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Finding Disjoint Trees in Planar Graphs in Linear Time

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ABSTRACT. We show that for each fixed k there exists a linear-time algorithm for the problem: *given:* an undirected plane graph $G = (V, E)$ and subsets $X_1, \ldots, X_p$ of V with $|X_1 \cup \cdots \cup X_p| \leq k$; *find:* pairwise vertex-disjoint trees $T_1, \ldots, T_p$ in G such that T_i covers X_i $(i = 1, \ldots, p)$.

1. Introduction

Consider the following *disjoint trees problem*:

given: an undirected graph $G = (V, E)$ and subsets $X_1, \ldots, X_p$ of V;

find: pairwise vertex-disjoint trees $T_1, \ldots, T_p$ in G such that T_i covers X_i $(i = 1, \ldots, p)$.

(We say that tree T_i *covers* X_i if each vertex in X_i is a vertex of T_i.)

Robertson and Seymour [5] gave an algorithm for this problem that runs, for each fixed k, in time $O(|V|^3)$ for inputs satisfying $|X_1 \cup \cdots \cup X_p| \leq k$. (Recently, Reed gave an improved version with running time $O(|V|^2 \log |V|)$.) In this paper we show that if we moreover restrict the input graphs to planar graphs there exists a *linear-time* algorithm:

THEOREM. *There exists an algorithm for the disjoint trees problem for planar graphs that runs, for each fixed k, in time $O(|V|)$ for inputs satisfying*

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$|X_1 \cup \cdots \cup X_p| \leq k$.

If we do not fix an upper bound k on $|X_1 \cup \cdots \cup X_p|$, the disjoint trees problem is NP-hard (D.E. Knuth, see [1]), even when we restrict ourselves to planar graphs and each X_i is a pair of vertices (Lynch [2]).

Our result extends a result of Suzuki, Akama, and Nishizeki [7] stating that the disjoint trees problem is solvable in linear time for planar graphs for each fixed upper bound k on $|X_1 \cup \cdots \cup X_p|$, when

(1) there exist two faces f_1 and f_2 such that each vertex in $X_1 \cup \cdots \cup X_p$ is incident with at least one of f_1 and f_2.

(In fact, they showed more strongly that the problem (for nonfixed k) is solvable in time $O(k|V|)$. Indeed, recently Ripphausen, Wagner, and Weihe [4] showed that it is solvable in time $O(|V|)$.)

Equivalent to a linear-time algorithm for the disjoint trees problem (for fixed k) is one for the following "realization problem". Let $G = (V, E)$ be a graph and let $X \subseteq V$. For any $E' \subseteq E$ let $\Pi(E')$ be the partition $\{K \cap X | K$ is a component of the graph (V, E') with $K \cap X \neq \emptyset\}$ of X. We say that E' *realizes* Π if $\Pi = \Pi(E')$. We call a partition of X *realizable in* G if it is realized by at least one subset E' of E. Now the *realization problem* is:

given: a graph $G = (V, E)$ and a subset X of V;

find: subsets $E_1, \ldots, E_N$ of E such that each realizable partition of X is realized by at least one of $E_1, \ldots, E_N$.

We give an algorithm for the realization problem for planar graphs that runs, for each fixed k, in time $O(|V|)$ for inputs satisfying $|X| \leq k$. In [3] we extend this result to graphs embedded on any fixed compact surface.

2. Realizable partitions

We will use the following lemma of Robertson and Seymour [6], saying that any vertex that is "far away" from X and also is not on any "short" curve separating X, is irrelevant for the realization problem and can be left out from the graph.

Let $G = (V, E)$ be a plane graph (that is, a graph embedded in the plane $\mathcal{R}^2$). For any curve C on $\mathcal{R}^2$, the *length* length(C) of C is the number of times C meets G (counting multiplicities). We say that a curve C *separates* a subset X of $\mathcal{R}^2$ if X is contained in none of the components of $\mathcal{R}^2 \setminus C$. (So C separates X if C intersects X.)

