_0k} |^2 + \sum_{k\in\mathbb{Z}, j\geq j_0} | d_{jk} |^2 .$$

Most important is the unconditional convergence of the series because it means that the associated algorithms for analysis and synthesis are stable (the constants A and B do not depend on j_0). Although orthonormal bases may be convenient, they are not necessary for the practical use as long as the stability of the associated algorithms is preserved. This is why one may use unconditional but nonorthogonal bases. This choice means that, instead of each W_j, one will consider a nonorthogonal complementary set of V_j in V_{j+1}, leading to another decomposition of $L^2(\mathbb{R})$:

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} X_j, \tag{4}$$

or

$$L^2(\mathbb{R}) = V_{j_0} \oplus \left(\bigoplus_{j\geq j_0} X_j \right), \tag{5}$$

where the sums are not orthogonal but only direct. We emphasise that the set of functions that form the bases of the spaces X_j does not directly form a basis of $L^2(\mathbb{R})$. This fact must now be proved in order to make (4) or (5) hold. We shall see that these oblique spaces are constructed by hand, and their bases are explicitly given by the construction. Thus, on the one hand we have the standard wavelet bases for the W_j spaces which trivially form a basis of $L^2(\mathbb{R})$ but are difficult to compute, and on the other hand we have explicit bases of the oblique X_j spaces, for which the property that they form a basis of $L^2(\mathbb{R})$ requires a specific proof. The second approach is clearly better adapted to numerical purposes and indeed used more and more.

Let us recall that, in order to prove the existence of an orthonormal basis of W_0 and to effectively construct it in the translation invariant case, the essential idea is to systematically work on the Fourier side. This is no longer possible in the inhomogeneous cases. However, we may still have access to the oblique complement X_j, of V_j in V_{j+1} and this allows us to construct W_j and its basis. More precisely, we can construct multiresolution analyses in such a way that an oblique complement of V_j in V_{j+1} is always accessible. Finally, the bases of the X_j spaces are interesting for themselves. We have already mentioned their numerical simplicity. In addition, by a suitable definition of X_j we construct a basis that is adapted to different situations. This procedure gives us some additional freedom.

We now have shown the strategy of the construction: a redefinition of the multiresolution analysis and the construction of oblique complementary sets X_j of V_j in V_{j+1}. Then we shall need specific mathematical tools to prove that the bases of these X_j spaces form an unconditional basis of $\overline{\bigcup_{j\in\mathbb{Z}} V_j}$.

The goal of this paper is not to show all the results and the details of the construction which one can find in the above mentioned references. We shall also willingly omit some aspects of motivation when they divert too far from our central point. Neither do we pretend to give an exhaustive overview of the results in this field of research (which is in rapid development) not even, in the above mentioned references. Our aim is to show the common skeleton of the construction of wavelet bases through three different examples which we assume to be representative of our purpose. These examples (1, a special case of 4 and 5 in the list above) are described in the next section. In the third section, we shall show the definition of the multiresolution analyses associated to them. Once the multiresolution analysis is given, the construction of wavelets may start. This is what is done in the section 4, while the section 5 is devoted to the mathematical tools which are needed to prove that the constructed wavelets form a Riesz basis of $\overline{\bigcup_j V_j}$ indeed. In these three sections, the approach is general and followed by the application to the specific examples.

§2 Examples of inhomogeneous problems

We denote by Λ the set of the wavelet labels. Λ is the set of all the numbers $\lambda = (k + \frac{1}{2})2^{-j}$ which have a one-to-one correspondence with the pairs $(k2^{-j}, 2^{-j})$, *i.e.,* points in the position-scale plane. In the $L^2(\mathbb{R})$ case, both j and k are taken from $\mathbb{Z}$, or if we choose the decomposition (1.3), $k \in \mathbb{Z}$ and $j \geq j_0$. This expresses the fact that the union of all dyadic numbers is dense in $\mathbb{R}$. It can easily be generalised to $\mathbb{R}^n$ by considering $k \in \mathbb{Z}^n$ and $j \in \mathbb{Z}$. When one considers an interval or *a fortiori* an irregular mesh, k and j will be taken only from a part of $\mathbb{Z}$. For these cases, Λ will be precisely defined later. For the present time, the wavelets are labeled by $\lambda \in \Lambda$ which we assume to be adapted to each particular case. In the examples below, we call 2^{-j_0} the largest scale taken in account. This implies that we shall consider decompositions of type (1.3) and (1.5).

Let us present the three examples.

The first is the construction of an orthonormal wavelet basis of $L^2(\Omega)$, where Ω is an arbitrary open set of $\mathbb{R}^n$, with homogeneous Dirichlet boundary conditions. The wavelets will belong to $C_0^{2m-2}(\Omega)$, $m \geq 1$, *i.e.,* they will be of regularity C^{2m-2} with support included in Ω (the last property reflects the Dirichlet conditions). In [8], these wavelets are used to analyze the Hölder and Sobolev spaces, $C_0^r(\Omega)$ and $H_0^s(\Omega)$, $0 < r < 2m - 2$ and $0 < s < 2m - 2$. Briefly, this means that the size of the wavelet coefficients, $\langle f, \psi_\lambda \rangle$, of $f \in L^2(\Omega)$ enables us to measure the degree of regularity of f (see [11] chapter VI). In what follows, we shall not describe how this is done. However, this motivation is of importance because it sets a condition to how the multiresolution analysis is constructed. Actually, the key of the characterisation of Hölder spaces by means of wavelet bases lies in the cancellation properties of the wavelets. It requires that the V_j spaces contain, in a certain sense, all the polynomials of degree less or equal to $2m-1$ in each variable.

The presentation of the second example requires more details. Let us consider the boundary value problem $D^\star aDu = f$ on $I=[0,1]$ with the Dirichlet conditions $u(0)=u(1)=0$ and $f\in L^2$, where
- $D=-i\frac{d}{dx}$ on the domain $V=H_0^1(0,1)=\{f\in H^1(0,1):\ f(0)=f(1)=0\}$;
- $D^\star=-i\frac{d}{dx}$ on the domain $V^\star=H^1(0,1)$;
- a is the operator of pointwise multiplication by the complex-valued function $a(x)$ which is bounded and accretive, *i.e.*,

$$| a(x) | \leq M = \| a \|_\infty ,$$

$$\text{Re } a(x) \geq 1.$$

Notice that no assumption of regularity is made for $a(x)$.

