A NEW SEMIDEFINITE PROGRAMMING HIERARCHY FOR CYCLES IN BINARY MATROIDS AND CUTS IN GRAPHS

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Abstract. The theta bodies of a polynomial ideal are a series of semidefinite programming relaxations of the convex hull of the real variety of the ideal. In this paper we construct the theta bodies of the vanishing ideal of cycles in a binary matroid. Applied to cuts in graphs, this yields a new hierarchy of semidefinite programming relaxations of the cut polytope of the graph. If the binary matroid avoids certain minors we can characterize when the first theta body in the hierarchy equals the cycle polytope of the matroid. Specialized to cuts in graphs, this result solves a problem posed by Lovász.

1. Introduction

A central question in combinatorial optimization is to understand the polyhedral structure of the convex hull, $\text{conv}(S)$, of a finite set $S \subseteq \mathbb{R}^n$. A typical instance is when S is the set of incidence vectors of a finite set of objects over which one is interested to optimize; think for instance of the problem of finding a shortest tour, a maximum independent set, or a maximum cut in a graph. As for hard combinatorial optimization problems one cannot hope in general to be able to find the complete linear description of the polytope $\text{conv}(S)$, the objective is then to find good and efficient approximations of this polytope. Such approximations could be polyhedra, obtained by considering classes of valid linear inequalities. In recent years more general convex semidefinite programming (SDP) relaxations have been considered, which sometimes yield much tighter approximations than those from LP methods. This was the case for instance for the approximation of stable sets and coloring in graphs via the theta number introduced by Lovász [18], and for the approximation of the max-cut problem by Goemans and Williamson [9]. See e.g. [17] for an overview. These results spurred intense

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research activity on constructing stronger SDP relaxations for combinatorial optimization problems (cf. [20, 23, 12, 22, 13, 17]). In this paper we revisit the hierarchy of SDP relaxations proposed by Gouveia et al. [10] which was inspired by a question of Lovász [19]. To present it we need some definitions.

Let $I \subseteq \mathbb{R}[\mathbf{x}]$ be an ideal and $V_{\mathbb{R}}(I) = \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) = 0 \ \forall f \in I\}$ be its real variety. Throughout $\mathbb{R}[\mathbf{x}]$ denotes the ring of multivariate polynomials in n variables $\mathbf{x} = (x_1, \ldots, x_n)$ over $\mathbb{R}$ and $\mathbb{R}[\mathbf{x}]_d$ its subspace of polynomials of degree at most $d \in \mathbb{N}$. As the convex hull of $V_{\mathbb{R}}(I)$ is completely described by the (linear) polynomials $f \in \mathbb{R}[\mathbf{x}]_1$ that are non-negative on $V_{\mathbb{R}}(I)$, relaxations of $\mathrm{conv}(V_{\mathbb{R}}(I))$ can be obtained by considering sufficient conditions for the non-negativity of linear polynomials on $V_{\mathbb{R}}(I)$.

A polynomial $f \in \mathbb{R}[\mathbf{x}]$ is said to be a *sum of squares* (*sos*, for short) if $f = \sum_{i=1}^t g_i^2$ for some polynomials $g_i \in \mathbb{R}[\mathbf{x}]$. Moreover, f is said to be *sos modulo the ideal* I if $f = \sum_{i=1}^t g_i^2 + h$ for some polynomials $g_i \in \mathbb{R}[\mathbf{x}]$ and $h \in I$. In addition, if each g_i has degree at most k, then we say that f is *k-sos modulo I*. Obviously any polynomial which is k-sos modulo I is non-negative over $V_{\mathbb{R}}(I)$. Following [10], for each $k \in \mathbb{N}$, define the set

$$\mathrm{TH}_k(I) := \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) \geq 0 \text{ for all } f \in \mathbb{R}[\mathbf{x}]_1 \ k\text{-sos modulo } I\}, \tag{1}$$

called the *k-th theta body* of the ideal I. Note that $\mathrm{TH}_k(I)$ is a (convex) relaxation of $\mathrm{conv}(V_{\mathbb{R}}(I))$, with

$$\mathrm{conv}(V_{\mathbb{R}}(I)) \subseteq \mathrm{TH}_{k+1}(I) \subseteq \mathrm{TH}_k(I).$$

The ideal I is said to be TH_k-*exact* if the equality $\overline{\mathrm{conv}(V_{\mathbb{R}}(I))} = \mathrm{TH}_k(I)$ holds. The theta bodies $\mathrm{TH}_k(I)$ were introduced in [10], inspired by a question of Lovász [19, Problem 8.3] asking to characterize TH_k-exact ideals, in particular when $k = 1$.

This question of Lovász was motivated by the following result about stable sets in graphs: The stable set ideal of a graph $G = (V, E)$ is TH_1-exact if and only if the graph G is perfect. Recall that a subset of V is *stable* in G if it contains no edge. The *stable set ideal* of G is the vanishing ideal of the 0/1 characteristic vectors of the stable sets in G and is generated by the binomials $x_i^2 - x_i$ ($i \in V$) and $x_i x_j$ ($\{i, j\} \in E$) (cf. [19] for details).

