Fast Computation of an Alternating Sum

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1. Introduction

In this note we present an algorithm to determine the sum

$$S_\alpha(n) = \sum_{j=1}^{n} (-1)^{\lfloor j\alpha \rfloor}, \qquad \alpha \text{ irrational},$$

where, as usual, $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. Our algorithm is simple and fast; it consists of two simple operations, and the number of operations needed to evaluate $S_\alpha(n)$ is of order $\log(n)$.

The sum $S_\alpha(n)$ has been studied before in [1], where it was shown that $S_\alpha(n)$ is unbounded for irrational α, and that on the other hand the equality $S_\alpha(n) = 0$ holds for infinitely many n. So this sum has some of the characteristics of a random walk.

However, this random-like sum incorporates some remarkable symmetry properties. For instance, if you calculate $S_{\sqrt{2}}(n)$ for increasing n, and keep track of those n for which $S_{\sqrt{2}}(n)$ attains a value for the first time, then a recurrence relation between the n is displayed. More specifically, $S_{\sqrt{2}}(0) = 0$ is the first new value. The next new value occurs at $n = 1$, for which $S_{\sqrt{2}}(1) = -1$, and then $S_{\sqrt{2}}(3) = 1$, $S_{\sqrt{2}}(8) = -2$, etc. The first few extremes occur at 0,1,3,8,20,49,119,288,... In [4] it was conjectured that these numbers satisfy the recurrence relation $n_{i+1} = 2n_i + n_{i-1} + 1$. At the end of this paper we will see that this is the case indeed.

In order to compute $S_\alpha(n)$ efficiently, we looked for patterns in the plus and minus signs of the terms $(-1)^{\lfloor j\alpha \rfloor}$. We observed two kinds of patterns: 'repetitions' and 'reflections'. Both patterns induce an operation in the algorithm.

- A *repetition* occurs for a number n if $(-1)^{\lfloor j\alpha \rfloor}$ is equal to $(-1)^{\lfloor (n+j)\alpha \rfloor}$ for all $1 \leq j \leq n$. This implies that $S_\alpha(n+k) = S_\alpha(n) + S_\alpha(k)$ for $1 \leq k \leq n$, which is one of the operations in the algorithm.
- A *reflection* occurs for a number n if $(-1)^{\lfloor j\alpha \rfloor}$ and $(-1)^{\lfloor (n-j)\alpha \rfloor}$ have opposite signs for all $1 \leq j < n/2$. This implies that $S_\alpha(n-1) = S_\alpha(n-k) - S_\alpha(k-1)$ for $1 \leq k \leq n/2$, which is the other operation.

For which n does a repetition or a reflection take place? Assume that, for some n, $n\alpha$ is very close to an even integer $2m$. Then $(n+j)\alpha \approx 2m + j\alpha$ and $(n-j)\alpha \approx 2m - j\alpha$, which makes it plausible that

$$\begin{aligned}
(-1)^{\lfloor (n+j)\alpha \rfloor} &= (-1)^{\lfloor 2m+j\alpha \rfloor} = (-1)^{2m+\lfloor j\alpha \rfloor} &&= (-1)^{\lfloor j\alpha \rfloor}, \\
(-1)^{\lfloor (n-j)\alpha \rfloor} &= (-1)^{\lfloor 2m-j\alpha \rfloor} = (-1)^{2m-\lfloor j\alpha \rfloor -1} &&= -(-1)^{\lfloor j\alpha \rfloor}.
\end{aligned}$$

Apparently, repetitions and reflections are likely to occur if $n\alpha \approx 2m$, or, in other words, if a rational m/n is a very good approximation of $\alpha/2$.

The best rational approximations of $\alpha/2$ are the so-called convergents of the continued fraction of $\alpha/2$. The next section contains a brief review of continued fractions. Since $S_{\alpha+2}(n)$ is equal to $S_\alpha(n)$, we may restrict ourselves to $-1 < \alpha < 1$. Furthermore, $S_{-\alpha}(n) = -S_\alpha(n)$ if α is irrational, so we may even assume that $0 < \alpha < 1$. Hence, we only consider continued fractions of irrationals between 0 and 1/2.