In [4], Tchamitchian and Auscher construct a Riesz basis $\{\tau_\lambda(x), \lambda\in\Lambda\}$ of $L^2(0,1)$, $V=H_0^1(0,1)$ and the domain of the operator $T=D^\star aD$, $\mathcal{D}(T)$. The starting point is the construction of a wavelet basis, $(\theta_\lambda)_{\lambda\in\Lambda}$, of $L^2(0,1)$ which has the following specific cancellation property:

$$\int_0^1 \theta_\lambda(x)\frac{dx}{a(x)} = 0. \tag{1}$$

Let us show how we proceed once these special wavelets are known: τ_λ is defined by

$$\tau_\lambda(0)=0\ , \quad D\tau_\lambda(x) = 2^j\frac{1}{a(x)}\theta_\lambda(x). \tag{2}$$

(Notice that $\tau_\lambda\in V$ when (1) holds.) Then we have

$$(\tau_\lambda)_{\lambda\in\Lambda}\ \text{ basis of } \mathcal{D}(T) \iff (\theta_\lambda)_{\lambda\in\Lambda}\ \text{ basis of } H^1(0,1)\ \ (=V^\star)\,.$$

Since $\theta_\lambda\in V^\star$, we may then define the functions σ_λ by

$$2^j\sigma_\lambda(x) = D^\star\theta_\lambda(x), \tag{3}$$

and finally we have

$$D^\star aD\tau_\lambda(x) = 4^j\sigma_\lambda(x). \tag{4}$$

So, when τ_λ and σ_λ have similar properties of size, decay and regularity, (4) can be seen as a "almost diagonalisation" of T. This is indeed the main motivation of the use of wavelets to characterise the domain of differential operators for which they constitute pseudo eigenfunctions.

The collection of functions $(\tau_\lambda(x))_{\lambda\in\Lambda}$ is a Riesz basis of $\mathcal{D}(T)$. The solution u of the boundary value problem above introduced belongs to $\mathcal{D}(T)$. Thus, we may represent $u(x)$ by $u(x) = \sum_{\lambda\in\Lambda} \alpha_\lambda 4^{-j}\tau_\lambda(x)$ as soon as f is represented by the sequence $(\alpha_\lambda)_{\lambda\in\Lambda}$ in the basis of $L^2(0,1)$ consisting of the functions $\sigma_\lambda(x)$, $\lambda \in \Lambda$, *i.e.*, as soon as $f(x) = \sum_{\lambda\in\Lambda} \alpha_\lambda\sigma_\lambda(x)$. In other words, one has decomposed the Green kernel of $D^\star aD$ with Dirichlet boundary conditions in a series $\sum_{\lambda\in\Lambda} 4^{-j}\tau_\lambda(x)\sigma_\lambda(y)$. Rigorously, one must complete the sets $(\tau_\lambda)_{\lambda\in\Lambda}$ and $(\sigma_\lambda)_{\lambda\in\Lambda}$ with two suitable bases of V_{j_0} in order to obtain Riesz bases of $L^2(0,1)$. For simplicity, we skip this part of the problem and refer to [4] for the exact formulation and results. We shall concentrate on the construction of the set of functions $(\theta_\lambda)_{\lambda\in\Lambda}$ satisfying (1), which yields a Riesz basis of $L^2(0,1)$ and $V^\star$ (when completed with a suitable basis of V_{j_0}) .

The third and last example concerns the construction of wavelets associated with a particular family of irregular meshes called Γ. These are defined as follows. We set $j_0 = 0$. Let j_p be a strictly positive integer. Then we have $\Gamma := \bigcup_{j\geq 0}^{j_p} \Gamma_j$, where the Γ_j are nested sequences of points satisfying the three following conditions:
1) $\Gamma_0 := \mathbb{Z}$ and $\Gamma_j \subset 2^{-j}\mathbb{Z}$;
2) if $\Lambda_j := \Gamma_{j+1} \setminus \Gamma_j$ then $\Lambda_j \subset 2^{-j-1}\mathbb{Z} \setminus 2^{-j}\mathbb{Z}$;
3) there exists $\rho \in \mathbb{N}^\star$ such that, if $\gamma = (2k+1)2^{-j-1} \in \Lambda_j$ then all points $\ell 2^{-j}$ with $|\ \ell - k\ | \leq \rho$, belong to Γ_j. This condition is called the *cone condition.*

One sees that the meshes Γ_{j+1} are constructed from Γ_j by addition of some points of $2^{-j-1}\mathbb{Z}$. Thus, Γ_{j_p} contains pieces of $2^{-j_p}\mathbb{Z} \setminus 2^{-j_p+1}\mathbb{Z}$ nested in pieces of $2^{-j}\mathbb{Z}$ for $0 < j < j_p$ and all of $\mathbb{Z}$. The cone condition ensures that the transition between a zone where Γ_j is a coarse mesh and one where Γ_j is refined is not too abrupt. Finally, when we represent the sets of points Γ_0 and Λ_j, $0 \leq j \leq j_p$, in the position-scale half plane, we see that they form cones which cluster in the intervals of finest meshsize. To understand the choice of these meshes, we first recall how a wavelet series converges in $L^2(\mathbb{R})$.

Let us consider $f \in L^2(\mathbb{R})$ represented by the wavelet series

$$f(x) = \sum_{k\in\mathbb{Z}} c_{0k} g_{j_0 k}(x) + \sum_{j\geq 0, k\in\mathbb{Z}} d_{jk}\psi_{jk}(x). \tag{5}$$

Let us assume that f belongs locally to a Sobolev space H^s, $s > 0$, for instance, on an open interval I. Then the wavelet coefficients of f will satisfy:

$$\sum_{k\in\mathbb{Z}} |\ c_{0k}\ |^2 + \sum_{j\geq 0, k\in\mathbb{Z}} |\ d_{jk}\ |^2 < \infty, \tag{6}$$

and

$$\sum_{j\geq 0, k\in \mathbb{Z}: k2^{-j}\in I} 4^{js} \mid d_{jk} \mid^2 < \infty. \tag{7}$$

The expression (6) is the characterisation of L^2, whereas (7) expresses the fact that f is more regular on the interval I (once more, we refer to chapter VI in [11] for the characterisation of functional spaces by wavelets). More generally, one knows that the more regular the function f is in x_0, the quicker the coefficients d_{jk} decrease when $j \to +\infty$ and $k2^{-j} \to x_0$ (see [6] and [7] for complete and precise results). Thus, when f is regular on the larger part of the domain under consideration and singular (or pseudo-singular) on the remaining part, the series (5) will be *sparse* and, in the scale-position half plane, the nonnegligible coefficients will be localised in cones condensing near the singularities.