LEMMA. *There exists a computable function $g : \mathcal{N} \longrightarrow \mathcal{N}$ with the following property. Let $G = (V, E)$ be a plane graph, let $X \subseteq V$ and let $v \in V$ be such that each closed curve C traversing v and separating X satisfies* length$(C) \geq g(|X|)$; *then each partition of X realizable in G is also realizable in $G - v$.*

[$G - v$ is the graph obtained from G by deleting v and all edges incident with v.]

Moreover, we use the following easy proposition, enabling us to reduce the realization problem to smaller problems.

PROPOSITION 1. *Let $G = (V, E)$ be an undirected graph and let $X \subseteq V$. Moreover, let $V_1, \ldots, V_n, Y$ be subsets of V such that*

(2) (i) *each edge of G is contained in at least one of $V_1, \ldots, V_n$;*
(ii) *$X \subseteq Y$ and $V_i \cap V_j \subseteq Y$ for each $i, j \in \{1, \ldots, t\}$ with $i \neq j$.*

Let $E_{i,1}, \ldots, E_{i,N_i}$ form a solution for the realization problem with input $\langle V_i \rangle, V_i \cap Y$ $(i = 1, \ldots, n)$. Then the sets $E_{1,j_1} \cup \cdots \cup E_{n,j_n}$, where j_i ranges over $1, \ldots, N_i$ (for $i = 1, \ldots, n$), form a solution for the realization problem with input G, X.

[$\langle W \rangle$ denotes the subgraph of G induced by W.]

3. Proof of the theorem

We show that, for each fixed k, there exists a linear-time algorithm for the realization problem for plane graphs $G = (V, E)$ and subsets X of V with $|X| \leq k$. We may assume that G is connected.

For any subset W of V let $\delta(W)$ be the set of vertices in W that are adjacent to at least one vertex in $V \setminus W$. Let $W^o := W \setminus \delta(W)$.

Let H be the graph with vertex set V, where two vertices v, v' are adjacent if and only if there exists a face of G that is incident with both v and v'. For any subset W of V, let $\kappa(W)$ denote the number of components of the subgraph of H induced by W. Note that $\kappa(W)$ can be computed in linear time.

We say that W is *linked* if $\kappa(W) = 1$. Observe that if $W \neq \emptyset$ then

(3) W is linked if and only if G does not contain a circuit C splitting W.

Here we say that C *splits* W if C does not intersect W and $\emptyset \neq W \cap \text{int}C \neq W$, where int$C$ denotes the (open) area of $\mathcal{R}^2$ enclosed by C.

We apply induction on $\kappa(X)$. If $\kappa(X) \leq 2$, the problem can be reduced to one satisfying (1). Indeed, if $\kappa(X) = 2$ we can find in linear time a collection F of faces of G such that the subspace $X \cup \bigcup_{f \in F} f$ of $\mathcal{R}^2$ has two connected components and such that $|F| \leq |X|$. Choose two faces $f, f' \in F$ and a vertex $v \in X$ incident with both f and f'. "Open" the graph at v, by splitting v into two new vertices, joining f and f' to form one new face. After this is repeated $|F| - 3$ times, the faces in F are replaced by two faces f_1 and f_2 and the vertices in X are split (or not) to a set X' of $|X| + |F| - 2$ vertices, such that each vertex in X' is incident with f_1 or f_2. By the result of Suzuki, Akama, and Nishizeki [7] we can solve the realization problem for the new graph and X' in linear time. This directly gives a solution for the realization problem for the original realization problem. We proceed similarly if $\kappa(X) = 1$.

If $\kappa(X) > 2$ we proceed as follows. Let $X_1, \ldots, X_t$ be the components of the subgraph of H induced by X. (So $t = \kappa(X) \leq k$.) We may assume that $\delta(X_i) = X_i$ for each $i = 1, \ldots, t$ (by attaching to each vertex in X_i a new vertex of valency 1). Let p be a nonnegative integer. A *p-neighbourhood* is a collection $W_1, \ldots, W_t$ of pairwise disjoint linked subsets of V with the following properties:

(4) (i) for $i = 1, \ldots, t$, $W_i \supseteq X_i$, and if $W_i \neq X_i$ then $|\delta(W_i)| = p$

(ii) for all distinct $i, j \in \{1, \ldots, t\}$, there are p vertex-disjoint paths in G between W_i and W_j.

