On the Foundations of Final Semantics: Non-Standard Sets, Metric Spaces, Partial Orders

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Abstract. Canonical solutions of domain equations are shown to be *final coalgebras*, not only in a category of non-standard sets (as already known), but also in categories of metric spaces and partial orders. Coalgebras are simple categorical structures generalizing the notion of post-fixed point. They are also used here for giving a new comprehensive presentation of the (still) non-standard theory of *non-well-founded sets* (as non-standard sets are usually called).

This paper is meant to provide a basis to a more general project aiming at a full exploitation of the finality of the domains in the semantics of programming languages — concurrent ones among them. Such a *final semantics* enjoys uniformity and generality. For instance, semantic observational equivalences like bisimulation can be derived as instances of a single 'coalgebraic' definition (introduced elsewhere), which is parametric of the functor appearing in the domain equation. Some properties of this general form of equivalence are also studied in this paper.

Keywords: final semantics, category, functor, coalgebra, domain equation, fixed point, non-well-founded sets, non-standard set theory, metric spaces, partial orders, concurrency, (F-)bisimulation, ordered F-bisimulation.

Contents

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0 Introduction

This work originates from an attempt to identify the common features of *partial orders*, *metric spaces*, and *non-standard sets*, that make these three different mathematical settings all suitable for defining semantic *domains* for concurrent programming languages. (To be precise, the distinctive feature of the domains under consideration is *non-determinism* rather than concurrency, the starting point being languages like *CCS* [Mil80] in which concurrency is reduced to sequentiality plus non-determinism.) It has resulted in a general semantic framework which could be called *final semantics*, as it is based on the observation that domains are final objects in a categorical sense.

This paper is a first account on this work, namely on its foundational part. It is shown that, regardless of the fact one is working with partial orders, metric spaces, or non-standard sets, domains are final objects in a suitable category of *coalgebras*. Moreover, some properties of final coalgebras are investigated in the abstract.

The categorical notion of coalgebra is quite elementary: given a category $\mathcal{C}$ (e.g., a category of complete metric spaces) and a functor $F : \mathcal{C} \to \mathcal{C}$, a coalgebra of F is a pair (A, α), with A an object in $\mathcal{C}$ and $\alpha : A \to F(A)$ an arrow in $\mathcal{C}$. Clearly, a solution to a domain equation $X \cong F(X)$ can be seen as a coalgebra (D, i), with i being an isomorphism between D and $F(D)$. The coalgebras of a given functor F over a category $\mathcal{C}$ form a category $\mathcal{C}_F$. Arrows are mappings of $\mathcal{C}$ which preserve the coalgebra structure (see the next section for a formal definition).

Semantic domains are usually obtained as solutions of recursive domain equations of the kind given above. There might be more than one such solution, but, for large classes of functors, a *canonical* one is taken. One of the starting points for the present work is a result in [Acz88], showing that, within a category of (classes over) non-standard sets, the canonical solution of a domain equation is a final coalgebra. (Non-standard sets are actually called *non-well-founded* sets in [Acz88], which is one of the standard references on the subject — but see also [FH83, FH92]. The word 'non-standard' has here a different meaning than in model theory.)

In this paper, it is shown that the canonical solutions of domain equations are final coalgebras, not only in that category of non-standard sets, but also in a category of complete metric spaces and in a category of complete partial orders. In other words, for these three different categories $\mathcal{C}$ and for large classes of functors F, the canonical solution to a domain equation $X \cong F(X)$ is a final object in the category $\mathcal{C}_F$.

0.1 Final Semantics

The finality of the domains is not only a unifying property. Final objects are the target of a unique arrow from any other object of the same category. This is a valuable property from a semantic point of view.

Recall that semantics can be given to a programming language by first defining a semantic domain and then associating a meaning to the programs of the language by mapping them onto elements of the chosen domain. The (by finality!) unique arrow from another coalgebra (of the same functor) into that domain is then a natural candidate for such an interpretation mapping. The problem is to give the class of programs of the language a coalgebra structure of the same functor used for the domain. Loosely speaking, syntax and semantics should live in the same category of coalgebras of this functor, the latter expressing the structure to be preserved under semantic mapping.

For instance, consider the language *CCS*. A semantic mapping should equate those programs which perform the same *computations* under a certain — informal — notion of *observation* (and keep the other distinct). As will become clear later, the choice of the functor for the domain amounts to making this notion of observation formal. Thus the functor defining the domain should be fixed according to the observation one has in mind. Further, computations are described by means of a transition system (induced by a set of structural rules) which is essentially a graph having programs as nodes and transitions as edges. Every program is the root of a tree obtained by unfolding the graph from that program. Such a tree gives the computations performable by the root program. Notice that there are many different ways of traversing a tree, each corresponding to a different notion of observation. The problem is thus, given a functor for a domain, to find a representation of the transition system as a coalgebra of that functor.

In general, the semantics shall depend on the observation one wants to perform on the computations or, more abstractly, on the functor one fixes. (Observations as functors!) For simplicity, it will be convenient that the functor be on some category of sets, possibly with some additional structure (e.g., metric or order), and leave to further developments generalizations to less concrete categories. More essentially, the existence of a final coalgebra for the functor will be needed, possibly to be shown via some limit construction. Then if one is able to find a representation of all the observable computations as a coalgebra of

the same functor, the (final) semantics of the language will immediately follow. (Ideally, this scheme would include not only concurrent languages, but also applicative ones — see, e.g., [Abr90]). Alternatively, the observable computations of the class of programs of the language under study might be directly defined as a coalgebra of the chosen functor.

Of the general methodology sketched above at least one instance is to be found in the literature: it is the final semantics for *CCS* given in [Acz88]. There, the semantics is based on a (straightforward) coalgebra representation of transition systems for a specific functor (see Example 1.4). The existence of other representations (for different functors and, thus, domains) of transition systems (and, possibly, of observable computations in general) will be treated in a forthcoming paper (*Observations as Functors: final semantics for programming languages*), together with other issues (like compositionality) involving the languages. Instead here, as already mentioned, the attention is rather focussed on foundational issues, independent from the languages, like the general properties of functors ensuring the construction of final coalgebras. Moreover, there is a 'coalgebraic' notion which can be studied in the abstract and which is of major interest for semantics: the kind of equivalence induced by a functor and its coalgebras. Some properties of such an equivalence are useful in clarifying the relationship between final semantics and 'equivalence-based' semantics.

Consider again *CCS*. An alternative approach to its semantics is to formalize the notion of observation in terms of an (observational) equivalence. The semantic mapping associates to each program its equivalence class and the domain is then simply defined as the image of that mapping. A popular example of such an observational equivalence is (strong) bisimulation as defined in [Par81]. Now, the functor used for the final semantics in [Acz88] can be shown to induce bisimulation equivalence in the sense that two programs are mapped (via the final semantics) into the same process if and only if they are bisimilar.

One of the advantage of working with final semantics is that there is a single 'coalgebraic' notion of (possibly observational) equivalence which is parametric of the functor: it is the definition of F-bisimulation as given in [AM89]. For a particular choice of the functor F, namely the one used in [Acz88] (but see also [BZ82]), F-bisimulation coincides with bisimulation in the traditional sense, as was observed above. Also other equivalences, like for instance trace equivalence, can be obtained by instantiating F-bisimulation to a certain functor (as will be shown in the above mentioned *Observations as Functors*). And even for the existing observational equivalences which do not fall under this scheme, it might still be useful to understand why they fail to be described in this way.

0.2 Contribution of this Paper

It is now possible to be more precise about the technical results in this paper. First of all it is shown that final coalgebras are *strongly extensional* in the sense that two elements of a final F-coalgebra are equal if and only if they are F-bisimilar. Also other abstract properties concerning F-bisimulations are studied. Then a *final coalgebra theorem* is given for each of the three categories under study, stating that the canonical solution of a domain equation is a final coalgebra.

As already mentioned, the (so-called *special*) final coalgebra theorem for non-standard sets is not a new result ([Acz88]). However, the proof given here is somewhat more transparent than the original one because of a different formulation of the definition of *unifor-*

mity on maps, which occurs in the conditions of the theorem. An extensive description of non-standard set theory is included as well, both because this theory (still) is non-standard indeed, and because the way it is presented here has some interest on its own. A uniform characterization of *standard* and *non-standard* set theory is introduced, showing that the latter theory is as natural as the former: the foundation and anti-foundation axioms are stated in terms of initial algebras and final coalgebras, respectively. The use of final coalgebras is particularly helpful to have a concise and uniform presentation of equivalent forms of the anti-foundation axiom, like, e.g., the Solution Lemma used in the proof of the final coalgebra theorem.

For metric spaces the final coalgebra theorem is a new result. It is shown that locally contracting functors on the category of complete metric spaces (with non-expansive mappings as arrows) have a final coalgebra. The proof is based on a theorem stating that such functors have fixed points. The latter theorem extends earlier results of [AR89] along the lines of [SP82], and is proved in full detail.

For partial orders an initial algebra theorem and the so-called limit-colimit coincidence are well-known (see [SP82]), but, apparently, it was never proved in detail that (in $CPO_\perp$) initial algebras and final coalgebras coincide. (Actually, the proof given here of the 'order-theoretic' final coalgebra theorem does not make direct use of the limit-colimit coincidence.) It is shown that the fixed point of a locally continuous functor on the category of complete partial orders (with strict and continuous mappings) is a final coalgebra in that category.

The main result about the category of cpo's is the study of a new notion, called *ordered F-bisimulation*, which is a generalization of the definition of F-bisimulation. Both the notions of partial bisimulation from [Abr91] and that of simulation from [Pit92] (for the functorial case) can be seen to be examples of ordered F-bisimulations. Corresponding to the notion of ordered F-bisimulation is a generalized notion of strong extensionality. A proof is given of the fact that the final coalgebras of locally continuous functors are strongly extensional in such a generalized sense. It implies the internal full abstractness result from [Abr91], and the extensionality results (for the functorial case) from [Pit92].

0.3 Overview of the Paper

In Section 2 (algebras and) coalgebras of functors are introduced. Examples are given showing that the powerset functor can be used for coalgebra representations of graphs and (labelled) transition system. A third example consists of a metric variant of the final semantics given in [Acz88] (and mentioned above).

Section 3 is dedicated to the notion of F-bisimulation. It is first shown that for the same kind of functor as in Examples 1.4 and 1.8 it corresponds to strong bisimulation. Then abstract properties are proved like strong extensionality and preservation of F-bisimulation in the category of F-coalgebras.

In the next three sections, final coalgebras in the categories of non-standard sets, complete metric spaces, and complete partial orders are treated. These sections can be read independently from each other (but presuppose Sections 2 and 3).

In the last section, a comparative analysis is made between the three different final coalgebra constructions discussed in the paper. Related and future work, including the relationship between final coalgebras and *coinduction* (the dual of induction), are also

discussed.

Although an extensive use of diagrams is made throughout the paper, no previous knowledge of category theory is required. Indeed, just a few (elementary) categorical notions are used.

1 Algebras and Coalgebras of Functors

Let $\mathcal{C}$ be a category and $F : \mathcal{C} \to \mathcal{C}$ be a functor from $\mathcal{C}$ to $\mathcal{C}$. (Such a functor is called an *endofunctor* on $\mathcal{C}$.)

Definition 1.1 An *F-coalgebra* is a pair (A, α), consisting of an object A and an arrow $\alpha : A \to F(A)$ in $\mathcal{C}$. It is dual to the notion of *F-algebra*: an *F-algebra* is a pair (A, α), consisting of an object A and an arrow $\alpha : F(A) \to A$ in $\mathcal{C}$. □

For instance, consider a preorder $(C, \leq)$. It can be interpreted as a category: the objects are the elements of C, and between any two elements $c, d \in C$ there is an arrow if and only if $c \leq d$. Any monotonic function $F : C \to C$ is then an endofunctor on C. Thus an F-coalgebra is a post-fixed point $x \in C$ with $x \leq F(x)$, and an F-algebra is a pre-fixed point $x \in C$ with $F(x) \leq x$.

Definition 1.2 *F-coalgebras* form a category, denoted by $\mathcal{C}_F$, by taking as arrows between coalgebras (A, α) and (A', α') those arrows $f : A \to A'$ in $\mathcal{C}$ such that $\alpha' \circ f = F(f) \circ \alpha$; that is, the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ {\scriptstyle\alpha}\downarrow & * & \downarrow{\scriptstyle\alpha'} \\ F(A) & \xrightarrow[F(f)]{} & F(A') \end{array}$$

Reversing the arrows one can easily define the category of F-algebras. □

Notice that in category theory the name F-(co)algebra is usually reserved for the case when F is the functor of a *(co)monad* (see, e.g, [Lan71]). F-(co)algebras have then some extra structure. They form a different category which, however, can be regarded as a subcategory of the above category of F-(co)algebras by simply forgetting the extra (co)monadic structure both in the objects and in the arrows.

As the name suggests, there is a relationship between algebras of functors and the more traditional Σ-algebras (sets with operations). For instance, the natural numbers together with the constant 0 and the successor function form a Σ-algebra (for any Σ consisting of a constant and a unary function symbol). Consider the functor $1 + -$ on the category *Set* of sets, where 1 is a one element set, and $+$ is the disjoint sum. An algebra of this functor is a pair (A, α), with $\alpha : 1 + A \to A$ defined as the sum of the functions

$$\begin{aligned} e &: \ 1 \to A \\ t &: \ A \to A. \end{aligned}$$

Now the natural numbers can be seen to be an algebra of the above functor by defining e and t as follows: e maps the only element of 1 to 0, and t is defined as the successor function.

Given this relationship between algebras of functors and algebras in the traditional sense, it is natural to look for a notion of coalgebra dual to the one of algebra. In other words, what is the dual of operations? An operation on a set A can be regarded as an action which, given some objects of A, *combines* them into a new object of A. Its dual is then an action which, given an object, *decomposes* it into several new components. A simple example is the following.

Example 1.3 *Graphs*
A graph is a pair $(N, \rightarrow)$ consisting of a set N of nodes and a collection $\rightarrow$ of (directed) arcs between nodes: $\rightarrow \subseteq N \times N$. A graph can be regarded as a coalgebra of the (covariant) powerset functor $\mathcal{P}$ on the category *Set* of sets as follows. Let $child : N \rightarrow \mathcal{P}(N)$ be defined by, for all $n \in N$,

$$child(n) \equiv \{m \mid n \rightarrow m\}.$$

□

A similar example is given by non-deterministic computations which can be said to be split at every state into a set of possible computations. To describe non-deterministic computations labelled transition systems in the style of [Plo81b] are often used:

Example 1.4 *Labelled Transition Systems*
A labelled transition system (*LTS*) is a triple $\mathcal{L} = (S, A, \rightarrow)$, consisting of a set S of states, a set A of labels, and a transition relation

$$\rightarrow \subseteq S \times A \times S$$

Often programs, given as closed terms over some signature, constitute the set S of states. Non-determinism is expressed by the fact that from a single state many different transitions are possible. Every LTS can be seen as a *labelled* graph: the nodes are the elements of S; there is an arc with label a between two nodes s and s' if and only if $(s, a, s') \in \rightarrow$ (also written as $s \xrightarrow{a} s'$). LTS's can be represented as coalgebras as follows. Let the functor

$$\mathcal{P}(A \times \text{-}) : Set \rightarrow Set$$

be defined, for any set X, by

$$\mathcal{P}(A \times X) \equiv \{U \mid U \subseteq A \times X\}.$$

A labelled transition system $(S, A, \rightarrow)$ can then be represented as a coalgebra (S, α) of the functor $\mathcal{P}(A \times \text{-})$ by defining $\alpha : S \rightarrow \mathcal{P}(A \times S)$, for all $s, s' \in S$, $a \in A$, by

$$< a, s' > \in \alpha(s) \iff s \xrightarrow{a} s'.$$

□

The above is the coalgebra representation of transition systems from [Acz88] (but see also [Hes88]) mentioned in the introduction. The LTS associated to a language like *CCS* has programs as states and atomic actions as labels. Transitions are given by the inductive closure of a set of structural rules. In Example 1.8, still along the lines of [Acz88], a final semantics based on this representation is illustrated. But first the definition of *final* objects in a category is needed:

Definition 1.5 An object A in $\mathcal{C}$ is called *final* if for any other object B in $\mathcal{C}$ there exists a unique arrow from B to A. It is the dual notion of initial object (unique arrow *from* the object). Final and initial objects are unique up to isomorphism. □

Consider again a preorder $(C, \leq)$ (viewed as a category) and a monotonic function $F : C \to C$. A final F-coalgebra is simply the greatest post-fixed point of F, which by a standard result is also the greatest fixed point. (Dually, an initial F-algebra is the least (pre-)fixed point of F.) Below, the notion of fixed point is generalized to functors and then a standard result is shown: final coalgebras are fixed points.

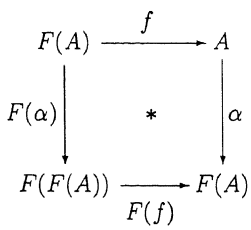
Definition 1.6 An F-coalgebra (A, α) is a *fixed point* for F (write $A \cong F(A)$) if α is an isomorphism between A and $F(A)$. That is, there exists an arrow $\alpha^{-1} : F(A) \to A$ such that

$$\alpha \circ \alpha^{-1} = \mathrm{id}_{F(A)} \text{ and } \alpha^{-1} \circ \alpha = \mathrm{id}_A.$$

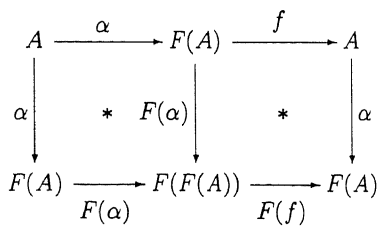
□

Proposition 1.7 A final F-coalgebra is a fixed point of F.