Let us consider such kind of functions and *assume* that f is well described by its sampling points on the above defined mesh Γ. The wavelets which are constructed in [13] are associated with the points of Λ_j, *i.e.*, they allow to represent f intrinsically by its wavelet coefficients computed from the sampled values on Γ.

In the following sections, we shall refer to the above cases as respectively E1,E2, and E3.

§3 Multiresolution analyses

The definition of the multiresolution analysis associated with any of the three above described problems is divided in two steps: the geometrical and the functional properties. In the first step, we define the sets of dyadic points which will approximate the geometry of the situation, *i.e.*, the sets Γ_j. The definition of the set of wavelet labels, Λ, will follow. In the second step, we define the functional spaces V_j associated with these meshes.

3.1 Geometrical aspects

We already gave a description of the construction of the meshes Γ_j (and Λ_j) for E3. Let us now do it for E1 and E2. In both these cases it consists of restricting the dyadic grids $2^{-j}\mathbb{Z}$.

E1- $\Gamma_j \subset 2^{-j}\mathbb{Z}^n$ is the set of all $\gamma = 2^{-j}k$ whose distance to $\partial\Omega$ is greater or equal to $(m+1)2^{-j}$.

E2- For $j \geq 0$, Γ_j is defined by: $\Gamma_j = \{k2^{-j} : 0 \leq k \leq 2^j\}$.

Thus, in all cases, the so defined meshes satisfy the following properties:

- $\Gamma_j \subset \Gamma_{j+1}$ for j, $j_0 \leq j \leq j_p$,
- $\bigcup_{j=j_0}^{j_p} \Gamma_j$ is dense in some domain Ω_p,

and the set of the wavelet labels Λ is given by:

$$\Lambda = \bigcup_{j=j_0}^{j_p} \Lambda_j \quad \text{with} \quad \Lambda_j = \Gamma_{j+1} \setminus \Gamma_j,$$

where, for E3: $j_0 = 0$ and j_p is a strictly positive integer given by the mesh Γ, Ω_p is Γ; for E2: $j_0 = 0$ and $j_p = +\infty$, Ω_p is the interval $[0, 1]$; for E1: Ω_p is the open set Ω, $j_p = +\infty$ and j_0 is an integer when Ω is bounded.

3.2 Functional properties

The main idea is to construct the V_j spaces in such a way that for any function $f \in V_j$, the ℓ^2-norm of its values on Γ_j, $(f(\gamma))_{\gamma\in\Gamma_j}$, is equivalent to the L^2-norm of f. This means that there should exist two positive constants, A and B, independent of j, such that

$$A \int |f|^2 \;\le\; \sum_{\gamma\in\Gamma_j} c_j(\gamma)\, |f(\gamma)|^2 \;\le\; B \int |f|^2,$$

where $c_j(\gamma)$ is a normalisation constant equal to: $2^{-\frac{j}{2}}$ for E2, $2^{-\frac{jn}{2}}$ for E1 and the distance to the nearest neighbour of γ in Γ_j for E3. In other words, we need to prove:

Propostion. *On V_j, the norm defined by the scalar product*

$$(f, g)_j = \sum_{\gamma\in\Gamma_j} c_j(\gamma) f(\gamma) \overline{g(\gamma)}$$

is equivalent to the L^2 norm, uniformly with respect to j.

This result is nothing but a theorem of sampling on Γ_j. It is obtained by the construction of the continuous functions $\Delta_{j,\gamma} \in V_j$, which have the following property: for any $\gamma, \gamma' \in \Gamma_j$, $\Delta_{j,\gamma}(\gamma') = 1$ if $\gamma = \gamma'$ and $\Delta_{j,\gamma}(\gamma') = 0$ if $\gamma \neq \gamma'$ (the so called Lagrangian property). The fact that the set $(\Delta_{j,\gamma}(x))_{\gamma\in\Gamma_j}$ forms an unconditional basis of V_j is equivalent to the proposition. The motivation of this idea will become clear in the next section. Then, we shall see that it allows us to define the required supplementary set of V_j in V_{j+1} very naturally.

Before we give the description of the multiresolution analysis in each particular case, let us mention which properties of the classical multiresolution analysis of $L^2(\mathbb{R})$ are preserved. Since the functions of V_j are completely defined by their values on the meshes Γ_j, the features of the new multiresolution analysis come very naturally from its geometrical properties as described above. We have, with the same notations as before,

(1) $V_j \subset V_{j+1}$,

(2) $\bigcup_{j=j_0}^{j_p} V_j$ is dense in $L^2(\Omega_p)$,

(3) for each V_j, there exists a family of regular and well localised functions, $(\Delta_{j,\gamma}(x))_{\gamma\in\Gamma_j}$, which constitutes a Riesz basis of V_j.

Neither the invariance under dyadic translation nor the scaling properties are preserved. Notice that for this reason, the bases of the V_j spaces are no longer deduced by scaling from the basis of V_0. Moreover, the basis of each V_j is not formed by a set of dilated and translated versions of a single function. However, their localisation is ensured by the following estimates, so-called *standard estimates*: for $\mid \alpha \mid \leq r$ and two constants C and ζ (depending only on n and r)

$$\mid \partial^\alpha \Delta_{j,\gamma}(x) \mid \leq C\ 2^{j|\alpha|}\ 2^{n\frac{j}{2}} \exp\left[-\zeta 2^j \mid x-\gamma \mid\right]. \tag{1}$$

Remark. Point (2) deserves some additional attention for E3. We shall return to this in the remarks at the end of this section.

E1- Let $Q(j,k)$ be the dyadic cubes defined by $2^j x - k \in [0,1]^n$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$. Let V_j be the space of compactly supported functions of regularity C^{2m-2}, with a support included in Ω and at a distance to $\partial\Omega$ of at least 2^{-j}, and of which the restriction to $Q(j,k)$ coincides with a polynomial of degree less or equal to $2m-1$ in each variable. We introduce the basic spline, s_m, by the convolution of the characteristic function on the unit cube: $s_m(x-m) = \chi \star \cdots \star \chi$ ($2m$ times). Then the set $\{s_{j,\gamma}(x),\ \gamma \in \Gamma_j\}$, defined by $s_{j,\gamma}(x) = 2^{n\frac{j}{2}} s_m(2^j x - k)$, where $\gamma = k2^{-j} \in \Gamma_j$, is a Riesz basis of V_j. From this basis one constructs the desired Lagrangian basis by application of a transfer matrix which preserves estimates of type 1.1(iii).