We note:

PROPOSITION 2. *Let $W_1, \ldots, W_t$ be a p-neighbourhood. Let $i, j \in \{1, \ldots, t\}$ be distinct, and let T be a set of vertices intersecting each path from W_i to W_j such that $|T| = p$. Then T is linked.*

Proof. Suppose not. Let C be a circuit in G splitting T. Let U_i and U_j be the sets of vertices that can be reached from W_i and W_j, respectively, without intersecting T. So $U_i \cap U_j = \emptyset$. Then $U_i \cap C = \emptyset$ or $U_j \cap C = \emptyset$, since otherwise all vertices in C belong both to U_i and U_j. We may assume that $U_i \cap C = \emptyset$. Hence we may assume moreover that U_i is contained in int$C$ (as U_i is linked). Then each path from W_i to W_j intersects $T \cap \text{int}C$, contradicting the facts that there exist p disjoint such paths and that $|T \cap \text{int}C| < |T| = p$. ∎

In particular, $\delta(W_i)$ is linked for all i. (If $W_i = X_i$ then $\delta(W_i) = \delta(X_i) = X_i$.)

Call a p-neighbourhood $W_1, \ldots, W_t$ *maximal* if for each $i = 1, \ldots, t$ and for each linked U satisfying $W_i \subset U \subseteq V \setminus \bigcup_{j \neq i} W_j$ one has $|\delta(U)| > p$.

First we describe an algorithm which, given a p-neighbourhood $W_1, \ldots, W_t$, finds a maximal p-neighbourhood:

1. Choose $i \in \{1, \ldots, t\}$. Determine an inclusionwise maximal set U satisfying $W_i \subseteq U \subseteq V \setminus \bigcup_{j \neq i} W_j$ and $|\delta(U)| = p$. Replace W_i by U. If no such U exists, we leave W_i unchanged.

2. Repeat for all $i \in \{1, \ldots, t\}$ in turn. This gives a maximal p-neighbourhood.

Note that by Proposition 2, $\delta(U)$ in Step 1 is linked, and hence U is linked. Note moreover that Step 1 can be performed in time $O(p|V|)$ with the Ford-Fulkerson augmenting path method (one augmenting path can be found in time $O(|V|)$). See also [4].

Second we give an algorithm which, given a maximal p-neighbourhood, finds either a $p+1$-neighbourhood or a reduction for the realization problem:

1. If there exist $i \neq j$ and a vertex v such that both $W_i \cup \{v\}$ and $W_j \cup \{v\}$ are linked, apply Proposition 1 to $V_1 := W_i \cup \{v\}, V_2 := W_j \cup \{v\}, V_3 := V \setminus (W_i^o \cup W_j^o)$ and $Y := X \cup \delta(W_i) \cup \delta(W_j) \cup \{v\}$.

Otherwise, for each $i = 1, \ldots, t$ with $|\delta(W_i)| = p$, choose a vertex $v_i \in V \setminus W_i$ such that $W_i \cup \{v_i\}$ is linked, and let $U_i := W_i \cup \{v_i\}$; for all other i let $U_i := W_i$.

2. If there exist $i \neq j$ such that there do not exist $p+1$ disjoint paths connecting U_i and U_j, find a subset U of V such that $U_i \subseteq U, U_j \subseteq U' := V \setminus U^o$ and $|\delta(U)| = p$. Apply Proposition 1 to $V_1 := W_1, \ldots, V_t := W_t, V_{t+1} := (U \setminus (W_1^o \cup \cdots \cup W_t^o)) \cup \delta(U), V_{t+2} := (U' \setminus (W_1^o \cup \cdots \cup W_t^o)) \cup \delta(U)$ and $Y := X \cup \delta(W_1) \cup \cdots \cup \delta(W_t) \cup \delta(U)$.

