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Percolation and the hard-core lattice gas model

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Recently a uniqueness condition for Gibbs measures in terms of disagreement percolation (a type of dependent percolation involving two realizations) has been obtained. In general this condition is sufficient but not necessary for uniqueness. In the present paper we study the hard-core lattice gas model which we abbreviate as hard-core model. This model is not only relevant in Statistical Physics, but was recently rediscovered in Operations Research in the context of certain communication networks.

First we show that the uniqueness result mentioned above implies that the critical activity for the hard-core model on a graph is at least $P_c/(1-P_c)$, where P_c is the critical probability for site percolation on that graph.

Then, for the hard-core model on bi-partite graphs, we study the probability that a given vertex v is occupied under the two extreme boundary conditions, and show that the difference can be written in terms of the probability of having a 'path of disagreement' from v to the boundary. This is the key to a proof that, for this case, the uniqueness condition mentioned above is also necessary, i.e. roughly speaking, phase transition is equivalent with disagreement percolation in the product space.

Finally, we discuss the hard-core model on $\mathbb{Z}^d$ with two different values of the activity, one for the even, and one for the odd vertices. It appears that the question whether this model has a unique Gibbs measure, can, in analogy with the standard ferromagnetic Ising model, be reduced to the question whether the third central moment of the surplus of odd occupied vertices for a certain class of finite boxes is negative.

1. Introduction

In this introduction we will briefly discuss the notions which play an important role in this paper: Percolation, Markov fields, Gibbs measures, and the hard-core lattice gas model. As abbreviation of th[illegible] we will just write 'hard-core model'. First of all we need some graph-theoretic terminology:

In this paper we deal with connected graphs which are finite or countably infinite and locally finite. The last means that every vertex has finitely many edges. A graph will typically be denoted by G, and the set of its vertices by V_G. Vertices are denoted by v, w, i, j, etc. Two vertices v and w are *adjacent* (notation $v \sim w$) if there is an edge between them.

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A *path* from v to w of *length* n is a sequence of vertices $v_0=v, v_1, v_2, \ldots, v_n=w$ with the property that $v_i \sim v_{i+1}$, $0 \leqslant i \leqslant n-1$. An *infinite path* is a sequence of vertices $v_0, v_1, \ldots$ of which infinitely many are distinct and which has the property $v_i \sim v_{i+1}$, $i=0, 1, 2, \ldots$. For $B \subset V_G$, δB denotes the boundary of B, i.e. the set of all vertices which are not in B but are adjacent to some vertex in B.

A graph G is *bipartite* if V_G is the union of two subclasses (called the *even* and the *odd* subclass), with the property that no vertex is adjacent to any vertex in the same subclass. Paths between vertices in the same subclass always have even length and paths between vertices in different subclasses have odd length. Vertices in the even (odd) subclass are called even vertices (odd vertices).

Two vertices are said to have the same *parity* if they belong to the same subclass.

• *Percolation*. Suppose each vertex i is, independent of all other vertices, *open* (i.e. accessible) with probability p_i and *closed* with probability $1-p_i$. Denote the corresponding probability measure by $P_{\{p_i\}}$. For a realization of the process a path is called open if all its vertices are open. We say that percolation occurs if $P_{\{p_i\}}$(there exists an infinite open path) >0 (in which case this probability is even 1 since the event is a tail event). In case all p_i's are equal, say p, we write P_p for the above probability measure and define the critical probability $p_c=\inf\{p: P_p(\text{there exists an infinite open path})>0\}$. This critical probability depends on G. One of the first results in percolation was to show that $p_c<1$ for a large class of graphs, including the square lattice (Broadbent and Hammersley, 1957). The above model is called *independent site percolation*. If the vertices do not behave independent of each other we speak of *dependent percolation*, and if the edges rather than the vertices are open or closed we speak of *bond percolation*. For further study, see Grimmett (1989) and Kesten (1982).

• *Markov fields and Gibbs measures*. Let S be a finite or countably infinite set and define $\Omega=S^{V_G}$. Elements of Ω will typically be denoted by $\omega \equiv (\omega_i, i \in V_G)$. We are interested in certain probability measures μ on Ω (equipped with the σ-algebra generated by the events $(\omega_i=s)$, $i \in V_G$, $s \in S$; we will call this the *obvious* σ-algebra). Roughly speaking, μ is called a *Markov field* if, for each *finite* set of vertices B, the conditional distribution of the configuration inside B, given the configuration outside B, depends only on the configuration on δ_B. A *specification* is a prescription of such conditional probabilities, and we say that μ is a *Gibbs measure* (or Gibbs state) for a given specification ρ if the conditional probabilities for μ, mentioned above, are (μ-a.s.) equal to the values prescribed by ρ. For more precise and general definitions see Georgii (1988) or Prum and Fort (1991). If G is infinite, a specification may have more than one Gibbs measure, in which case we say that there is a *phase transition*. A central problem in the theory is to determine if a given specification has a unique Gibbs measure. Several uniqueness results, with references, can be found in the bibliographical notes to chapter 8 in Georgii (1988). In van den Berg (1991) a uniqueness result is given which involves a special kind of percolation in the product space. In the present paper we will (among other things) develop that result in more detail for the hard-core model.

• *The hard-core model*. Let a_i, $i \in V_G$ be positive real numbers. If G is finite, the *hard-*

core measure on G with activities a_i, $i \in V_G$ is defined as the following probability measure on $\Omega \equiv \{0, 1\}^{V_G}$:

$$\mu(\omega) = \frac{I(\omega \text{ is feasible}) \prod_{i \in V_G} a_i^{\omega_i}}{Z}, \tag{1.1}$$

where I denotes the indicator function, *feasible* means $\omega_i \omega_j = 0$ for all $i \sim j$, and Z is the normalizing constant (called *partition function*). It is not difficult to see that μ is Markov and that it can also be characterized as follows: denote, for $i \in V_G$, $p_i = a_i / (a_i + 1)$ and let $P_{\{p_i\}}$ be the probability measure under which the ω_i, $i \in V_G$, are independent r.v.'s with $P(\omega_i = 1) = p_i$. Then μ is just $P_{\{p_i\}}$, conditioned on "having no adjacent 1's".

Now we turn to the case that G is infinite. In that case we say that a probability measure μ on $\{0, 1\}^{V_G}$ is a hard-core measure on G with activities a_i, $i \in V_G$ if for all finite $B \subset V_G$ and all $\alpha \in \{0, 1\}^B$,

$$\mu(\omega \equiv \alpha \text{ on } B \mid \omega_{V_G \setminus B}) = \frac{I(\alpha \text{ is feasible}) \prod_{i \in B} a_i^{\alpha_i}}{Z} \quad (\mu\text{-a.s.}),$$

where, this time, feasible means $\alpha_i \alpha_j = 0$ for all $i, j \in B$ with $i \sim j$ *and* $\alpha_i \omega_j = 0$ for all $i \in B$, $j \in \delta B$ with $i \sim j$, and where Z is the appropriate normalizing constant, and $\omega_{V_G \setminus B}$ denotes the collection (vector) of r.v.'s ω_i, $i \in V_G \setminus B$.