E2- V_j, $j \geq 0$, is the space of square integrable piecewise linear splines $f(x)$ of the form

$$f(x) = \sum_{\gamma\in\Gamma_j} 2^{-\frac{j}{2}} f(\gamma)\Delta_{j,\gamma}(x), \quad x \in [0,1].$$

The so called hat-function $\Delta(x)$ is defined by $\Delta(x) = \sup\{1-|x|,\ 0\}$, $x \in \mathbb{R}$. The $\Delta_{j,\gamma}(x)$, $\gamma \in \Gamma_j$ are the restrictions to $[0,1]$ of the functions $\Delta_{j,k}(x) = 2^{\frac{j}{2}}\Delta(2^j x - k)$ with $k \in 2^j\Gamma_j$.

E3- V_j consists of the continuous functions of $L^2(\mathbb{R})$ which are completely determined by their values on Γ_j according to the following procedure. The first part of the procedure consists of defining the values of f on the whole set $2^{-j}\mathbb{Z}$ by "completing" $\frac{\mathbb{Z}}{2}$, then $\frac{\mathbb{Z}}{4}$ and so until $2^{-j}\mathbb{Z}$. Let us consider $\frac{k}{2} \in \frac{\mathbb{Z}}{2}$. If $\frac{k}{2} \in \Gamma_j$, $f(\frac{k}{2})$ is given. If $\frac{k}{2} \notin \Gamma_j$, then $f(\frac{k}{2})$ is *defined* by dyadic interpolation from its values on $\mathbb{Z}$ ($\mathbb{Z}$ always belongs to Γ_j by definition). We continue until $f(k2^{-j})$ is defined for all $k \in \mathbb{Z}$. Then, in the second part of the procedure,

f is defined on the whole real line by successive dyadic interpolation from its values on $2^{-j}\mathbb{Z}$. Of course, Δ is the fundamental interpolant of the process of dyadic interpolation. The functions $\Delta_{j,\gamma}(x)$ are compactly supported and, due to the cone condition, their support is uniformly controlled by $c_j(\gamma)$, *i.e.*, by the local density of the irregular mesh Γ_j.

Remarks. Our first remark concerns the treatment of the boundary conditions in E2. As we already said, V_j must be in $V^\star = H^1(0,1)$. This implies that the functions of V_j must not satisfy any condition at the boundary of the interval. This is ensured by keeping the end points of the interval, 0 and 1, in Γ_j : *the boundary values of* Γ_j *imply boundary conditions on the functions of* V_j. According to this rule, Auscher and Tchamitchian propose all the possible boundary conditions by choosing a suitable definition of Γ_j :

$$\Gamma_j = \{k2^{-j} : \operatorname{dist}(k2^{-j}, \partial I) \geq 2^{-j}\} \cup BV, \ \forall\, j \geq 0,$$

where BV denotes the set of boundary points in Γ_j adapted to each different situation. In our example, $BV = \{0,1\}$ whereas the example where $V = H^1(0,1)$ and $V^\star = H^1_0(0,1)$, BV is the empty set. The latter ensures that the functions of V_j are equal to 0 at the end points of the interval.

Our second remark, still about E2, concerns the choice of the spaces V_j. The spaces of piecewise linear splines are the simplest one can imagine. The authors of [4] chose them for this reason. It appears that the piecewise linear splines are sufficient for their purpose, which is not to construct wavelets adapted to the differential operator T for their own sake but to use them to prove Kato's conjecture on open sets of $\mathbb{R}$. As usual, the very low regularity of the multiresolution analysis is the price to pay for the simplicity of the piecewise linear splines. There is however no reason to think that the construction could not hold for more complicated (and more regular) spaces V_j.

In contrast to the above, the regularity of the multiresolution analysis constructed for E1 is essential, because the wavelet basis which is developed from it is used for the characterisation of the Hölder spaces. Let us remark that only the splines of odd order are considered. Actually, it is known (see [11], p.24-25) that the splines of even order cannot lead to cardinal (or Lagrangian) splines.

Finally, we return to point (2) for case E3. The functional space $\overline{\bigcup_j V_j}$ is not a classical space. For convenience, we denoted it by $L^2(\Gamma)$ without any ambition of a precise description. For the present time, $L^2(\Gamma)$ has no other definition than this one which is derived from the Lagrangian multiresolution analysis.

§4 Construction of the wavelets

Now the multiresolution analysis has been defined, the construction of wavelets may start. We have divided it in two parts. The first part is devoted to the construction of the wavelets which are called "classical" because they are obtained by considering the usual orthogonal supplement, W_j, of V_j in V_{j+1}. This will lead to the desired orthonormal bases of W_j for E1 and E2 and the not less desired Riesz basis of W_j for E3.

In the second stage, we describe the construction of the "non-classical" wavelets, which are called so because they are obtained by considering oblique supplements of V_j in V_{j+1}. We shall explain how the wavelets $\theta_\lambda(x)$, $\lambda \in \Lambda_j$, which satisfy the special cancellation property (2.1), are constructed and we shall propose an alternative for the Riesz basis defined for E3 earlier.

4.1 The "classical" wavelets

Let us consider the space W_j, the orthogonal supplement of V_j in V_{j+1} with respect to the L^2-scalar product, *i.e.*,

$$W_j := \{f \in V_{j+1} \ : \ \langle f, v\rangle = 0 \quad \forall\, v \in V_j\}\,.$$

As we have already pointed it out, we do not have direct access to the spaces W_j. However, we have defined the spaces V_j of the multiresolution analysis in such a way that we perfectly know how to construct an other supplement of V_j in V_{j+1}, which we call U_j. Let us now give the basic recipe for their construction. Since, from the theorem of sampling on Γ_{j+1}, the functions of V_{j+1} are completely determined by their values on Γ_{j+1}, and since Γ_{j+1} is the disjoint union of Γ_j and Λ_j, we can define U_j as a closed subspace of V_{j+1} consisting of the functions that are completely determined by their values on Λ_j. Let us call $(u_\lambda(x))_{\lambda\in\Lambda_j}$ a Riesz basis of U_j, then we have

$$\forall\, f \in U_j \quad f(x) = \sum_{\lambda\in\Lambda_j} f(\lambda) u_\lambda(x).$$

This ensures that V_{j+1} is the direct sum of V_j and U_j. Later we shall see the precise definition of the two different U_j, constructed for E1 and E2 on the one hand and for E3 on the other hand. From this definition, we shall see that we know $(u_\lambda(x))_{\lambda\in\Lambda_j}$ explicitly.