For a graph G, let IG be the vanishing ideal of the incidence vectors of cuts in G, and the cut polytope, $\mathrm{CUT}(G)$, be the convex hull of the incidence vectors of cuts in G. Following Problem 8.3, Problem 8.4 in [19] asks for a characterization of "cut-perfect" graphs which are precisely those graphs G for which IG is TH_1-exact. We answer this question (Corollary 4.12) by studying theta bodies in the more general setting of cycles in binary matroids. As an intermediate step we derive the theta bodies of IG which give rise to a new hierarchy of semidefinite programming relaxations of $\mathrm{CUT}(G)$.

Some notation. Let E be a finite set. For a subset $F \subseteq E$, let $\mathbf{1}^F \in \{0,1\}^E$ denote its 0/1-incidence vector and $\chi^F \in \{\pm 1\}^E$ its ± 1-incidence vector, defined by $\mathbf{1}^F_e = 1$, $\chi^F_e = -1$ if $e \in F$ and $\mathbf{1}^F_e = 0$, $\chi^F_e = 1$ otherwise.

Throughout $\mathbb{R}E := \mathbb{R}[x_e \mid e \in E]$ denotes the polynomial ring with variables indexed by E. If $F \subseteq E$, we set $\mathbf{x}^F := \prod_{e\in F} x_e$. For a symmetric matrix $X \in \mathbb{R}^{n\times n}$, $X \succeq 0$ means that X is positive semidefinite, or equivalently, $\mathbf{u}^T X\mathbf{u} \geq 0$ for all $\mathbf{u} \in \mathbb{R}^n$.

Contents of the paper. Section 2 contains various preliminaries and some results of [10] needed in this paper. In Section 3 we introduce binary matroids, which provide the natural setting to present our results for cuts in graphs. A binary matroid is a pair $\mathcal{M} = (E, \mathcal{C})$ where E is a finite set and $\mathcal{C}$ is a collection of subsets of E (the *cycles* of $\mathcal{M}$) closed under taking symmetric differences; for instance, cuts (resp., cycles) in a graph form binary matroids. In Section 3.1 we present a generating set for the cycle ideal $I\mathcal{M}$ (i.e. the vanishing ideal of the incidence vectors of the cycles $C \in \mathcal{C}$) and a linear basis $\mathcal{B}$ of its quotient space $\mathbb{R}E/I\mathcal{M}$ (cf. Theorem 3.4). Using this, we can explicitly describe the series of theta bodies $\mathrm{TH}_k(I\mathcal{M})$ that approximate the cycle polytope $\mathrm{CYC}(\mathcal{M})$ (i.e. the convex hull of the incidence vectors of the cycles in $\mathcal{C}$). In Section 3.2, we specialize these results to cuts in a graph G and show that $\mathcal{B}$ can then be indexed by T-joins of G. This enables a combinatorial description of the theta bodies $\mathrm{TH}_k(IG)$ that converge to the cut polytope $\mathrm{CUT}(G)$ of G. Section 3.3 compares the semidefinite relaxations $\mathrm{TH}_k(IG)$ to some known semidefinite relaxations of the cut polytope. In Section 3.4 the results from Section 3.1 are specialized to cycles in a graph. Section 4 studies the binary matroids $\mathcal{M}$ whose cycle ideal $I\mathcal{M}$ is TH_1-exact (i.e., $\mathrm{TH}_1(I\mathcal{M}) = \mathrm{CYC}(\mathcal{M})$). Theorem 4.6 characterizes the TH_1-exact cycle ideals $I\mathcal{M}$ when $\mathcal{M}$ does not have the three special minors F_7^*, R_{10} and $\mathcal{M}_{K_5}^*$. As an application, we obtain characterizations of TH_1-exact graphic and cographic matroids, and the latter answers Problem 8.4 in [19]. The paper contains several examples of binary matroids for which we exhibit the least k for which $I\mathcal{M}$ is TH_k-exact. In Section 5 we do this computation for an infinite family of graphs; if C_n is the circuit with n edges, then the smallest k for which $\mathrm{TH}_k(IC_n) = \mathrm{CUT}(C_n)$ is $k = \lceil n/4 \rceil$.

2. Preliminaries

2.1. **Ideals and combinatorial moment matrices.** Let $\mathbb{R}[\mathbf{x}]$ be the polynomial ring over $\mathbb{R}$ in the variables $\mathbf{x} = (x_1, \ldots, x_n)$. A non-empty subset $I \subseteq \mathbb{R}[\mathbf{x}]$ is an *ideal* if I is closed under addition, and multiplication by elements of $\mathbb{R}[\mathbf{x}]$. The ideal generated by $\{f_1, \ldots, f_s\} \subseteq \mathbb{R}[\mathbf{x}]$ is the set $I = \{\sum_{i=1}^s h_i f_i \,:\, h_i \in \mathbb{R}[\mathbf{x}]\}$, denoted as $I = (f_1, \ldots, f_s)$. For $S \subseteq \mathbb{R}^n$, the *vanishing ideal* of S is $\mathcal{I}(S) := \{f \in \mathbb{R}[\mathbf{x}] \mid f(\mathbf{x}) = 0 \ \forall \mathbf{x} \in S\}$. For $W \subseteq [n]$, $I_W := I \cap \mathbb{R}[x_i \mid i \in W]$ is the *elimination ideal* of I with respect to W.

