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Acyclic edge-colouring of planar graphs. Extended abstract

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Abstract

A proper edge-colouring with the property that every cycle contains edges of at least three distinct colours is called an *acyclic edge-colouring*. The *acyclic chromatic index* of a graph G, denoted $\chi'_a(G)$ is the minimum k such that G admits an *acyclic edge-colouring* with k colours. We conjecture that if G is planar and $\Delta(G)$ is large enough then $\chi'_a(G) = \Delta(G)$. We settle this conjecture for planar graphs with girth at least 5 and outerplanar graphs. We also show that if G is planar then $\chi'_a(G) \leq \Delta(G) + 25$.

Keywords: edge-colouring, graphs of bounded density, planar graph, outerplanar graph

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1 Introduction

A proper edge-colouring with the property that every cycle contains edges of at least three distinct colours is called an *acyclic edge-colouring*. The *acyclic chromatic index* of a graph G, denoted $\chi'_a(G)$ is the minimum k such that G admits an *acyclic edge-colouring* with k colours.

Conjecture 1.1 (Alon et al. [1]) *For every graph G, $\chi'_a(G) \leq \Delta(G) + 2$.*

This conjecture would be tight as there are cases where more than $\Delta+1$ colours are needed. However, the only known such graphs G are subgraphs of K_{2n}. Therefore the following conjecture might even be true:

Conjecture 1.2 (Alon et al. [1]) *If G is a Δ-regular graph then $\chi'_a(G) = \Delta(G)+1$ unless $G = K_{2n}$.*

Molloy and Reed [4] showed $\chi'_a(G) \leq 16\Delta(G)$, which is the best general upper bound so far. For graph with large girth, better upper bounds are known. For example, Alon et al. also showed that Conjecture 1.1 is true for graphs with girth at least $C\Delta\log(\Delta)$ for some fixed constant C.

Muthu et al [5] proved that $\chi'_a(G) \leq 2\Delta(G) + 29$ if G is planar and $\chi'_a(G) \leq \Delta(G) + 6$ if G is planar and triangle-free. In this paper, we improve the first of these two results.

Theorem 1.3 $\chi'_a(G) \leq \Delta(G) + 25$ *for all planar graphs G.*

It is known (see [7]) that a planar graph G is $\Delta(G)$-edge-colourable if $\Delta(G)$ is large. We conjecture that the same is true for acyclic edge-colouring.

Conjecture 1.4 There exists Δ_0 such that every planar graph with maximum degree $\Delta \geq \Delta_0$ has an acyclic edge-colouring with Δ colours.

As evidences to this conjecture, we show that it holds for planar graphs of girth at least 5. In fact we prove the following more general result regarding graphs with bounded maximum average degree. Recall that the *maximum average degree* of G is $Mad(G) = \max\{\frac{2|E(H)|}{|V(H)|} \mid H$ is a subgraph of $G\}$. It is well known that a planar graph of girth g has maximum average degree less than $2 + \frac{4}{g-2}$.

Theorem 1.5 *For any $\epsilon > 0$, there exists an integer Δ_ϵ such that if $\Delta(G) \geq \Delta_\epsilon$ and $Mad(G) \leq 4 - \epsilon$ then $\chi'_a(G) = \Delta(G)$.*

Moreover, we show that Conjecture 1.4 holds for outerplanar graphs. This improve the upper bound $\Delta+1$ for the acyclic edge-chromatic number of such graphs obtained by Muthu et al. [5]. Note that $\sup\{Mad(G) \mid G$ is outerplanar$\} = 4$.

The detailed proofs of all these results may be found in [3]. We just give here sketches of the proofs of Theorem 1.3 and 1.5.

2 Sketch of proofs

The proof of Theorem 1.3 relies heavily on a structural result that can be derived from a theorem of Borodin et al. [2]. Before stating this structural result we need to introduce some notation and terminology. Let G be a graph. A vertex v is said to be *good* if its degree is at most 5 and it has a neighbour v^* such that $\sum_{u \in N(v)\setminus v^*} d(u) \leq 38$ and for all $u \in N(v)\setminus v^*$, $d(u) \leq 25$. We say that G has a bunch of length $m \geq 3$ with *poles* the vertices p and q (where $p \neq q$), if G contains a sequence of paths $(P_1, P_2, \ldots P_m)$ with the following properties. Each P_i has length 1 or 2 and joins p with q. Furthermore, for each $i = 1, \ldots, m-1$, the cycle formed by P_i and P_{i+1} is not separating in G (i.e., has no vertex of G inside). If a path P_i in the bunch has length 2, i.e., $P_i = pv_iq$, then the vertex v_i will be called a *bunch vertex*. A path $P_i = pq$ of length 1 in the bunch will be referred to as a *parental edge*.

Theorem 2.1 below is a relatively straightforward corollary to a theorem in [2], and it will play an important role in our proof of Theorem 1.3 below.

Theorem 2.1 *For every plane graph at least one of the following holds:*

(i) *G has a good vertex v;*

(ii) *G has a vertex v that is a pole for $1 \leq k \leq 6$ bunches of length at least 6, and has at most $24 - 4k$ neighbours that are not part of these bunches.*

Proof of Theorem 1.3 (Sketch) For the sake of presentation we sketch the proof that $\chi'_a(G) \leq \Delta + 38$ for all planar G. The proof of $\Delta + 25$ is very similar but slightly more involved. We consider a minimal counterexample G. We first show that G has the following property:

- If $(P_1, \ldots, P_m)$ is a bunch in G with $m \geq 6$ and if for some $2 \leq i \leq m-2$ neither of the paths P_i, P_{i+1} is a parental edge, then v_iv_{i+1} is not an edge.