Now we have at our disposal the three spaces, V_{j+1}, V_j, and U_j, such that

$$V_{j+1} \ = \ V_j \ \oplus \ U_j \ ,$$

where $\oplus$ denotes the direct sum. Moreover, we know a Riesz basis for the different parts:

- $(u_\lambda(x))_{\lambda\in\Lambda_j}$ is a Riesz basis of U_j;
- $(\Delta_{j,\gamma}(x))_{\lambda\in\Gamma_j}$ is a Riesz basis of V_j.

We aim at the construction of a Riesz basis for W_j, the orthogonal supplement of V_j in V_{j+1}. Therefore, we introduce Π_j, the orthogonal projection from V_{j+1} onto V_j; then $I - \Pi_j$ is the orthogonal projection from V_{j+1} onto W_j, and it is an isomorphism between U_j and W_j. Further, it follows that
The family of functions, $(w_\lambda(x))_{\lambda\in\Lambda_j}$, defined by

$$w_\lambda \;=\; (I - \Pi_j)\, u_\lambda \quad \forall\lambda \in \Lambda_j, \tag{1}$$

forms a Riesz basis of W_j. All $w_\lambda(x)$ satisfy standard estimates (3.1).

Let us now return to the definition of U_j.

E1,
E2- For E1, as well as for E2, U_j is the space of functions of V_{j+1} which are equal to 0 on Γ_j. The collection of functions $(u_\lambda(x))_{\lambda\in\Lambda_j}$ where $u_\lambda(x) = \Delta_{j+1,2k+1}(x)$, $\lambda = (2k+1)2^{-j-1} \in \Lambda_j$, forms a Riesz basis of U_j.
E3- For E3, U_j is the supplement of V_j in V_{j+1}, oblique with respect to the L^2-scalar product but orthogonal with respect to the scalar product on V_{j+1}, $(\ ,\ )_{j+1}$. Together with the fact that the functions of U_j are completely determined by their values on Λ_j, this definition allows to construct a Riesz basis of U_j: for $\lambda \in \Lambda_j$, $u_\lambda(x)$ is defined by

$$\begin{cases} u_\lambda \in V_{j+1}; \\ u_\lambda(\lambda') = \delta_{\lambda,\lambda'}, \quad \text{when} \quad \lambda' \in \Lambda_j; \\ u_\lambda(\gamma) = -\frac{c_{j+1}(\lambda)}{c_{j+1}(\gamma)}\, \overline{\Delta_{j,\gamma}(\lambda)}, \quad \text{when} \quad \gamma \in \Gamma_j. \end{cases} \tag{2}$$

Then, it is easy to show that the functions $(u_\lambda(x))_{\lambda\in\Lambda_j}$ form a Riesz basis of U_j.

We see that the space U_j, so-defined for this example, is fundamentally different from the previous one used for E1 and E2. Here U_j is a "wavelet space" since it is orthogonal to V_j with respect to a scalar product which is equivalent, on V_{j+1}, to the L^2-scalar product. We shall return to this in the next section.

Before this, however, we give few comments about the projection Π_j. This projection is explicitly known but its expression deserves a little attention.

E1,
E2- For E1 and E2, one obtains the explicit expression of Π_j by orthonormalisation the Riesz basis $(\Delta_{j,\gamma})_{\gamma\in\Gamma_j}$ of V_j. The orthonormal basis of V_j is denoted $(v_\gamma(x))_{\gamma\in\Gamma_j}$ and, hence, the projection Π_j is given by

$$\forall\, f \in V_{j+1} \quad (\Pi_j f)\,(x) = \sum_{\gamma\in\Gamma_j} \langle f, v_\gamma\rangle\; v_\gamma(x),$$

which gives for $(w_\lambda(x))_{\lambda\in\Lambda_j}$:

$$w_\lambda(x) = u_\lambda(x) - \sum_{\gamma\in\Gamma_j} \langle u_\lambda, v_\gamma\rangle\, v_\gamma(x).$$

Then, by orthonormalisation of $(w_\lambda(x))_{\lambda\in\Lambda_j}$, we obtain the orthonormal basis $(\psi_\lambda(x))_{\lambda\in\Lambda_j}$ of W_j.

E3- For E3, one shows that the biorthogonal family $\left(\tilde{\Delta}_{j,\gamma}\right)_{\gamma\in\Gamma_j}$ of $(\Delta_{j,\gamma})_{\gamma\in\Gamma_j}$ forms also a Riesz basis of V_j. Thus, Π_j is written in this case as:

$$\forall\, f \in V_{j+1} \quad (\Pi_j f)(x) = \sum_{\gamma\in\Gamma_j} \langle f, \tilde{\Delta}_{j,\gamma}\rangle\, \Delta_{j,\gamma}(x),$$

or equivalently by,

$$(\Pi_j f)(x) = \sum_{\gamma\in\Gamma_j} \langle f, \Delta_{j,\gamma}\rangle\, \tilde{\Delta}_{j,\gamma}(x),$$

which gives for $(w_\lambda(x))_{\lambda\in\Lambda_j}$:

$$w_\lambda(x) = u_\lambda(x) - \sum_{\gamma\in\Gamma_j} \langle u_\lambda, \Delta_{j,\gamma}\rangle \tilde{\Delta}_{j,\gamma}(x).$$

The family $(w_\lambda(x))_{\lambda\in\Lambda_j}$ as well as its biorthogonal system $(\tilde{w}_\lambda(x))_{\lambda\in\Lambda_j}$ is a Riesz basis of W_j.

Thus, we have constructed an orthonormal basis of W_j for E1 and E2 consisting of the corresponding orthonormalised $(\psi_\lambda(x))_{\lambda\in\Lambda_j}$ and a system of biorthogonal bases of W_j for E3, consisting of the collections of functions $(w_\lambda(x))_{\lambda\in\Lambda_j}$ and $(\tilde{w}_\lambda(x))_{\lambda\in\Lambda_j}$.