An ideal $I \subseteq \mathbb{R}[\mathbf{x}]$ is said to be *zero-dimensional* if its (complex) variety:

$$V_{\mathbb{C}}(I) := \{\mathbf{x} \in \mathbb{C}^n \mid f(x) = 0 \ \ \forall f \in I\},$$

is finite, I is *radical* if $f^m \in I$ implies $f \in I$ for any $f \in \mathbb{R}[\mathbf{x}]$, and I is *real radical* if $f^{2m} + \sum_{i=1}^t g_i^2 \in I$ implies $f \in I$ for all $f, g_i \in \mathbb{R}[\mathbf{x}]$. By the

Real Nullstellensatz (cf. [4]), I is real radical if and only if $I = \mathcal{I}(V_{\mathbb{R}}(I))$. Therefore, I is zero-dimensional and real radical if and only if $I = \mathcal{I}(S)$ for a finite set $S \subseteq \mathbb{R}^n$. If I is real radical, and π_W denotes the projection from $\mathbb{R}^{[n]}$ to $\mathbb{R}^W$, then the elimination ideal I_W is the vanishing ideal of $\pi_W(V_{\mathbb{R}}(I))$, and there is a simple relationship between the k-th theta body of I and that of its elimination ideal I_W:

$$\pi_W(\mathrm{TH}_k(I)) \subseteq \mathrm{TH}_k(I_W). \tag{2}$$

The quotient space $\mathbb{R}[\mathbf{x}]/I$ is a $\mathbb{R}$-vector space whose elements, called the *cosets* of I, are denoted as $f + I$ ($f \in \mathbb{R}[\mathbf{x}]$). For $f, g \in \mathbb{R}[\mathbf{x}]$, $f + I = g + I$ if and only if $f - g \in I$. The degree of $f + I$ is defined as the smallest possible degree of $g \in \mathbb{R}[\mathbf{x}]$ such that $f - g \in I$. The vector space $\mathbb{R}[\mathbf{x}]/I$ has finite dimension if and only if I is zero-dimensional; moreover, $|V_{\mathbb{C}}(I)| \leq \dim \mathbb{R}[\mathbf{x}]/I$, with equality if and only if I is radical.

Gouveia et al. [10] give a geometric characterization of zero-dimensional real radical ideals that are TH_1-exact.

Definition 2.1. For $k \in \mathbb{N}$, a finite set $S \subseteq \mathbb{R}^n$ is said to be *k-level* if $|\{f(\mathbf{x}) \mid \mathbf{x} \in S\}| \leq k$ for all $f \in \mathbb{R}[\mathbf{x}]_1$ for which the linear inequality $f(\mathbf{x}) \geq 0$ induces a facet of the polytope $\mathrm{conv}(S)$.

Theorem 2.2. [10] *Let $S \subseteq \mathbb{R}^n$ be a finite set. The ideal $\mathcal{I}(S)$ is TH_1-exact (i.e., $\mathrm{conv}(S) = \mathrm{TH}_1(\mathcal{I}(S))$) if and only if S is a 2-level set.*

More generally, Gouveia et al. [10, Section 4] show the implication:

$$S \text{ is } (k+1)\text{-level} \implies \mathcal{I}(S) \text{ is } \mathrm{TH}_k\text{-exact}; \tag{3}$$

the reverse implication however does not hold for $k \geq 2$ (see e.g. Remark 5.8 for a counterexample).

We now mention an alternative more explicit formulation for the theta body $\mathrm{TH}_k(I)$ of an ideal I in terms of positive semidefinite combinatorial moment matrices. We first recall this class of matrices (introduced in [16]) which amounts to using the equations defining I to reduce the number of variables. Let $\mathcal{B} = \{b_0 + I, b_1 + I, \ldots\}$ be a basis of $\mathbb{R}[\mathbf{x}]/I$ and, for $k \in \mathbb{N}$, let $\mathcal{B}_k := \{b + I \in \mathcal{B} \mid \deg(b + I) \leq k\}$. Then any polynomial $f \in \mathbb{R}[\mathbf{x}]$ has a unique decomposition $f = \sum_{l \geq 0} \lambda_l^{(f)} b_l$ modulo I; we let $\lambda^{(f)} = (\lambda_l^{(f)})_l$ denote the vector of coordinates of the coset $f + I$ in the basis $\mathcal{B}$ (which has only finitely many non-zero coordinates).

Definition 2.3. Let $\mathbf{y} \in \mathbb{R}^{\mathcal{B}}$. The *combinatorial moment matrix* $M_{\mathcal{B}}(\mathbf{y})$ is the (possibly infinite) matrix indexed by $\mathcal{B}$ whose (i, j)-th entry is

$$\sum_{l \geq 0} \lambda_l^{(b_i b_j)} y_l.$$

The k-th truncated combinatorial moment matrix $M_{\mathcal{B}_k}(\mathbf{y})$ is the principal submatrix of $M_{\mathcal{B}}(\mathbf{y})$ indexed by $\mathcal{B}_k$.