Proof. Let (A, α) be a final F-coalgebra. Since $(F(A), F(\alpha))$ is also an F-coalgebra, there exists a unique $f : F(A) \to A$ such that the following diagram commutes:



By finality, the only arrow from (A, α) into itself is the identity. Since both squares of the following diagram commute, $f \circ \alpha$ is the identity on A:



But then it also follows that $\alpha \circ f$ is the identity on $F(A)$:

$$\alpha \circ f \;=\; F(f) \circ F(\alpha) \;=\; F(f \circ \alpha) \;=\; F(\mathrm{id}_A) \;=\; \mathrm{id}_{F(A)}.$$

Therefore f is the inverse of α. □

Dually, an initial F-algebra is also a fixed point of F. Notice that a fixed point of a functor F can be regarded both as an F-coalgebra and as an F-algebra.

Example 1.8 *A Final Semantics*
Consider the category *CMS* of complete metric spaces (with non-expansive mappings as arrows). On this category, the usual constructions of disjoint sum and product are defined. Moreover, the powerset functor $\mathcal{P}_{comp}(\text{-})$, yielding all (metrically) compact subsets is well-defined on *CMS*. (Details on these constructions are omitted here; they are given in Section 4.) Similarly to Example 1.4, a LTS $(S, A, \rightarrow)$ can be represented as a coalgebra as follows. Let $\mathcal{P}_{comp}(A \times \text{-}) : \mathit{CMS} \rightarrow \mathit{CMS}$ be defined, for any metric space X, by

$$\mathcal{P}_{comp}(A \times X) \equiv \{U \subseteq A \times X \mid U \text{ is compact}\}.$$

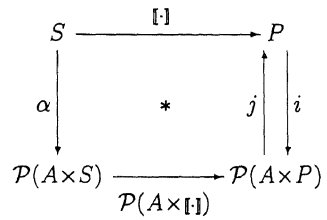
The above LTS can be seen to be a coalgebra of this functor by supplying S with the discrete metric (any two different states in S have distance 1), and defining, for all $s, s' \in S$ and $a \in A$,

$$< a, s' > \in \alpha(s) \iff s \xrightarrow{a} s'.$$

(For $\alpha(s)$ to be well defined, the transition relation $\rightarrow$ should be finitely branching. For LTS's not having this property, other choices for the functor can be made.) As will be shown in Section 5, the functor $\mathcal{P}_{comp}(A \times \text{-})$ has a final coalgebra (P, i), which by Proposition 1.7 is a fixed point:

$$P \cong \mathcal{P}_{comp}(A \times P).$$

Let j be the inverse of the isomorphism i. A semantic mapping $[\![\cdot]\!]$ from S into P can now be defined as the unique mapping from the coalgebra (S, α) into the final coalgebra (P, i):



Thus $[\![\cdot]\!]$ satisfies the following recursive equation:

$$[\![s]\!] = j(\{< a, [\![s']\!] > \mid s \xrightarrow{a} s'\}).$$

This semantics mapping is precisely the same given in [BM88, Rut92] as the fixed point of a contracting function $\Phi : (S \rightarrow P) \rightarrow (S \rightarrow P)$, using Banach's fixed-point theorem. (There the domain is the same, but its finality is not recognized.) □

A final remark. There is a notion which generalizes and combines both algebras and coalgebras of functors: An *F, G-dialgebra* [Hag87] of two functors F and G from a category D to a category C is still a pair (A, α), but with α an arrow in C from $F(A)$ to $G(A)$. It is a notion useful in type theory.

2 F-Bisimulation

The final semantics example in the previous section has the property that it maps two states into the same process if and only if they are (strongly) *bisimilar* in the following sense: A relation $R \subseteq S \times S$ on the set of states S of a LTS $(S, A, \rightarrow)$ is called a *(strong) bisimulation* ([Par81]) if for all $a \in A$ and $s, t \in S$ with $s R\, t$,

$$s \xrightarrow{a} s' \Rightarrow \exists t' \in S,\ t \xrightarrow{a} t' \text{ and } s' R\, t'$$

and

$$t \xrightarrow{a} t' \Rightarrow \exists s' \in S,\ s \xrightarrow{a} s' \text{ and } s' R\, t'.$$

Next $\sim$ is defined as the union of all bisimulations and two states s and t are called *bisimilar* when $s \sim t$.

In [AM89] it was noticed that coalgebras can be used for a natural generalization of the above notion of bisimilarity: For every functor F on the category of classes, a relation on F-coalgebras is defined, called F-bisimulation. This definition is here (generalized to other categories and) repeated, and some of its properties are analyzed. It is shown that final coalgebras are strongly extensional, that is, any two elements of a final F-coalgebra are equivalent if and only if they are F-bisimilar. Moreover, arrows between F-coalgebras preserve F-bisimulation. Together, these facts imply that (F-)bisimilar states are semantically mapped into the same process by the final semantics given in 1.8. Also the converse is proved here, under the condition that F weakly preserve kernel pairs.

For sake of simplicity, the (F-bisimulation) relations considered here are of a set-theoretic nature. That is, relations are defined as subsets of a cartesian product. A more general categorical formulation would, on one hand, allow defining F-bisimulations for all categories of coalgebras, but, on the other hand, it would bring unnecessary complications, since the categorical product of the three categories under study here amounts to a cartesian product. In categorical words, for each of the categories $\mathcal{C}$ considered here, there exists a faithful forgetful functor U from $\mathcal{C}$ into a category of (possibly large) sets and, moreover, for every object A in $\mathcal{C}$, $U(A \times A) = U(A) \times U(A)$. To be more specific, in the case of complete partial orders, the product $A \times A$ of a cpo $A = (|A|, \sqsubseteq_A)$ with itself is given by the cartesian (i.e., set-theoretic) product $|A| \times |A|$ together with the following order: for all $\langle x_1, y_1 \rangle$, $\langle x_2, y_2 \rangle \in |A| \times |A|$,

$$\langle x_1, y_1 \rangle \sqsubseteq \langle x_2, y_2 \rangle \equiv x_1 \sqsubseteq_A x_2 \text{ and } y_1 \sqsubseteq_A y_2.$$

Similarly, if $A = (|A|, d_A)$ is a complete metric space, the following metric is to be added to the cartesian product $|A| \times |A|$: for all $\langle x_1, y_1 \rangle$, $\langle x_2, y_2 \rangle \in |A| \times |A|$,

$$d(\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle) \equiv \max\{d_A(x_1, x_2), d_A(y_1, y_2)\}.$$

(All this can be more synthetically and generally rephrased as: $\mathcal{C}$ is a category for which the forgetful functor into *Set* exists and *creates products*.) The notation $|A|$ will be used also in the sequel to denote the set in a cpo or metric space A (i.e., $|A| \equiv U(A)$). If A is a (possibly large) set then $|A|$ will simply be A itself (U is the identity functor).

Definition 2.1 Let $\mathcal{C}$, throughout the rest of this section, be a category of (possibly large) sets possibly with an additional metric or order-theoretic structure. For any object A in $\mathcal{C}$, a *relation* R on A is an object R of $\mathcal{C}$ such that $|R| \subseteq |A| \times |A|$. If A is either a complete metric space or a cpo, then R inherits the metric or the order from $A \times A$. By abuse of notation, $R \subseteq A \times A$ will be used in the sequel to denote that R is a relation on A. □

Definition 2.2 Let $F : \mathcal{C} \to \mathcal{C}$ be a functor. Let (A, α) be an F-coalgebra. Let R be a relation on A. Then R is called an *F-bisimulation* on (A, α) if there exists an arrow $\beta : R \to F(R)$ such that the projections $\pi_1, \pi_2 : R \to A$ are arrows in $\mathcal{C}_F$ from (R, β) to (A, α). That is, both squares of the following diagram should commute:

$$\begin{array}{ccccc} R & \xrightarrow{\pi_1} & A & \xleftarrow{\pi_2} & R \\ \downarrow{\scriptstyle\beta} & * & \downarrow{\scriptstyle\alpha} & * & \downarrow{\scriptstyle\beta} \\ F(R) & \xrightarrow[F(\pi_1)]{} & F(A) & \xleftarrow[F(\pi_2)]{} & F(R) \end{array}$$

Two elements a and a' in A are called *F-bisimilar* (notation $a \overset{F}{\sim} a'$) if there exists a bisimulation relation R on (A, α) with aRa'; thus

$$\overset{F}{\sim} \equiv \bigcup\{R \subseteq A \times A \mid R \text{ is an } F\text{-bisimulation on } (A, \alpha)\}.$$

□

Definition 2.2 indeed generalizes the standard notion of strong bisimulation:

Example 2.3 *Bisimulation*
Recall from Example 1.4 that the functor

$$\mathcal{P}(A \times \text{-}) : Set \to Set$$

is used for representing LTS's. Consider a LTS $(S, A, \to)$ and let (S, α) be the corresponding $\mathcal{P}(A \times \text{-})$-coalgebra. It is shown that there is a one-to-one correspondence between the strong bisimulations and the $\mathcal{P}(A \times \text{-})$-bisimulations on S.

Let $R \subseteq S \times S$ be a strong bisimulation on S. Define $\beta : R \to \mathcal{P}(A \times R)$ by, for all $s R\, t$,

$$\beta((s,t)) \equiv \{< a, (s', t') > \mid s \xrightarrow{a} s' \wedge t \xrightarrow{a} t' \wedge s' R\, t'\}$$

It is straightforward to check that (R, β) satisfies the conditions of Definition 2.2.

Conversely, let R be an $\mathcal{P}(A \times \text{-})$-bisimulation, with corresponding coalgebra (R, β). Consider s and t such that $s R\, t$. By symmetry, it suffices to prove that, for all $s' \in S$, $a \in A$,

$$s \xrightarrow{a} s' \Rightarrow \exists t',\ s' R\, t' \text{ and } t \xrightarrow{a} t'.$$

That is, for all $s' \in S$, $a \in A$,

$$< a, s' >\in \alpha(s) \Rightarrow \exists t',\ s' R\ t' \text{ and } < a, t' >\in \alpha(t).$$

Suppose $< a, s' >\in \alpha(s)$. Since

$$\begin{aligned} \alpha(s) &= \alpha(\pi_1((s,t))) \\ &= \mathcal{P}(A \times \pi_1) \circ \beta((s,t)) \\ &= \{< a, u > |\ u \in S \text{ and } \exists v \in S,\ < a, (u,v) >\in \beta((s,t))\} \end{aligned}$$

there exists $t' \in S$ with $< a, (s', t') >\in \beta((s,t))$, and hence $s' R t'$. Because

$$\begin{aligned} \alpha(t) &= \alpha(\pi_2((s,t))) \\ &= \mathcal{P}(A \times \pi_2) \circ \beta((s,t)) \end{aligned}$$

it follows that $< a, t' >\in \alpha(t)$. □

The above definition of F-bisimulation paves the way for a uniform treatment of different kinds of observational equivalence. Other observational equivalences can be described by choosing a different functor.

The rest of this section describes some semantically interesting properties of F-bisimulation, starting from strong extensionality:

Theorem 2.4 Any final F-coalgebra (A, α) is *strongly extensional*: for all $a_1, a_2 \in A$,

$$a_1 = a_2 \iff a_1 \overset{F}{\sim} a_2$$

(Recall that $\overset{F}{\sim}$ is the union of all F-bisimulations on (A, α).)

Proof. Let $=_A$ be the identity relation on A. The inclusion from left to right follows from the fact that $=_A$ can be seen to be an F-bisimulation on (A, α) as follows. Define $\Delta : A \rightarrow =_A$ by, for all $a \in A$, $\Delta(a) \equiv < a, a >$, and $\beta : =_A \rightarrow F(=_A)$ by $\beta \equiv F(\Delta) \circ \alpha \circ \pi_1$. Then $(=_A, \beta)$ is an F-bisimulation on (A, α):

$$\begin{array}{ccccc} & \overset{\pi_1}{\longrightarrow} & & \overset{\pi_2}{\longleftarrow} & \\ =_A & \underset{\Delta}{\longleftarrow} & A & \underset{\Delta}{\longrightarrow} & =_A \\ \beta \downarrow & * & \alpha \downarrow & * & \downarrow \beta \\ F(=_A) & \underset{F(\Delta)}{\longleftarrow} \overset{F(\pi_1)}{\longrightarrow} & F(A) & \underset{F(\Delta)}{\longrightarrow} \overset{F(\pi_2)}{\longleftarrow} & F(=_A) \end{array}$$

Conversely, let $R \subseteq A \times A$ be an F-bisimulation with (R, β) as in Definition 2.2. Since both π_1 and π_2 are arrows in $\mathcal{C}_F$ from (R, β) to the final F-coalgebra (A, α), it follows that $\pi_1 = \pi_2$. Thus $R \subseteq =_A$. □

Theorem 2.5 Let (B, β) be an F-coalgebra and (A, α) a final F-coalgebra. Let $[\![\cdot]\!] : (B, \beta) \rightarrow (A, \alpha)$ be the unique arrow from (B, β) to (A, α). For all b_1, b_2 in B,

$$b_1 \overset{F}{\sim} b_2 \Rightarrow [\![b_1]\!] = [\![b_2]\!].$$

Proof. Let (R, γ) be an F-bisimulation on B. Since both $[\![\cdot]\!] \circ \pi_1$ and $[\![\cdot]\!] \circ \pi_2$ are arrows between the F-coalgebras (R, γ) and (A, α), and since (A, α) is final it follows that $[\![\cdot]\!] \circ \pi_1 = [\![\cdot]\!] \circ \pi_2$. □

In general, in categories of (possibly large) sets one can prove that certain arrows between F-coalgebras preserve F-bisimulation. More precisely, this holds for arrows that have a right inverse (also called split epis). (In *Set* every surjective mapping has, by the axiom of choice, a right inverse.) The idea is that one would like to show that, given an arrow f between F-coalgebras (A, α) and (A', α'), and given an F-bisimulation (R, β) on (A, α), the following relation

$$R^f \equiv \{\langle f(a), f(a')\rangle \in |A'| \times |A'| \mid a R a'\}$$

is an F-bisimulation on (A', α'). If F is an endofunctor on a category either of complete partial orders or of complete metric spaces, one needs first of all to show that R^f is a complete partial order or a complete metric space, respectively. This can be shown under the assumption that f has a right inverse as follows. Let $\mathcal{C}$ be, for instance, $CPO_\perp$ (see Section 5 for the formal definition of $CPO_\perp$) and assume the existence of a right inverse h to f. Then one can show that R^f is a cpo: $\langle \perp_{A'}, \perp_{A'}\rangle$ is the minimal element, since f is (an arrow in $CPO_\perp$ and hence) strict, and $\langle \perp_A, \perp_A\rangle \in R$. Further suppose that $\langle f(a_n), f(a'_n)\rangle_n$ is an ω-chain in R^f. By monotonicity of h, $\langle h \circ f(a_n), h \circ f(a'_n)\rangle_n$ is a chain in R. Because R is a cpo this chain has a limit in R, say $\langle a, a'\rangle$. By continuity of f it follows that

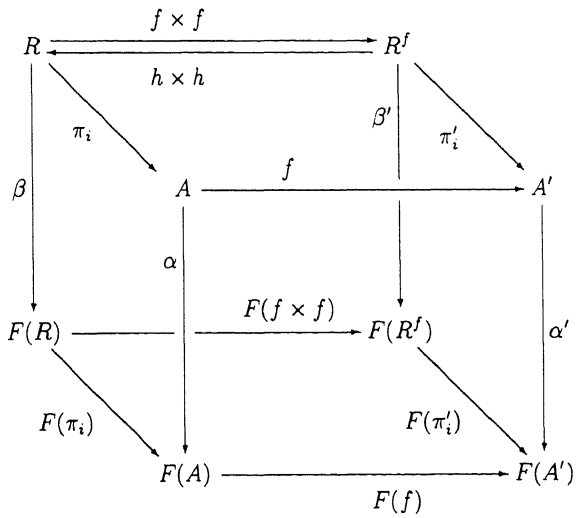
$$\langle f \circ h \circ f(a_n), f \circ h \circ f(a'_n)\rangle_n = \langle f(a_n), f(a'_n)\rangle_n$$

converges to $\langle f(a), f(a')\rangle$, which is in R^f.