4.2 The "non-classical" wavelets

The non-classical wavelets are obtained by considering the proper oblique supplements of V_j in V_{j+1}. These supplements are not orthogonal to V_j with respect to the L^2-scalar product, but they are orthogonal with respect to another suitable scalar product (E3) or even, with respect to a bilinear (E2) or sesquilinear form (see another example of boundary conditions in [4]). Depending on the choice of scalar product, or bi- (sesqui-) linear form, this specific orthogonality may imply, for the wavelets, specific cancellation properties. It may also allow to construct wavelets suitable for practical use in the sense that the associated analysis and synthesis algorithms are simple and fast. The two families of wavelets that we want to construct illustrate both possibilities.

We have already seen an example of construction of these "non-classical" wavelets for E3. They were obtained by considering the space U_j, supplement of V_j in V_{j+1}, nonorthogonal with respect with the L^2-scalar product but orthogonal with respect to the scalar product on V_{j+1}, $(\, , \,)_{j+1}$, which is equivalent, on V_{j+1} to the L^2-scalar product, *i.e.*,

$$U_j := \{f \in V_{j+1} \; : \; (f, v)_{j+1} = 0 \quad \forall \, v \in V_j\} .$$

The set of functions $(u_\lambda)_{\lambda \in \Lambda_j}$ defined by (2) forms a Riesz basis of U_j, of which the elements are mutually orthogonal for the scalar product $(\, , \,)_{j+1}$. The very simple definition of the functions $u_\lambda(x)$, $\lambda \in \Lambda_j$, and the simplicity of the associated analysis and synthesis algorithms (see [13]) make this basis a good candidate for practical applications.

To construct the wavelets $(\theta_\lambda(x))_{\lambda \in \Lambda_j}$ of the example E2, we define the space X_j, supplement of V_j in V_{j+1}, nonorthogonal with respect to the L^2-scalar product but orthogonal with respect to the bilinear form $B(\, , \,)$, symmetric and continuous on $L^2(0,1)$:

$$B(f,g) = \int_0^1 f(x) \frac{1}{a(x)} g(x) dx.$$

Thus X_j is defined by

$$X_j \; := \; \{f \in V_{j+1} \; : \; B(f,v) = 0 \quad \forall \, v \in V_j\} ,$$

and we have

$$V_{j+1} = V_j \oplus X_j ,$$

where $\oplus$ denotes the direct sum. Following the same approach as before, we define P_j, the projection associated with this direct sum from V_{j+1} onto V_j, the null-space of which is X_j. Then $I - P_j$ is the projection from V_{j+1} onto X_j which is an isomorphism from W_j to X_j, and functions defined by

$$\chi_\lambda(x) = \psi_\lambda(x) - P_j(\psi_\lambda)(x), \quad \forall \, \lambda \in \Lambda_j$$

form a Riesz basis of X_j. The functions $\theta_\lambda(x)$ are obtained by orthonormalisation with respect to the form B, as follows. Let M be the matrix of coefficients $B(\chi_\lambda, \chi_{\lambda'})$, M is bounded and accretive on $\ell^2(\Lambda_j)$. Let $\beta(\lambda, \lambda')$ be the coefficients of the matrix $M^{-\frac{1}{2}}$. From the estimates of $\psi_\lambda(x)$, it is shown in [4] that these coefficients satisfy the following estimates for $\lambda' = (\ell + \frac{1}{2})2^{-j}$ and $\lambda = (k + \frac{1}{2})2^{-j} \in \Lambda_j$:

$$| \, \beta(\lambda, \lambda') \leq C \exp[-\zeta \mid k - \ell \mid] . \tag{3}$$

Then the functions defined by

$$\theta_\lambda(x) = \sum_{\lambda' \in \Lambda_j} \beta(\lambda, \lambda')\chi_{\lambda'}(x), \quad \forall\ \lambda \in \Lambda_j, \tag{4}$$

form a Riesz basis of X_j, orthogonal for B. We shall say that $(\theta_\lambda(x))_{\lambda\in\Lambda_j}$ is a B-orthogonal Riesz basis of X_j.

Notice that we can describe the projection P_j by the same procedure applied to the Riesz basis of V_j. If we denote the B-orthogonal Riesz basis of V_j by $\left(v_\gamma^\star(x)\right)_{\lambda\in\Gamma_j}$, we have

$$\left(P_j f\right)(x) = \sum_{\gamma\in\Gamma_j} B(f, v_\gamma^\star)v_\gamma^\star(x), \quad \forall\ f \in V_{j+1}.$$

It is easy to see that the spaces V_j of the multiresolution analysis of $L^2(0,1)$ defined for E2, are generated by the monomials 1 and x. Thus, the B-orthogonality of the spaces X_j and V_j implies the desired cancellation property for the wavelets $\theta_\lambda(x)$, $\lambda \in \Lambda_j$, that form a basis of X_j. We have

$$\int_0^1 \frac{1}{a(x)}\theta_\lambda(x)dx = \int_0^1 x\frac{1}{a(x)}\theta_\lambda(x) = 0.$$

Notice that $\theta_\lambda(x) \in V^\star$. Effectively, since the points 0 and 1 belong to Γ_j for all $j \geq 0$, and since $(\theta_\lambda(x))_{\lambda\in\Lambda_j} \in V_{j+1}$, from the first remark in Section 3, it follows that $(\theta_\lambda(x))_{\lambda\in\Lambda_j} \in V^\star$.

Remark. Notice that E2 combines two distinct goals. The first goal is the construction of a basis of $L^2(0,1)$, consisting of wavelets which show precise boundary values. This is achieved by the definition of the multiresolution analysis. More precisely, once the idea of Lagrangian multiresolution analysis is adopted, the problem is reduced to the suitable choice of the geometrical aspects of the multiresolution analysis. The second goal consists of constructing wavelets adapted to a differential operator. It is achieved by the right choice of the supplement of V_j in V_{j+1}.

This part is really separable from the first one. Indeed the construction of such wavelets was performed in [3], independently of any interest for the interval. In the same way the construction of wavelets on an interval which show conditions at the boundary of the interval can be found in [1].