In other words, the matrix $M_{\mathcal{B}}(\mathbf{y})$ is obtained as follows. The coordinates y_l's correspond to the elements b_l+I of $\mathcal{B}$; expand the product b_ib_j in terms of the basis $\mathcal{B}$ as $b_ib_j = \sum_l \lambda_l^{(b_ib_j)} b_l$ modulo I; then the (b_i,b_j)-th entry of $M_{\mathcal{B}}(\mathbf{y})$ is its 'linearization': $\sum_l \lambda_l^{(b_ib_j)} y_l$.

To control which entries of $\mathbf{y}$ are involved in the truncated matrix $M_{\mathcal{B}_k}(\mathbf{y})$, it is useful to suitably choose the basis $\mathcal{B}$. Namely, we choose $\mathcal{B}$ satisfying the following property:

$$\deg(f+I) \le k \Longrightarrow f+I \in \mathrm{span}(\mathcal{B}_k). \tag{4}$$

This is true, for instance, when $\mathcal{B}$ is the set of *standard monomials* of a *term order* that respects degree. (See [5, Chapter 2] for these notions that come from Gröbner basis theory.) If $\mathcal{B}$ satisfies (4), then the entries of $M_{\mathcal{B}_k}(\mathbf{y})$ depend only on the entries of $\mathbf{y}$ indexed by $\mathcal{B}_{2k}$. Moreover, Gouveia et al. [10] show that $\mathrm{TH}_k(I)$ can then be defined using the matrices $M_{\mathcal{B}_k}(\mathbf{y})$, up to closure and a technical condition on $\mathcal{B}$. This technical condition, which states that $\{1+I, x_1+I, \dots, x_n+I\}$ is linearly independent in $\mathbb{R}[\mathbf{x}]/I$, is however quite mild since if there is a linear dependency then it can be used to eliminate variables.

Example 2.4. Consider the ideal $I = (x_1^2x_2-1) \subset \mathbb{R}[x_1,x_2]$. Note that $\mathcal{B} = \bigcup_{k\in\mathbb{N}}\{x_1^k+I, x_2^k+I, x_1x_2^k+I\}$ is a monomial basis for $\mathbb{R}[x_1,x_2]/I$ satisfying (4) for which

$$\mathcal{B}_4 = \{1, x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^3, x_1x_2^2, x_2^3, x_1^4, x_1x_2^3, x_2^4\} + I.$$

The combinatorial moment matrix $M_{\mathcal{B}_2}(\mathbf{y})$ for $\mathbf{y} = (y_0, y_1, \dots, y_{11}) \in \mathbb{R}^{\mathcal{B}_4}$ is

	1	x_1	x_2	x_1^2	x_1x_2	x_2^2
1	y_0	y_1	y_2	y_3	y_4	y_5
x_1	y_1	y_3	y_4	y_6	1	y_7
x_2	y_2	y_4	y_5	1	y_7	y_8
x_1^2	y_3	y_6	1	y_9	y_1	y_2
x_1x_2	y_4	1	y_7	y_1	y_2	y_{10}
x_2^2	y_5	y_7	y_8	y_2	y_{10}	y_{11}

.

Theorem 2.5. [10] *Assume $\mathcal{B}$ satisfies (4) and $\mathcal{B}_1 = \{1+I, x_1+I, \dots, x_n+I\}$, and let the coordinates of $\mathbf{y} \in \mathbb{R}^{\mathcal{B}_{2k}}$ indexed by $\mathcal{B}_1$ be $y_0, y_1, \dots, y_n$. Then $\mathrm{TH}_k(I)$ is equal to the closure of the set*

$$\{(y_1,\dots,y_n) \mid \mathbf{y} \in \mathbb{R}^{\mathcal{B}_{2k}} \text{ with } M_{\mathcal{B}_k}(\mathbf{y}) \succeq 0 \text{ and } y_0 = 1\}. \tag{5}$$

When $I = \mathcal{I}(S)$ where $S \subseteq \{0,1\}^n$, the closure is not needed and $\mathrm{TH}_k(I)$ equals the set (5).

Theorem 2.5 implies that optimizing a linear objective function over $\mathrm{TH}_k(I)$ can be reformulated as a semidefinite program with the constraints $M_{\mathcal{B}_k}(\mathbf{y}) \succeq 0$ and $y_0 = 1$ which, for fixed k, can thus be solved in polynomial time (to any precision).

2.2. **Graphs, cuts and cycles.** Let $G = (V, E)$ be a graph. Throughout, the vertex set is $V = [n]$, the edge set of the complete graph K_n is denoted by E_n, so that E is a subset of E_n, and the edges of E_n correspond to pairs $\{i, j\}$ of distinct vertices $i, j \in V$. For $F \subseteq E$, $\deg_F(v)$ denotes the number of edges of F incident to $v \in V$. A *circuit* is a set of edges $\{\{i_1, i_2\}, \{i_2, i_3\}, \ldots, \{i_{t-1}, i_t\}, \{i_t, i_1\}\}$ where $i_1, \ldots, i_t \in V$ are pairwise distinct vertices. A set $C \subseteq E$ is a *cycle* (or Eulerian subgraph) if $\deg_C(v)$ is even for all $v \in V$; every non-empty cycle is an edge-disjoint union of circuits. For $S \subseteq V$, the *cut* D corresponding to the partition $(S, V \setminus S)$ of V is the set of edges $\{i, j\} \in E$ with $|\{i, j\} \cap S| = 1$. A basic property is that each cut intersects each cycle in an even number of edges; this is in fact a property of binary matroids which is why we will present some of our results later in the more general setting of binary matroids (cf. Section 3).