Now, the above right inverse can also be used to define the following arrow

$$\beta' \equiv F(f \times f) \circ \beta \circ (h \times h).$$

This β' turns R^f into an F-bisimulation. Indeed, consider the cube below:



All sides commute but the back and the right one. One has to prove the commutativity of the latter. That is, $\alpha' \circ \pi_i' = F(\pi_i') \circ \beta'$, for $i = 1, 2$. Chasing the diagrams, it follows

$$\begin{aligned}
\alpha' \circ \pi_i' &= \alpha' \circ \pi_i' \circ (f \times f) \circ (h \times h) \\
&= \alpha' \circ f \circ \pi_i \circ (h \times h) \\
&= F(f) \circ \alpha \circ \pi_i \circ (h \times h) \\
&= F(f) \circ F(\pi_i) \circ \beta \circ (h \times h) \\
&= F(\pi_i') \circ F(f \times f) \circ \beta \circ (h \times h) \\
&= F(\pi_i') \circ \beta'.
\end{aligned}$$

All this proves the following:

Theorem 2.6 Let $f : (A, \alpha) \to (A', \alpha')$ be an arrow in $\mathcal{C}_F$ with a right inverse. For all $a, a' \in A$,

$$a \overset{F}{\sim} a' \Rightarrow f(a) \overset{F}{\sim} f(a').$$

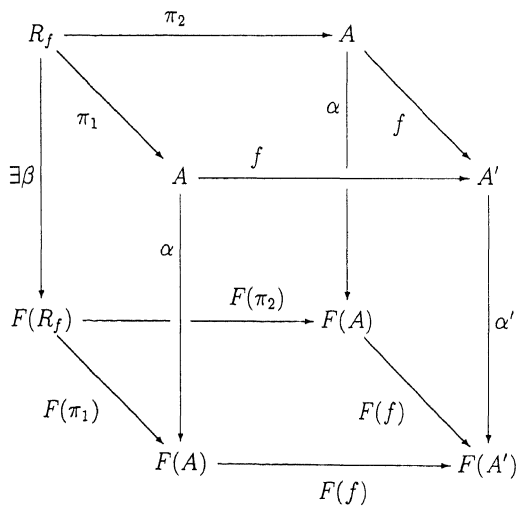
For the converse of Theorem 2.5, it is sufficient to prove that for any arrow f between any two F-coalgebras (A, α) and (A', α'), the following relation

$$R_f \equiv \{(a, a') \in |A| \times |A| \mid f(a) = f(a')\}$$

is an F-bisimulation on (A, α). Again, it is not difficult to prove that R_f is an object of the category: E.g., if $\mathcal{C}$ is $CPO_\perp$ then the fact that R_f is closed (i.e., all ω-chains have a least upper bound) follows from the continuity of f and the observation that R_f is the inverse image of the diagonal in $|A'| \times |A'|$, which is trivially closed:

$$R_f = (f^{-1} \times f^{-1})\{\langle x, x \rangle \in |A'| \times |A'|\}.$$

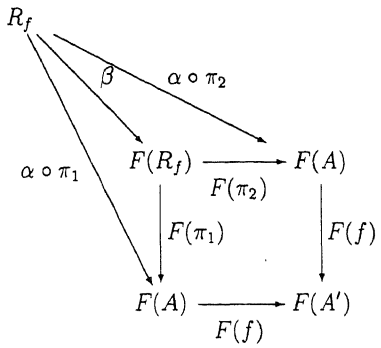
Now for R_f to be an F-bisimulation, there should exist an arrow $\beta : R_f \to F(R_f)$ making both the back and the left side squares of the following cube commute:



Note that the front and right squares are equal and commute, because f is an arrow between coalgebras. The top square also commutes; thus, by functoriality, the bottom one does as well. Further observe that

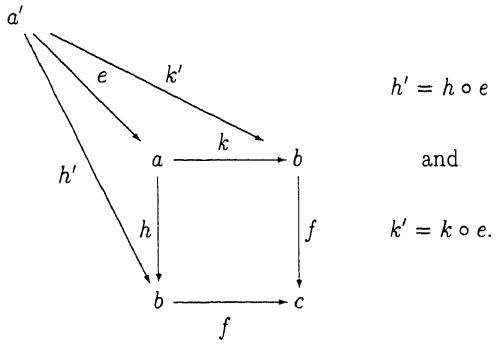
$$\begin{aligned} F(f) \circ \alpha \circ \pi_1 &= \alpha' \circ f \circ \pi_1 \\ &= \alpha' \circ f \circ \pi_2 \\ &= F(f) \circ \alpha \circ \pi_2 \end{aligned}$$

One needs the existence of an arrow β



such that $\alpha \circ \pi_1 = F(\pi_1) \circ \beta$ and $\alpha \circ \pi_2 = F(\pi_2) \circ \beta$. It is sufficient for the existence of such an arrow that the inner square of the above diagram is a *weak kernel pair* for $F(f)$:

Definition 2.7 Consider an arrow $f : b \to c$ in a category $\mathcal{C}$. A *kernel pair* for f is an object a and arrows $h : a \to b$ and $k : a \to b$ in $\mathcal{C}$ such that $f \circ h = f \circ k$, and such that for any other such triple (a', h', k') there exists a unique arrow e from a' to a such that



The object a, with arrows h and k is called a *weak kernel pair* if in the preceding formulation the requirement of uniqueness is dropped. □

It is not difficult to prove that R_f and its two projections form a kernel pair for f. Thus for the existence of an appropriate arrow β it is sufficient if the functor F weakly preserves kernel pairs. We have proved:

Theorem 2.8 Let F be a functor weakly preserving kernel pairs. That is, the image under F of a kernel pair for an arrow f is a weak kernel pair for the arrow $F(f)$. For every arrow f between any two F-coalgebras (A, α) and (A', α'), the kernel pair R_f of f is an F-bisimulation on (A, α).

The above proof is motivated by [AM89], were it is shown that for functors F that preserve weak pullbacks, the notions of F-bisimulation and *congruence* coincide. Many standard functors (built from sum, product etc.) weakly preserve kernel pairs.

The following corollary generalizes the fact mentioned at the beginning of this section that two states are $(\mathcal{P}(A \times \text{-})\text{-})$bisimilar if and only if they are mapped into the same process:

Corollary 2.9 Let F be a functor weakly preserving kernel pairs. Let (B, β) be an F-coalgebra and (A, α) a final F-coalgebra. Let $[\![\cdot]\!] : (B, \beta) \to (A, \alpha)$ be the unique arrow from (B, β) to (A, α). For all b_1, b_2 in B,

$$b_1 \overset{F}{\sim} b_2 \iff [\![b_1]\!] = [\![b_2]\!].$$

The Rest of this Paper

In the rest of this paper, the categories *Class**, *CMS* and $CPO_\perp$ will be treated in great detail. For each of these, a family of functors having a final coalgebra will be identified. In other words, three final coalgebra theorems will be proved for functors satisfying certain conditions. The three next sections can be read independently from each other.

3 Non-Standard Set Theory

In this section, a first concrete category is presented in which a final coalgebra theorem holds. It is the category *Class**: objects are classes, possibly containing non-standard (or *non-well-founded*, [Acz88]) sets, and arrows are functions between classes. This (so-called special) final coalgebra theorem goes as follow: Consider an endofunctor F over *Class** which has a greatest fixed point $J_F = F(J_F)$. Then, if this functor preserves inclusions and is *uniform on maps*, the fixed point J_F, together with its identity mapping, is a final F-coalgebra.

The section is divided in four parts. The first recalls the basic set theory ZFC^- of which both standard and non-standard set theory are extensions (obtained by adding respectively foundation and anti-foundation axioms). For this, no previous knowledge of set-theory is required. This part also describes fixed points of class functors (needed in the main theorem).

The second part introduces a new formulation of foundation and anti-foundation axioms in terms of initial algebras and final coalgebras (of a powerset functor). A comparison with the standard formulations then follows. The anti-foundation axiom as formulated in [Acz88] is here called Decoration Lemma.

The third part recalls the Solution Lemma from [Acz88]. It is yet another formulation of the anti-foundation axiom. It is used in the proof of the main theorem. The Solution Lemma is stated using coalgebras and this makes its proof trivial.

In the last part, about the special final coalgebra theorem, a new definition of uniformity on maps is given and then the special final coalgebra theorem is proved.

3.1 Basic Set Theory

The intuitive idea of a set is that of a collection of objects which have a certain property φ. Moreover, two sets should be equal if and only if they have the same elements. A first step towards a formalization of such an idea is to fix a language to express these properties. A natural candidate is a first order predicate calculus with equality. The only primitive relation needed seems to be that of membership, which is a binary predicate usually denoted by "$\in$". For instance the usual notion of subset can be expressed as follows:

$$x \subseteq y \equiv \forall v\,(v \in x \Rightarrow v \in y).$$

Constant symbols for denoting the elements of a set will turn out not to be necessary, as every object of interest can be represented as a set.

Following this intuition, the only axioms would then be:

Extensionality:

$$x = y \Leftrightarrow x \subseteq y \wedge y \subseteq x.$$

Strong Comprehension:

$\forall$ property φ, $\{x \mid \varphi(x)\}$ is a set.

However, Russel's paradoxical set $\{x \mid x \notin x\}$ shows that such a strong comprehension axiom cannot be stated in its full generality. Strong comprehension is thus to be replaced by the following axiom:

Comprehension:

$\forall$ property φ, $\forall$ set v, $\{x \mid \varphi(x) \wedge x \in v\}$ is a set.

As comprehension can be applied only to members of already defined sets, it is necessary to postulate the existence of some sets, either primitive or derived by applying some basic operators:

Empty Set:

There exists a set $\emptyset$ with no elements.

Paring, Union, Power Set:

$\{x, y\}$, $\bigcup x$, $\mathcal{P}(x)$ are all sets.

(As usual, $\bigcup x$ and $\mathcal{P}(x)$ stand respectively for the collection of all members of members of x and the collection of all subsets of x.) By means of the union operator one can define an operator s acting as successor as follows: $s(x) = x \cup \{x\}$. Regarding the empty set as 0, the existence of an infinite set can be stated by postulating the existence of a set containing the natural numbers. That is:

Infinity:

There exists a set containing 0 and closed under the successor operator s.

(The axioms above, as well as those given in the sequel, are written for convenience in natural language but note that they can also be expressed in the language of set theory – see, e.g., [Lev79].)

Further useful notions can be derived from the above axioms, like, for instance, that of *ordered pair*:

$$< x, y > \equiv \{x, \{x, y\}\}.$$

A formal definition of function can then be given as a collection f of ordered pairs such that for every x there exists a unique y with $< x, y > \in f$. (This was also the first formal definition of function.) Two more axioms about functions are then usually added:

Replacement:

The image of a set under a function is a set.

Choice:

Every surjective function has a right inverse.

A *right inverse* of a function $f : a \rightarrow b$ is a function $g : b \rightarrow a$ such that $f \circ g$ is the identity on b. The above axiom of choice is equivalent to postulate that for every set a there exists a *choice function*, that is, a function f such that, for every $x \in a$, $f(x) \in x$.

Even though the collection $\{x \mid \varphi(x)\}$ of all sets x having a given property φ might not be a set it can still be of interest for set theory. Such 'specifiable' collections are called *classes*. Clearly, a set is a class, but the converse is not true, in which case one speaks of a *proper class*. For this reason classes are also called *large sets*. Extensionality can be applied also to classes, but the restriction has to be imposed that an element of a class is a set. Thus the classes specified by two properties φ and ψ are equal if and only if φ and ψ hold for the same sets. In the sequel, lower case letters will denote sets while capital letters will be used to denote classes.

An example of a proper class is the so-called *universe* of sets, namely the collection of all sets:

$$V \equiv \{x \mid x = x\}.$$

(V is indeed the collection of all sets as the property $x = x$ trivially holds for all sets!) Notice that different properties may specify the same class. For instance, any property other than '$x = x$' which holds for all sets can be used to specify the universe.

The theory associated with (i.e., the collection of all sentences derivable from) the above axioms (extensionality, comprehension, empty set, pairing, union, power set, infinity, replacement, choice) is usually denoted by ZFC^- in the literature (e.g., [Lev79, Lan86]). In the sequel it will be also called *basic set theory*.

From the axioms of basic set theory alone it is not possible to draw a canonical picture of how the universe looks like, a picture independent of the specific interpretation one might give to the theory. This was felt as a problem already in the early developments of set theory. The solution was found in the so-called *foundation axiom*, which was then added to basic set theory. This axiom restricts the universe to the 'smallest' of all possible

ones. Then the picture arises of a universe in which sets are hereditarily constructed from the empty set, by iterative applications of the powerset operator. Every set has a *rank*, namely the stage at which it appears in such a *cumulative hierarchy*. This intuitive structure, together with the fact that all existing mathematics discovered at that time could still be carried out inside this restricted universe, made the axiom easily accepted. However, recent applications in computer science have raised interest in the dual choice, namely in postulating that the universe be the 'largest' possible one (*anti-foundation axiom*).

In the sequel, this duality between foundation and anti-foundation axiom will be expressed formally in terms of the categorical dualities between algebras and coalgebras and initiality and finality, the latter providing a formal definition of 'smallest' and 'greatest'. This makes the qualitative descriptive improvement in adding a foundational axiom to basic set theory quite transparent: the universe is described as a universal object in a suitable—that is, rich enough—category. Therefore, the above two extensions of basic set theory will be both called *categorical set theories*. The classical one (basic set theory with the foundation axiom) will be called *standard set theory*, while the other (basic set theory with the anti-foundation axiom) will be called *non-standard set theory*. Notice that here the use of the word 'non-standard' differs from the use of the same word in model theory: here non-standard is the postulated presence of non-well-founded sets in the universe, rather than a model of the universe.

Before introducing categorical set theories, it is useful to discuss some fixed point theory of functions within basic set theory. Notice that it is customary in set theory to consider strict equalities rather than isomorphisms as fixed points of functors:

Definition 3.1 A *fixed point* of an endofunctor F in a category of sets (or classes) is a set (or a class) X satisfying the equality $X = F(X)$. That is, X is a fixed point of F w.r.t. set-inclusion. □

The definitions and results in the rest of this subsection are from [Acz88].

Definition 3.2 Let F be a class function. Then:

F is *set-based* if

$$\forall \text{ class } A\ \forall x \in F(A) \ \Rightarrow\ \exists \text{ a set } a \subseteq A \text{ such that } x \in F(a).$$

F is *monotone* if

$$\forall A, B:\ A \subseteq B \ \Rightarrow\ F(A) \subseteq F(B).$$

F is *set-continuous* if it is both monotone and set-based. □

Theorem 3.3 If a class function F is set-continuous then:

1. There exists a class I_F which is the *least pre-fixed point* of F. As usual, it can be shown that I_F is also the *least fixed point* of F.
2. There exists a class J_F which is the *greatest post-fixed point* of F. It can be shown that J_F is also the *greatest fixed point* of F.

There is a characterization of least and greatest fixed points in terms of iterations. For this purpose the class On of all *ordinals* is needed. An ordinal is a *transitive set* (a set x is transitive if every element y of x is also a subset of x) x which is *well-ordered* by $\in$, that is, $\in$ totally orders x and every non-empty subset of x has a least element w.r.t. $\in$. If α and β are two ordinals such that $\beta \in \alpha$, one usually writes $\beta < \alpha$. The first ordinals are: $\emptyset$, $s(\emptyset)$, $s^2(\emptyset)$, etc. The first limit ordinal is $\omega \equiv \bigcup_{n \in N} s^n(\emptyset)$, which, by the infinity axiom, is indeed a set.

Corollary 3.4 If a class function F is set-continuous then the following definitions are sound:

$$F \uparrow \alpha \equiv F(\bigcup_{\beta<\alpha} F \uparrow \beta) \quad \text{and} \quad F \downarrow \alpha \equiv F(\bigcap_{\beta<\alpha} F \downarrow \beta).$$

Moreover,

$$I_F = \bigcup_{\alpha \in On} F \uparrow \alpha \quad \text{and} \quad J_F = \bigcap_{\alpha \in On} F \downarrow \alpha.$$

There is yet another characterization of J_F as union of sets (thus not arbitrary classes!) which are pre-fixed points of F:

$$J_F = \bigcup \{x \mid x \subseteq F(x)\}.$$

3.2 Categorical Set Theory: Standard vs Non-Standard

Classes form the objects of a category, having as arrows *class functions*, that is, mappings assigning to every class a class. Actually, to every set theory a different category of classes is associated.

Definition 3.5 The category of classes of (sets defined in terms of) basic set theory is denoted by *Class*. □

The powerset constructor can be turned into a (covariant) functor from *Class* to *Class* as follows: for every class A,

$$\mathcal{P}(A) \equiv \{x \mid x \text{ is a set } \wedge\ x \subseteq A\};$$

for every function $f : A \to B$ and every set $x \subseteq A$,

$$\mathcal{P}(f)(x) \equiv \{f(y) \mid y \in x\}.$$

Notice that only sub*sets* are taken into consideration. This makes possible that V be a fixed point of the powerset functor (which, by cardinality reasons, would not be the case if one would consider the collection of all sub*classes* of a given class):

Proposition 3.6 $V = \mathcal{P}(V)$.

Proof. V is the largest class. Thus, since $\mathcal{P}(V)$ is itself a class, $\mathcal{P}(V) \subseteq V$. For the converse it is sufficient to prove that every set x is a subset of V. That is, for every $y \in x$, y is also in V. This is immediate from the fact that y is a set. □

Since V is the largest class one also has:

Corollary 3.7 The universe V is the greatest fixed point of the powerset functor.

Notice that $\mathcal{P}$ is set-continuous, thus, by Corollary 3.4, $V = J_{\mathcal{P}}$.

Moreover, the identity mapping id_V of V can be seen both as a mapping from $\mathcal{P}(V)$ to V and as mapping from V to $\mathcal{P}(V)$:

Corollary 3.8 (V, id_V) is both a $\mathcal{P}$-algebra and a $\mathcal{P}$-coalgebra.

Notice that the categories of $\mathcal{P}$-algebras and a $\mathcal{P}$-coalgebras are very rich categories. For instance, every class function $f : A \to f(A)$ can be seen as an arrow between the $\mathcal{P}$-coalgebras $(A, sing_A)$ and $(f(A), sing_{f(A)})$, where the function *sing* maps every set x into $\{x\}$.

The notions of 'initial' and 'final' are the categorical abstraction of the notions of 'smallest' and 'largest'. Therefore, one could categorically express that the universe is the smallest or the largest possible one, respectively, as:

Foundation Axiom
(V, id_V) is an initial $\mathcal{P}$-algebra.

Anti-Foundation Axiom
(V, id_V) is a final $\mathcal{P}$-coalgebra.