In words: the conditional distribution on a finite set B, given the configuration outside B is just the distribution under which those vertices of B which are adjacent to a vertex of δB with value 1, have, with probability 1, value 0, and whose restriction to the remaining subset A of B is just the (finite-case) hard-core measure for A with activities a_i, $i \in A$. (Note that the measure μ of (1.1) also satisfies this property, so that this gives indeed a natural extension to infinite graphs.) The fact that at least one such measure exists follows from standard arguments (see Georgii, 1988). It is also clear from the definitions that all such measures are Markov, and are Gibbs states for the same specification. It has been shown by Dobrushin (1968b) (for the d-dimensional cubic lattice ($d \geq 2$) with all a_i's equal and sufficiently large) that phase transition can indeed occur for this model, i.e. there are cases where there is more than one hard-core measure with the same activities. (Dobrushin's result was recently rediscovered by Louth (1990) in the context of communication models mentioned below.) For a more general class of lattices phase transition for the hard-core model was proved by Runnels (1975).

Motivation of the hard-core model

The hard-core model is relevant in statistical physics as a simple model of a gas whose particles have a non-negligible size: $\omega_i = 1$ (0) means that the vertex i is occupied by a particle (empty); the condition that two adjacent vertices are not both occupied prevents particles from overlapping. The model also arises by taking certain limits for the Ising antiferromagnet (see e.g. Dobrushin, Kolafa and Shlosman, 1985). Recently, the model was rediscovered in Operations Research in the context of communication networks (see Kelly, 1991, and Louth, 1990): consider (the finite graph) G as a communication network where calls arrive at the vertices according to independent Poisson streams with intensities λ_i, $i \in V_G$. The durations of the calls are assumed to be independent, exponentially distributed r.v.'s with mean T_i, $i \in V_G$. If, upon arrival of a call at a vertex i, this vertex and all its neighbours are idle, the call is transmitted and i will be busy for the duration of the call. However, if upon arrival of the call, i or at least one of its neighbours is busy, the call is lost. (Generalizations of this communication model have been studied by Kelbert and Suhov (1990).) The evolution of the system in time (as a $\{0, 1\}^{V_G}$-valued process, where 1 means busy and 0 means free) is a (continuous-time) Markov Chain, and it is not difficult to see (by checking the detailed balance equations) that the stationary distribution is given by (1.1) with $a_i = \lambda_i T_i$, $i \in V_G$. Using the principles of the theory of infinite interacting particle systems, see e.g. Liggett (1985), the time evolution described above makes also sense for infinite graphs: then the time-reversible equilibria are the Gibbs measures for the hard-core model with activities $a_i = \lambda_i T_i$, $i \in V_G$.

In Section 2 we present some general results (i.e. for general Markov fields and for hard-core models on general graphs). In Section 3 we show that, for hard-core models on *bipartite* graphs, phase transition is equivalent to a certain type of dependent percolation. Even for non-experts these sections should be accessible after having read this introduction.

The last section, Section 4, is somewhat separate from the others. In this section, for a good understanding of which familiarity with Ising model theory is desirable but not essential, we specialize even further and restrict to the d-dimensional cubic lattice. Using similarities with the standard Ising model, we shed some light on the question whether the hard-core model with two different values of the activities (one for the even class and one for the odd class) has a unique Gibbs measure.

2. General results

We start this section with the following definition: a *path of disagreement* for the pair (ω, ω') is a path in G on which all vertices i have $\omega_i \neq \omega'_i$.

In van den Berg (1991) the following uniqueness condition for Gibbs measures, in terms of a type of *dependent* percolation is given:

Theorem 2.1. *Let G be a countable, locally finite, connected graph, V_G its set of vertices, and S a finite or countably infinite set. Let the probability measures μ and μ' on S^{V_G} (with the obvious σ-algebra) be Markov fields with the same specification. Consider two independent realizations, one under μ, the other under μ'. If $(\mu \times \mu')((\omega, \omega')$ has an infinite path of disagreement$) = 0$, then $\mu = \mu'$.* □

As a corollary, that paper also gives a uniqueness condition in terms of *independent* percolation. That corollary is a little too weak for our purpose and we give instead the following, whose proof is very similar:

Corollary 2.2. *Let G, S, μ and μ' as defined in Theorem 2.1. Consider again two independent realizations, one under μ, the other under μ'. Let, for each vertex i, N_i be the set of neighbours of i, and define*

$$p_i = \sup_{\alpha,\alpha' \in S^{N_i};\, \alpha \neq \alpha'} (\mu \times \mu')(\omega_i \neq \omega_i' \mid \omega_j = \alpha_j \text{ and } \omega_j' = \alpha_j' \text{ for all } j \in N_i)\ . \tag{2.1}$$

Consider the percolation process where each vertex of G, independently of all others, is open with probability p_i and closed with probability $1-p_i$. If $P_{\{p_i\}}$ (there exists an infinite open path) $=0$, then $\mu = \mu'$.

Proof. Let, for each i, $\mathscr{J}_i$ be the σ-field generated by the variables $\omega_j, j \neq i$ and $\omega_j', j \neq i$. Let O be an arbitrary vertex. Since μ and μ' are Markov, it is easy to see that, for $i \neq O$, $(\mu \times \mu')((\omega, \omega')$ has a path of disagreement from O to $i \mid \mathscr{J}_i)$ equals 0 if, for all $j \sim i$, $\omega_j = \omega_j'$, and $\leqslant p_i$ (a.s.) otherwise. So in any case it is $\leqslant p_i$ (a.s.). Hence the process $(I($there is a path of disagreement from O to $i))_{i \in V_G \setminus \{O\}}$ is stochastically dominated by the process $(I(i \text{ is open}))_{i \in V_G \setminus \{O\}}$. So we have the following:

$$\begin{aligned}
&(\mu \times \mu')(\exists \text{ an infinite path of disagreement containing } O) \\
&\quad = (\mu \times \mu')(\exists \text{ infinite path } \Pi, \text{ not containing } O, \text{ such that for each } i \text{ on } \Pi \\
&\qquad\qquad \text{there exists a path of disagreement from } O \text{ to } i) \\
&\quad \leqslant P_{\{p_i\}}(\exists \text{ infinite path } \Pi, \text{ not containing } O, \text{ such that each } i \text{ on } \Pi \text{ is open}) \\
&\quad = P_{\{p_i\}}(\exists \text{ infinite open path}) \\
&\quad = 0 \quad (\text{by assumption})\ .
\end{aligned} \tag{2.2}$$

The first equality is trivial: the event on the left-hand side is just rewritten in a more complicated way to make it suitable for application of the stochastic dominance inequality. The reverse happens in the second equality.

Since the above holds for any vertex O, and V_G is countable, we have

$$(\mu \times \mu')(\exists \text{ an infinite path of disagreement}) = 0\ .$$

Now apply Theorem 2.1. □

We will apply Corollary 2.2 to the hard-core model. First we define the critical activity a_c of a graph by

$$\begin{aligned}
a_c = \inf\{a:\ &\text{the hard-core model with activities } a_i \equiv a \\
&\text{has more than one Gibbs measure}\}\ .
\end{aligned}$$

Theorem 2.3. *Let G be a countably infinite, locally finite, connected graph. Let* a_i, $i \in V_G$ *be non-negative. Let* $P_{\{a_i/(a_i+1)\}}$ *be the probability measure under which each i, independent of the others is open with probability* $a_i/(a_i+1)$ *and closed with probability* $1/(a_i+1)$.

