An obstacle to a decomposition theorem for near-regular matroids*

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Abstract

Seymour's Decomposition Theorem for regular matroids states that any matroid representable over both GF(2) and GF(3) can be obtained from matroids that are graphic, cographic, or isomorphic to R_{10} by 1-, 2-, and 3-sums. It is hoped that similar characterizations hold for other classes of matroids, notably for the class of near-regular matroids. Suppose that all near-regular matroids can be obtained from matroids that belong to a few basic classes through k-sums. Also suppose that these basic classes are such that, whenever a class contains all graphic matroids, it does not contain all cographic matroids. We show that in that case 3-sums will not suffice.

1 Introduction

A regular matroid is a matroid representable over every field. Much is known about this class, the deepest result being Seymour's Decomposition Theorem:

Theorem 1.1 (Seymour [Sey80]). *Let M be a regular matroid. Then M can be obtained from matroids that are graphic, cographic, or equal to R_{10} through* 1-*,* 2-*, and* 3-*sums.*

A class $\mathscr{C}$ of matroids is *polynomial-time recognizable* if there exists an algorithm that decides, for any matroid M, in time $f(|E(M)|, \tau)$ whether or not $M \in \mathscr{C}$, where τ is the time of one rank evaluation, and $f(x, y)$ a polynomial. Seymour [Sey81] showed that the class of graphic matroids is polynomial-time recognizable. Also every finite class is polynomial-time recognizable. Using these facts Truemper [Tru82] (see also Schrijver [Sch86, Chapter 20]) showed the following:

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Theorem 1.2. *The class of regular matroids is polynomial-time recognizable.*

A *near-regular matroid* is a matroid representable over every field, except possibly GF(2). Near-regular matroids were introduced by Whittle [Whi95, Whi97], one result of which is the following:

Theorem 1.3 (Whittle [Whi97])**.** *Let M be a matroid. The following are equivalent:*

(i) *M is representable over* GF(3)*,* GF(4)*, and* GF(5)*;*

(ii) *M is representable over $\mathbb{Q}(\alpha)$ by a near-unimodular matrix;*

(iii) *M is near-regular.*

A *near-unimodular matrix* is a matrix over $\mathbb{Q}(\alpha)$ such that the determinant of every square submatrix is either zero or equal to $(-1)^s\alpha^i(1-\alpha)^j$ for some $s, i, j \in \mathbb{Z}$. Whittle [Whi97, Whi05] wondered if an analogue of Theorem 1.1 would hold for the class of near-regular matroids. The following conjecture was made:

Conjecture 1.4. *Let M be a near-regular matroid. Then M can be obtained from matroids that are signed-graphic, their duals, or members of some finite set through* 1-*,* 2-*, and* 3-*sums.*

A matroid is *signed-graphic* if it can be represented by a GF(3)-matrix with at most two nonzero entries in each column (see Zaslavsky [Zas82] for more on these matroids). One difference with the regular case is that not every signed-graphic matroid is near-regular.

Several people have made an effort to understand the structure of near-regular matroids. Oxley et al. [OVW98] studied maximum-sized near-regular matroids. Hliněný [Hli04] and Pendavingh [Pen04] have both written software to investigate all 3-connected near-regular matroids up to a certain size. Pagano [Pag98] studied signed-graphic near-regular matroids, and Pendavingh and Van Zwam [PZb] studied a closely related class of matroids which they call near-regular-graphic.

Despite these efforts, an analogue to Theorem 1.1 is still not in sight. In this paper we record an obstacle we found, that will have to be taken into account in any structure theorem. Our result is the following:

Theorem 1.5. *Let G_1, G_2 be graphs. There exists an internally 4-connected near-regular matroid M having both $M(G_1)$ and $M(G_2)^*$ as a minor.*

It follows immediately that Conjecture 1.4 is false. More generally, suppose we want to find a decomposition theorem for near-regular matroids, such that each basic class that contains all graphic matroids, does not contain all cographic matroids. Theorem 1.5 implies that such a characterization must employ at least 4-sums.