This is shown by extending an acyclic edge colouring of $G \setminus v_iv_{i+1}$ with $\Delta + 38$ colours (which exists by minimality), as follows. If $3 \leq i \leq m-3$ then we can find a colour for v_iv_{i+1} that is distinct from the colours used on edges incident with either p, v_{i-1}, v_i, v_{i+1} or v_{i+2} (there are at most $\Delta + 8$ such edges) to obtain an acyclic edge colouring of G. Now consider $i = 2$. As v_3v_4 is not an edge, if v_1v_2 has a colour different from those of pv_3, qv_3 any colour different from the colours of edges indicent with p, v_2 or v_3 (and there are at most $\Delta + 3$ of those) will result in an acyclic edge colouring of G with $\Delta + 38$ colours. If v_1v_2 has the same colour as pv_3 we also pick a colour for v_2v_3 that differs from the colours of edges indicent with p, v_2 or v_3, and if v_1v_2 has the same colour as qv_3 then we pick a colour for v_2v_3 that differs from the colours of edges indicent with q, v_2 or v_3. The case $i = m - 2$ is analogous to the case $i = 2$.

Next, we show:

- G does not have a good vertex.

To prove this, suppose v is a good vertex. We extend an acyclic edge colouring of $G \setminus v$ with $\Delta + 38$ colours as follows. Assume $d(v) = 5$ (the case when $d(v) < 5$ is similar) and write $N(v) = \{v^*, u_1, u_2, u_3, u_4\}$ where $d(v^*) \geq d(u_i)$ for $i = 1, \dots, 4$. We colour vv^* with a colour different from those of edges incident with $v^*, u_1, \dots, u_4$ (there are at most $\Delta - 1 + 38 - 4$ such edges in $G \setminus v$ by definition of a good vertex), and we colour the vu_i with colours different from those of edges incident with $u_1, \dots, u_4$.

Since there is no good vertex, by Theorem 2.1, there exists a vertex p that is a pole for $1 \leq k \leq 6$ bunches of length at least 6 and has at most $24 - 4k$ neighbours not in those bunches. Let us denote those bunches by $B_1 = (P^1_1, \dots, P^1_{m_1}), \dots, B_k = (P^k_1, \dots, P^k_{m_k})$. Let us denote the other pole of B_i by q_i and we will denote the middle vertex of P^i_j by v^i_j if it exists. We have $d(v^i_j) = 2$ for all $3 \leq j \leq m_i - 2$ and $1 \leq i \leq k$. We can assume (w.l.o.g.) that P^1_3 is not a parental edge. By minimality of G there is an acyclic edge colouring of $G \setminus v^1_3$ with $\Delta + 38$ colours. We extend it as follows. For pv^1_3 we choose a colour different from those used on the edges $v^i_1v^i_2, v^i_{m_i-1}v^i_{m_i}$ and the edges incident with p (there are at most $\Delta - 1 + 12$ such edges). Next we colour $q_1v^1_3$ with a colour different from those used on pv^1_3, the edges incident with q_1, the edges $pv^i_1, pv^i_{m_i}, pq_i$, the edges incident with p that do not belong to the bunches and those edges pv^i_j for which $q_iv^i_j$ has the same colour as pv^1_3 (there are at most $1 + \Delta - 1 + 2k + 24 - 4k + k + k = \Delta + 24$ such edges). It can be checked that this indeed gives an acyclic edge colouring of G with at most $\Delta + 38$ colours. □

Proof of Theorem 1.5 (Sketch) By considering a minimal counterexample G.

A *thread* is a path of length two whose internal vertex has degree 2. We first show that G has the following properties:

- G is 2-connected. In particular, $\delta(G) \geq 2$.
- For every vertex $v \in V(G)$, $\sum_{u \in N(v)} d(u) \geq \Delta + 1$.
- A Δ-vertex is the end of at most k threads whose other endvertex has degree at most k.
- A $(\Delta - l)$-vertex is the end of at most $k - 1 - l$ threads whose other endvertex has degree at most k.

We then use the discharging method. We assign an initial charge of $d(v)$ to each vertex v and discharge according to the following rules where $d_\epsilon = \lceil \frac{8}{\epsilon} - 2 \rceil$.

R1: for $4 \leq d < d_\epsilon$, every d-vertex sends $a(d) = 1 - \frac{4-\epsilon}{d}$ to each neighbour.

R2: for $d_\epsilon \leq d \leq \Delta + 1 - d_\epsilon$ then every d-vertex sends $1 - \frac{\epsilon}{2}$ to each neighbour.

R3: for $\Delta + 2 - d_\epsilon \leq d \leq \Delta$ then every d-vertex sends

- $1 - \epsilon$ to each 3-neighbour;
- $2 - \epsilon$ to each 2-neighbour whose second neighbour has degree 2 or 3;
- $b(d) = 2 - \epsilon - a(d)$ to each 2-neighbour whose second neighbour has degree d with $4 \leq d < d_\epsilon$;
- $1 - \frac{\epsilon}{2}$ to each 2-neighbour whose second neighbour has degree $d \geq d_\epsilon$.

We then check that every vertex v has final charge $f(v)$ at least $4 - \epsilon$ if $\Delta \geq \Delta_\epsilon = \left\lceil \frac{2}{\epsilon} \left(\left[d_\epsilon - 2 + 3(2-\epsilon) + \sum_{d=4}^{d_\epsilon - 1} b(d) - (1 - \frac{\epsilon}{2})(d_\epsilon - 1) \right] + 4 - \epsilon \right) \right\rceil$. Hence G has average degree at least $4 - \epsilon$, which is a contradiction. □

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