A comparison of the above formulation of foundation and anti-foundation axioms with the standard one is made below, so that it will become clear that equivalent formulations of these axioms are expressible in the language of set theory. But first the answer is given to a question which might naturally arises here. Namely, whether initial $\mathcal{P}$-algebras and final $\mathcal{P}$-coalgebras exist at all in basic set theory. The following two theorems are from [AM89] and [Acz88], respectively:

Theorem 3.9 Every set-based functor $F : \mathit{Class} \to \mathit{Class}$ has a final coalgebra.

Proof. See [AM89], where the theorem is called *Final Coalgebra Theorem*. (The proof is actually based upon a definition of set-based functor which is even more liberal than the one given above.) □

From the above theorem one can (although not directly) prove that there exists a function α from V to $\mathcal{P}(V)$ such that (V, α) is a final $\mathcal{P}$-coalgebra. What cannot be proved is that the identity function is one such α which makes V final, which is in fact the content of the anti-foundation axiom as formulated above.

Set theory deals with strict equalities rather than just isomorphisms. If one postulates the anti-foundation axiom then one can prove that, under some rather liberal hypotheses, the greatest fixed point of an endofunctor F, together with the identity mapping, is a final F-coalgebra (i.e., the special final coalgebra theorem). The dual theorem, instead, can be proved without further assumptions, that is, within basic set theory:

Theorem 3.10 The least fixed point $I_F = F(I_F)$ of a set-continuous functor $F : \mathit{Class} \to \mathit{Class}$ which preserves inclusion mappings (see definition below) is an initial F-algebra.

Proof. See [Acz88]. □

An inclusion mapping is a function associated with two classes A and B such that $A \subseteq B$. It has A as domain, B as codomain and maps every element a of A in the same a which, by inclusion, is also in B. It is denoted by $\iota_{A,B}$ and the subscript is dropped whenever clear from the context. An endofunctor F on *Class preserves inclusion mappings* when, for all classes A and B with $A \subseteq B$, if $F(A) \subseteq F(B)$ then $F(\iota_{A,B}) = \iota_{F(A),F(B)}$. The powerset functor is easily provable to preserve inclusion mappings, as well as being set-continuous. Thus its least fixed point is an initial algebra.

3.2.1 Well-Founded Sets

The formulation of the two axioms above is not quite standard. Usually, by *foundation axiom* the following is intended:

$$V \text{ is the least fixed point of } \mathcal{P}. \tag{1}$$

Since $\mathcal{P}$ is set-continuous, its least fixed point is, by Corollary 3.4, the so-called *cumulative hierarchy*

$$\bigcup_{\alpha \in On} \mathcal{P} \uparrow \alpha.$$

Thus, assuming V is such a class, a *rank* can be associated with every set, namely the stage α at which the set first appears in the hierarchy. This ranking function allows one to prove that (1) is equivalent to the following statement:

Every set is well-founded w.r.t. $\in$

which amounts to saying that every non empty set has an $\in$-least element. This can be easily expressed in the language of set theory as follows:

$$\forall x\,(x \neq \emptyset \Rightarrow \exists v\,(v \in x \wedge \neg\exists y\,(y \in x \wedge y \in v))).$$

In other words, there is no infinitely descending chain of sets w.r.t. $\in$. This explains why the universe of basic set theory together with the foundation axiom is called *universe of well-founded sets*.

Theorem 3.11 (V, id_V) is an initial $\mathcal{P}$-algebra $\iff$ V is the least fixed point of $\mathcal{P}$.

Proof. Since $\mathcal{P}$ is set-continuous and preserves inclusion mappings, the implication from right to left follows from Theorem 3.10. For the implication from left to right consider an arbitrary fixed point $X = \mathcal{P}(X)$. Since:

1. $X \subseteq V$,
2. $\mathcal{P}$ preserves inclusion mappings,
3. (X, id_X) is a $\mathcal{P}$-algebra,
4. (V, id_V) is initial,

the unique arrow f from (V, id_V) to (X, id_X) is such that

$$\iota_{X,V} \circ f = \mathrm{id}_V.$$

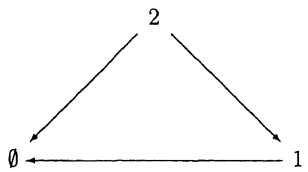
From this, it easily follows that f itself is the identity on V and thus $V \subseteq X$. □

Basic set theory together with the foundation axiom is the *standard set theory*. Virtually all known mathematics can be carried out inside such a theory and therefore for many decades only well-founded sets were considered to be sets. It was computer science that provided non-well-founded sets with one of the first significant applications: semantic processes are non-well-founded sets. (But see also [FH83] for a - previous - purely mathematical application.)

3.2.2 Decoration Lemma

In [Acz88] the *anti-foundation* axiom is formulated in terms of graphs and their "decorations". Corollary 3.8 shows that, already in basic set theory, the universe of sets is a $\mathcal{P}$-coalgebra. In Example 1.3 it is shown that graphs are $\mathcal{P}$-coalgebras as well. On the other hand every $\mathcal{P}$-coalgebra (A, α) can be seen as a (possibly large) graph, by interpreting A as a set (or class) of nodes and α as the *child* function. Therefore, the universe of sets can be interpreted as the class of nodes of a (large) graph. The childhood relation in such a graph is given by the membership relation between sets.

At a more local level one can observe that every set x can be "pictured" as a graph: nodes are the sets in the transitive closure w.r.t. $\in$ of x. The same membership relation gives also the childhood relation. For instance, the set $2 \equiv \{\emptyset, 1\}$, with $1 \equiv \{\emptyset\}$, can be pictured as:



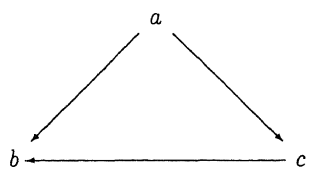
The converse of the notion of picture of a set by a graph is the "decoration" of a graph by a set:

Definition 3.12 Given a graph G, let $G_{\mathcal{P}}$ denote its $\mathcal{P}$-coalgebra representation (see Example 1.3). A *decoration* of a graph G is an arrow from the $\mathcal{P}$-coalgebra representation $G_{\mathcal{P}}$ of the graph into the $\mathcal{P}$-coalgebra (V, id_V). □

For instance the mapping

$$a \mapsto 2 \qquad b \mapsto \emptyset \qquad c \mapsto 1$$

is a decoration of the graph:



Moreover, it is the unique such decoration. In general, it can be proved within basic set theory that for every graph which contains no infinite path there exists a unique decoration. (Mostowski's Collapsing Lemma.) Notice that a graph has no infinite path if and only if its childhood relation is well-founded. Thus:

Proposition 3.13 For every well-founded graph there exists a unique decoration.

Clearly, every graph which is picture of a well-founded set is itself well-founded. And the (unique) decoration of a well-founded graph is a well-founded set.

Many graphs of interest, especially in computer science, are not well-founded, like, for instance, the cyclic graph with one node and one arc:

(2)

One might therefore consider a set theory in which the following generalization of the above proposition holds:

Decoration Lemma
For every graph there exists a unique decoration.

In fact, the above statement, expressible in the language of set theory is the formulation of the anti-foundation axiom as given in [Acz88]. It turns out to be equivalent to the anti-foundation axiom formulated in terms of finality:

Theorem 3.14 (V, id_V) is a final $\mathcal{P}$-coalgebra if and only if for every graph there exists a unique decoration.

Proof. The implication from left to right is immediate: if (V, id_V) is final, from any $\mathcal{P}$-coalgebra there exists a unique arrow into it; in particular this holds for coalgebras representing graphs. The implication from right to left follows by applying the Special Final Coalgebra Theorem (see below) to the powerset functor, as that theorem can be proved assuming the decoration lemma instead of the anti-foundation axiom in terms of finality (see [Acz88]). □

The unique decoration of the graph in (2) is thus then the unique arrow from the coalgebra $(\{\bullet\}, \alpha)$, with $\alpha(\bullet) = \{\bullet\}$, into (V, id_V):

$$\begin{array}{ccc} \{\bullet\} & \longrightarrow & V \\ {\scriptstyle\alpha}\downarrow & * & \Vert \\ \mathcal{P}(\{\bullet\}) & \longrightarrow & \mathcal{P}(V) \end{array}$$

Chasing the diagram, the (only) node of the graph will be uniquely associated to a (non-well-founded) set, say Ω, such that $\Omega = \{\Omega\}$. (This example shows that non-well-founded sets can also be finite.)

Notice that, since the relation $\in$ is not any more well-founded, more than extensionality is needed in order to establish equality between sets. But a criterion for establishing

equality of sets arises from the postulated finality of the universe and from Theorem 2.4, stating that final coalgebras are strongly extensional:

Theorem 3.15 Two sets are equal if and only if they are in a $\mathcal{P}$-bisimulation relation.

By applying Definition 2.2 to the powerset functor, one obtains:

Definition 3.16 A relation R on V is a *$\mathcal{P}$-bisimulation* if, for every set x and y such that xRy,

$$\forall x' \in x,\ \exists y' \in y,\ x' R\, y'$$

and

$$\forall y' \in y,\ \exists x' \in x,\ x' R\, y'.$$

□

Regarding sets as graphs, and thus edges going from sets into their members, this definition is just the standard definition of bisimulation as given in [Par81], abstracting from the fact that there graphs are labelled.

In the rest of this section only *non-standard set theory*, that is, basic set theory together with the anti-foundation axiom, will be considered. In particular:

Definition 3.17 The category denoted by *Class** is the category with objects the classes of non standard set theory and with arrows the functions between these classes. □

3.3 Solution Lemma

The finality of the universe V can be exploited not only to regard sets as decorations of graphs but also as *solutions of systems of set-equations*. This is the content of the *solution lemma*, illustrated in this subsection, which is yet another formulation of the anti-foundation axiom. This lemma is used in the special final coalgebra theorem.

Let x_1 and x_2 be two 'indeterminates'. Then the following is an example of a system of set-equations in $\{x_1, x_2\}$:

$$\begin{aligned} x_1 &= \{x_2, \{x_1, 0\}\}, \\ x_2 &= \{0, 2\}. \end{aligned}$$

In general, a set-equation has an indeterminate in its left hand side and a collection in its right hand side. The collection in the rhs is a set, apart from the fact that it might contain not only sets but also indeterminates as elements, and as elements of its elements, and so on. (It is thus important to keep the symbols used for indeterminates distinct from those used for 'pure' sets.) The collection of all these sets which might contain indeterminates in their transitive closure forms an 'expanded' universe:

Definition 3.18 Given a class X, the *expanded universe w.r.t. X* — denoted by V_X — is defined as the greatest fixed point of the (set-continuous) functor $\mathcal{P}(X + \text{-})$. Thus:

$$V_X = \mathcal{P}(X + V_X).$$

□

Clearly, the universe V is isomorphic to $V_\emptyset$ and can be embedded into any V_X.

The formal definition of a system of set-equations can now be given:

Definition 3.19 Given a class X, a *system of set-equations in* X is a function

$$\nu : X \to V_X.$$

That is, a collection of equations of the form

$$x = \nu_x,$$

with $x \in X$ and $\nu_x \in V_X$. □

Consider again the above example of a system of set-equations. A solution to that system would simply be a function $f : \{x_1, x_2\} \to V$ such that

$$f(x_1) = \{f(x_2), \{f(x_1), 0\}\}, \qquad f(x_2) = \{0, 2\}.$$

In general, a solution to a system $\{x = \nu_x\}_{x \in X}$ is a function $f : X \to V$ such that, for all $x \in X$,

$$f(x) = \hat{f}(\nu_x) \tag{3}$$

where, informally, $\hat{f}(\nu_x)$ is obtained by replacing every x_i in the transitive closure of ν_x by the corresponding $f(x_i)$. That is, if $x_0, x_1, \ldots$, are the variables appearing in the transitive closure of ν_x, and denoting ν_x by $\nu_x[x_0, x_1, \ldots]$, then

$$\hat{f}(\nu_x) = \nu_x[x_0/f(x_0),\ x_1/f(x_1), \ldots].$$

This intuitive idea has a formal definition:

Definition 3.20 A *solution to a system of set-equations* $\nu : X \to V_X$ is a composed arrow $\pi \circ \nu$, where $\pi : V_X \to V$ is any arrow making the square in the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{\ \nu\ } & V_X \\ & & \Big\downarrow{\scriptstyle \Theta_\nu} \\ & & \mathcal{P}(V_X) \end{array} \qquad \begin{array}{ccc} V_X & \xrightarrow{\ \pi\ } & V \\ & * & \Big\| \\ \mathcal{P}(V_X) & \xrightarrow[\mathcal{P}(\pi)]{} & \mathcal{P}(V) \end{array}$$

where, for every v in V_X, that is, for every $v \subseteq X + V_X$ (since $V_X = \mathcal{P}(X + V_X)$),

$$\Theta_\nu(v) \equiv \{\nu_x \mid x \in v \cap X\} \cup \{v' \mid v' \in v \cap V_X\}.$$

□

If one puts $f = \pi \circ \nu$ and $\hat{f} = \pi = \mathcal{P}(\pi) \circ \Theta_\nu$, then, for every v in V_X,

$$\hat{f}(v) = \{f(x) \mid x \in v \cap X\} \cup \{\hat{f}(v') \mid v' \in v \cap V_X\} \tag{4}$$

and, in particular, (3) holds.

Since solutions are defined in terms of $\mathcal{P}$-coalgebra arrows between (V_X, Θ_ν) and the universe, the finality of the latter immediately gives the following

Solution Lemma [Acz88]
For every system of set-equations there exists a unique solution.

This lemma provides thus sets with another representation, describing them as unique solutions of systems of equations; moreover, it is an important tool in proving properties of non-standard sets, as the next section will illustrate. Since its proof relies on the finality of the universe, the solution lemma holds only in non-standard set theory. In fact, it can be proved that the solution lemma is equivalent to the anti-foundation axiom.

Notice that the use of coalgebras makes the presentation of the solution lemma much simpler than in [Acz88]. In particular, its proof becomes here trivial, while the following is needed there:

Substitution Lemma [Acz88]
For every function $f : X \to V$ there exists a unique function $\hat{f} : V_X \to V$ such that, for every $v \in V_X$,

$$\hat{f}(v) = \{f(x) \mid x \in v \cap X\} \cup \{\hat{f}(v') \mid v' \in v \cap V_X\}.$$

Although the above lemma is not needed here for proving the solution lemma, the existence of such a unique extension of any function on X to a function on V_X is needed in the sequel (in the definition of *uniformity on maps*). Notice that it simply generalizes (4) to any function on X.

One final remark. Here, the definition of the expanded universe is carried out within the language of set theory, but, alternatively, indeterminates could also be added as new symbols in the language. For instance, in [BE88] indeterminates are indeed treated as primitive elements (*Urelemente*) of a set theory like the one in [Bar75]. But in order to carry out this extension of the language formally, an extension of the axioms of the theory is also required.

3.4 Special Final Coalgebra Theorem

The assumption that the universe (greatest fixed point of $\mathcal{P}$) be a final coalgebra of the powerset functor is strong enough to make the greatest fixed points of a large class of other functors be final coalgebras of the respective functors too. This is the content of the *special final coalgebra theorem* illustrated in this subsection.

The finality of the greatest fixed point of (certain) functors is proved here by means of the solution lemma. Arrows into such candidate final coalgebras are associated to solutions of systems of set equations (having the class in the source coalgebra as indeterminates). This is best illustrated by means of the powerset functor:

For any function $f : A \to V$ and any $\{a_0, a_1, \ldots\}$ in $\mathcal{P}(A)$,

$$\mathcal{P}(f)(\{a_0, a_1, \ldots\}) = \{f(a_0), f(a_1), \ldots\}.$$

Regard now A as a class of indeterminates and $\{a_0, a_1, \ldots\}$ as a set in V_A, that is, associate to A the obvious embedding function $\varphi_A : \mathcal{P}(A) \to V_A$. Then:

$$\{f(a_0), f(a_1), \ldots\} = \hat{f} \circ \varphi_A(\{a_0, a_1, \ldots\}).$$

Loosely speaking, this shows that the powerset functor behaves on maps as it behaves on objects (*uniform on maps*).

The mapping φ_A is described above as an embedding and this is indeed the case for most of functors of interest for semantics. In general, other mappings can be considered as well, so that what above generalizes to the following:

Definition 3.21 An endofunctor $F : \mathit{Class}^* \to \mathit{Class}^*$ is *uniform on maps* if for every class A there exists a V_A*-translation for* F, that is, a mapping $\varphi_A : F(A) \to V_A$ such that, for every function $f : A \to V$, the square in the following diagram commutes:

$$\begin{array}{ccc} A & & F(A) \xrightarrow{\varphi_A} V_A \\ \downarrow f & & \downarrow F(f) \quad * \quad \downarrow \hat{f} \\ V & & F(V) \xrightarrow[\iota]{} V \end{array}$$

Briefly:

$$\forall A\ \exists \varphi_A : F(A) \to V_A \text{ such that } \forall f : A \to V \text{ and } \forall \sigma \in F(A)$$
$$F(f)(\sigma) = \hat{f} \circ \varphi_A(\sigma).$$

□

Theorem 3.22 (*Special Final Coalgebra Theorem*)
Let $F : \mathit{Class}^* \to \mathit{Class}^*$ be a functor uniform on maps and inclusion preserving. If, w.r.t. set-inclusion, F has a greatest fixed (as well as postfixed) point J_F, then (J_F, id) is a final F-coalgebra.