Each cut D can be encoded by its ± 1-incidence vector $\chi^D \in \{\pm 1\}^E$, called the *cut vector* of D. The *cut ideal* of G, denoted as IG, is the vanishing ideal of the set of cut vectors of G. The *cut polytope* of G is

$$\mathrm{CUT}(G) := \mathrm{conv}\{\chi^D \mid D \text{ is a cut in } G\} = \pi_E(\mathrm{CUT}(K_n)) \subseteq \mathbb{R}^E, \tag{6}$$

where π_E is the projection from $\mathbb{R}^{E_n}$ onto $\mathbb{R}^E$. (Cf. e.g. [6] for an overview on the cut polytope.) The cuts of K_n can also be encoded by the *cut matrices* $X := \mathbf{x}\mathbf{x}^T$ for $\mathbf{x} \in \{\pm 1\}^n$ indexing the partitions of $[n]$ corresponding to the cuts. Thus the set

$$\{\mathbf{y} \in \mathbb{R}^E \mid \exists X \in \mathbb{R}^{V \times V}, X \succeq 0,\ X_{ii} = 1\ (i \in V),\ X_{ij} = y_{\{i,j\}}\ (\{i, j\} \in E)\} \tag{7}$$

is a relaxation of the cut polytope $\mathrm{CUT}(G)$, over which one can optimize any linear objective function in polynomial time (to any precision), using semidefinite optimization.

Given edge weights $\mathbf{w} \in \mathbb{R}^E$, the max-cut problem asks for a cut D in G of maximum total weight $\sum_{e \in D} w_e$; thus it can be formulated as

$$\max \left\{ \frac{1}{2} \sum_{e \in E} w_e (1 - y_e) \mid \mathbf{y} \in \mathrm{CUT}(G) \right\}, \tag{8}$$

where the variable can alternatively be assumed to lie in $\mathrm{CUT}(K_n)$. This is a well-known NP-hard problem [8]. Thus one is interested in finding tight efficient relaxations of the cut polytope, potentially leading to good approximations for the max-cut problem. It turns out that the simple semidefinite programming relaxation (7) has led to the celebrated 0.878-approximation algorithm of Goemans and Williamson [9] which, as of today, still gives the best known performance guarantee for max-cut.

3. Theta bodies for cuts and matroids

In this section we study in detail the hierarchy of SDP relaxations for the cut polytope arising from the theta bodies of the cut ideal. As is well-known, cuts in graphs form a special class of binary matroids. It is thus natural to consider the theta bodies in the more general setting of binary matroids, where the results become more transparent. Then we will apply the results to cuts in graphs (the case of cographic matroids) and also to cycles in graphs (the case of graphic matroids).

3.1. **The cycle ideal of a binary matroid and its theta bodies.** Let $\mathcal{M} = (E, \mathcal{C})$ be a binary matroid; that is, E is a finite set and $\mathcal{C}$ is a collection of subsets of E that is closed under taking symmetric differences. Members of $\mathcal{C}$ are called the *cycles* of $\mathcal{M}$, and members of the set

$$\mathcal{C}^* := \{D \subseteq E : |D \cap C| \text{ even } \ \forall C \in \mathcal{C}\}$$

are called the *cocycles* of $\mathcal{M}$. Then, $\mathcal{M}^* = (E, \mathcal{C}^*)$ is again a binary matroid, known as the *dual matroid* of $\mathcal{M}$, and $(\mathcal{M}^*)^* = \mathcal{M}$. The (inclusion-wise) minimal non-empty cycles (cocycles) of $\mathcal{M}$ are called the *circuits* (*cocircuits*) of $\mathcal{M}$. An element $e \in E$ is a *loop* (*coloop*) of $\mathcal{M}$ if $\{e\}$ is a circuit (cocircuit) of $\mathcal{M}$. Two distinct elements $e, f \in E$ are *parallel* (*coparallel*) if $\{e, f\}$ is a circuit (cocircuit) of $\mathcal{M}$. Every non-empty cycle is a disjoint union of circuits. Given $C \in \mathcal{C}$, an element $e \in E \setminus C$ is called a *chord* of C if there exist $C_1, C_2 \in \mathcal{C}$ such that $C_1 \cap C_2 = \{e\}$ and $C = C_1 \Delta C_2$ (if C is a circuit then C_1, C_2 are in fact circuits); C is said to be *chordless* if it has no chord. Here is a property of chords that we will use later.