2. Continued fractions

Every irrational β, with $0 < \beta < 1/2$, can be represented as an infinite continued fraction

$$\beta = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{\ddots}}}} \qquad n_1 \in \mathbb{Z}_{>1}, \qquad n_i \in \mathbb{Z}_{>0} \quad \text{for } i \geq 2,$$

which is denoted by $[0; n_1, n_2, n_3, \ldots]$. The truncation $r_i = [0; n_1, n_2, \ldots, n_i]$ is called the ith convergent of β. The r_i are rational numbers, and their numerators and denominators can be constructed from simple recurrence relations. If we define

$$\begin{array}{lll}
p_{-1} = 1 & p_0 = 0 & p_{i+2} = n_{i+2}p_{i+1} + p_i \\
q_{-1} = 0 & q_0 = 1 & q_{i+2} = n_{i+2}q_{i+1} + q_i
\end{array}$$

then $r_i = p_i/q_i$. By induction one can prove the equality

$$p_i q_{i+1} - p_{i+1} q_i = (-1)^{i-1},$$

which implies that p_i and q_i are relatively prime. Moreover, the recurrence relation for the denominators q_i implies that $q_i < q_{i+1}$ for $i \geq 0$.

The following classical result indicates that convergents of β provide good rational approximations of β with relatively small denominators. For a proof see [3, page 58].

Proposition 1. *If a rational m/n lies between β and one of its convergents $r_i = p_i/q_i$, then $n \geq q_{i+1} + q_i$.*

Proposition 1 will be an important ingredient of the proofs for the equations that constitute the algorithm.

3. THE ALGORITHM

Consider an irrational α, with $0 < \alpha < 1$. Let $q_0, q_1, q_2, \ldots$ be the successive denominators of the convergents of $\alpha/2$. The following equation is based on repetitions.

EQUATION 1

$$S_\alpha(mq_i + l) = mS_\alpha(q_i) + S_\alpha(l), \quad q_i < mq_i + l < \frac{q_{i+1} + q_i}{2}, \quad 0 \le l < q_i.$$

PROOF. It is sufficient to prove that

$$(-1)^{\lfloor (kq_i + j)\alpha \rfloor} = (-1)^{\lfloor j\alpha \rfloor}, \quad q_i < kq_i + j < \frac{q_{i+1} + q_i}{2}, \quad 0 \le j < q_i.$$

Suppose that this equation does not hold for some k and j. Then we have to prove that $kq_i + j$ is greater than or equal to $(q_{i+1} + q_i)/2$. Let $kq_i + j$ be the smallest number for which the repetitive pattern breaks down. In this case the equation still holds for $(k-1)q_i + j$, and we conclude that

$$(-1)^{\lfloor ((k-1)q_i + j)\alpha \rfloor} \ne (-1)^{\lfloor (kq_i + j)\alpha \rfloor}.$$

We rewrite this inequality. Since p_i/q_i is a convergent of $\alpha/2$, the difference between $\alpha/2$ and p_i/q_i is small. Putting $\epsilon = \alpha/2 - p_i/q_i$, we get $((k-1)q_i + j)\alpha = (kq_i + j)\alpha - 2q_i\epsilon - 2p_i$. Since $2p_i$ is even, it follows that

$$(-1)^{\lfloor (kq_i + j)\alpha - 2q_i\epsilon \rfloor} \ne (-1)^{\lfloor (kq_i + j)\alpha \rfloor}.$$

Hence, there must be an integer m between $(kq_i + j)\alpha - 2q_i\epsilon$ and $(kq_i + j)\alpha$. This implies that $m/2(kq_i + j)$ lies between $\alpha/2 - \epsilon = p_i/q_i$ and $\alpha/2$. According to Proposition 1 we then have $2(kq_i + j) \ge q_{i+1} + q_i$, which is what we wanted to prove. □

Equation 1 reduces the effort to compute $S_\alpha(n)$ considerably. If one knows $S_\alpha(n)$ for $n \le q_{i-1}$, then by Equation 1 $S_\alpha(n)$ is known for $n < (q_i + q_{i-1})/2$. However, Equation 1 by itself does not yet constitute a fast algorithm for calculating the $S_\alpha(n)$. For this purpose it should produce, from the values $S_\alpha(n)$ for $n \le q_{i-1}$, the values for $n \le q_i$.

The following equation nearly closes the gap between $(q_i + q_{i-1})/2$ and q_i. Equation 2 relates $S_\alpha(n)$ to $S_\alpha(q_i - n - 1)$ for $q_i/2 \le n < q_i$. The equation is based on reflections.