Proof. For every F-coalgebra (A, α) one needs to find a function $f : A \to J_F$ such that, for all a in A,

$$f(a) = F(f)(\alpha(a)) \tag{5}$$

and then show that it is unique. By uniformity on maps, there exists a V_A-translation for F. Since $\alpha(a)$ belongs to $F(A)$, one can rewrite (5) as

$$f(a) = \hat{f} \circ \varphi_A(\alpha(a)).$$

But then the unique solution of the system $\{a = \varphi_A(\alpha(a))\}_{a \in A}$ of set-equations in A is a function $f : A \to V$ for which (5) holds. Now it remains to be proved that the image of this function f is contained in J_F, that is, f is a function into J_F as well. From equation (5) one can derive that:

$$\begin{aligned} f(A) &= F(f)(\alpha(A)) \\ &\subseteq F(f)(F(A)) \\ &\subseteq F(f(A)), \end{aligned}$$

that is, $f(A)$ is a postfixed point of F. From all this follows that f is an arrow which makes the following diagram commute:

$$\begin{array}{ccc} A & \xrightarrow{f} & f(A) \\ {\scriptstyle\alpha}\downarrow & * & \downarrow{\scriptstyle\iota} \\ F(A) & \xrightarrow[F(f)]{} & F(f(A)) \end{array}$$

Since J_F is the greatest postfixed point of F w.r.t. set-inclusion, $f(A)$ is included in J_F and $F(f(A))$ is included in $F(J_F)$. Moreover, since F is an inclusion preserving functor, the inclusion mapping from $F(f(A))$ into $F(J_F)$ is equal to the F image of the inclusion mapping from $f(A)$ into J_F. Therefore, the following diagram commutes:

$$\begin{array}{ccc} f(A) & \xrightarrow{\iota} & J_F \\ {\scriptstyle\iota}\downarrow & * & \| \\ F(f(A)) & \xrightarrow[F(\iota)]{} & F(J_F) \end{array}$$

Combining the last two diagrams, f can be regarded as an arrow from A into J_F which makes the following diagram commute:

$$\begin{array}{ccc} A & \xrightarrow{f} & J_F \\ {\scriptstyle\alpha}\downarrow & * & \| \\ F(A) & \xrightarrow[F(f)]{} & F(J_F) \end{array}$$

This shows the existence of an arrow from (A, α) into (J_F, id). Uniqueness follows from the fact that any such an arrow is also a solution of $\{a = \varphi_A(\alpha(a))\}_{a \in A}$, which by the solution lemma is unique. □

Corollary 3.23 The greatest fixed point of a set-continuous functor which is uniform on maps and inclusion preserving is, together with the identity mapping, a final coalgebra.

4 Complete Metric Spaces

Let *CMS* be the category with complete metric spaces (D, d_D) as objects and non-expansive (non-distance-increasing) functions as arrows. That is, functions $f : D \to E$ such that, for all $x, y \in D$,

$$d_E(f(x), (f(y)) \leq d_D(x, y).$$

(For basic facts on metric spaces see, e.g., [Dug66].) For any two complete metric spaces D and E, the set of arrows between D and E,

$$\hom(D, E) \equiv \{f : D \to E \mid f \text{ is non-expansive}\}$$

is itself a complete metric space, with metric, for all $f, g \in \hom(D, E)$,

$$d(f, g) \equiv \sup_{x \in D}\{d_E(f(x), g(x))\}.$$

In analogy to the so-called order-enriched (or **O**-) categories of [SP82], *CMS* is called a metric-enriched category.

Definition 4.1 A category $\mathcal{C}$ is called *metric-enriched* if every hom-set is a complete metric space and composition of arrows is continuous with respect to this metric. □

In the sequel, only metric-enriched categories like *CMS* will be considered, in which the objects themselves are metric spaces (from which the hom-sets inherit their metric structure). Nevertheless, it will turn out to be convenient to formulate some definitions and results about metric-enriched categories in general.

The fact that hom sets are metric spaces allows the following characterization of families of functors in terms of how they act on arrows.

Definition 4.2 Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor on metric-enriched categories. It is called *locally continuous (non-expansive)* if, for any two objects $D, E \in \mathcal{C}$, the mapping

$$F_{D,E} : \hom(D, E) \to \hom(F(D), F(E)) \qquad f \mapsto F(f)$$

is continuous (non-expansive). The functor F is called *locally contracting* (or hom-contracting) if there exists ϵ with $0 \leq \epsilon < 1$ such that, for all D, E, the mapping $F_{D,E}$ is a contraction with factor ϵ: for all $f, g \in \hom(D, E)$,

$$d_{\hom(F(D),F(E))}(F(f), F(g)) \leq \epsilon \cdot d_{\hom(D,E)}(f, g).$$

□

Example 4.3 Let $\mathcal{P}_{comp} : CMS \to CMS$ be the metric powerset functor defined on objects by, for all $(D, d_D) \in CMS$,

$$\mathcal{P}_{comp}(D) \equiv \{X \mid X \text{ is a compact (w.r.t. } d_D) \text{ subset of } D\}.$$

The metric on $\mathcal{P}_{comp}(D)$ is the so-called Hausdorff metric d_H, given by, for $X, Y \in \mathcal{P}_{comp}(D)$,

$$d_H(X,Y) = \max\{\sup_{x\in X}\{d(x,Y)\}, \sup_{y\in Y}\{d(y,X)\}\},$$

where $d(x,Z) = \inf_{z\in Z}\{d_D(x,z)\}$ for every $Z \subseteq M$, $x \in M$. (by convention, $\sup\emptyset = 0$ and $\inf\emptyset = 1$.) One can show that if D is complete then $\mathcal{P}_{comp}(D)$ is complete as well. On arrows $f : D \to E$, we have

$$\mathcal{P}_{comp}(f) : \mathcal{P}_{comp}(D) \to \mathcal{P}_{comp}(E), \quad X \mapsto \{f(x) \mid x \in X\}.$$

It is not difficult to prove that $\mathcal{P}_{comp}$ is locally non-expansive. □

Example 4.4 For every ϵ with $0 \leq \epsilon < 1$, the "shrinking" functor $id_\epsilon : CMS \to CMS$ is defined as the identity on arrows and, for any (D, d_D),

$$id_\epsilon((D,d_D)) \equiv (D, \epsilon \cdot d_D).$$

Clearly id_ϵ is locally contracting. □

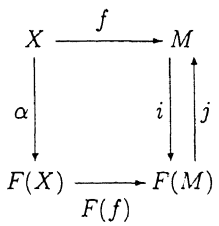
4.1 A 'Metric' Final Coalgebra Theorem

The final coalgebra theorem below will be based on the following.

Theorem 4.5 Every fixed point of a locally contracting functor $F : CMS \to CMS$ is a final F-coalgebra.

Proof. Suppose that M is a fixed point for F, that is, $M \cong F(M)$. Let $i : M \to F(M)$ and $j : F(M) \to M$ be the two components of such an isomorphism. Thus $j \circ i = \mathrm{id}_M$ and $i \circ j = \mathrm{id}_{F(M)}$. Let (X,α) be an F-coalgebra. Define $\Phi : \hom(X,M) \to \hom(X,M)$ by, for all f,

$$\Phi(f) \equiv j \circ F(f) \circ \alpha$$



Let F be locally contracting with factor ϵ. Then Φ is a contraction with factor ϵ. That is, for all $f_1, f_2 \in \hom(X,M)$,

$$\begin{aligned}
d(\Phi(f_1),\Phi(f_2)) &= \sup_{x\in X}\{d_M(\Phi(f_1)(x),\Phi(f_2)(x))\} \\
&= \sup_{x\in X}\{d_M(j\circ F(f_1)\circ\alpha(x), j\circ F(f_2)\circ\alpha(x))\} \\
&\leq \sup_{y\in F(X)}\{d_M(j\circ F(f_1)(y), j\circ F(f_2)(y))\} \\
&\leq \sup_{y\in F(X)}\{d_{F(M)}(F(f_1)(y),F(f_2)(y))\} \ (j \text{ is non-expansive}) \\
&= d(F(f_1),F(f_2)) \\
&\leq \epsilon\cdot d(f_1,f_2) \ (F \text{ is locally contracting}).
\end{aligned}$$

By Banach's theorem F has a unique fixed point $\pi : X \to M$. Moreover:

$$i \circ \pi = i \circ \Phi(\pi) = i \circ j \circ F(f) \circ \alpha = F(f) \circ \alpha,$$

which shows that π is the unique arrow from (X, α) into (M, i). □

The dual of this theorem can be proved similarly:

Theorem 4.6 Every fixed point of a locally contracting functor $F : \mathit{CMS} \to \mathit{CMS}$ is an initial F-algebra.

In subsection 4.3, the following theorem will be proved.

Theorem 4.7 Every locally contracting functor $F : \mathit{CMS} \to \mathit{CMS}$ has a fixed point.

From Theorem 4.5 and Theorem 4.7, the following final coalgebra theorem for *CMS* is immediate.

Theorem 4.8 Every locally contracting functor $F : \mathit{CMS} \to \mathit{CMS}$ has a final F-coalgebra.

Since final coalgebras are unique (up to isomorphism) the following is immediate.

Corollary 4.9 Every locally contracting functor $F : \mathit{CMS} \to \mathit{CMS}$ has a unique fixed point (which is at the same time a final F-coalgebra and an initial F-algebra).

4.2 F-Bisimulation in *CMS*

According to the definition of bisimulation (Definition 2.2), F-bisimulations have to be objects in the category under consideration. For the category *CMS* this implies that they have to be complete metric spaces: that is, an F-bisimulation on an F-coalgebra (A, α) in *CMS* is a closed subset of $A \times A$, satisfying the conditions of Definition 2.2.

The following theorem is an instantiation of Theorem 2.4 to the category *CMS*.

Theorem 4.10 The unique fixed point (M, i) of a locally contracting functor $F : \mathit{CMS} \to \mathit{CMS}$ is strongly extensional; that is, for all $x, y \in M$,

$$x = y \Leftrightarrow x \overset{F}{\sim} y.$$

(Recall that $\overset{F}{\sim} = \bigcup\{R \subseteq M \times M \mid R \text{ is an } F\text{-bisimulation on } (M, i)\}$.) □

Next the construction of a metric domain for strong bisimulation (as used in Example 1.8 and [BM88, Rut90]) will be described in detail.

Let A be an arbitrary set supplied with the discrete metric. The constant functor $F_A : \mathit{CMS} \to \mathit{CMS}$ assigns to all objects the complete metric space A, and to all arrows the identity arrow id_A. Let I be the identity functor on *CMS*. The product functor $\times : \mathit{CMS} \times \mathit{CMS} \to \mathit{CMS}$ gives for any two objects D and E in *CMS* the Cartesian product $D \times E$, with metric, for all $x_1, x_2 \in D$ and $y_1, y_2 \in E$,

$$d_{D \times E}(\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle) \equiv \max\{d_D(x_1, x_2), d_E(y_1, y_2)\}$$

On arrows $\times$ is defined as usual.

Let F_1 and F_2 be two functor from *CMS* to *CMS*. The functor $< F_1, F_2 >$: *CMS* $\rightarrow$ *CMS* $\times$ *CMS* (the *tupling* of F_1 and F_2) is defined on objects D by

$$< F_1, F_2 > (D) \equiv < F_1(D), F_2(D) >$$

and on arrows $f : D \rightarrow E$ by

$$< F_1, F_2 > (f) \equiv < F_1(f), F_2(f) >$$

Let the functor F : *CMS* $\rightarrow$ *CMS* be defined as a composition of the above functors as follows:

$$F \equiv \mathcal{P}_{comp} \circ \times \circ < F_A, I > .$$

It has already been observed that $\mathcal{P}_{comp}$ is locally continuous, and the same applies to the other constructs. Composition of functors preserves local continuity, hence F is locally continuous. Next define, for some ϵ with $0 \leq \epsilon < 1$, a functor F_ϵ by

$$F_\epsilon \equiv id_\epsilon \circ F.$$

It is immediate that F_ϵ is locally contracting since id_ϵ is locally contracting and F is locally continuous. Finally we are ready for the following.

Definition 4.11 Let the metric domain for bisimulation P_M be the unique fixed point of the locally contracting functor F_ϵ. That is, P_M is the unique complete metric space satisfying

$$P_M \cong \mathcal{P}_{comp}(A \times P_M).$$

□

By Theorem 4.5 P_M is a final coalgebra. Recall that it is used in Example 1.8 for representing finitely branching labelled transition systems.

(For LTS's that are image finite (a weaker notion than finitely branching), one could replace in the above definition the functor $\mathcal{P}_{comp}$ by another powerset functor: $\mathcal{P}_{closed}$, which yields all metrically closed subsets. In [Bre92], domains are given suited for LTS's that satisfy even more general "branching" properties.)

4.3 Fixed Points in *CMS*

In this subsection, it will be shown that every locally contracting functor has a fixed point, thus proving Theorem 4.7. In [AR89], a similar theorem is proved: so-called contracting functors on a category of complete metric spaces (with double arrows) have a fixed point (see also below). Here the results of [AR89] are generalized; in summary, a reconstruction of that paper is given along the lines of [SP82] and [Plo81a].

A standard way of constructing fixed points of functors on a category of complete partial orders, as described in [SP82], can be seen as a category-theoretic generalization of the least fixed point construction of monotone functions on complete partial orders. In metric-enriched categories, the construction of fixed points of functors can be better

compared to Banach's fixed point theorem: any contracting function f from a complete metric space to itself has a unique fixed point, which can be obtained as the limit of all finite iterations of f starting in an arbitrary element. (See also the remark following Theorem 4.23.)

As in [SP82], fixed points will be constructed in a category with so-called embedding-projection pairs as arrows. One of the reasons for this is that certain constructions, like the function space construction, are not functorial. However, such constructions can be turned into functors on this category with double arrows, which is introduced next.

Definition 4.12 Let $\mathcal{C}$ be a metric-enriched category. A subcategory $\mathcal{C}^E$ (of embeddings) can be defined by taking as objects the same objects as $\mathcal{C}$. Arrows $\alpha : D \to E$ in $\mathcal{C}^E$ are pairs $\alpha = \langle \alpha^e, \alpha^p \rangle$ such that

$$\alpha^e : D \to E, \quad \alpha^p : E \to D$$

are arrows in $\mathcal{C}$ with

$$\alpha^p \circ \alpha^e = id_D.$$

The first component α^e is called an embedding and the second component α^p a projection. Identity arrows in $\mathcal{C}^E$ on objects D are $\langle id_D, id_D \rangle$, and composition of two arrows α and β is defined by

$$\beta \circ \alpha \equiv \langle \beta^e \circ \alpha^e, \alpha^p \circ \beta^p \rangle.$$

□

Note that for arrows $\alpha : D \to E$ in CMS^E the facts that α^e and α^p are non-expansive and $\alpha^p \circ \alpha^e = id_D$ imply that α^e is a distance-preserving embedding.

It is illustrative to compare the above definition to the standard example of an order-enriched category, namely the category $CPO_\perp$ of complete partial orders with strict continuous mappings. If D and E are cpo's and $i : D \to E$ and $j : E \to D$ are arrows in $CPO_\perp$ then $\langle i, j \rangle$ is called a projection pair from D to E provided that

$$j \circ i = id_D \text{ and } i \circ j \sqsubseteq_{\hom(E,E)} id_E.$$

Note that the one half of such projection pairs determines the other. For the metric case this does not hold. For instance, in CMS the trivial one point metric space can be embedded in different ways into any other metric space containing more than one element.

Though the latter condition of projection pairs ($i \circ j \sqsubseteq_{\hom(E,E)} id_E$) does not seem to have a direct corresponding metric counterpart, it is possible, due to the fact that hom-sets are complete metric spaces, to define a function on projection pairs that technically will play a similar role.

Definition 4.13 Let $\alpha : D \to E$ be an arrow in $\mathcal{C}^E$. Then

$$\delta(\alpha) \equiv d_{\hom(E,E)}(\alpha^e \circ \alpha^p, id_E).$$

More generally, let

$$< \alpha_1, \ldots, \alpha_n >:< D_1, \ldots, D_n > \rightarrow < E_1, \ldots, E_n >$$

be an arrow in $(\mathcal{C}^E)^n$. Then

$$\delta(< \alpha_1, \ldots, \alpha_n >) \equiv \max\{\delta(\alpha_1), \ldots, \delta(\alpha_n)\}.$$

□

The above $\delta(\alpha)$ is called the *approximation degree* of α: it can be understood as a measure of the quality with which E is approximated by D. (Note that $\delta(\alpha) = 0$ implies that D and E are isomorphic.) The approximation degree can be conveniently used in characterizing colimits in the category CMS^E. But let us first explain what a colimit is.