§5 The proof that the wavelets form a basis of $\overline{\bigcup_j V_j}$

The bases of the spaces W_j form a basis of $\overline{\bigcup_j V_j}$ since the W_j are mutually orthogonal and since their union is dense in $\overline{\bigcup_j V_j}$ by construction. Hence, the family of wavelets

$(\psi_\lambda(x))_{\lambda\in\Lambda}$ defined for E1 (E2) forms an orthonormal basis of $L^2(\Omega)$ $(L^2(0,1))$. The functions $(w_\lambda(x))_{\lambda\in\Lambda}$ and $(\tilde{w}_\lambda(x))_{\lambda\in\Lambda}$ form a system of biorthogonal bases of $L^2(\Gamma)$. Thus, the problem addressed in E1 has been solved, as well as part of the problem described in E3.

In this paragraph, we shall present the strategy to determine whether the bases $(u_\lambda(x))_{\lambda\in\Lambda}$ and $(\theta_\lambda(x))_{\lambda\in\Lambda}$ form a basis of $L^2(\Gamma)$ and $L^2(0,1)$ respectively. We must warn the reader that the result for E3 is incomplete, *i.e.*, whether the functions $(u_\lambda(x))_{\lambda\in\Lambda}$ form a Riesz basis of $L^2(\Gamma)$ remains an open question. We shall comment on this at the end of this section. As we noticed in the introduction, this fact does not follow immediately from the construction of the wavelets, as it does for the classical wavelets, but it requires a specific proof. As before, we shall first present a general approach and later, we shall mention the particularities for each example. By $\theta_\lambda(x)$, $\lambda\in\Lambda$, we denote the wavelets which are considered (in particular $\theta_\lambda(x)$ and $u_\lambda(x)$ constructed in the previous section). In the following we denote by I an interval of $\mathbb{R}$, finite or infinite. The wavelets we consider satisfy the standard estimates:

$$|\,\theta_\lambda^{(m)}(x)\,| \leq C2^{m\frac{j}{2}}\exp\{-\zeta 2^j\,|\,x-\lambda\,|\} \tag{1}$$

for all $x\in I$ and $\lambda\in\Lambda$ and for all m such that $|m|\leq r$, where r is the regularity of the multiresolution analysis from which the wavelets are derived (C and ζ are two positive constants). Moreover, the wavelets satisfy one of the following cancellation properties:

$$\begin{aligned} &\int_I \theta_\lambda(x)dx \quad \text{vanishes} \\ &\qquad \text{or} \\ &\int_I \theta_\lambda(x)dx \quad \text{is small in a sense which will be made precise later.}\end{aligned} \tag{2}$$

Thus, we need to prove that two strictly positive constants A and B exist such that for any sequence (α_λ) we have

$$A\sum_{\lambda\in\Lambda}|\,\alpha_\lambda\,|^2 \;\leq\; \|\sum_{\lambda\in\Lambda}\alpha_\lambda\theta_\lambda\,\|^2 \tag{3}$$

and

$$\|\sum_{\lambda\in\Lambda}\alpha_\lambda\theta_\lambda\,\|^2 \;\leq\; B\sum_{\lambda\in\Lambda}|\,\alpha_\lambda\,|^2\,. \tag{4}$$

The two inequalities will be treated separately. We start with (4).

5.1 Inequality (4)

The proof of (4) follows from the combination of the properties (1) and (2). We shall need the following concepts of *vaguelettes* and *Carleson condition* to formulate it.

Definition 1. *Let $m_\lambda(x)$ be a sequence of functions defined on I. Then, (m_λ) is said to be a collection of vaguelettes on I if the functions $m_\lambda(x)$ satisfy the standard estimates (1) and the relation*

$$\int_I m_\lambda(x)dx = 0. \tag{5}$$

It is clear that the classical wavelets of $L^2(\mathbb{R})$ form a family of vaguelettes on $\mathbb{R}$. But, where -on the one hand- the wavelets are translated and dilated versions of one single function (the mother wavelet), -on the other hand- by the standard estimates, the vaguelettes only have approximately the same shape as obtained from an initial pattern by dilations and translations. In the regular case where $I = \mathbb{R}$, it is known that (4) holds as soon as the functions $\theta_\lambda(x)$ defined on I are a family of vaguelettes on I [12]. For the case when I is a finite interval of $\mathbb{R}$ it is shown in [4].

In fact, the condition (5) is somewhat too strong. To satisfy (4), it is sufficient that the numbers $\left(\int_I m_\lambda(x)dx\right)$, $\lambda \in \Lambda$, are small, in a sense that is quantified by the *Carleson condition*. Let I_μ be the dyadic interval centered on $\mu = (k+\frac{1}{2})2^{-j} \in \Lambda_j$,*i.e.*, $I_\mu = [k2^{-j}, (k+1)2^{-j}]$. We have the following definition.

Definition 2. *A complex-valued sequence (c_λ) is said to satisfy a Carleson condition if there exists a constant $C \geq 0$ such that for all $\lambda \in \Lambda$, one has*

$$\sum_{\mu\in\Lambda, I_\mu \subset I_\lambda} |\, c_\mu\,|^2 \;\leq C\; |\, I_\lambda\,|\,. \tag{6}$$

We now have the result (see [4]):

Theorem 1. *Let θ_λ, $\lambda \in \Lambda$, be a sequence of functions which satisfies the standard estimates (1). Then, the following two assertions are equivalent:*

(i) $\|\sum \alpha_\lambda \theta_\lambda \|^2 \leq \; C \sum |\, \alpha_\lambda\,|^2$;

(ii) $\left(\int_I \theta_\lambda(x)dx\right)$ *satisfies a Carleson condition.*

E2- We give the main arguments used to prove that $\left(\int_I \theta_\lambda(x)dx\right)$ satisfies (ii). From the definition (4.4) of the functions $\theta_\lambda(x)$, we just have to notice that

$\left(\int_I \chi_\lambda(x)dx\right)$ satisfies a Carleson condition (see the estimates (4.3) of the matrix elements). We recall that,

$$\int_I \chi_\lambda(x)dx = \int_I \psi_\lambda(x)dx - \int_I \frac{1}{a(x)} q_\lambda(x)$$

where $q_\lambda(x) = \psi_\lambda(x) \sum_{\gamma\in\Gamma_j} \left(\int_I v^\star_\gamma(x)dx\right) v^\star_\gamma(x)$. With the standard estimates of the functions $v^\star_\gamma(x)$ and the orthogonality of V_j and W_j (which implies that $\psi_\lambda(x)$, $\lambda \in \Lambda_j$, is a family of vaguelettes on $L^2(0,1)$), we conclude that $q_\lambda(x)$, $\lambda \in \Lambda_j$ is a collection of vaguelettes on $L^2(0,1)$. This allows us to conclude (see Theorem 6, page 168 [4]) that $\left(\frac{1}{a}q_\lambda\right)$ satisfies a Carleson condition.