3. Otherwise, $U_1, \ldots, U_t$ form a $p+1$-neighbourhood.

PROPOSITION 3. *In Step 1, if there exist i and j as stated, then $\kappa(V_h \cap Y) < t$ for $h = 1, 2, 3$.*

Proof. Without loss of generality, $i = 1$ and $j = 2$. We have $\kappa(V_1 \cap Y) = \kappa(X_1 \cup \delta(W_1) \cup \{v\}) \leq 2 < t$, since both X_1 and $\delta(W_1) \cup \{v\}$ are linked. Similarly, $\kappa(V_2 \cap Y) \leq 2 < t$.

Finally, $\kappa(V_3 \cap Y) < t$, since $V_3 \cap Y = X_3 \cup \cdots \cup X_t \cup \delta(W_1) \cup \delta(W_2) \cup \{v\}$, where $X_3, \ldots, X_t$ and $\delta(W_1) \cup \delta(W_2) \cup \{v\}$ are linked (as $\delta(W_1) \cup \{v\}$ and $\delta(W_2) \cup \{v\}$ are linked). ∎

PROPOSITION 4. *Let $A, B \subseteq V$ such that $\delta(A)$ and $\delta(B)$ are linked, and such that $B \not\subseteq A^o$ and $A^o \cup B^o \neq VG$. Then $\delta(A) \cup (A \cap \delta(B))$ is linked.*

Proof. Suppose $\delta(A) \cup (A \cap \delta(B))$ is not linked. Let C be a circuit in G splitting $\delta(A) \cup (A \cap \delta(B))$. Since $\delta(A)$ is linked, we may assume that $\delta(A) \subset \mathrm{int}C$. Since C splits $\delta(A) \cup (A \cap \delta(B))$, we know that there are vertices in $A \cap \delta(B)$ that are in the exterior of C.

Since G is connected, there exists a path in G from a vertex in A in the exterior of C to a vertex of C disjoint from $\delta(A)$, and hence C intersects A. Therefore, $VC \subseteq A$. Hence every vertex of G in the exterior of C belongs to A. As $\delta(B)$ is linked and as $\delta(B)$ does not intersect C (because $A \cap \delta(B)$ does not intersect C), we have that $\delta(B)$ is contained in the exterior of C. As $B \not\subseteq A^o$ this implies that each vertex in $\mathrm{int}C$ is contained in B. So $A^o \cup B^o = VG$, contradicting the assumption. ∎

PROPOSITION 5. *In Step 2, if there exist i and j as stated, then $\kappa(V_h \cap Y) < t$ for $h = 1, \ldots, t+2$.*

Proof. Without loss of generality, $i = 1$ and $j = 2$. By the maximality of W_1 we know that U intersects at least one of $W_2, W_3 \ldots, W_t$. So U intersects at least two of $W_1, \ldots, W_t$. Similarly, U' intersects at least two of $W_1, \ldots, W_t$.

For each $h = 1, \ldots, t$ we have $\kappa(V_h \cap Y) \leq 2 < t$, since $V_h \cap Y = X_h \cup \delta(W_h) \cup (W_h \cap \delta(U))$ and since $\delta(W_h) \cup (W_h \cap \delta(U))$ is linked by Proposition 4. (Note that $U \not\subseteq W_h^o$ since U intersects at least two of $W_1, \ldots, W_t$, and that $U^o \cup W_h^o \neq VG$ since U' intersects at least two of $W_1, \ldots, W_t$.)

Next we show $\kappa(V_{t+1} \cap Y) < t$. Note that $V_{t+1} \cap Y = \delta(U) \cup (U \cap (\delta(W_1) \cup \cdots \cup \delta(W_t)))$. Since U' intersects at least two of $W_1, \ldots, W_t$, it suffices to show that if U' intersects W_h then $\delta(U) \cup (U \cap \delta(W_h))$ is linked.