The paper is organized as follows. In Section 2 we give some preliminary definitions and prove a lemma on generalized parallel connection. In Section 3 we prove Theorem 1.5. We conclude in Section 4 with some updated conjectures.

Throughout this paper we assume familiarity with matroid theory as set out in Oxley [Oxl92].

2 Preliminaries

2.1 Connectivity

Definition 2.1. *A matroid is* internally 4-connected *if it is 3-connected and, for every 3-separation* (X, Y)*,* $\min(|X|, |Y|) = 3$.

This notion of connectivity is useful in our context. For instance, Theorem 1.1 can be rephrased as follows:

Theorem 2.2. *Let M be an internally 4-connected regular matroid. Then M is graphic, cographic, or equal to R_{10}.*

Intuitively, separations (X, Y) where both $|X|$ and $|Y|$ are big should give rise to a decomposition into smaller matroids.

Definition 2.3. *Let M be a matroid, and N a minor of M. Let (X', Y') be a k-separation of N. We say that (X', Y') is* induced *in M if M has a k-separation (X, Y) such that $X' \subseteq X$ and $Y' \subseteq Y$.*

2.2 Partial fields

Our main tool in the proof of Theorem 1.5 is useful outside the scope of this paper. Hence we have stated it in the general framework of partial fields. For that purpose we need a few definitions. More on the theory of partial fields can be found in Semple and Whittle [SW96] and in Pendavingh and Van Zwam [PZa].

Definition 2.4. *A* partial field *is a pair (R, G), where R is a commutative ring with identity, and G is a subgroup of the group of units of R such that $-1 \in G$.*

For example, the near-regular partial field is $\big(\mathbb{Q}(\alpha), \langle -1, \alpha, 1-\alpha \rangle\big)$, where $\langle S \rangle$ denotes the multiplicative group generated by S.

We will adopt the convention that matrices have labelled rows and columns, so an $X \times Y$ matrix is a matrix whose rows are labelled by the (ordered) set X and whose columns are labelled by the (ordered) set Y.

Definition 2.5. *Let $\mathbb{P} := (R, G)$ be a partial field, and let A be a matrix with entries in R. Then A is a $\mathbb{P}$-matrix if, for every square submatrix A' of A, either $\det(A') = 0$ or $\det(A') \in G$.*

Suppose the columns of a $\mathbb{P}$-matrix A are labelled by a set E. If $B \subseteq E$ then we denote by $A[B]$ the submatrix of A obtained by discarding all columns except those in B. If A is an $X \times Y$ matrix, where $X \cap Y = \emptyset$, then we denote by $[I\, A]$ the $X \times (X \cup Y)$ matrix obtained from A by prepending an identity matrix whose rows and columns are both labelled by X.

Theorem 2.6. *Let $\mathbb{P} = (R, G)$ be a partial field, let A be an $X \times Y$ $\mathbb{P}$-matrix for disjoint sets X, Y. Let $E := X \cup Y$, and define the $X \times E$ matrix $A' := [I\, A]$. Let $\mathscr{B} := \{B \subseteq E \mid |B| = r, \det(A[B]) \neq 0\}$. Then $\mathscr{B}$ is the set of bases of a matroid.*

We denote this matroid by $M[A']$.

2.3 Generalized parallel connection

Recall the generalized parallel connection of two matroids M, N along a common restriction X, denoted by $P_X(M,N)$. This construction was introduced by Brylawski [Bry75] (see also Oxley [Oxl92, Section 12.4]). Lee [Lee90] generalized Brylawski's result to matroids representable over a field such that each subdeterminant is in a multiplicatively closed set. We generalize Brylawski's result further to matroids representable over a partial field.