E3- The proof of (ii) starts by noticing that most of the wavelets $u_\lambda(x)$, $\lambda \in \Lambda$ have zero mean value. Roughly speaking, the wavelets of which the support is included in the piece taken from $2^{-j-1}\mathbb{Z}$ of Γ_{j+1} are not perturbed by the transition zones between two scales of Γ_{j+1}. Indeed, they are just the wavelets which would be obtained from a Lagrangian multiresolution analysis of $L^2(\mathbb{R})$. The last step of the proof consists of showing that the number of wavelets which do not satisfy the cancellation property (5) is sufficiently small.

5.2 Inequality (3)

The inequality (3) is obtained by duality, due to the following well known result.

Lemma. *Let H be a separable Hilbert space and let (e_k) and (f_k) be two biorthonormal sequences of H, i.e., $\langle e_k, f_\ell\rangle = \delta_{k,l}$ for all k and ℓ. If for every sequence of complex numbers (α_k)*

$$\|\sum \alpha_k e_k\|^2 \le C_1 \sum |\alpha_k|^2, \tag{7}$$

$$\|\sum \alpha_k f_k\|^2 \le C_2 \sum |\alpha_k|^2, \tag{8}$$

then

$$\sum |\alpha_k|^2 \le C_2 \|\sum \alpha_k e_k\|^2,$$
$$\sum |\alpha_k|^2 \le C_1 \|\sum \alpha_k f_k\|^2 .$$

E2- We apply this lemma to the two families $(\overline{\theta_\lambda})$ and $\left(\frac{1}{a}\theta_\lambda\right)$ which are biorthogonal with respect to the L^2-scalar product. We recall that the functions $\frac{1}{a(x)}\theta_\lambda(x)$, $\lambda \in \Lambda$, form a family of vaguelettes (see the end of the Section 3) and we proved that $\left(\int_I \theta_\lambda(x)dx\right)$ satisfy a Carleson condition. Thus, the inequalities (7) and (8) hold. Finally, we proved that the collection of functions $(\theta_\lambda(x))_{\lambda\in\Lambda}$ is a Riesz basis for $L^2(0,1)$, when completed with the Riesz basis $\left(v^\star_\gamma(x)\right)_{\gamma\in\Gamma_0}$ of

V_0. That is, any function $f \in L^2(0,1)$ can be represented by the converging series in $L^2(0,1)$ as:

$$f(x) = \sum_{\lambda \in \Lambda} \alpha_\lambda \theta_\lambda(x) + \sum_{\gamma \in \Gamma_0} \beta_\lambda v_\gamma^\star(x),$$

where $\alpha_\lambda = \int_0^1 f(x) \frac{1}{a(x)} \theta_\lambda(x) dx$ and $\beta_\lambda = \int_0^1 f(x) \frac{1}{a(x)} \; v_\gamma^\star(x) dx$.

E3- We see that the proof of inequality (4) requires knowledge about the system biorthogonal to the basis $(u_\lambda(x))_{\lambda \in \Lambda}$. Although we know that it exist, it is not sufficient because we must be able to estimate it. As it is shown in [13], the access to the biorthogonal system for $(u_\lambda(x))_{\lambda \in \Lambda}$ is missing and this prevents us from proving (8) and hence also (3). In other words, it has not yet been proved that the analysis algorithm associated with $(u_\lambda(x))_{\lambda \in \Lambda}$ is unconditionally stable. However, as was shown by (4), we know that the synthesis algorithm is unconditionally stable.

§6 Final remarks

The method treated in this chapter is not the only alternative for the use of Fourier techniques to construct the wavelets. To conclude this chapter, we want to make some remarks about an other approach, that was initiated by Meyer [10] and followed by Auscher, Cohen, Daubechies and Vial [2,5]. All this work is devoted to the construction of a basis of $L^2(I)$ consisting of Daubechies' compactly supported wavelets. Here, the main idea is to complete the set of wavelets with support inside the interval (which do not generate the whole $L^2(I)$) with a set of suitable functions. These functions show the advantage of being deduced from each other by dyadic dilation. The wavelets that are constructed in [10] do not satisfy any restrictive condition at the boundary of the interval. The construction is improved for numerical purposes in [5]. Finally, in [2] Auscher shows that one can prescribe boundary values for such wavelets. These wavelets also allow us to characterise the Sobolev spaces related to some homogeneous boundary value problem. For instance, consider the Dirichlet problem $-u'' + u = f$ on $[0, +\infty[$, $f \in L^2[0, +\infty[$, $u(0) = 0$ and $u(x) \to 0$ when $x \to +\infty$. This problem has a unique solution in $H^2 \cap H_0^1 = \{g \in H^2[0, +\infty[\; ; \; g(0) = 0\}$. In [2] a Riesz basis for $H^2 \cap H_0^1$ is constructed. Notice that the restriction to the same interval of the wavelet basis constructed in E1 does not lead to such a basis (it only forms an orthonormal basis for $H_0^s[0, +\infty[$, $0 < s < 2m - 2$).

One of the most important issues in wavelet theory is to construct wavelet bases on open subsets of $\mathbb{R}^n$, which can be used for boundary value problems in partial differential equations. As we explained, this is already achieved on the interval (*i.e.*, for $n = 1$) where the simplicity of the geometry enables us to use a natural and simple construction. However, the extension to higher dimensional cases is much more difficult (the complexity of the geometry is only one of the additional difficulties). We expect that the flexibility of the general construction exposed in this chapter will lead to such desired results.

A last important issue is the development of adaptive wavelet decompositions. The construction of wavelets associated with a locally refined mesh of points, as is described in E3, was motivated by the development of adaptive schemes for the numerical approximation of the solution of partial differential equations in the presence of local singularities. The starting point of the construction in [13] was the adaptive algorithm for the periodic Burger equation, constructed by Liandrat and Tchamitchian [9]. This, however, is not the only direction to start investigations to apply wavelet research to the construction of adaptive codes for the numerical treatment of partial differential equations. Recently, much work has also been done in the field of adaptive multigrid methods, where the bases used for approximation are very much related to wavelet bases. It seems that a complete new field of research opens itself here at the interface of wavelet research and multigrid methods. We believe that this research area will generate many interesting results in the next few years.

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