Lemma 3.1. *Let C be a circuit of $\mathcal{M}$, let $e \in E \setminus C$ be a chord of C and C_1, C_2 be circuits with $C = C_1 \Delta C_2$ and $C_1 \cap C_2 = \{e\}$. Then each C_i has strictly fewer chords than C.*

Proof: It suffices to show that each chord e' of C_1 is also a chord of C. For this let C_1', C_1'' be two circuits with $C_1' \cap C_1'' = \{e'\}$ and $C_1 = C_1' \Delta C_1''$. Say, $e \in C_1'$, and thus $e \notin C_1''$. Suppose first that $e' \in C_2$. Then we have $C_1'' \cap C_2 = \{e'\}$ and $C_1'' \Delta C_2 \subseteq C$. As C is a circuit and $C_1'' \neq C_2$, we deduce that $C = C_1'' \Delta C_2$, which shows that e' is a chord of C.

Suppose now that $e' \notin C_2$. Then, $C = C_1 \Delta C_2 = (C_1' \Delta C_2) \Delta C_1''$ with $(C_1' \Delta C_2) \cap C_1'' = \{e'\}$, which shows again that e' is a chord of C. □

The binary matroids on E correspond to the GF(2)-vector subspaces of $\mathrm{GF}(2)^E$, where GF(2) is the two-element field $\{0, 1\}$ with addition modulo 2. Namely, identifying a set $F \subseteq E$ with its 0/1-incidence vector $\mathbf{1}^F \in \mathrm{GF}(2)^E$, the set of cycles $\mathcal{C}$ is a vector subspace of $\mathrm{GF}(2)^E$ and the set of cocycles $\mathcal{C}^*$ is its orthogonal complement. Thus the cycles of a binary matroid also arise as the solutions in $\mathrm{GF}(2)^E$ of a linear system $M\mathbf{x} = 0$, where M is a matrix with columns indexed by E, called a *representation matrix* of the matroid. In what follows we will use $\mathcal{C}$ (and $\mathcal{C}^*$) both as a collection of subsets of E and as a $GF(2)$-vector space.

As before let $\mathbb{R}E := \mathbb{R}[x_e \mid e \in E]$ and, for $C \in \mathcal{C}$, let $\chi^C \in \{\pm 1\}^E$ denote its ± 1-incidence vector, called its *cycle vector*. Then,

$$\mathrm{CYC}(\mathcal{M}) := \mathrm{conv}(\chi^C \mid C \in \mathcal{C})$$

is the *cycle polytope* of $\mathcal{M}$ and

$$I\mathcal{M} := \mathcal{I}(\chi^C \mid C \in \mathcal{C})$$

is the vanishing ideal of the cycle vectors of $\mathcal{M}$, called the *cycle ideal* of $\mathcal{M}$. Thus $I\mathcal{M}$ is a real radical zero-dimensional ideal in $\mathbb{R}E$.

We first study the quotient space $\mathbb{R}E/I\mathcal{M}$. For this consider the set

$$\mathcal{H} := \{x_e^2 - 1 \ (e \in E),\ 1 - \mathbf{x}^D \ (D \text{ chordless cocircuit of } \mathcal{M})\}. \tag{9}$$

Obviously, $\mathcal{H} \subseteq I\mathcal{M}$; Theorem 3.4 below shows that $\mathcal{H}$ in fact generates the ideal $I\mathcal{M}$. First we observe that $\mathcal{H}$ also generates all binomials $\mathbf{x}^A - \mathbf{x}^B$ where $A \cup B$ partitions any cocycle of $\mathcal{M}$.

Lemma 3.2. *Let $D \in \mathcal{C}^*$ be partitioned as $D = A \cup B$. Then, $\mathbf{x}^A - \mathbf{x}^B \in (\mathcal{H})$.*

Proof: First we note that it suffices to show that $1 - \mathbf{x}^D \in (\mathcal{H})$ for all $D \in \mathcal{C}^*$. Indeed, for any partition $A \cup B = D$, $\mathbf{x}^A(1 - \mathbf{x}^D) = \mathbf{x}^A - (\mathbf{x}^A)^2\mathbf{x}^B \equiv \mathbf{x}^A - \mathbf{x}^B$ modulo $(\mathcal{H})$. Thus $1 - \mathbf{x}^D \in (\mathcal{H})$ implies $\mathbf{x}^A - \mathbf{x}^B \in (\mathcal{H})$.

Next, we show the lemma for the case when D is a cocircuit, using induction on the number p of its chords. If $p = 0$ then $1 - \mathbf{x}^D \in \mathcal{H}$ by definition. So let $p \geq 1$, let e be a chord of D and let D_1, D_2 be cocircuits with $D = D_1 \Delta D_2$ and $D_1 \cap D_2 = \{e\}$. Then, $1 - \mathbf{x}^{D_1}, 1 - \mathbf{x}^{D_2} \in (\mathcal{H})$, using the induction assumption, since each D_i has at most $p - 1$ chords by Lemma 3.1. We have: $1 - \mathbf{x}^D \equiv 1 - (x_e)^2 \mathbf{x}^{D_1 \setminus \{e\}} \mathbf{x}^{D_2 \setminus \{e\}} = 1 - \mathbf{x}^{D_1}\mathbf{x}^{D_2} = \mathbf{x}^{D_1}(1 - \mathbf{x}^{D_2}) + 1 - \mathbf{x}^{D_1}$, where the first equality is modulo $(\mathcal{H})$. This shows that $1 - \mathbf{x}^D \in (\mathcal{H})$.