Definition 4.14 An ω-chain Δ in a category $\mathcal{C}$ is a sequence of objects and arrows like

$$\Delta = D_0 \rightarrow^{\alpha_0} D_1 \rightarrow^{\alpha_1} \cdots.$$

Given an object D in $\mathcal{C}$, a *cone* $\mu : \Delta \rightarrow D$ from Δ to D is a sequence of arrows $\mu_n : D_n \rightarrow D$ such that for all $n \geq 0$,

$$\mu_n = \mu_{n+1} \circ \alpha_n.$$

A *colimit* of Δ is an *initial* cone from Δ, that is, a cone $\mu : \Delta \rightarrow D$ such that for every other cone $\gamma : \Delta \rightarrow E$ there exists a unique arrow $\iota : D \rightarrow E$ satisfying, for all $n \geq 0$,

$$\iota \circ \mu_n = \gamma_n.$$

□

Theorem 4.15 Let $\mathcal{C}$ be a metric-enriched category and let Δ be an ω-chain in $\mathcal{C}$. Let $\mu : \Delta \rightarrow D$ be a cone from Δ. Then

$$\mu : \Delta \rightarrow D \text{ is initial (a colimit) for } \Delta \Leftrightarrow \lim_{n\to\infty} \delta(\mu_n) = 0.$$

Proof. The theorem generalizes the metric version of the 'initiality lemma' given in [AR89]. There the theorem is formulated for the category CMS and assumes, more importantly, Δ to be a so-called converging ω-chain. An inspection of the proof given there shows that this condition is superfluous. □

Observing that

$$\lim_{n\to\infty} \delta(\mu_n) = 0 \Leftrightarrow \lim_{n\to\infty} {\mu_n}^e \circ {\mu_n}^p = id_D$$

shows the correspondence with the order-theoretic version of the initiality lemma,

$$\mu : \Delta \rightarrow D \text{ is initial (a colimit) for } \Delta \Leftrightarrow \bigsqcup_n {\mu_n}^e \circ {\mu_n}^p = id_D,$$

interpreting Δ and μ over the category CPO^E.

In the sequel, also products of metric-enriched categories will be considered.

Definition 4.16 Let $\mathcal{C}$ and $\mathcal{C}'$ be two metric-enriched categories. The product category $\mathcal{C} \times \mathcal{C}'$ has as objects pairs $< D, E >$ of objects D in $\mathcal{C}$ and E in $\mathcal{C}'$. Arrows are pairs of arrows as usual: For any two pairs $< D, E >$ and $< D', E' >$,

$$\hom(< D, E >, < D', E' >) =$$

$$\{< f, g > | \, f : D \to D' \text{ in } \mathcal{C} \text{ and } g : E \to E' \text{ in } \mathcal{C}'\}.$$

Clearly, $\mathcal{C} \times \mathcal{C}'$ is again a metric-enriched category, by putting for arrows $< f_1, g_1 >$ and $< f_2, g_2 >$ in the above hom-set,

$$d(< f_1, g_1 >, < f_2, g_2 >) \equiv \max\{d_{\hom(D,D')}(f_1, f_2),\ d_{\hom(E,E')}(g_1, g_2)\}.$$

□

Let $\mathcal{C}$ be a metric-enriched category. It is next shown how in general every functor $F : \mathcal{C}^{m+n} \to \mathcal{C}$, which is contravariant in its first m and covariant in its last n arguments (with $m + n \geq 1$) induces a functor

$$F^E : (\mathcal{C}^E)^{m+n} \to \mathcal{C}^E.$$

(Note that the general case includes, e.g., covariant functors of one argument.) A typical example of such a functor F is the function space constructor:

Example 4.17 The function space constructor $\to$: $CMS \times CMS \to CMS$ gives for any two objects D and E the set $D \to E$ of non-expansive mappings from D to E: $D \to E \equiv \hom(D, E)$. (The metric on $D \to E$ is as on $\hom(D, E)$.) Consider the category $CMS \times CMS$ with arrows

$$< f, g >:< D, E > \to < D', E' >,$$

where $f : D' \to D$ and $g : E \to E'$ are arrows in CMS. Note the different directions: $\to$ is called contravariant in its first argument and covariant in its second. (Formally, $\to$ is a functor (covariant in both arguments) from $CMS^{op} \times CMS$ to CMS.) The image under $\to$ of such an arrow is given by

$$f \to g : (D \to E) \to (D' \to E'), \quad h \mapsto g \circ h \circ f.$$

□

Definition 4.18 Let $\mathcal{C}$ be a metric-enriched category and let $F : \mathcal{C}^{m+n} \to \mathcal{C}$ be contravariant in its first m arguments and covariant in its last n arguments. For convenience take $m = 1$ and $n = 1$. The functor

$$F^E : (\mathcal{C}^E)^{1+1} \to \mathcal{C}^E$$

is defined on objects by, for any $< D, E > \in (\mathcal{C}^E)^{1+1}$,

$$F^E(< D, E >) \equiv F(< D, E >).$$

On arrows $< \alpha, \beta >:< D, E > \to < D', E' >$ in $(\mathcal{C}^E)^{1+1}$ (with $\alpha : D \to D'$ and $\beta : E \to E'$ arrows in $\mathcal{C}^E$), F^E is defined by

$$F^E(<\alpha,\beta>) \equiv \langle F(<\alpha^p,\beta^e>), F(<\alpha^e,\beta^p>)\rangle.$$

Note that F^E is covariant in both arguments. If F and G are functors and $G \circ F$ is defined then $(G \circ F)^E = G^E \circ F^E$. □

It is easy to show that F^E is a functor. In particular,

$$\begin{aligned} F(<\alpha^e,\beta^p>) \circ F(<\alpha^p,\beta^e>) &= (F \text{ is contravariant in its first argument}) \\ & \quad F(<\alpha^p \circ \alpha^e, \beta^p \circ \beta^e>) \\ &= F(<id_D, id_E>) \\ &= F(id_{<D,E>}) \\ &= id_{F(<D,E>)}. \end{aligned}$$

Example 4.17 (continued) According to the above definition, the functor $\rightarrow$: $CMS \times CMS \rightarrow CMS$ induces a functor $\rightarrow^E$ defined on objects $< D, E >$ by

$$D \rightarrow^E E \equiv D \rightarrow E$$

and on arrows $< \alpha, \beta >:< D, E > \rightarrow < D', E' >$ by

$$\alpha \rightarrow^E \beta \equiv \langle \alpha^p \rightarrow \beta^e,\ \alpha^e \rightarrow \beta^p \rangle.$$

□

Starting with a locally continuous functor F will yield an ω-continuous functor F^E:

Definition 4.19 Let $\mathcal{C}$ be a metric-enriched category. A (covariant) functor $F : \mathcal{C}^E \rightarrow \mathcal{C}^E$ is *ω-continuous* if for every ω-chain Δ and every colimit (initial cone) $\mu : \Delta \rightarrow D$ of Δ the cone $F(\mu) : F(\Delta) \rightarrow F(D)$ is again initial. (This definition can be straightforwardly generalized to functors from $(\mathcal{C}^E)^n$ to $\mathcal{C}^E$.) □

In other words, F preserves colimits of ω-chains.

Theorem 4.20 Let $\mathcal{C}$ be a metric-enriched category and let $F : (\mathcal{C})^{m+n} \rightarrow \mathcal{C}$ be contravariant in its first m arguments and covariant in its last n arguments. If F is locally continuous then F^E is ω-continuous.

Proof. The proof mimics that of [Plo81a]. For simplicity let $m = 1 = n$. Consider $F : (\mathcal{C})^{1+1} \rightarrow \mathcal{C}$ and let $\mu : \Delta \rightarrow D$ and $\nu : \Gamma \rightarrow E$ be two initial cones. It has to be proved that $F^E(\mu,\nu) : F^E(\Delta,\Gamma) \rightarrow F^E(D,E)$ is again initial. Theorem 4.15 will be used:

$$\begin{aligned} & \lim_{n\to\infty} (F^E(<\mu,\nu>)_n)^e \circ (F^E(<\mu,\nu>)_n)^p \\ = & \lim_{n\to\infty} (F^E(<\mu_n,\nu_n>))^e \circ (F^E(<\mu_n,\nu_n>))^p \\ = & \lim_{n\to\infty} F(<\mu_n^p,\nu_n^e>) \circ F^E(<\mu_n^e,\nu_n^p>) \\ = & \lim_{n\to\infty} F(<\mu_n^e \circ \mu_n^p, \nu_n^e \circ \nu_n^p>) \\ = & \ (F \text{ is locally continuous}) \\ & F(<\lim_{n\to\infty} \mu_n^e \circ \mu_n^p, \lim_{n\to\infty} \nu_n^e \circ \nu_n^p>) \\ = & \ (\text{Theorem 4.15}) \\ & F(<id_D, id_E>) \\ = & \ F^E(<id_D, id_E>) \\ = & \ id_{F^E(<D,E>)}. \end{aligned}$$

Thus, again by Theorem 4.15, $F^E(\mu, \nu)$ is initial. □

There is also a property of functors on $\mathcal{C}^E$ that corresponds with the notion of local contractivity.

Definition 4.21 Let $\mathcal{C}$ be a metric-enriched category. A (covariant) functor $F : \mathcal{C}^E \to \mathcal{C}^E$ is *contracting* if there exists $0 \leq \epsilon < 1$ such that, for every arrow $\alpha : D \to E$ in $\mathcal{C}^E$,

$$\delta(F(\alpha)) \leq \epsilon \cdot \delta(\alpha).$$

(Again the definition can be easily generalized to functors from $(\mathcal{C}^E)^n$ to $\mathcal{C}^E$.) □

The value of $\delta(\alpha)$ can be seen as a measure of the quality with which E is approximated, and hence contractivity of a functor amounts to the property that it strictly improves such approximations. Using the initiality lemma (Theorem 4.15), one can easily show that contractivity implies ω-continuity. There is also a relation between local contractivity and contractivity, as pointed out by Gordon Plotkin (personal communication):

Theorem 4.22 Let $\mathcal{C}$ be a metric-enriched category and let $F : (\mathcal{C})^{m+n} \to \mathcal{C}$ be contravariant in its first m arguments and covariant in its last n arguments. If F is locally contracting then F^E is contracting.

Proof. Again restrict to the convenient case that $m = n = 1$. Let F be locally contracting with factor ϵ. Consider an arrow $< \alpha, \beta >$ from $< D, E >$ to $< D', E' >$ in $\mathcal{C}^E \times \mathcal{C}^E$. Then

$$\begin{aligned}
\delta(F^E(<\alpha,\beta>)) &= \text{(definition } F^E\text{)} \\
&\quad \delta(\langle F(<\alpha^p,\beta^e>), F(<\alpha^e,\beta^p>)) \\
&= \text{(definition } \delta\text{)} \\
&\quad d(F(<\alpha^p,\beta^e>) \circ F(<\alpha^e,\beta^p>), id_{F(<D',E'>)}) \\
&= d(F(<\alpha^e \circ \alpha^p, \beta^e \circ \beta^p>), F(id_{<D',E'>})) \\
&\leq (F \text{ is locally contracting}) \\
&\quad \epsilon \cdot d(<\alpha^e \circ \alpha^p, \beta^e \circ \beta^p>, id_{<D',E'>}) \\
&= \epsilon \cdot d(<\alpha^e \circ \alpha^p, \beta^e \circ \beta^p>, <id_{D'}, id_{E'}>) \\
&= \epsilon \cdot \max\{d(\alpha^e \circ \alpha^p, id_{D'}), d(\beta^e \circ \beta^p, id_{E'})\} \\
&= \epsilon \cdot \max\{\delta(\alpha), \delta(\beta)\} \\
&= \epsilon \cdot \delta(<\alpha,\beta>).
\end{aligned}$$

□

Contracting functors on CMS^E are particularly interesting.

Theorem 4.23 Every contracting functor $F : CMS^E \to CMS^E$ has a fixed point.

Proof. The proof is given in [AR89]. It consists of a metric variant of the standard construction for cpo's. An important difference however is the use of the metric version of the 'initiality lemma', as formulated in Theorem 4.15. We give a sketch of the proof.

Let D_0 be the trivial one point metric space and let $\alpha_0 : D_0 \to F(D_0)$ be an arbitrary arrow embedding D_0 into $F(D_0)$. Define an ω-chain $\Delta \equiv (D_n, \alpha_n)_n$ by putting $D_{n+1} \equiv F(D_n)$ and $\alpha_{n+1} \equiv F(\alpha_n)$, for $n \geq 0$. The so-called direct (or projective) limit of Δ,

$$D \equiv \{(x_n)_n \mid \forall n \geq 0[x_n \in D_n \wedge \alpha_n^p(x_{n+1}) = x_n]\}$$

can be seen to be a complete metric space with metric d_D on D given by, for all $(x_n)_n, (y_n)_n$ in D,

$$d_D((x_n)_n, (y_m)_m) \equiv \sup_{n \geq 0}\{d_{D_n}(x_n, y_n)\}.$$

(It is assumed that the metrics d_{D_n} have a common upper bound.) Next D can be turned into a cone $\mu : \Delta \to D$ with arrows $\mu_n : D_n \to D$, for all $n \geq 0$, by defining for all $x \in D_n$, and $(x_m)_m \in D$,

$$\begin{aligned} \mu_n^e(x) &\equiv (\alpha_{n-1}^p \circ \cdots \circ \alpha_0^p(x),\ \alpha_{n-1}^p \circ \cdots \circ \alpha_1^p(x),\ \ldots, \\ &\quad \alpha_{n-1}^p(x),\ x,\ \alpha_n^e(x),\ \alpha_{n+1}^e \circ \alpha_n^e(x),\ \ldots), \\ \mu_n^p((x_m)_m) &\equiv x_n. \end{aligned}$$

So far the fact that F is a contracting functor has not been used. An easy argument shows that the contractivity of F implies $\lim_{n\to\infty} \delta(\mu_n) = 0$, whence D is a colimit for Δ. Contractivity of F also implies that F preserves ω-chains and their colimits: $F(\mu) : F(\Delta) \to F(D)$ is again a colimit. Since Δ and $F(\Delta)$ are equal but for the first element and colimits are unique (up to isomorphism), it follows that $D \cong F(D)$. □

Remark: Contractivity of F implies $\lim_{n\to\infty} \delta(\mu_n) = 0$. Another way of describing this fact is to observe that the ω-chain Δ is *Cauchy* (in [AR89], it is called converging):

$$\forall \epsilon > 0\ \exists N > 0\ \forall m > n \geq N,\ \delta(\alpha_{m-1} \circ \cdots \circ \alpha_n) < \epsilon$$

Implicit in the above construction is the following fact: every ω-chain that is Cauchy has a colimit. (Thus the category CMS^E could be called Cauchy-ω-complete.) The parallel with Banach's fixed point theorem is now clear: iterating F from the one point metric space yields (by F's contractivity) an ω-chain that is Cauchy. By Cauchy-completeness of CMS^E, this chain has a colimit, which is a fixed point of F.

Combining the results of this subsection now yields a proof of Theorem 4.7.

Theorem 4.7 Every locally contracting functor $F : CMS \to CMS$ has a fixed point.

Proof. Let $F : CMS \to CMS$ be locally contracting. By Definition 4.18, it can be extended to a functor $F^E : CMS^E \to CMS^E$, which is by Theorem 4.22 contracting. Thus F^E has a fixed point, by Theorem 4.23, which is also a fixed point of F, since both functors act identically on objects. □

Example 4.24 Let $+ : (CMS)^2 \to CMS$ be defined, for D and E, by

$$D + E \equiv \{0\} \times D + \{1\} \times E,$$

the disjoint union of D and E (with the disjoint sum of their metrics); on arrows $+$ is defined as usual. Let $I = \{0\}$ be the one-point metric space. Let the functor $\Omega : CMS \to CMS$ be defined by, for objects D,

$$\Omega(D) \equiv I + D$$

Next define Ω_ϵ, for some ϵ with $0 \leq \epsilon < 1$, by $\Omega_\epsilon \equiv id_\epsilon \circ \Omega$. It is easy to see that $+$ is locally continuous and thus Ω_ϵ is locally contracting. Hence, by Theorem 4.22, Ω_ϵ^E is a contracting functor. Starting in I and embedding I into $\Omega_\epsilon^E(I)$ by α_0, the above construction yields a chain

$$I \rightarrow^{\alpha_0} I + I \rightarrow^{\alpha_1} I + (I + I) \rightarrow^{\alpha_2} \ldots$$

The n-th element $(\Omega_\epsilon^E)^n(I)$ in this chain contains from left to right $n-1$ copies of 0, which will be called $0, 1, 2, \ldots, n-1$, respectively. Note that for $i, j \in (\Omega_\epsilon^E)^n(I)$ their distance is given by $d(i,j) = \epsilon^{\min\{i,j\}}$, whenever $i \neq j$. Let ∞ denote the colimit as constructed above; it looks like

$$\infty = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots, \omega\}$$

where, for all $n \geq 0$,

$$\mathbf{n} \equiv (0, 1, 2, \ldots, n-1, n, n, n, \ldots)$$

and

$$\omega \equiv (0, 1, 2, 3, \ldots)$$

From Corollary 4.9 it follows that ∞ is the unique fixed point of Ω_ϵ. □

5 Complete Partial Orders

Let $CPO_\perp$ be the category with complete partial orders $(D, \sqsubseteq_D)$ as objects and strict and continuous functions as arrows. For any two cpo's D and E, the set $\hom(D, E)$ of arrows between D and E is itself a cpo, with the usual order: for all $f, g \in \hom(D, E)$,

$$f \sqsubseteq g \equiv \forall x \in D,\ f(x) \sqsubseteq_E g(x).$$

Moreover composition of arrows is continuous with respect to this ordering. Therefore the category $CPO_\perp$ is called an order-enriched (or **O**-) category ([SP82]).

As in the previous section, the structure on hom sets can be used to characterize a class of functors.

Definition 5.1 A functor $F : CPO_\perp \rightarrow CPO_\perp$ is called *locally continuous* if, for any two objects $D, E \in CPO_\perp$, the mapping

$$F_{D,E} : \hom(D, E) \rightarrow \hom(F(D), F(E)) \qquad f \mapsto F(f)$$

is continuous. □

Next the subcategory CPO^E of $CPO_\perp$ is introduced. If D and E are cpo's and $\mu^e : D \rightarrow E$ and $\mu^p : E \rightarrow D$ are arrows in $CPO_\perp$ then $\langle \mu^e, \mu^p \rangle$ is called an *embedding-projection* pair from D to E provided that

$$\mu^p \circ \mu^e = id_D \text{ and } \mu^e \circ \mu^p \sqsubseteq_{\hom(E,E)} id_E.$$

Note that the one half of such projection pairs determines the other. Let CPO^E denote the subcategory of $CPO_\perp$ that has cpo's as objects and embedding-projection pairs as arrows. Note that also CPO^E is an order-enriched category. The following theorem is standard.