Suppose U' intersects W_h and $\delta(U) \cup (U \cap \delta(W_h))$ is not linked. As $\delta(U)$ and $\delta(W_h)$ are linked (by Proposition 2), Proposition 4 implies that $W_h \subseteq U^o$ or $W_h^o \cup U^o = VG$. However, $W_h \subseteq U^o$ contradicts the fact that W_h intersects U'. Moreover, $W_h^o \cup U^o = VG$ contradicts the fact that there is another $W_{h'}$ intersecting U'.

This shows $\kappa(V_{t+1} \cap Y) < t$. Similarly, $\kappa(V_{t+2} \cap Y) < t$. ∎

Finally we give the algorithm which finds a reduction:

> Starting with the 0-neighbourhood $X_1, \ldots, X_t$, for $p = 0, 1, .., 2g(k) - 1$ apply the above algorithms to find a reduction or a $2g(k)$-neighbourhood.
>
> If we find a $2g(k)$-neighbourhood $W_1, \ldots, W_t$, then for all distinct $i, j \in \{1, \ldots, t\}$, find a shortest path $P_{i,j}$ in H between W_i and W_j. Among all $P_{i,j}$ choose one, $P := P_{1,2}$ say, of minimum length.
>
> If length$(P) > 2g(k)$, delete from G all vertices of P except the first $g(k)$ and the last $g(k)$. If length$(P) \leq 2g(k)$ leave G unchanged. Call the new graph G'.
>
> Let R be the set of vertices in P that are not deleted. Apply Proposition 1 to G' and $V_1 := W_1, V_2 := W_2, V_3 := V \setminus (W_1^o \cup W_2^o)$ and $Y := X \cup \delta(W_1) \cup \delta(W_2) \cup R$.

PROPOSITION 6. *In G', $\kappa(V_h \cap Y) < t$ for $h = 1, 2, 3$.*

Proof. $\kappa(V_1 \cap Y) = \kappa(X_1 \cup \delta(W_1)) \leq 2 < t$. Similarly, $\kappa(V_2 \cap Y) < t$. Finally, $\kappa(V_3 \cap Y) = \kappa(X_3 \cup \cdots \cup X_t \cup \delta(W_1) \cup \delta(W_2) \cup R) < t$ since $\delta(W_1) \cup \delta(W_2) \cup R$ is linked. ∎

PROPOSITION 7. *Deleting the vertices does not affect realizability.*

Proof. Let Q be the set of vertices deleted. We must show that for any vertex $v \in Q$, any closed curve C traversing v and separating X has at least $g(k)$ intersections with $G - (Q \setminus \{v\})$ (since it means by the lemma that we can delete v, even after having deleted all other vertices in Q). In other words, any closed curve in $\mathcal{R}^2$ intersecting Q and separating X should have at least $g(k) - 1$ intersections with $G - Q$.

Let C be a closed curve intersecting Q and separating X, having a minimum number p of intersections with $G - Q$. We may assume that C intersects G only in vertices of G. Suppose $p \leq g(k) - 2$. It is not difficult to see that, by the minimality of p, there exist $x, y \in Q$ on C (possibly $x = y$) such that, if we denote by K and K' the two (closed) $x - y$ parts of C, then one of these parts, K say, intersects G only in Q, while K' intersects Q only in the end points x and y of K'. We may assume that K is part of P. Hence as P is a shortest path, length$(K) \leq$ length$(K') = p + 2$. So length$(C) =$ length$(K) +$ length$(K') - 2 \leq 2p + 2 \leq 2g(k) - 2$.

Hence C does not intersect any face incident with any point in any W_i, since otherwise C would contain a curve of length at most $g(k)-1$ connecting Q and W_i, contradicting the minimality of P. As C separates X, there exist $i \neq j$ such that W_i and W_j are in different components of $\mathcal{R}^2 \setminus C$. This contradicts the facts that there exist $2g(k)$ pairwise disjoint paths from W_i to W_j and that $\mathrm{length}(C) < 2g(k)$. ∎

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