Lemma 2.7. *Suppose A_1, A_2 are $\mathbb{P}$-matrices with the following structure:*

$$A_1 = \begin{array}{c} \\ X_1 \\ X \end{array} \begin{array}{c} \begin{array}{cc} Y_1 & Y \end{array} \\ \left[\begin{array}{cc} D_1' & 0 \\ D_1 & D_X \end{array}\right] \end{array}, \qquad A_2 = \begin{array}{c} \\ X \\ X_2 \end{array} \begin{array}{c} \begin{array}{cc} Y & Y_2 \end{array} \\ \left[\begin{array}{cc} D_X & D_2 \\ 0 & D_2' \end{array}\right] \end{array},$$

where X, Y, X_1, Y_1, X_2, Y_2 are pairwise disjoint sets. If X is a modular flat of $M[I\,A_1]$ then

$$A := \begin{array}{c} \\ X_1 \\ X \\ X_2 \end{array} \begin{array}{c} \begin{array}{ccc} Y_1 & Y & Y_2 \end{array} \\ \left[\begin{array}{ccc} D_1' & 0 & 0 \\ D_1 & D_X & D_2 \\ 0 & 0 & D_2' \end{array}\right] \end{array}$$

is a $\mathbb{P}$-matrix. Moreover, if $M_1 = M[IA_1]$ and $M_2 = M[IA_2]$, then $M[IA] = P_{X\cup Y}(M_1, M_2)$.

Proof sketch. Suppose $\widetilde{A}$ was obtained from A by a number of pivots in the submatrix indexed by X_1 and Y_1, say

$$\widetilde{A} = \begin{array}{c} \\ \widetilde{X}_1 \\ X \\ X_2 \end{array} \begin{array}{c} \begin{array}{ccc} \widetilde{Y}_1 & Y & Y_2 \end{array} \\ \left[\begin{array}{ccc} \widetilde{D}_1' & 0 & 0 \\ \widetilde{D}_1 & D_X & D_2 \\ 0 & 0 & D_2' \end{array}\right] \end{array}.$$

Since $X \cup Y$ is a modular flat of M_1, each column in $\widetilde{D}_1$ is parallel to a column in $[I\ D_X]$. Since each subdeterminant can be computed by a number of pivots, one can show that each subdeterminant of A is the product of a subdeterminant of A_1 and a subdeterminant of A_2. The result follows. □

The special cases $X = \emptyset$ and $X = \{p\}$ were previously proven by Semple and Whittle [SW96].

3 The need for 4-sums

We sketch a proof of Theorem 1.5 from the introduction.

Proof sketch of Theorem 1.5. It suffices to prove the theorem for $G_1 = G_2 = K_n$, where $n \geq 4$. Let $M_{12} := M[I\, A_{12}]$, where

$$A_{12} = \begin{array}{c} \\ a \\ b \\ c \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} d & e & f & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & \alpha \\ 1 & 1 & 0 & 0 & \alpha & -\alpha \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right] \end{array}. \tag{1}$$

A bit of work, which is best left to a computer, verifies that A_{12} is near-unimodular. It is easily verified that M_{12} is internally 4-connected. Furthermore $M_{12} \backslash \{1,2,3,4,5,6\} \cong M(K_4)$, and $M_{12}/\{a,b,c,d,e,f\} \cong M(K_4)^*$. Now $M(K_4)$ is isomorphic to a modular flat in $M(G_1)$. Label the edges of some K_4-restriction of G_1 by $\{a,b,c,d,e,f\}$, and label the edges of some K_4-restriction of G_2 by $\{1,2,3,4,5,6\}$. Let

$$M := P_{\{a,b,c,d,e,f\}} \left(S_{\{1,2,3,4,5,6\}} (M_{12}, M(G_2)^*), M(G_1) \right). \tag{2}$$

By Lemma 2.7, M is near-regular. Moreover, since $G_1 = G_2 = K_n$, it is not difficult to verify that M is internally 4-connected. The result follows. □

Matroid M_{12} was found while studying the 3-separations of R_{12}. The unique 3-separation (X, Y) of R_{12} with $|X| = |Y| = 6$ is induced in the class of regular matroids. Pendavingh and Van Zwam had found, using a computer search for blocking sequences, that it is not induced in the class of near-regular matroids.