(i) *If* $P_{\{a_i/(a_i+1)\}}$ *(there exists an infinite open path)* $=0$, *then the hard-core model on G with activities* a_i, $i \in V_G$ *has a unique Gibbs measure.*

(ii) *The critical activity of G satisfies* $a_c \geqslant P_c/(1-P_c)$, *where* P_c *is the critical probability for site percolation on G.*

Remarks. A classical result for the square lattice is that at $p=\frac{1}{2}$, there is no percolation and hence $P_c \geqslant \frac{1}{2}$ (see Harris, 1960, and Hammersley, 1961), which according to the above theorem, immediately yields $a_c \geqslant 1$. Using the result $P_c > \frac{1}{2}$ (Higuchi, 1982; for a more modern and general proof see Aizenman and Grimmett, 1991) yields $a_c > 1$. This was first proved by Dobrushin, Kolafa and Shlosman (1985) who were motivated by the significance of this special lower bound for the 2-dimensional Ising antiferromagnet. However, their proof was computer-assisted. Radulescu and Styer (1987) and Kirillov, Radulescu and Styer (1989) have made several simplifications, considerably reducing the amount of computations. However, their proof is still laborious (involving polynomials of degree 15) and, although their bound (1.185) is better than ours, we think our method gives the most elegant proof of the important inequality $a_c > 1$.

Another reason, besides the significance for the 2-dimensional Ising antiferromagnet, why this special bound is important is its interpretation in terms of subshifts of finite type (see Burton and Steif, 1992). A subshift of finite type is a set of configurations on the d-dimensional lattice given by disallowing certain finite configurations. For example, in our case, we would have 0's and 1's on the d-dimensional lattice with adjacent 1's forbidden. It is of interest in ergodic theory to know if such a set supports more than 1 measure of maximal entropy, where a measure of maximal entropy corresponds in most cases (in particular for the hard-core model) to a measure which has uniform conditional probabilities on finite sets when conditioning outside. The fact that $a_c > 1$ in 2-dimensions tells us there is only one Gibbs state when $a_i \equiv 1$ which translates to the fact that the hard-core subshift of finite type has a unique measure of maximal entropy in 2-dimensions. In fact, we only need the original Harris–Hammersley result (together with Theorem 2.3 (i)) to conclude this.

Finally, we mention that by using the latest bounds for P_c, and by refining our stochastic dominance arguments (thus yielding smaller p_i's in (2.1)), we can further improve the lower bound for a_c.

Proof of Theorem 2.3. Part (ii) follows immediately from part (i) and the definitions. As to part (i), let i be an arbitrary vertex. We will calculate p_i as defined in (2.1): first define, for $\alpha, \alpha' \in \{0,1\}^{N_i}$,

$$f(\alpha, \alpha') = (\mu \times \mu')(\omega_i \neq \omega'_i \mid \omega \equiv \alpha \text{ on } N_i \text{ and } \omega' \equiv \alpha' \text{ on } N_i) .$$

In case $\alpha_k = \alpha'_l = 1$ for some $k, l \in N_i$, the conditional probability that $\omega_i = \omega'_i = 0$ is 1, and hence $f(\alpha, \alpha') = 0$. Hence, since by (2.1) we may assume $\alpha \neq \alpha'$, we are left with the cases $\alpha \equiv 0$, $\alpha' \not\equiv 0$ and the case $\alpha' \equiv 0$, $\alpha \not\equiv 0$. By symmetry it suffices to take the first, in which case it is easily seen that

$$f(\alpha, \alpha') = \mu(\omega_i = 1 \mid \omega \equiv 0 \text{ on } N_i) ,$$

which, by definition equals $a_i/(a_i + 1)$. Hence $\rho_i = a_i/(a_i + 1)$. Now apply Corollary 2.2. □

We finish this section with the following theorem. First some notation: if μ is a probability measure and X and Y are r.v.'s then $E_\mu(X)$ denotes the expectation of X w.r.t. μ, and $\mathrm{Cov}_\mu(X, Y)$ the covariance of X and Y w.r.t. μ, i.e. $E_\mu(XY) - E_\mu(X)E_\mu(Y)$.

Theorem 2.4. *Let G be a finite graph and μ a hard-core measure for G. Then, for any pair of vertices i, j:*

$$\begin{aligned} 2\,\mathrm{Cov}_\mu(\omega_i, \omega_j) &= (\mu \times \mu)((\omega, \omega') \textit{ has a path of disagreement,} \\ &\qquad \textit{with even length, from } i \textit{ to } j) \\ &\quad - (\mu \times \mu)((\omega, \omega') \textit{ has a path of disagreement,} \\ &\qquad \textit{with odd length, from } i \textit{ to } j) \,. \end{aligned} \tag{2.3}$$

In particular, if G is bipartite, then

$$\mathrm{Cov}_\mu(\omega_i, \omega_j) = \pm \tfrac{1}{2}(\mu \times \mu)((\omega, \omega') \textit{ has a path of disagreement from } i \textit{ to } j), \tag{2.4}$$

with + if i and j have the same parity and − if they have different parity.

Remarks. (i) This resembles a result obtained by Fortuin and Kasteleyn for spin–spin covariances in the Ising model (see Fortuin (1972), Fortuin and Kasteleyn (1972) and Newman (1987)). However, their result does not involve the product space, and deals with bond percolation instead of site percolation.

(ii) Using the same dominance arguments as in Corollary 2.2, Theorem 2.4 provides upper bounds for the covariances of the hard-core model in terms of connection probabilities for "ordinary" site percolation.

Proof of Theorem 2.4. The second part (2.4) follows immediately from the first. As to the first, let ω and ω' be two independent $\{0, 1\}^{V_G}$-valued r.v.'s with distribution μ. It is easy to check (just expand the right-hand side), that

$$\mathrm{Cov}_\mu(\omega_i, \omega_j) = \tfrac{1}{2}E_{\mu \times \mu}[(\omega_i - \omega'_i)(\omega_j - \omega'_j)] \,. \tag{2.5}$$

(This equation holds for any pair of r.v.'s and has been used before, see e.g. Lebowitz (1974) and Liggett (1985, p. 78).)

For each pair (ω, ω') define the set of vertices

$$C(i) = \{l \in V_G : (\omega, \omega') \text{ has a path of disagreement from } i \text{ to } l\} .$$

Define the transformation $T : \Omega\times\Omega \to \Omega\times\Omega$ as follows.

If $j \in C(i)$, then $T(\omega, \omega') = (\omega, \omega')$.

If $j \notin C(i)$, then T exchanges ω and ω' on $C(i)$, i.e. $T(\omega, \omega') = (\sigma, \sigma')$, where $\sigma_k = \omega_k$ and $\sigma'_k = \omega'_k$ for all $k \notin C(i)$, and $\sigma_k = \omega'_k$ and $\sigma'_k = \omega_k$ for all $k \in C(i)$.