EQUATION 2

$$S_\alpha(q_i - k) = S_\alpha(k - 1) + S_\alpha(q_i - 1), \qquad 1 \le k \le \frac{q_i}{2}.$$

PROOF. First we show that

$$(-1)^{\lfloor j\alpha \rfloor} = (-1)^{\lfloor j\frac{2p_i}{q_i} \rfloor} \qquad \text{for } 1 \le j < q_i \text{ and } j \ne \frac{q_i}{2}.$$

Here, the argument is similar to that for Equation 1. Suppose that the equation is not true for some particular j:

$$(-1)^{\lfloor j\alpha \rfloor} \neq (-1)^{\lfloor j \frac{2p_i}{q_i} \rfloor}$$

Since both $j\alpha$ and $j2p_i/q_i$ are non-integral (because $j \neq q_i/2$ and $j \neq q_i$), there must be an integer m in between. In other words, $m/2j$ lies between $\alpha/2$ and p_i/q_i. Proposition 1 then tells us that $2j \geq q_{i+1} + q_i > 2q_i$, and we have a contradiction.

Now we can prove Equation 2. Since $(q_i - j)2p_i/q_i = 2p_i - j2p_i/q_i$, we have

$$(-1)^{\lfloor (q_i-j)\frac{2p_i}{q_i} \rfloor} = -(-1)^{\lfloor j\frac{2p_i}{q_i} \rfloor}, \qquad \text{for } 1 \leq j < \frac{q_i}{2}.$$

By the equality that has just been deduced, we may replace $2p_i/q_i$ by α:

$$(-1)^{\lfloor (q_i-j)\alpha \rfloor} = -(-1)^{\lfloor j\alpha \rfloor}, \qquad \text{for } 1 \leq j < \frac{q_i}{2},$$

which immediately implies Equation 2. □

The algorithm is nearly complete. We already know the operations to reduce the n in $S_\alpha(n)$, but in order to compute $S_\alpha(n)$ we still need to know its values at the denominators q_i and $q_i - 1$. The values $S_\alpha(q_i)$ can be obtained efficiently from the reflection principle.

EQUATION 3

$$\begin{array}{ll} S_\alpha(q_i) = (-1)^i & \text{if } q_i \text{ is odd,} \\ S_\alpha(q_i) = 0 & \text{if } q_i \text{ is even.} \end{array}$$

PROOF. In the proof of Equation 2, it was shown that

$$(-1)^{\lfloor (q_i-j)\alpha \rfloor} = -(-1)^{\lfloor j\alpha \rfloor}, \qquad \text{for } 1 \leq j < \frac{q_i}{2}.$$

(This equality says that almost all terms of $S_\alpha(q_i)$ cancel.)

First, assume that q_i is odd. Then it follows that $S_\alpha(q_i) = (-1)^{\lfloor q_i\alpha \rfloor}$. If we put $\epsilon = \alpha/2 - p_i/q_i$, then $q_i\alpha = 2p_i + 2q_i\epsilon$. We will show that $2q_i|\epsilon| < 1$.

Suppose that $j|\epsilon| \geq 1$ for some j. Then there lies an integer m between $j\alpha/2$ and $j\alpha/2 - j\epsilon$. So m/j lies between $\alpha/2$ and $\alpha/2 - \epsilon = p_i/q_i$. According to Proposition 1 we then have $j \geq q_{i+1} + q_i > 2q_i$. Thus $2q_i|\epsilon| < 1$.

Hence, $\lfloor q_i\alpha \rfloor = 2p_i$ is even if $\epsilon > 0$, and $\lfloor q_i\alpha \rfloor = 2p_i - 1$ is odd if $\epsilon < 0$. If i is even, then the convergent p_i/q_i approximates $\alpha/2$ from below, so in that case $\epsilon > 0$. If i is odd, then p_i/q_i approximates $\alpha/2$ from above, so that $\epsilon < 0$. This proves that $S_\alpha(q_i) = (-1)^i$.

Next, assume that q_i is even. Then $S_\alpha(q_i) = (-1)^{\lfloor q_i\alpha/2 \rfloor} + (-1)^{\lfloor q_i\alpha \rfloor}$. We claim that these remaining two terms have opposite signs. As above, we have $q_i\alpha/2 = p_i + q_i\epsilon/2$ and $q_i\alpha = 2p_i + q_i\epsilon$. The numerator p_i is odd, because q_i is even. It follows for $\epsilon > 0$ (i.e., for even i) that $\lfloor q_i\alpha/2 \rfloor$ and $\lfloor q_i\alpha \rfloor$ are odd and even respectively. Similarly, if $\epsilon < 0$ (i.e., if i is odd), then they are even and odd respectively. So $S_\alpha(q_i) = 0$. □

To complete the algorithm, we calculate $S_\alpha(q_i - 1)$, in order to reduce Equation 2 to a more suitable form. We have $S_\alpha(q_i - 1) = S_\alpha(q_i) - (-1)^{\lfloor q_i\alpha \rfloor}$. Using equalities that have been deduced in the proof of Equation 3, we obtain that $S_\alpha(q_i - 1) = 0$ if q_i is odd and $S_\alpha(q_i - 1) = (-1)^{\lfloor q_i\alpha/2 \rfloor} = (-1)^{i-1}$ if q_i is even. Hence, the following equation is equivalent to Equation 2.