Surprisingly, the matroid M_{12} appears quite inconspicuous by itself. A natural class of near-regular matroids is the class of near-regular signed-graphic matroids. It turns out that M_{12} is a member of this class. A signed-graphic representation is given in Figure 1. The K_4-restriction is readily identified. M_{12} is self-dual and has an automorphism group of size 6, generated by $(c,e)(d,f)(1,5)(3,6)$ and $(a,d)(b,e)(1,4)(2,3)$.

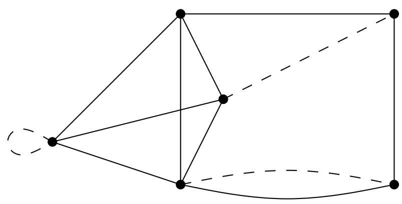

Figure 1: *Signed-graphic representation of M_{12}. Negative edges are dashed; positive edges are solid.*

4 Conjectures

While Theorem 1.5 is a bit of a setback, we remain hopeful that a satisfactory decomposition theory for near-regular matroids can be found.

First of all, the construction in Section 3 employs only graphic matroids. In fact, it seems difficult to extend the regular side of the 4-sum to some strictly near-regular matroid. The proof of Theorem 1.5 suggests the following construction:

Definition 4.1. *Let M_1, M_2 be matroids such that $E(M_1) \cap E(M_2) = X$, $M_1|X = M_2|X \cong M(K_k)$, and M_1 is graphic. Then the* graph k-clique sum *of M_1 and M_2 is $P_X(M_1, M_2) \backslash X$.*

Now we offer the following update of Conjecture 1.4:

Conjecture 4.2. *Let M be a near-regular matroid. Then M can be obtained from matroids that are signed-graphic, the dual of a signed-graphic matroid, or that belong to a finite set $\mathscr{C}$, by applying the following operations:*

(i) 1-, 2-, and 3-sums;

(ii) Graph k-clique sums and their duals, where $k \leq 4$.

Note that the work of Geelen et al. [GGW07], when finished, should imply a decomposition into parts that are bounded-rank perturbations of signed-graphic matroids and their duals. However, the bounds they require on connectivity are huge. Conjecture 4.2 expresses our hope that for near-regular matroids specialized methods will give much more refined results.

As noted in the introduction, Seymour's Decomposition Theorem is not the only ingredient in the proof of Theorem 1.2. Another requirement is that the basic classes can be recognized in polynomial time. The following result suggests that this may not hold for the basic classes of near-regular matroids:

Theorem 4.3. *Let M be a signed-graphic matroid. Let N be a matroid on $E(M)$ given by a rank oracle. It is not possible to decide if $M = N$ using a polynomial number of rank evaluations.*

A matroid is *dyadic* if it is representable over GF(p) for all primes $p > 2$. Since all signed-graphic matroids are dyadic (which was first observed by Dowling [Dow73]), this in turn implies that dyadic matroids are not polynomial-time recognizable.

A proof of Theorem 4.3, analogous to the proof by Seymour [Sey81] that binary matroids are not polynomial-time recognizable, was found by Jim Geelen and, independently, by the first author. It involves ternary swirls, which have a number of circuit-hyperplanes that is exponential in the rank. To test if the matroid under consideration is really the ternary swirl, all these circuit-hyperplanes have to be examined, since relaxing any one of them again yields a matroid.

However, this family of signed-graphic matroids is not near-regular for $n \geq 4$. Hence the complexity of recognizing near-regular signed-graphic matroids is still open. The techniques used by Seymour [Sey81] do not seem to extend, but perhaps some new idea can yield

Conjecture 4.4. *Let $\mathscr{C}$ be the class of near-regular signed-graphic matroids. Then $\mathscr{C}$ is polynomial-time recognizable.*

In fact, we still have some hope for the following:

Conjecture 4.5. *The class of near-regular matroids is polynomial-time recognizable.*

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