T is clearly 1-1 and, since μ is Markov, T is also measure preserving. Using this and (2.5), and setting $f(\omega, \omega') = (\omega_i - \omega'_i)(\omega_j - \omega'_j)$, we get

$$\mathrm{Cov}_\mu(\omega_i, \omega_j) = \tfrac{1}{2}E_{\mu\times\mu}(f(\omega, \omega')) = \tfrac{1}{4}E_{\mu\times\mu}[f(\omega, \omega') + f(T(\omega, \omega'))] .$$

From the definition of T and f we have that, if $j \in C(i)$ then $f(T(\omega, \omega')) = f(\omega, \omega')$, and if $j \notin C(i)$, then $f(\omega, \omega') + f(T(\omega, \omega')) = 0$. (Note that this holds even when $C(i) = \emptyset$.) Hence

$$\mathrm{Cov}_\mu(\omega_i, \omega_j) = \tfrac{1}{2}E_{\mu\times\mu}[(f(\omega, \omega') I((\omega, \omega') \text{ has a path of disagreement from } i \text{ to } j)] . \tag{2.6}$$

(Note that so far we only used the Markov property of μ. Hence (2.6) holds for any Markov field on a finite graph.)

Suppose Π is a path of disagreement from i to j. Since we may assume that neither ω nor ω' has adjacent 1's, the values of ω, as well as these of ω', must be alternating along Π. Hence, if the length of Π is even, then $\omega_i = \omega_j = 1 - \omega'_i = 1 - \omega'_j$ and thus $f(\omega, \omega') = 1$. Similarly, if the length of Π is odd, then $f(\omega, \omega') = -1$. This, together with (2.6) yields the desired result. $\square$

Remark. As to the case $i = j$, note that the probability of a path of disagreement of odd length from i to itself is 0, and that the probability of a path of disagreement of even length from i to itself is just the probability that $\omega_i \neq \omega'_i$, which is $2P(\omega_i = 1)P(\omega_i = 0)$. This is indeed twice the variance of ω_i.

3. Results for bipartite graphs

First we restrict to *finite* bipartite graphs. Let G be such graph and μ a hard-core measure for G. Define the partial order $<$ on $\Omega = \{0, 1\}^{V_G}$ by: $\omega < \omega'$ iff $\omega_i \leqslant \omega'_i$ for all even i and $\omega_i \geqslant \omega'_i$ for all odd i. It is not difficult to see that Ω, with this partial order, is a so-called distributive lattice. Moreover, using 1.1 it is easily seen that μ satisfies the FKG condition (Fortuin, Kasteleyn and Ginibre, 1971) w.r.t. this partial order:

$$\mu(\omega\wedge\omega')\,\mu(\omega\vee\omega') \geqslant \mu(\omega)\mu(\omega') , \tag{3.1}$$

for all ω and $\omega' \in \Omega$, where $\omega \wedge \omega'$ is defined by $(\omega \wedge \omega')_i = \min(\omega_i, \omega_i')$ for even i, and $\max(\omega_i, \omega_i')$ for odd i, and $\omega \vee \omega'$ by $(\omega \vee \omega')_i = \max(\omega_i, \omega_i')$ for even i and $\min(\omega_i, \omega_i')$ for odd i.

Hence, by the FKG theorem, μ is *associated*, i.e.

$$\mu(A \cap B) \geqslant \mu(A)\mu(B) \tag{3.2}$$

for all increasing $A, B \subset \Omega$.

Remark. $A \subset \Omega$ is *increasing* if $\omega \in A$, $\omega < \omega'$ implies $\omega' \in A$.

If ν and ν' are two probability measures on Ω, then we say ν is *dominated* by ν', notation $\nu \leqslant \nu'$, if $\nu(A) \leqslant \nu'(A)$ for every increasing $A \subset \Omega$.

The following dominance property is also a standard consequence of the FKG condition:

Lemma 3.1 (dominance property). *Let $W \subset V_G$ and $\alpha, \alpha' \in \{0, 1\}^W$ feasible (i.e. they have no adjacent 1's). If $\alpha < \alpha'$ then the probability measure μ, conditioned on the event ($\omega \equiv \alpha$ on W), is dominated by the prob. measure μ conditioned on the event ($\omega \equiv \alpha'$ on W).* □

Now we turn to the case that G is an infinite bipartite graph. Let a_i, $i \in V_G$ be non-negative real numbers. We are interested in hard-core measures for G with activities a_i, $i \in V_G$. First some terminology: if $W \subset V_G$, we say $\omega \equiv$ *even* (*odd*) on W if $\omega_i = 1$ for all even (odd) $i \in W$ and $\omega_i = 0$ for all odd (even) $i \in W$.

Let Λ_n, $n = 1, 2, \ldots$, be a nested sequence of finite sets of vertices such that

$$\bigcup_{n=1}^{\infty} \Lambda_n = V_G\,.$$

For each n, denote by $\mu^{\mathrm{e}}_{\Lambda_n}$ the hard-core measure for Λ_n with *even boundary condition*. More precisely, $\mu^{\mathrm{e}}_{\Lambda_n}$ is the hard-core measure for $\Lambda_n \cup \delta\Lambda_n$, with activities a_i, $i \in \Lambda_n \cup \delta\Lambda_n$, conditioned on the event ($\omega \equiv$ *even on* $\delta\Lambda_n$). Similarly, define $\mu^{\mathrm{o}}_{\Lambda_n}$ as the hard-core measure for Λ_n with *odd boundary condition*.

The following result is completely analogous to a similar result for the standard ferromagnetic Ising model. Using μ is Markov and the dominance property Lemma 3.1 the reader may, possibly after consultation of the proof for the Ising model (see e.g. Liggett, 1985, Chapter IV), obtain a detailed proof.

Lemma 3.2. *The sequence $\mu^{\mathrm{e}}_{\Lambda_n}$, $n = 1, 2, \ldots$ is decreasing. More precisely if $n < n'$ then the restriction of $\mu^{\mathrm{e}}_{\Lambda_{n'}}$ to $\Lambda_n \cup \delta\Lambda_n$ is dominated by $\mu^{\mathrm{e}}_{\Lambda_n}$. Hence the sequence has a weak limit, which we call μ^{e}. Similarly, the sequence $\mu^{\mathrm{o}}_{\Lambda_n}$, $n = 1, 2, \ldots$, is increasing and hence has a weak limit, which we denote by μ^{o}. The measures μ^{e} and μ^{o} do not depend on the choice of the sequence Λ_n, $n = 1, 2, \ldots$. For each n, $\mu^{\mathrm{o}}_{\Lambda_n} \leqslant \mu^{\mathrm{e}}_{\Lambda_n}$. Hence $\mu^{\mathrm{o}} \leqslant \mu^{\mathrm{e}}$. Both μ^{e} and μ^{o} are Gibbs measures for the hard-core model on G with activities a_i, $i \in V_G$. Moreover, each Gibbs measure ν for this model satisfies $\mu^{\mathrm{o}} \leqslant \nu \leqslant \mu^{\mathrm{e}}$. Hence this model has a unique Gibbs measure if and only if $\mu^{\mathrm{e}} = \mu^{\mathrm{o}}$.* □

Our aim is to show that, roughly speaking, for hard-core models on bipartite graphs, the reverse of Theorem 2.1 also holds. The key to this result is the following proposition, which is of the same spirit as Theorem 2.4.