EQUATION 2′
For $1 \le k \le q_i/2$ we have

$$\begin{aligned} S_\alpha(q_i - k) &= S_\alpha(k-1) && \text{if } q_i \text{ is odd,} \\ S_\alpha(q_i - k) &= S_\alpha(k-1) + (-1)^{i-1} && \text{if } q_i \text{ is even.} \end{aligned}$$

Combining Equations 1, 2′ and 3, we obtain the promised fast algorithm for $S_\alpha(n)$.

4. AN EXAMPLE
We demonstrate the use of the algorithm by calculating $S_e(1,000,000)$. Since $0 < e - 2 < 1$, we replace e by $e - 2$. The continued fraction of $(e-2)/2$ is

$$[0; 2, 1, 3, 1, 1, 1, 3, 3, 3, 1, 3, 1, 3, 5, 3, 1, 5, \ldots],$$

so that the denominators of the first convergents are

1, 2, 3, 11, 14, 25, 39, 142, 465, 1537, 2002, 7543,
9545, 36178, 190435, 607483, 797918, 4597073.

This is all we need to know in order to apply the algorithm to $S_e(1,000,000)$. From Equations 1 and 3 it follows that

$$\begin{array}{lclclclcl} S_e(1,000,000) &=& S_e(797,918) &+& S_e(202,082) &=& & & S_e(202,082) \\ S_e(202,082) &=& S_e(190,435) &+& S_e(11,647) &=& 1 &+& S_e(11,647) \\ S_e(11,647) &=& S_e(9,545) &+& S_e(2,102) &=& 1 &+& S_e(2,102) \\ S_e(2,102) &=& S_e(2,002) &+& S_e(100) &=& & & S_e(100). \end{array}$$

According to Equation 2′, reflection with respect to 142 yields

$$S_e(100) = S_e(41) + 1 = S_e(39) + S_e(2) + 1 = 2.$$

Since we picked up four ones on the way, we find $S_e(1,000,000) = 4$.

5. A RECURRENCE RELATION
Using the results from Section 3, we can prove the conjecture from [4], saying that the numbers n where $S_{\sqrt{2}}(n)$ attains a new value satisfy the recurrence relation $n_{i+1} = 2n_i + n_{i-1} + 1$.

Since $0 < 2 - \sqrt{2} < 1$, we replace $\sqrt{2}$ by $2 - \sqrt{2}$. The continued fraction of $(2-\sqrt{2})/2$ is $[0; 3, 2, 2, 2, \ldots]$, so that the denominators $q_0, q_1, q_2, \ldots$ of the convergents are found by the recurrence relation $q_{i+1} = 2q_i + q_{i-1}$ with $q_0 = 1$ and $q_1 = 3$. This implies that all q_i are odd, so according to Equation 3 we have $S_\alpha(q_i) = (-1)^i$. Then Equations 2′ and 1 yield

$$\begin{aligned} S_\alpha(q_{i+1} - k) &= S_\alpha(k-1), && k \le q_{i+1}/2, \\[1ex] S_\alpha(q_i + l) &= (-1)^i + S_\alpha(l), && q_i + l < q_{i+1}/2. \end{aligned}$$

The first equation implies that extremes do not occur between $q_{i+1}/2$ and q_{i+1}: the value of S_α at $q_{i+1} - k$ has already been attained at $k - 1$. The second equation implies that, if j_i and k_i denote the numbers where the ith new minimum and maximum of S_α are attained, then we have the recurrence relations

$$\begin{aligned} j_i &= q_{2i-1} + j_{i-1}, \qquad & j_0 = 0, \\ k_i &= q_{2i} + k_{i-1}, \qquad & k_0 = 0. \end{aligned}$$

Hence, $j_i = q_{2i-1} + q_{2i-3} + \cdots + q_1$ and $k_i = q_{2i} + q_{2i-2} + \cdots + q_2$, from which it is clear that each new minimum is followed by a new maximum and vice versa. It is now straightforward to check that the recurrence relation for the n_i (with $n_{2i-1} = j_i$ and $n_{2i} = k_i$) reads $n_{i+1} = 2n_i + n_{i-1} + 1$. □

In fact, along the same lines we can deduce a similar result for all *quadratic* irrationals α, because these are exactly the irrationals that have a periodic continued fraction (see e.g. [2]).

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