Theorem 5.2 Every $F : CPO_\perp \to CPO_\perp$ that is locally continuous can be extended to a functor $F^E : CPO^E \to CPO^E$ that is ω-continuous. A fixed point of F is obtained by constructing an initial F^E-algebra D in CPO^E.

The proof can be found in [SP82] and is similar to that for the metric case (since the latter mimics the original proof). Some parts of the proof are repeated next since they are needed later.

Let $D_0 \equiv \{\perp\}$ be the trivial one point cpo and let $\alpha_0 : D_0 \to F(D_0)$ be the unique arrow embedding D_0 into $F(D_0)$. Define an ω-chain $\Delta \equiv (D_n, \alpha_n)_n$ by putting $D_{n+1} \equiv F(D_n)$ and $\alpha_{n+1} \equiv F(\alpha_n)$, for $n \geq 0$. The direct (or projective) limit of Δ,

$$D \equiv \{(x_n)_n \mid \forall n \geq 0[x_n \in D_n \wedge \alpha_n^p(x_{n+1}) = x_n]\}$$

can be seen to be a cpo with order $\sqsubseteq_D$ on D given by, for all $(x_n)_n, (y_n)_n \in D$,

$$(x_n)_n \sqsubseteq_D (y_m)_m \equiv \forall n \geq 0,\ x_n \sqsubseteq_{D_n} y_n.$$

Now D can be turned into a cone $\mu : \Delta \to D$ with arrows $\mu_n : D_n \to D$, for all $n \geq 0$, as usual. The fact that F is locally continuous implies $\bigsqcup_n \mu_n^e \circ \mu_n^p = id_D$. By the initiality lemma for cpo's (which is similar to the one for metric spaces—see the previous section), D is a colimit for Δ. It follows that $D \cong F(D)$, say with $i : D \to F(D)$ as the isomorphism. It satisfies the following fact (which will be used below): for all $n \geq 0$,

$$F(\mu_n^p) \circ i = \mu_{n+1}^p.$$

It is not difficult to prove that (D, i^{-1}) is an initial F^E-algebra in CPO^E.

5.1 An 'Order-Theoretic' Final Coalgebra Theorem

The fixed point D constructed above is an initial F-algebra (D, i^{-1}) in the category CPO^E. Moreover, it can also be seen to be initial in $CPO_\perp$: the fact that D is a colimit (of its defining chain) in CPO^E implies, by a small exercise, that it is a colimit in $CPO_\perp$ as well; then the 'Basic Lemma', from [SP82], immediately yields the result. For completeness, a direct proof is given below.

By the so-called "limit-colimit coincidence" for **O**-categories, which is extensively discussed in [SP82], the dual of these facts also holds. Thus (D, i) is a final F-coalgebra in CPO^P, which is defined as the opposite category of CPO^E: $CPO^P \equiv (CPO^E)^{op}$. (Thus arrows in CPO^P are projections μ^p for which there exists a (unique) μ^e such that $\langle \mu^e, \mu^p \rangle$ is an embedding-projection pair.) Again, (D, i) is a final coalgebra in $CPO_\perp$ as well, which can be shown by dualizing the little argument above. For completeness, and because we have never seen this fact stated explicitly in the literature, a direct proof is given next. A minor variation will also prove that (D, i^{-1}) is an initial F-algebra in the category $CPO_\perp$. (A direct proof of the latter can be found in [Plo81a].)

Theorem 5.3 Let $F : CPO_\perp \to CPO_\perp$ be a locally continuous functor and let (D, i^{-1}) be the (in CPO^E) initial F-algebra as described above. Then (D, i) is a final F-coalgebra in $CPO_\perp$ and (D, i^{-1}) is an initial F-algebra in $CPO_\perp$.

Proof. First it is shown that (D, i) is a final F-coalgebra in $CPO_\perp$. Let (A, α) be any F-coalgebra. The existence of an arrow in $(CPO_\perp)_F$ from (A, α) to (D, i) can be established similarly to the metric case (Theorem 4.5): Define a function $\Phi : \hom(A, D) \to \hom(A, D)$ by, for all $f \in \hom(A, D)$,

$$\Phi(f) \equiv i^{-1} \circ F(f) \circ \alpha.$$

Since F is locally continuous, it follows that Φ is a continuous function. The existence of a least fixed point for Φ provides an arrow from (A, α) to (D, i).

The uniqueness of such an arrow has still to be demonstrated. (Recall that in the metric case—for locally contracting functors—existence and uniqueness are established simultaneously.) Consider two arrows f_1 and f_2 from (A, α) to (D, i):

$$\begin{array}{ccc} & \overset{f_1}{\underset{f_2}{\rightrightarrows}} & \\ A & & D \\ \alpha \downarrow & * & \downarrow i \\ & \underset{F(f_2)}{\overset{F(f_1)}{\rightrightarrows}} & \\ F(A) & & F(D) \end{array}$$

The equality of f_1 and f_2 is proved next. Let $(\mu_n : D_n \to D)_n$ be the cone used in the construction of D. It will be sufficient to prove, for all $n \geq 0$,

$$\mu_n^p \circ f_1 = \mu_n^p \circ f_2$$

because each of the following formulas implies the next one:

$$\mu_n^p \circ f_1 = \mu_n^p \circ f_2$$

$$\mu_n^e \circ \mu_n^p \circ f_1 = \mu_n^e \circ \mu_n^p \circ f_2$$

$$\bigsqcup_n \mu_n^e \circ \mu_n^p \circ f_1 = \bigsqcup_n \mu_n^e \circ \mu_n^p \circ f_2$$

$$f_1 = f_2$$

(The latter implication follows from the initiality lemma and the continuity of $\circ$.) Use induction on n. The case $n = 0$ is trivial because μ_0^p is the constant function $\lambda d.\ \perp$. Suppose next that $\mu_n^p \circ f_1 = \mu_n^p \circ f_2$. Then

$$\begin{aligned} \mu_{n+1}^p \circ f_1 &= \text{(by the fact stated at the end of Theorem 5.2)} \\ & \quad F(\mu_n^p) \circ i \circ f_1 \\ &= F(\mu_n^p) \circ F(f_1) \circ \alpha \\ &= F(\mu_n^p \circ f_1) \circ \alpha \end{aligned}$$

$$
\begin{aligned}
&= \text{(by the induction hypothesis)} \\
&\quad F(\mu_n^p \circ f_2) \circ \alpha \\
&= F(\mu_n^p) \circ F(f_2) \circ \alpha \\
&= F(\mu_n^p) \circ i \circ f_2 \\
&= \mu_{n+1}^p \circ f_2
\end{aligned}
$$

By a similar proof, (D, i^{-1}) can be shown to be an initial F-algebra in $CPO_\perp$. Existence of an arrow from (D, i^{-1}) to an arbitrary (A, α) is established by taking the least fixed point of a function $\Psi : \hom(D, A) \to \hom(D, A)$ defined by, for all $f \in \hom(D, A)$,

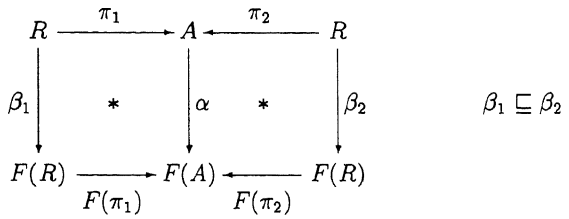
$$\Psi(f) \equiv \alpha \circ F(f) \circ i.$$

Uniqueness of such an arrow is proved as above, now using the fact that for all n, $\mu_{n+1}^e = i^{-1} \circ F(\mu_n^e)$.

5.2 Ordered F-Bisimulation

The order on hom sets makes the following generalization of the definition of F-bisimulation (Definition 2.2) possible.

Definition 5.4 Consider a functor $F : CPO_\perp \to CPO_\perp$ and let (A, α) be an F-coalgebra. A relation $R \subseteq A \times A$ is called an *ordered F-bisimulation* on (A, α) if there exist arrows $\beta_1 : R \to F(R)$ and $\beta_2 : R \to F(R)$ such that $\beta_1 \sqsubseteq \beta_2$, and the projections $\pi_1, \pi_2 : R \to A$ make both squares of the following diagram commute:



Note that the relation R should be an object in $CPO_\perp$. Thus it should be an ω-complete subset of $A \times A$ (that is, R should be closed under taking the least upper bound of ω-chains). The ordered F-bisimilarity relation is defined by

$$\sqsubseteq_F \equiv \bigcup \{ R \subseteq A \times A \mid \ R \text{ is an ordered } F\text{-bisimulation on } (A, \alpha) \}.$$

□

Example 5.5 *Divergence and partial bisimulation*
In [Abr91] transition systems with divergence are considered (see also [Mil80]). A labelled transition system *with divergence* is a four tuple $< S, A, \to, \uparrow >$ consisting of a set S of states, a set A of actions (or action labels), a transition relation $\to \subseteq S \times A \times S$, and a divergence set $\uparrow \subseteq S$. The interpretation of $s \in \uparrow$ (notation: $s \uparrow$) is that in the state s there is the possibility of divergence. Similarly $s \downarrow$ is used to indicate that s converges, that is, $s \notin \uparrow$.

Also labelled transition systems with divergence can be represented in terms of coalgebras: let $\mathcal{P}^0(A \times \text{-}) : CPO_\perp \to CPO_\perp$ be defined by, for all $< D, \sqsubseteq_D > \in CPO_\perp$,

$$\mathcal{P}^0(A \times D) \equiv \{\emptyset\} \cup \{X \subseteq (A \times D)_\perp \mid X \text{ is both Lawson and convex closed }\}$$

(where the ordering on $A \times D$ is determined by taking the discrete ordering on A, and the ordering on D.) Though formulated slightly differently—using the lifted version of the Cartesian product rather than sum—this is Abramsky's version of the standard Plotkin powerdomain, to which the empty set has been added. The ordering is such that the empty set is greater than the bottom element $\{\perp\}$, and incomparable to all other elements; non-empty sets are ordered as usual by, for all sets $X, Y \in \mathcal{P}^0(A \times D)$,

$$X \sqsubseteq Y \equiv X = \{\perp\} \vee X \sqsubseteq_{EM} Y,$$

where $\sqsubseteq_{EM}$ is the Egli-Milner order. Now any labelled transition system with divergence $< S, A, \rightarrow, \uparrow >$ can be represented as a coalgebra of the above functor by supplying S with the discrete order (define $S_\perp \equiv S \cup \{\perp_S\}$) and defining

$$\alpha : S_\perp \rightarrow \mathcal{P}^0(A \times S_\perp)$$

by $\alpha(\perp_S) \equiv \{\perp\}$ and, for all $s \in S$,

$$\alpha(s) \equiv \{< a, s' > \mid s \xrightarrow{a} s'\} \cup \{\perp \mid s \uparrow\}.$$

Following [Abr91], a relation $R \subseteq S \times S$ is called a *partial bisimulation* if, for all states $s, t \in S$ with sRt, and actions $a \in A$,

$$s \xrightarrow{a} s' \Rightarrow \exists t',\ t \xrightarrow{a} t' \ \wedge\ s'Rt'$$

and

$$s \downarrow \ \Rightarrow \ t \downarrow \ \wedge (t \xrightarrow{a} t' \Rightarrow \ \exists s',\ s \xrightarrow{a} s' \ \wedge\ s'Rt').$$

Similar to Example 2.3, it is shown next that these partial bisimulations correspond precisely to the ordered bisimulations of Definition 5.4 for the functor $\mathcal{P}^0(A \times \text{-})$.

Let $R \subseteq S \times S$ be a partial bisimulation. It can be seen to be an ordered $\mathcal{P}^0(A \times \text{-})$-bisimulation as follows. Define $T \subseteq S_\perp \times S_\perp$ by

$$T \equiv R \cup (\{\perp_S\} \times S_\perp).$$

Next define, for $i = 1, 2$, $\beta_i : T \rightarrow \mathcal{P}^0(A \times T)$ as follows. For $t \in S$, define

$$\beta_1((\perp_S, t)) \equiv \{\perp\},$$

$$\beta_2((\perp_S, t)) \equiv \{< a, (\perp_S, t') > \mid < a, t' > \in \alpha(t)\} \ \cup\ \{\perp \mid \perp \in \alpha(t)\}.$$

For $(s, t) \in R$, put

$$\begin{aligned} \beta_1((s,t)) \ &\equiv \ \{< a, (s', t') > \mid < a, s' > \in \alpha(s) \wedge \ < a, t' > \in \alpha(t) \ \wedge \ s'Rt'\} \\ &\cup \ \{\perp \mid \perp \in \alpha(s)\}, \end{aligned}$$

$$\begin{aligned} \beta_2((s,t)) \ &\equiv \ \{< a, (s', t') > \mid < a, s' > \in \alpha(s) \wedge \ < a, t' > \in \alpha(t) \ \wedge \ s'Rt'\} \\ &\cup \ \{< a, (\perp_S, t') > \mid \perp \in \alpha(s) \wedge \ < a, t' > \in \alpha(t)\} \\ &\cup \ \{\perp \mid \perp \in \alpha(t)\} \end{aligned}$$

It is readily checked that β_1 and β_2 are (monotonic and thus) continuous and satisfy the conditions of Definition 5.4. In particular, $\alpha \circ \pi_2 = \mathcal{P}^0(A \times \pi_2) \circ \beta_2$ because for all pairs $(s,t) \in R$ with $\perp \in \alpha(s)$, the set $\beta_2((s,t))$ contains elements $< a, (\perp_S, t') >$, for every $< a, t' > \in \alpha(t)$. This will ensure the presence of $< a, t' >$ in $\mathcal{P}^0(A \times \pi_2) \circ \beta_2((s,t))$, even if there exist no $s' \in S$ with $< a, s' > \in \alpha(s)$. (Similarly for $(\perp_S, t) \in T$.)

Conversely, every ordered $\mathcal{P}^0(A \times \text{-})$-bisimulation can be seen to correspond to a partial bisimulation: Let $R \subseteq S_\perp \times S_\perp$ be an ordered $\mathcal{P}^0(A \times \text{-})$-bisimulation. Define

$$T \equiv R \cap (S \times S)$$

and let $(s,t) \in T$. Suppose $s \xrightarrow{a} s'$. Then there exists $t' \in S$ such that $< a, (s',t') > \in \beta_1((s,t))$. Since $\beta_1 \sqsubseteq \beta_2$, also $< a, (s',t') > \in \beta_2((s,t))$. Thus $t \xrightarrow{a} t'$ and $s'Tt'$.

Next suppose $s \downarrow$. It follows from $\beta_1 \sqsubseteq \beta_2$ that $t \downarrow$. Suppose moreover that $t \xrightarrow{a} t'$. Then there exists $s' \in S$ such that $< a, (s',t') > \in \beta_2((s,t))$. It follows from $s \downarrow$ and $\beta_1 \sqsubseteq \beta_2$ that $< a, (s',t') > \in \beta_1((s,t))$. Thus $s \xrightarrow{a} s'$ and $s'Tt'$. □

Example 5.6 *Simulation*
The above definition of ordered F-bisimulation was motivated by [Pit92]. Ordered F-bisimulations can be equivalently defined as follows: Let $F : CPO_\perp \to CPO_\perp$ be a functor and let (A, α) be an F-coalgebra. Consider a relation $R \subseteq A \times A$ with projections π_1 and π_2 as usual. A relation $R^F \subseteq F(A) \times F(A)$ is defined by

$$R^F \equiv \{< F(\pi_1)(x_1), F(\pi_2)(x_2) > \mid x_1, x_2 \in F(R) \ \wedge \ x_1 \sqsubseteq_{F(R)} x_2\}.$$

Then R is an ordered F-bisimulation on (A, α) if and only if, for all $(a, a') \in A \times A$,

$$aRa' \Rightarrow \ \alpha(a) R^F \alpha(a').$$

Now, in this shape, ordered F-bisimulations can be easily seen to generalize the *simulations* (for the functorial case) of [Pit92]. □

5.3 Strong Extensionality in $CPO_\perp$

Because the definition of F-bisimulation has been generalized to that of ordered F-bisimulation, the fact that final F-coalgebras are strongly extensional is not immediate from Theorem 2.4. In fact, a somewhat stronger property can be proved (again referred to as strong extensionality):

Theorem 5.7 The initial fixed point (D, i) of a locally continuous functor $F : CPO_\perp \to CPO_\perp$ is strongly extensional; that is, for all $d, e \in D$,

$$d \sqsubseteq_D e \Leftrightarrow d \sqsubseteq_F e$$

(where $\sqsubseteq_F \equiv \bigcup\{R \subseteq D \times D \mid \ R \text{ is an ordered } F\text{-bisimulation on } (D,i)\ \}$).