Proposition 3.3. *Let $i \in \Lambda_n$ be an even vertex.*

$$\mu^{e}_{\Lambda_n}(\omega_i = 1) - \mu^{o}_{\Lambda_n}(\omega_i = 1)$$
$$= (\mu^{e}_{\Lambda_n} \times \mu^{o}_{\Lambda_n})((\omega, \omega') \textit{ has a path of disagreement from } i \textit{ to } \delta\Lambda_n) . \quad (3.3)$$

Proof. Let $\Omega_n \equiv \{0, 1\}^{\Lambda_n \cup \delta\Lambda_n}$ and let $C(i)$ be as defined in the proof of Theorem 2.4. This time let $T: \Omega_n \times \Omega_n \to \Omega_n \times \Omega_n$ be the transformation given by $T(\omega, \omega') = (\omega, \omega')$ if $C(i) \cap \delta\Lambda_n \neq \emptyset$ (i.e. if (ω, ω') has a path of disagreement from i to $\delta\Lambda_n$), and $T(\omega, \omega') = (\sigma, \sigma')$ otherwise, where $\sigma_j = \omega_j$ and $\sigma'_j = \omega'_j$ for $j \notin C(i)$, and $\sigma_j = \omega'_j$ and $\sigma'_j = \omega_j$ for $j \in C(i)$. Again it is clear that T is $1-1$ and preserves the measure $\mu^{e}_{\Lambda_n} \times \mu^{o}_{\Lambda_n}$. Hence

$$\mu^{e}_{\Lambda_n}(\omega_i = 1) = (\mu^{e}_{\Lambda_n} \times \mu^{o}_{\Lambda_n})(\{(\omega, \omega') : \omega_i = 1\})$$
$$= (\mu^{e}_{\Lambda_n} \times \mu^{o}_{\Lambda_n})(\{(\omega, \omega') : C(i) \cap \delta\Lambda_n = \emptyset \text{ and } \omega_i = 1\})$$
$$+ (\mu^{e}_{\Lambda_n} \times \mu^{o}_{\Lambda_n})(\{(\omega, \omega') : C(i) \cap \delta\Lambda_n \neq \emptyset \text{ and } \omega_i = 1\}) . \quad (3.4)$$

Similarly,

$$\mu^{o}_{\Lambda_n}(\omega_i = 1) = (\mu^{e}_{\Lambda_n} \times \mu^{o}_{\Lambda_n})(\{(\omega, \omega') : C(i) \cap \delta\Lambda_n = \emptyset \text{ and } \omega'_i = 1\})$$
$$+ (\mu^{e}_{\Lambda_n} \times \mu^{o}_{\Lambda_n})(\{(\omega, \omega') : C(i) \cap \delta\Lambda_n \neq \emptyset \text{ and } \omega'_i = 1\}) . \quad (3.5)$$

It is easy to see that T maps the event in the first term of (3.5) to the event in the first term of (3.4). Hence these two terms are equal. Moreover, arguments similar to those at the end of the proof of Theorem 2.4 show that if ω and $\omega' \in \Omega_n$ are feasible and $\omega \equiv$ even on $\delta\Lambda_n$ and $\omega' \equiv$ odd on $\delta\Lambda_n$, and (ω, ω') has a path of disagreement from i to $\delta\Lambda_n$, then $\omega'_i = 1 - \omega_i = 0$. Hence the second term in (3.5) equals 0. The result now follows immediately. $\square$

We are now ready for the main result of this section:

Theorem 3.4. *Let G be a countably infinite, connected, locally finite, bipartite graph, V_G its set of vertices, and a_i, $i \in V_G$ non-negative numbers. Let μ^e and μ^o be as defined in Lemma 3.2. Then the hard-core model on G with activities a_i, $i \in V_G$ has a unique Gibbs measure if and only if*

$$(\mu^e \times \mu^o)((\omega, \omega') \textit{ has an infinite path of disagreement}) = 0 .$$

Proof. If $(\mu^e \times \mu^o)((\omega, \omega')$ has an infinite path of disagreement$) = 0$, then, by Theorem 2.1, $\mu^e = \mu^o$ and hence, by Lemma 3.2, the Gibbs measure is unique.

Conversely, suppose there is a unique Gibbs measure, i.e. $\mu^e = \mu^o$. We say that a path Π is *perfect* for (ω, ω') if every even vertex j on Π has $\omega_j = 1 - \omega'_j = 1$, and every odd vertex j on Π has $\omega_j = 1 - \omega'_j = 0$. (Note that if Π is a path of disagreement for (ω, ω'), and neither ω nor ω' has adjacent 1's, then Π is either perfect for (ω, ω') or for (ω', ω).)

Let Λ_n, $\mu^e_{\Lambda_n}$ and $\mu^o_{\Lambda_n}$ be as in Lemma 3.2. Let i be an arbitrary even vertex. Since $\mu^e = \mu^o$, and by symmetry, we have

$$(\mu^e \times \mu^o)((\omega, \omega') \text{ has an infinite path of disagreement containing } i)$$

$$= 2(\mu^e \times \mu^o)((\omega, \omega') \text{ has an infinite perfect path containing } i). \qquad (3.6)$$

Further, for $m \leqslant n$ with $i \in \Lambda_m$ we have

$$(\mu^e \times \mu^o)(\exists \text{ a perfect path from } i \text{ to } \delta\Lambda_n)$$

$$\leqslant (\mu^e_{\Lambda_n} \times \mu^o_{\Lambda_n})(\exists \text{ a perfect path from } i \text{ to } \delta\Lambda_n)$$

$$\leqslant (\mu^e_{\Lambda_n} \times \mu^o_{\Lambda_n})(\exists \text{ a perfect path from } i \text{ to } \delta\Lambda_m) . \qquad (3.7)$$

The first inequality follows, after some reflection, from the dominance property in the beginning of this section, and the second is trivial. Now let in (3.7) first $n \to \infty$ and then $m \to \infty$. This yields

$$(\mu^e \times \mu^o)(\exists \text{ an infinite perfect path containing } i)$$

$$= \lim_{n \to \infty} (\mu^e_{\Lambda_n} \times \mu^o_{\Lambda_n})(\exists \text{ a perfect path from } i \text{ to } \delta\Lambda_n) . \qquad (3.8)$$

However, if $\omega \equiv$ even on $\delta\Lambda_n$ and $\omega' \equiv$ odd on $\delta\Lambda_n$, then a path from i to $\delta\Lambda_n$ is perfect if and only if it is a path of disagreement. Using this, (3.8), (3.6) and (3.3) we get

$$(\mu^e \times \mu^o)(\exists \text{ an infinite path of disagreement containing } i)$$

$$= 2(\mu^e(\omega_i = 1) - \mu^o(\omega_i = 1)) , \qquad (3.9)$$

which, by assumption, equals 0. Since an infinite path must contain an even vertex, we are done. □

4. The cubic lattice with two activity parameters

In this section, we restrict to a special bipartite graph, namely the cubic lattice $\mathbb{Z}^d$. Recall we mentioned earlier that it is possible that $\mu^o \neq \mu^e$ when the activity is constant on the entire lattice and sufficiently high (Dobrushin, 1968b). Let T be the transformation on $\mathbb{Z}^d$ given by $T(x_1, x_2, \ldots, x_d) = (x_1 + 1, x_2, \ldots, x_d)$. This induces a transformation on $\{0, 1\}^{\mathbb{Z}^d}$ and thereby a transformation on probability measures on $\{0, 1\}^{\mathbb{Z}^d}$. It should be clear from Section 3 that $T\mu^o = \mu^e$ (and of course $T\mu^e = \mu^o$). In particular if $\mu^o \neq \mu^e$, then μ^o and μ^e are not translation invariant. We say that a 'translation symmetry breaking' has occurred since the specification was translation invariant but there are associated Gibbs measures which are not translation invariant.