Finally we show the lemma for $D \in \mathcal{C}^*$, using induction on the number p of cocircuits in a partition of D. For this, let $D = D_1 \cup D_2$, where D_1 is a cocircuit and D_2 is a cocycle partitioned into $p - 1$ cocircuits. Then, by the previous case, $1 - \mathbf{x}^{D_1} \in (\mathcal{H})$, and $1 - \mathbf{x}^{D_2} \in (\mathcal{H})$ by the induction assumption. Then, $1 - \mathbf{x}^D \equiv (\mathbf{x}^{D_2})^2 - \mathbf{x}^{D_1}\mathbf{x}^{D_2} = \mathbf{x}^{D_2}(1 - \mathbf{x}^{D_1}) - \mathbf{x}^{D_2}(1 - \mathbf{x}^{D_2})$, where the first equality is modulo $(\mathcal{H})$. This implies $1 - \mathbf{x}^D \in (\mathcal{H})$. $\square$

Define the relation '$\sim$' on $\mathcal{P}(E)$, the collection of all subsets of E, by

$$F \sim F' \ \text{ if } F \Delta F' \in \mathcal{C}^*; \tag{10}$$

this is an equivalence relation, since $\mathcal{C}^*$ is closed under taking symmetric differences. The next lemma characterizes the equivalence classes.

Lemma 3.3. *For $F, F' \subseteq E$, we have:*

$$F \Delta F' \in \mathcal{C}^* \Longleftrightarrow \mathbf{x}^F - \mathbf{x}^{F'} \in (\mathcal{H}) \Longleftrightarrow \mathbf{x}^F - \mathbf{x}^{F'} \in I\mathcal{M}.$$

Proof: If $F \Delta F' \in \mathcal{C}^*$, then $\mathbf{x}^F - \mathbf{x}^{F'} = \mathbf{x}^{F \cap F'}(\mathbf{x}^{F \setminus F'} - \mathbf{x}^{F' \setminus F}) \in (\mathcal{H})$, using Lemma 3.2; $\mathbf{x}^F - \mathbf{x}^{F'} \in (\mathcal{H}) \Longrightarrow \mathbf{x}^F - \mathbf{x}^{F'} \in I\mathcal{M}$ follows from $\mathcal{H} \subseteq I\mathcal{M}$. Conversely, if $\mathbf{x}^F - \mathbf{x}^{F'} \in I\mathcal{M}$ then, for any $C \in \mathcal{C}$, $\mathbf{x}^F - \mathbf{x}^{F'}$ vanishes at

χ^C and thus $|C \cap F|$ and $|C \cap F'|$ have the same parity, which implies that $|C \cap (F \Delta F')|$ is even and thus $F \Delta F' \in \mathcal{C}^*$. □

Let

$$\mathcal{F} := \{F_1, \ldots, F_N\} \tag{11}$$

be a set of distinct representatives of the equivalence classes of $\mathcal{P}(E)/\sim$ and set

$$\mathcal{B} := \{\mathbf{x}^F + I\mathcal{M} \mid F \in \mathcal{F}\}. \tag{12}$$

Theorem 3.4. *The set $\mathcal{B}$ is a basis of the vector space $\mathbb{R}E/I\mathcal{M}$ and the set $\mathcal{H}$ generates the ideal $I\mathcal{M}$.*

Proof: First, we show that $\mathcal{B}$ spans the space $\mathbb{R}E/(\mathcal{H})$. As $x_e^2 - 1 \in \mathcal{H}$ ($\forall e \in E$), it suffices to show that $\mathcal{B}$ spans all cosets of square-free monomials. For this, let $F \subseteq E$ and, say, $F \sim F_1$; then, $\mathbf{x}^F - \mathbf{x}^{F_1} \in I\mathcal{M}$ by Lemma 3.3, which shows that $\mathbf{x}^F + I\mathcal{M} \in \mathrm{span}(\mathcal{B})$. Therefore, we obtain:

$$|\mathcal{C}| = \dim \mathbb{R}E/I\mathcal{M} \leq \dim \mathbb{R}E/(\mathcal{H}) \leq |\mathcal{B}| = N.$$

To conclude the proof it now suffices to show that $|\mathcal{C}| = N$. For this, fix a basis $\{C_1, \ldots, C_m\}$ of the GF(2)-vector space $\mathcal{C}$, so that $|\mathcal{C}| = 2^m$. Let M be the $m \times |E|$ matrix whose rows are the 0/1-incidence vectors of $C_1, \ldots, C_m$. Then $M\mathbf{x}$ takes 2^m distinct values for all $\mathbf{x} \in \mathrm{GF}(2)^E$. As, for $F, F' \subseteq E$, $F \sim F'$ if and only if $M\mathbf{1}^F = M\mathbf{1}^{F'}$, we deduce that the equivalence relation (10) has $N = 2^m$ equivalence classes. □