Proof. The inclusion from left to right follows from the observation that $\sqsubseteq_D$ is an ordered F-bisimulation on D: First observe that $\sqsubseteq_D$, with the inherited order from $D \times D$, is a cpo. Next define $\Delta : D \to \sqsubseteq_D$ by , for all $d \in D$,

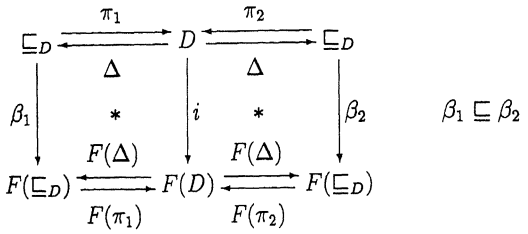
$$\Delta(d) \equiv < d, d >$$

and $\beta_1, \beta_2 : \sqsubseteq_D \to F(\sqsubseteq_D)$ by

$$\beta_1 \equiv F(\Delta) \circ i \circ \pi_1$$

$$\beta_2 \equiv F(\Delta) \circ i \circ \pi_2.$$

Then $\sqsubseteq_D$ is an ordered F-bisimulation on D with β_1 and β_2:



Conversely, let $R \subseteq D \times D$ be an ordered F-bisimulation with $\beta_1 \sqsubseteq \beta_2$. As usual, let π_1 and π_2 be the projections from R on D. We want to show $\pi_1 \sqsubseteq \pi_2$ (from which $R \subseteq \sqsubseteq_D$ follows). The proof is very similar to that of Theorem 5.3. Let $(\mu_n : D_n \to D)_n$ be the cone used in the construction of D. It will be sufficient to prove, for all $n \geq 0$,

$$\mu_n^p \circ \pi_1 \sqsubseteq \mu_n^p \circ \pi_2$$

because (as in Theorem 5.3) each of the following formulas implies the next one:

$$\mu_n^p \circ \pi_1 \sqsubseteq \mu_n^p \circ \pi_2$$

$$\mu_n^e \circ \mu_n^p \circ \pi_1 \sqsubseteq \mu_n^e \circ \mu_n^p \circ \pi_2$$

$$\bigsqcup_n \mu_n^e \circ \mu_n^p \circ \pi_1 \sqsubseteq \bigsqcup_n \mu_n^e \circ \mu_n^p \circ \pi_2$$

$$\pi_1 \sqsubseteq \pi_2$$

(The latter implication follows from the initiality lemma and the continuity of $\circ$.)

Use induction on n. The case $n = 0$ is trivial because μ_0^p is the constant function $\lambda d.\ \bot$. Suppose next that $\mu_n^p \circ \pi_1 \sqsubseteq \mu_n^p \circ \pi_2$. Then $\mu_{n+1}^p \circ \pi_1 \sqsubseteq \mu_{n+1}^p \circ \pi_2$ is proved as follows:

$$\mu_n^p \circ \pi_1 \sqsubseteq \mu_n^p \circ \pi_2$$

implies

$$F(\mu_n^p) \circ F(\pi_1) \sqsubseteq F(\mu_n^p) \circ F(\pi_2)$$

because F is a locally (continuous and thus) monotonic functor. Since $\beta_1 \sqsubseteq \beta_2$ this implies

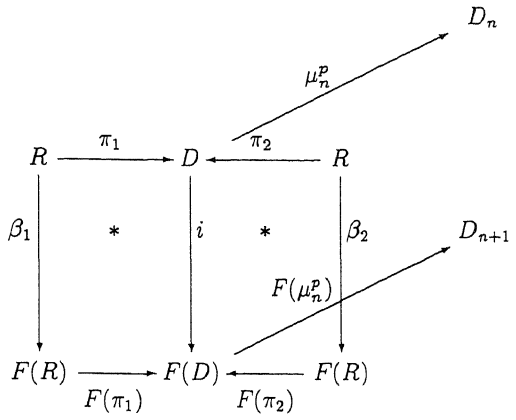
$$F(\mu_n^p) \circ F(\pi_1) \circ \beta_1 \sqsubseteq F(\mu_n^p) \circ F(\pi_2) \circ \beta_2$$

Using the commutativity properties of β_1 and β_2, it follows that

$$F(\mu_n^p) \circ i \circ \pi_1 \sqsubseteq F(\mu_n^p) \circ i \circ \pi_2.$$

Finally the fact stated at the end of Theorem 5.2 yields

$$\mu^{\mathcal{P}}_{n+1} \circ \pi_1 \sqsubseteq \mu^{\mathcal{P}}_{n+1} \circ \pi_2.$$



□

Corollary 5.8 Let us call an ordered F-bisimulation R on (D, i) *symmetric* if $\beta_1 = \beta_2$. Define $\overset{F}{\sim} \equiv \bigcup\{R \subseteq D \times D \mid R \text{ is a symmetric } F\text{-bisimulation on } (D,i)\}$. For all $d, e \in D$,

$$d \overset{F}{\sim} e \Leftrightarrow d = e.$$

Example 5.5 (continued) The fact that the initial fixed point of the functor $\mathcal{P}^0(A \times \text{-}) : CPO_\perp \to CPO_\perp$ is "internally fully abstract"—Proposition 3.10 of [Abr91]—follows from Theorem 5.7 and the observation that this functor is locally continuous.

Example 5.6 (continued) The extensionality results of [Pit92] (for the functorial case) can all be obtained as instantiations of Theorem 5.7.

6 Conclusion

The final coalgebra theorems discussed in this paper show that standard domain constructions are in fact final coalgebra constructions. A more categorical approach could be taken in the sense that only categorical properties, like the existence of colimits, would be taken into account in the construction of final coalgebras.

Recall that algebras and coalgebras can be regarded as abstractions of the notions of pre- and post-fixed points, respectively. It would then be natural to look for a generalization of the following standard fixed point theorems from lattice theory:

Let $\mathcal{L} = (L, \leq)$ be a complete lattice, with $\perp$ and $\top$ as bottom and top elements, and $\coprod$ and $\prod$ as join and meet operators. Let $f : \mathcal{L} \to \mathcal{L}$ be a monotone function and consider the following chains:

$$\perp \leq f(\perp) \equiv f\uparrow 1 \leq f^2(\perp) \equiv f\uparrow 2 \leq \cdots \leq \coprod_{n<\omega} f\uparrow n \equiv f\uparrow\omega \leq f\uparrow\omega + 1 \leq \cdots \qquad (6)$$

$$\top \geq f(\top) \equiv f{\downarrow}1 \geq f^2(\top) \equiv f{\downarrow}2 \geq \cdots \geq \coprod_{n<\omega} f{\downarrow}n \equiv f{\downarrow}\omega \geq f{\downarrow}\omega + 1 \geq \cdots \tag{7}$$

Then the least and the greatest fixed point (w.r.t. $\leq$) of f are

$$\coprod_{\alpha \in On} f{\uparrow}\alpha \qquad \text{and} \qquad \prod_{\alpha \in On} f{\downarrow}\alpha.$$

The generalization of the above theorem from least fixed points to initial algebras has already been worked out in [AK79]. Lattices (as pre-orders) generalize to categories, bottom elements to initial objects, monotone functions to endofunctors, least upper bounds to colimits. One has then the following diagram:

$$0 \xrightarrow{!} F(0) \xrightarrow{F(!)} F^2(0) \xrightarrow{F^2(!)} \cdots \longrightarrow \mathrm{Colim}_{n<\omega} F^n(0) = F^\omega \xrightarrow{!} F(F^\omega)) \longrightarrow \cdots \tag{8}$$

Here the fact is used that a unique arrow (denoted by '!') exists from the initial object to any other object of the category, and from a colimit of a diagram to any other cone over that diagram. In [AK79] conditions are given for the existence of an ordinal at which this construction stops and then shown that it yields an initial F-algebra.

A dual result would then be phrased in terms of final objects and limits, generalizing top elements and greatest lower bounds:

$$1 \xleftarrow{!} F(1) \xleftarrow{F(!)} F^2(1) \xleftarrow{F^2(!)} \cdots \longleftarrow \mathrm{Lim}_{n<\omega} F^n(1) = F^\omega \xleftarrow{!} F(F^\omega) \longleftarrow \cdots \tag{9}$$

This has not been fully investigated so far, although a 'schematological' approach to domain equations as in (9) is sketched in [Abr88].

A more abstract approach is taken in [Bar91] when dealing with the existence of final coalgebras in the category *Set* of sets (it is not immediately clear whether standard set theory or just basic set theory is assumed there). The existence of final coalgebras in such category is proved for a certain class of functors F (so-called *accessible*) by showing that the evident forgetful functor from the category of F-coalgebras Set_F to the category *Set* has a right adjoint. Moreover, if the functor F preserves limits of countable chains (i.e., it is ω-*continuous*) then this final coalgebra is the limit of the chain

$$1 \xleftarrow{!} F(1) \xleftarrow{F(!)} F^2(1) \xleftarrow{F^2(!)} \cdots \xleftarrow{F^n(!)} F^n(1) \xleftarrow{F^{n+1}(!)} F^{n+1}(1) \longleftarrow \cdots \tag{10}$$

where 1 is an arbitrary one element set (indeed final object in *Set*). In the same paper it is shown that, under the further assumption that $F(\emptyset) \neq \emptyset$, the final F-coalgebra is the Cauchy completion of the initial F-algebra.

As already mentioned in the section about non-standard set theory, the existence of final coalgebras in the category *Class* of classes over basic set theory has been proved in [AM89]. Also there the construction is of a categorical nature, but of a different character. It amounts to a "quotient construction": given a notion of F-congruence (of which F-bisimulation is a special case) the final F-coalgebra is obtained by taking the quotient under the (existing) maximal F-congruence of the (disjoint) union of all small F-coalgebras. A quotient construction is also carried out in [Bar91].

6.1 A Comparative Analysis

To come back to the constructions discussed in this paper, they can be regarded as instances of (9) and (10) (and even (8)). The construction in $CPO_\perp$ is the one which better fits into those schemata. By instantiating (10) in $CPO_\perp$, where the final object is $\{\perp\}$, and taking F to be a locally continuous endofunctor, one obtains a diagram which is both in $CPO_\perp$ and in CPO^P, the subcategory having projections as arrows. The latter category can be considered as a cpo itself and this structure can be used in order to find that the limit of that diagram is a final F-coalgebra in CPO^P. As indicated at the beginning of Section 4.1 a 'lifting lemma' can be proved which ensures that limits of ω-chains in CPO^P are limits in $CPO_\perp$ as well. By applying the dual of the Basic Lemma from [SP82] it follows that this limit is a final F-coalgebra in $CPO_\perp$.

Notice that the final object in $CPO_\perp$ is a final object in CPO^P as well. Moreover it is an initial object in both $CPO_\perp$ and CPO^E, the category of embeddings which is dual to CPO^P. This duality arises from an adjunction between the embedding and the projection in an embedding-projection pair. It implies that the dual of the diagram in CPO^P is a diagram in CPO^E with reversed arrows, which has as *colimit* the *limit* of the original diagram in CPO^P. A lifting lemma can be applied also to CPO^E so that initial and final coalgebras of a locally continuous endofunctor coincide. (See Theorem 5.3.)

For CMS there is a similar passage from the original category to a subcategory of embedding-projection pairs. However, the adjunction property between embeddings and projections which holds in $CPO_\perp$ is not available here. Therefore, the limit-colimit coincidence does not hold in this setting. The category CMS^P of projections can be defined as the subcategory of CMS with as arrows those non-expansive mappings which have a right inverse. This right inverse is an embedding (not unique!) making f part of an embedding-projection pair. Notice that singleton sets are final objects both in CMS and in CMS^P. Instantiating diagram (10) to CMS yields, for every locally contracting endofunctor F, a diagram in CMS^P whose limit is a limit in CMS as well. Although initial and final objects in CMS do not coincide and, more in general, the limit-colimit coincidence does not apply, in CMS final coalgebras are initial algebras as well.

For $Class^*$ the situation is rather different. The limit is still taken in a subcategory, but this is not a category of embedding-projection pairs. It is rather the subcategory, say $Class^I$, having inclusion mappings as arrows (and therefore with the extra structure of a lattice). The final object (top element) is the universe V, which is clearly not final in $Class^*$, while the initial object is the empty set, which is also initial in $Class^*$. Set-continuous functors have both a final coalgebra (greatest fixed point) and an initial algebra (least fixed point) in $Class^I$. These will in general be distinct (in contrast to what happens in $CPO_\perp$ and CMS). Set-continuous functors are not ω-continuous, hence these constructions cannot be seen as instances of (10) and its dual, but rather of (9) and (8) (as well as of (7) and (6)). Now, for functors which preserve inclusions, the initial algebra in $Class^I$ is also an initial algebra in $Class^*$. For final coalgebras an extra requirement is needed, namely that the functor be uniform on maps as well. This asymmetry has to be better understood.

6.2 Final Semantics (continued)

As suggested by the title and mentioned in the introduction, this paper is meant to provide a basis to a final coalgebra semantics. Two distinctive features of such semantics are the definition of *semantic mappings as final arrows* (which implies that the domain itself is final) and the use of coalgebras in order to express the *structure to be preserved* under (not necessarily semantic) transformations.

Semantic mappings as arrows into a final object are not an exclusive feature of final coalgebra semantics, apart from the fact that, as already mentioned, several semantics in the style of [BZ82] can be seen as final coalgebra semantics. For instance, in [Abr91] there is a 'Final algebra theorem' which says that the given semantic mapping associated to a specific domain for bisimulation is the unique morphism (in which category?) from a transition system into the domain, the latter regarded as a transition system itself. Here, 'algebra' presumably stands for the *Lindenbaum algebra* which is associated with a certain *domain logic* introduced in that paper. The definition of that semantic mapping makes use of the fact that that domain is the *Stone dual* of the finitary fragment of such logic. (By the way, the fact that the same (final) semantic construction in [Abr91] for *CCS*-like languages has been carried out in [Abr90] for the *lazy lambda-calculus* makes it plausible that final coalgebra semantics might be given to applicative languages as well.)

However, in the above as well as in other examples the recognized finality of the domain is not systematically exploited, except for the final coalgebra semantics for *CCS* given in [Acz88]. As mentioned in the introduction, in the forthcoming paper *Observations as Functors* other instances of final coalgebra semantics will be given, starting from the idea that observations can be formalized as functors. Other equivalences than bisimulation will be treated, like, for instance, trace equivalence. The coalgebraic approach will give a particular insight into the problem of *full abstraction* and other issues related to compositionality (see also below).

Notice that the specific domain defined in [Abr91] as an initial algebra not only is recognized there to be a final transition system, but also indicated to be a final coalgebra as a consequence of the limit-colimit duality. The latter has been used also in [Smy92] to prove that, for so-called *information categories* (general order-theoretic frameworks for solving domain equations) and suitable endofunctors over them, initial algebras and final coalgebras coincide. Finally, it should be mentioned that an early reference to finality as a definition method for semantic mappings can be found in [Ole82].

Consider now the other distinctive characteristic of final coalgebra semantics mentioned above. An extra coalgebraic structure is added to programs (as a function from programs to their observable computations) and arrows from the coalgebra associated with a program are transformations which preserve this extra structure — together with the information contained in it. Part of this information is, for instance, F-bisimilarity, which is indeed preserved by (certain) arrows between F-coalgebras. This addition of a categorical structure, together with its preservation under transformation, again is not exclusive of final coalgebra semantics. Another example of such an approach is the classical initial Σ-algebra semantics. The extra algebraic structure is used there in order to preserve the operators (of the signature Σ of the language) under transformation. The semantic mapping is again a unique arrow, only it is initial, instead of final: it is the unique arrow from the programs regarded as the (free and thus) initial Σ-algebra into the

chosen domain. Since operators are preserved by transformations, the semantic will be by definition compositional. The problem there is to define suitable semantic operators, that is, to turn the domain into a suitable Σ-algebra.

The issue of defining semantic operators within the context of final coalgebra semantics has already been treated in [Acz88]. There, the finality of the domain is exploited for defining semantic operators for *CCS*, but by means of a rather ad hoc construction. Instead, in the forthcoming paper *Observations as functors*, a systematic method for deriving semantic operators from transition system specifications given in [Rut92] is rephrased in terms of final coalgebra semantics. This amounts to deriving a Σ-algebra for the domain by means of finality properties. It can be then proved that the original final semantics is compositional if and only if it coincides with the initial Σ-algebra semantics associated to that construction, which is also unique, but now w.r.t. a different category.

As already mentioned, the categories of F-coalgebras considered in this paper are not the standard ones in category theory. Usually, the endofunctor F is to be part of a *comonad* and the arrows between F-coalgebras have to preserve also this extra comonadic structure. Semantics by means of comonads has been investigated in [BG91]. (But see also [Mog89] for semantics in terms of the dual notion — *monads*.) It would be interesting to understand whether some connections can be established with that work.

6.3 Coinduction

For F-algebras the following induction principle can be easily proved: let (A, α) be an initial F-algebra and let (B, β) be any F-algebra. If $\pi : (A, \alpha) \to (B, \beta)$ is a mapping between F-algebras and π is monic (the category-theoretical generalization of injective), then π is an isomorphism. An immediate consequence is, for instance, the induction principle for natural numbers (viewed as initial algebra of a suitably chosen functor). (E.g., see [Plo81a] and [LS81].) The dualization of the induction principle yields what could be called a *coinduction* principle for final F-coalgebras: let (A, α) be a final F-coalgebra and let (B, β) be any F-coalgebra. If $\pi : (B, \beta) \to (A, \alpha)$ is a mapping between F-algebras and π is epic (the generalization of surjective), then π is an isomorphism. (See also [Smy92].) In [MT91], this principle is used in the basic case where the category under consideration is a lattice and the functor F a monotonic operation.

At the same time, the fact that an F-coalgebra (A, α) is final implies the principle of strong extensionality (stating that on (A, α) equality and F-bisimulation coincide— Theorem 2.4). (See also the remark about [Pit92] in Example 5.6.) And for many functors it is possible to deduce from the principle of strong extensionality the coinduction principle mentioned above. In a forthcoming paper, these different formulations of coinduction will be compared.

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