The following discussion is heuristic and for motivational purposes only. It presupposes a knowledge of the Ising model but will not be used when we return to our results.

The specification for the Ising model on $\{-1, 1\}^{\mathbb{Z}^d}$ when there is no external field is invariant under ± 1 interchange. Nonetheless, when the coupling interaction becomes sufficiently large, there is a ± 1 symmetry breaking in that there are Gibbs measures for the Ising model which are not invariant under ± 1 interchange. It is known that when a nonzero external field is added, there is always a unique Gibbs measure. By adding an external field, one is breaking the ± 1 symmetry at the level of the specification; i.e., the specification is no longer invariant under ± 1 interchange. This is precisely the symmetry which can be broken when there is no external field. In view of this heuristic (and Dobrushin's result mentioned above) it is natural to expect to have a unique Gibbs measure for the hard-core model if the *translation* symmetry of the hard-core specification is broken.

We therefore should consider the hard-core model where the activity at the odd lattice points is λ_1 and that at the even lattice points is $\lambda_2 \neq \lambda_1$. Throughout this section, we let $\Lambda_n = \{-n, \ldots, n\}^d$ and $\mu_n^e(\lambda_1, \lambda_2)$ and $\mu_n^o(\lambda_1, \lambda_2)$ be $\mu_{\Lambda_n}^e(\lambda_1, \lambda_2)$ and $\mu_{\Lambda_n}^o(\lambda_1, \lambda_2)$ as defined in accordance with Section 3, where λ_1 is the activity on the odd lattice points and λ_2 the activity on the even lattice points. We now include the two parameters λ_1 and λ_2 in our model in the notation. We also let $\mu_n(\lambda_1, \lambda_2)$ simply be the hard-core model on Λ_n (i.e., with free boundary conditions). Finally, we let Λ_n^T be Λ_n with boundaries identified making it a discrete torus and $\mu_n^T(\lambda_1, \lambda_2)$ be the hard-core model on Λ_n^T. We should therefore conjecture that whenever $\lambda_1 \neq \lambda_2$ the hard-core model has a unique Gibbs measure. The two theorems below give partial results in this direction.

The approach is very similar to one of the proofs that the Ising model has a unique Gibbs measure whenever there is a non-zero external field. One of the key lemmas in this proof for the Ising model is a correlation inequality (involving three spins) called the GHS inequality. This inequality implies negativity of the third central moment (and therefore concavity) of the magnetization in finite boxes, which in turn implies differentiability of the pressure (w.r.t. the external field), which, finally, implies uniqueness. The proof of the GHS inequality requires a certain monotonicity in the model which is only known to hold when the interaction is what is called ferromagnetic. Unfortunately, the hard-core model has an intrinsic antimonotonicity, which prevents us from obtaining an analogue of the GHS inequality. We have seen of course that by flipping the spins on the odd sublattice, we introduce a certain degree of monotonicity which proves helpful. However, this flipping gives us a model which is in some sense analogous to an Ising model which has a positive external field on the even sublattice and a negative external field on the odd sublattice. For such an Ising model, no GHS inequality is known. Nevertheless, we believe (but have not proved) that the analogue of the negativity of the third central moment of the magnetization does hold for the hard-core model and we show that the remaining part of the uniqueness proof for the Ising model can be adapted to the hard-core model.

First we let $E = E(\omega)$ be the number of even vertices in a certain finite set which have value 1, and $O = O(\omega)$ be the number of odd vertices in that set which have value 1. The choice of the set will always be clear from the context. Further, let $s = O - E$ (i.e. the 'surplus' of odd vertices with value 1). It appears that s is the proper analogue of the magnetization and our theorems are the following. Theorem 4.1 tells us that for most parameter values, we get uniqueness while Theorem 4.2 tells us that the third central moment condition mentioned above is enough to give uniqueness when $\lambda_1 > 1$. An analogue of Theorem 4.1 for the Ising model is essentially contained in Lebowitz and Martin-Löf (1972). See Ruelle (1972) for related results on the Ising model.

Theorem 4.1. *For each x, the hard-core model with parameters* e^{x+h} *and* e^{x-h} *has a unique Gibbs measure for all but at most countably many values of h.*

Theorem 4.2. *Let s be 'the surplus of odd vertices with value 1' as defined above. Assume that* $E_{\mu_n^T(\lambda_1,\lambda_2)}[(s - E_{\mu_n^T(\lambda_1,\lambda_2)}(s))^3] \leqslant 0$ *for all n and for all* $\lambda_1 > \lambda_2$ *with* $\lambda_1 > 1$. *Then the hard-core model with parameters* λ_1 *and* λ_2 *has a unique Gibbs measure whenever* $\lambda_1 \neq \lambda_2$ *and* $\lambda_1 > 1$.

Remarks. (i) As we mentioned before, we have not (yet) been able to prove the assumption in this theorem. That is why we called it a partial result. In the 2-dimensional case, we have checked this negative third central moment assumption for all $\lambda_1 > \lambda_2$ with $\lambda_1 \geqslant 1$ on a number of toroids including 6×6 and 6×8. Computer capability makes it quite hard to check larger examples. It turned out that the third central moment of the relevant quantity was in fact nonpositive in all cases we checked. In fact the terms appeared negative in such a very systematic way that we believe it gives good support for this conjecture in general.

(ii) If $\lambda_1 < 1$, the third central moment is not always negative. However, for the 2-dimensional case, uniqueness for $\lambda_1 \leqslant 1$ is already guaranteed by Theorem 2.3 (since $P_c > \frac{1}{2}$; see the remark after that theorem). In general, if every vertex has an activity smaller than 1, the configuration (in a finite box) which has maximal probability is the one where every vertex has value 0. In view of this it is not surprising that this case needs a separate approach.

We give the proofs of these theorems later but first give some development. We can assume without loss of generality that $\lambda_1 > \lambda_2$ and we can clearly reparametrize λ_1 and λ_2 using real numbers x and h with $h > 0$ by $\lambda_1 = e^{x+h}$ and $\lambda_2 = e^{x-h}$. What we need to do is to analyze the partition function of the hard-core model.

Let $A_n = \{\omega \in \{0, 1\}^{\Lambda_n}$ with no adjacent 1's$\}$. Let O and E be as defined just before Theorem 4.1. We then let $Z(n, h, x) = \sum_{\omega \in A_n} \lambda_1^O \lambda_2^E = \sum_{\omega \in A_n} e^{(x+h)O} e^{(x-h)E}$ and call this the partition function. Recall that this is just the normalization used in the definition of the hard-core model measure. It is also necessary to define this concept for 'even' and 'odd' boundary conditions. We therefore $A_n^e = \{\omega \in \{0, 1\}^{\Lambda_n}$ with no adjacent 1's and with no 1's at any odd points adjacent to $\delta\Lambda_n\}$. (In words, these are of course the configurations which are compatible with the even boundary condition.) Similarly, let $A_n^o = \{\omega \in \{0, 1\}^{\Lambda_n}$ with no adjacent 1's and with no 1's at any even points adjacent to $\delta\Lambda_n\}$. (Similarly, these are the configurations which are compatible with the odd boundary condition.) We also let $A_n^T = \{\omega \in \{0, 1\}^{\Lambda_n^T}$ with no adjacent 1's$\}$.

Finally, let

$$Z^e(n, h, x) = \sum_{\omega \in A_n^e} \lambda_1^O \lambda_2^E = \sum_{\omega \in A_n^e} e^{(x+h)O} e^{(x-h)E},$$

$$Z^o(n, h, x) = \sum_{\omega \in A_n^o} \lambda_1^O \lambda_2^E = \sum_{\omega \in A_n^o} e^{(x+h)O} e^{(x-h)E},$$

and

$$Z^T(n, h, x) = \sum_{\omega \in A_n^T} \lambda_1^O \lambda_2^E = \sum_{\omega \in A_n^T} e^{(x+h)O} e^{(x-h)E}.$$

Proposition 4.3.

$$\frac{\log(Z(n, h, x))}{(2n+1)^d}, \qquad \frac{\log(Z^e(n, h, x))}{(2n+1)^d},$$

$$\frac{\log(Z^o(n, h, x))}{(2n+1)^d} \quad \textit{and} \quad \frac{\log(Z^T(n, h, x))}{(2n+1)^d}$$

all converge as $n \to \infty$ to the same limit, which we denote by $P(h, x)$. □

We do not give the proof of this. It follows almost verbatim the proof of Theorem D.1.1 on p. 333 in Ellis (1985) with only some minor modifications needed. We call $P(h, x)$ the *pressure* in analogy with the Ising model.

Proposition 4.4. *$P(h, x)$ is for all x a convex function of h.*

Proof. Since a limit of convex functions is convex, it suffices, by Proposition 4.3, to show that for all n and for all x,

$$\frac{\log(Z(n, h, x))}{(2n+1)^d}$$

is convex in h. Computing the first derivative with respect to h gives

$$\frac{\partial}{\partial h} \frac{\log(Z(n, h, x))}{(2n+1)^d} = \frac{\sum_{\omega \in A_n}((O-E)/(2n+1)^d) e^{(x+h)O} e^{(x-h)E}}{Z(n, h, x)}.$$

We note for later purposes that this is nothing but

$$E[s/(2n+1)^d]$$

with respect to the probability measure $\mu_n(\lambda_1, \lambda_2)$.

Computing the second derivative yields

$$\frac{\partial^2}{\partial h^2}\frac{\log(Z(n,h,x))}{(2n+1)^d}$$

$$= \frac{\partial}{\partial h}\frac{\sum_{\omega\in A_n}((O-E)/(2n+1)^d)e^{(x+h)O}e^{(x-h)E}}{Z(n,h,x)}$$

$$= \frac{\sum_{\omega\in A_n}((O-E)^2/(2n+1)^d)e^{(x+h)O}e^{(x-h)E}}{Z(n,h,x)}$$

$$- \frac{(\sum_{\omega\in A_n}((O-E)/(2n+1)^d)e^{(x+h)O}e^{(x-h)E})(\sum_{\omega\in A_n}(O-E)e^{(x+h)O}e^{(x-h)E})}{Z^2(n,h,x)}$$

$$= \frac{\mathrm{Var}(s)}{(2n+1)^d},$$

where the variance is computed with respect to the probability measure $\mu_{(\lambda_1,\lambda_2)}$. Since the second derivative is therefore nonnegative, we obtain the desired convexity. □

Proposition 4.5. *Let $f_n(x)$ be a sequence of differentiable convex functions defined on an open interval I containing x_0. Assume that $f_n(x)\to f(x)$ as $n\to\infty$ for all x in I and that f is differentiable at x_0. Then $f'_n(x_0)\to f'(x_0)$ as $n\to\infty$.* □

This is a well-known result for convex functions (see e.g. Lemma IV.6.3 on p. 114 in Ellis, 1985).

Proposition 4.6. *If $P(h,x)$ is differentiable at $h=h_0$, then there is a unique Gibbs measure for $(\lambda_1,\lambda_2)=(e^{x+h_0},e^{x-h_0})$.*

Proof. As in the proof of Proposition 4.4, one can show that

$$\frac{\partial}{\partial h}\frac{\log(Z^{e}(n,h,x))}{(2n+1)^d} = E\left[\frac{s}{(2n+1)^d}\right]$$

with respect to the measure $\mu_n^{e}(e^{x+h},e^{x-h})$ and that

$$\frac{\partial}{\partial h}\frac{\log(Z^{o}(n,h,x))}{(2n+1)^d} = E\left[\frac{s}{(2n+1)^d}\right]$$

with respect to the measure $\mu_n^{o}(e^{x+h},e^{x-h})$.

Next, we recall that by Proposition 4.3, both

$$\frac{\log(Z^{e}(n,h,x))}{(2n+1)^d} \quad\text{and}\quad \frac{\log(Z^{o}(n,h,x))}{(2n+1)^d} \quad\to\quad P(h,x)$$

for all x and h as $n\to\infty$.

As $(\partial/\partial h)P(h,x)\,|_{h=h_0}$ exists by assumption, Proposition 4.5 implies that

$$E_{\mu_n^{\mathrm{e}}(\mathrm{e}^{x+h},\mathrm{e}^{x-h})}\left[\frac{s}{(2n+1)^d}\right] \to \frac{\partial}{\partial h} P(h, x)\bigg|_{h=h_0}$$

and

$$E_{\mu_n^{\mathrm{o}}(\mathrm{e}^{x+h},\mathrm{e}^{x-h})}\left[\frac{s}{(2n+1)^d}\right] \to \frac{\partial}{\partial h} P(h, x)\bigg|_{h=h_0}$$

as $n \to \infty$.

It can be shown that (somewhat analogous to the Ising model)

$$E_{\mu_n^{\mathrm{e}}(\mathrm{e}^{x+h},\mathrm{e}^{x-h})}\left[\frac{s}{(2n+1)^d}\right]$$
$$\to \mu^{\mathrm{e}}(\mathrm{e}^{x+h}, \mathrm{e}^{x-h})(\omega\colon \omega(\mathbf{1}) = 1) - \mu^{\mathrm{e}}(\mathrm{e}^{x+h}, \mathrm{e}^{x-h})(\omega\colon \omega(\mathbf{0}) = 1) ,$$

as $n \to \infty$, where $\mathbf{0}$ and $\mathbf{1}$ denote the vertices $(0, 0, \ldots, 0)$ and $(1, 0, \ldots, 0)$, respectively.

However, we do not need this and all we need is the weaker

$$\limsup_{n\to\infty} E_{\mu_n^{\mathrm{e}}(\mathrm{e}^{x+h},\mathrm{e}^{x-h})}\left[\frac{s}{(2n+1)^d}\right]$$
$$\leqslant \mu^{\mathrm{e}}(\mathrm{e}^{x+h}, \mathrm{e}^{x-h})(\omega\colon \omega(\mathbf{1}) = 1) - \mu^{\mathrm{e}}(\mathrm{e}^{x+h}, \mathrm{e}^{x-h})(\omega\colon \omega(\mathbf{0}) = 1) , \tag{4.1}$$

which follows easily from $s = O - E$ and monotonicity (Lemma 3.2).

Similarly

$$\liminf_{n\to\infty} E_{\mu_n^{\mathrm{o}}(\mathrm{e}^{x+h},\mathrm{e}^{x-h})}\left[\frac{s}{(2n+1)^d}\right]$$
$$\geqslant \mu^{\mathrm{o}}(\mathrm{e}^{x+h}, \mathrm{e}^{x-h})(\omega\colon \omega(\mathbf{1}) = 1) - \mu^{\mathrm{o}}(\mathrm{e}^{x+h}, \mathrm{e}^{x-h})(\omega\colon \omega(\mathbf{0}) = 1) . \tag{4.2}$$

In view of the above, this gives us

$$\mu^{\mathrm{o}}(\mathrm{e}^{x+h}, \mathrm{e}^{x-h})(\omega\colon \omega(\mathbf{1}) = 1) - \mu^{\mathrm{o}}(\mathrm{e}^{x+h}, \mathrm{e}^{x-h})(\omega\colon \omega(\mathbf{0}) = 1)$$
$$\leqslant \mu^{\mathrm{e}}(\mathrm{e}^{x+h}, \mathrm{e}^{x-h})(\omega\colon \omega(\mathbf{1}) = 1) - \mu^{\mathrm{e}}(\mathrm{e}^{x+h}, \mathrm{e}^{x-h})(\omega\colon \omega(\mathbf{0}) = 1) . \tag{4.3}$$

However, since $\mu^{\mathrm{o}} \leqslant \mu^{\mathrm{e}}$ (Lemma 3.2), we also have

$$\mu^{\mathrm{o}}(\mathrm{e}^{x+h}, \mathrm{e}^{x-h})(\omega\colon \omega(\mathbf{0}) = 1) \leqslant \mu^{\mathrm{e}}(\mathrm{e}^{x+h}, \mathrm{e}^{x-h})(\omega\colon \omega(\mathbf{0}) = 1) , \tag{4.4}$$

and

$$\mu^{\mathrm{o}}(\mathrm{e}^{x+h}, \mathrm{e}^{x-h})(\omega\colon \omega(\mathbf{1}) = 1) \geqslant \mu^{\mathrm{e}}(\mathrm{e}^{x+h}, \mathrm{e}^{x-h})(\omega\colon \omega(\mathbf{1}) = 1) . \tag{4.5}$$

From (4.3)–(4.5) we clearly obtain

$$\mu^{\mathrm{o}}(\mathrm{e}^{x+h}, \mathrm{e}^{x-h})(\omega\colon \omega(\mathbf{1}) = 1) = \mu^{\mathrm{e}}(\mathrm{e}^{x+h}, \mathrm{e}^{x-h})(\omega\colon \omega(\mathbf{1}) = 1) ,$$

and

$$\mu^{\mathrm{o}}(\mathrm{e}^{x+h}, \mathrm{e}^{x-h})(\omega\colon \omega(\mathbf{0}) = 1) = \mu^{\mathrm{e}}(\mathrm{e}^{x+h}, \mathrm{e}^{x-h})(\omega\colon \omega(\mathbf{0}) = 1) .$$

Lastly, the fact that $\mu^e \geqslant \mu^o$ and the equality of the 1-dimensional marginals immediately gives that $\mu^e = \mu^o$ as desired. This last step is fairly clear and is obtained by a simple modification of Corollary 2.8 on p. 75 of Liggett (1985). □

We are now in a position to prove Theorems 4.1 and 4.2.

Proof of Theorem 4.1. Choose x such that $e^x = \lambda_1$. It is well-known that a convex function from $\mathbb{R}$ to $\mathbb{R}$ is differentiable at all but at most countably many points. Hence for fixed x $P(h, x)$ is differentiable at all but at most countably many values of h. By Proposition 4.6, the hard-core model with parameters (e^{x+h}, e^{x-h}) has a unique Gibbs measure for all but at most countably many values of h. □

Proof of Theorem 4.2. Fix λ_1 and λ_2 with $\lambda_1 > \lambda_2$ and $\lambda_1 > 1$. We want to show that the hard-core model with parameters (λ_1, λ_2) has a unique Gibbs measure.

Take h_0 and x so that $\lambda = e^{x+h_0}$ and $\lambda_2 = e^{x-h_0}$ (note that $h_0 > 0$). To apply Proposition 4.6, we want to show that $(\partial/\partial h)P(h, x)|_{h=h_0}$ exists.

Let

$$P_n^T(h, x) = \frac{\log(Z^T(n, h, x))}{(2n+1)^d}.$$

Recall that the proof of Proposition 4.4 showed that

$$\frac{\partial}{\partial h} P_n^T(h, x) = E\left[\frac{S}{(2n+1)^d}\right]$$

with respect to the measure $\mu_n^T(e^{x+h}, e^{x-h})$. We call this last quantity $M(n, h, x)$.

Choose $h' \in (0, h_0)$ with $e^{x+h'} > 1$. Then, for all $h > h'$,

$$P_n^T(h, x) - P_n^T(h', x) = \int_{h'}^{h} M(n, y, x)\, dy. \tag{4.6}$$

Assuming for the moment that $M(n, y, x)$ is concave in y for $y > h'$, we proceed as follows. Since $-1 \leqslant M(n, y, x) \leqslant 1$ for all n, y and x, we can choose $n_k \to \infty$ such that for all rational $y \in (h', \infty)$,

$$\lim_{k\to\infty} M(n_k, y, x) \equiv M(y, x) \text{ exists}.$$

Moreover, since $M(n, y, x)$ is concave as a function of y on (h', ∞), it even follows (see Theorem VI.3.3 on p. 214 of Ellis, 1985) that

$$\lim_{k\to\infty} M(n_k, y, x) \equiv M(y, x) \text{ exists}$$

for all y in (h', ∞). Letting $n_k \to \infty$ in (4.6) and using bounded convergence gives

$$P(h,x)-P(h',x)=\int_{h'}^{h} M(y,x)\,\mathrm{d}y\,.$$

Next, $M(y,x)$ is a concave function of y on (h',∞) since it is a limit of concave functions. It is therefore also continuous on (h',∞). Since $P(h,x)$ can be expressed as an integral of a continuous function on (h',∞), it follows that $P(h,x)$ is differentiable on (h',∞) and so in particular at h_0 as desired.

Finally, we show that, under the assumption in the theorem, $M(n,y,x)$ is indeed concave in y on (h',∞). It suffices to show that its second derivative is $\leqslant 0$ for all $y>h'$. The first derivative of $M(n,y,x)$, or, equivalently, the second derivative of $\log(Z^T(n,y,x))/(2n+1)^d$, can be obtained in complete analogy to the computation in the proof of Proposition 4.4 and equals

$$\frac{\sum_{\omega\in A_n^T}((O-E)^2/(2n+1)^d)\mathrm{e}^{(x+y)O}\mathrm{e}^{(x-y)E}}{Z^T(n,y,x)} - \frac{(\sum_{\omega\in A_n^T}((O-E)/(2n+1)^d)\mathrm{e}^{(x+y)O}\mathrm{e}^{(x-y)E})(\sum_{\omega\in A_n^T}(O-E)\mathrm{e}^{(x+y)O}\mathrm{e}^{(x-y)E}}{(Z^T(n,y,x))^2}\,.$$

Differentiating this last expression with respect to y and simplifying yields

$$\frac{\partial}{\partial y^2}M(n,y,x)=\frac{1}{(2n+1)^d}E_{\mu_n^T(\mathrm{e}^{x+y},\mathrm{e}^{x-y})}[(s-E_{\mu_n^T(\mathrm{e}^{x+y},\mathrm{e}^{x-y})}(s))^3]$$

which is $\leqslant 0$ by assumption. $\square$

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