We now consider the combinatorial moment matrices for the cycle ideal $I\mathcal{M}$. For any integer k define the set

$$\mathcal{F}_k := \{F \in \mathcal{F} \mid \exists D \in \mathcal{C}^* \text{ with } |F \Delta D| \leq k\}$$

corresponding to the equivalence classes of $\sim$ having a representative of cardinality at most k. Then $\mathcal{B}_k = \{\mathbf{x}^F + I\mathcal{M} \mid F \in \mathcal{F}_k\}$ can be identified with the set $\mathcal{F}_k$. Moreover relation (4) holds, so that the entries of the truncated moment matrix $M_{\mathcal{B}_k}(\mathbf{y})$ depend only on the entries of $\mathbf{y}$ indexed by $\mathcal{B}_{2k}$. For instance, $\mathcal{F}_1$ can be any maximal subset of E containing no coloops or coparallel elements of $\mathcal{M}$, along with $\emptyset$. Indeed, $e \in E$ is a coloop precisely if $\{e\} \sim \emptyset$, and two elements $e \neq f \in E$ are coparallel precisely if $e \sim f$. Thus, $\mathcal{F}_0 = \{\emptyset\}$ and $\mathcal{F}_1 \setminus \mathcal{F}_0 = E$ if $\mathcal{M}$ has no coloops and no coparallel elements.

When $\mathcal{M}$ has no coloops and no coparallel elements, its k-th theta body $\mathrm{TH}_k(I\mathcal{M})$ consists of the vectors $\mathbf{y} \in \mathbb{R}^E$ for which there exists a positive semidefinite $|\mathcal{F}_k| \times |\mathcal{F}_k|$ matrix X satisfying $X_{\emptyset,e} = y_e$ for all $e \in E$ and

$$\begin{array}{ll} \text{(i)} & X_{\emptyset,\emptyset} = 1, \\ \text{(ii)} & X_{F_1,F_2} = X_{F_3,F_4} \text{ if } F_1 \Delta F_2 \Delta F_3 \Delta F_4 \in \mathcal{C}^*. \end{array} \tag{13}$$

Remark 3.5. The constraints (13)(ii) contain in particular the constraints

$$X_{F_1,F_2} = X_{F_3,F_4} \text{ if } F_1 \Delta F_2 = F_3 \Delta F_4. \tag{14}$$

Note that the above constraints are the basic 'moment constraints', which are satisfied by all ± 1 vectors. Indeed, if $\mathbf{y} = \chi^F \in \{-1,1\}^E$, define the $|\mathcal{F}_k| \times |\mathcal{F}_k|$ matrix X by $X_{F_1,F_2} := (-1)^{|F\cap F_1|}(-1)^{|F\cap F_2)|}$, so that $y_e = X_{\emptyset,e}$ ($e \in E$). Then $X \succeq 0$ since $X = \mathbf{u}\mathbf{u}^T$ where $\mathbf{u} = ((-1)^{|F\cap F_i|})_{F_i \in \mathcal{F}_k}$, and X satisfies (13)(i) and (14). Therefore the constraints (14) do not cut off any point of the cube $[-1,1]^E$. Non-trivial constraints that cut off points of $[-1,1]^E$ that do not lie in $\mathrm{CYC}(\mathcal{M})$ come from those constraints (13)(ii) where $F_1 \Delta F_2 \Delta F_3 \Delta F_4$ is a non-empty cocycle.

Remark 3.6. Checking whether $F \in \mathcal{F}_k$ amounts to finding a minimum cardinality representative in the equivalence class of F for (10) which might be a hard problem. Indeed, this amounts to solving

$$\min |F \Delta D| \text{ such that } D \in \mathcal{C}^*$$

or equivalently

$$\max \mathbf{w}^T \mathbf{x} \text{ such that } \mathbf{x} \in \mathrm{CYC}(\mathcal{M}^*), \tag{15}$$

after defining $\mathbf{w} \in \mathbb{R}^E$ by $w_e = -1$ for $e \in F$ and $w_e = 1$ for $e \in E \setminus F$ (and noting that $\mathbf{w}^T \chi^D = |E| - 2|F\Delta D|$). As we find in Sections 3.2 and 3.4, (15) is the (polynomial-time solvable) maximum T-join problem when $\mathcal{M}$ is a cographic matroid and the (NP hard) maximum cut problem when $\mathcal{M}$ is a graphic matroid.

However if we fix the cardinality of F, then the problem becomes easy (by enumeration), so that it is still possible to construct the truncated combinatorial moment matrix $M_{\mathcal{B}_k}(\mathbf{y})$ (for fixed k).

3.2. **Application to cuts in graphs.** Binary matroids arise naturally from graphs in the following way. Let $G = ([n], E)$ be a graph, let $\mathcal{C}_G$ denote its collection of cycles, and $\mathcal{D}_G$ its collection of cuts. Since $\mathcal{C}_G$ and $\mathcal{D}_G$ are closed under symmetric difference, both $\mathcal{M}_G := (E, \mathcal{C}_G)$ and $\mathcal{M}_G^* := (E, \mathcal{D}_G)$ are binary matroids, and since each cut has an even intersection with each cycle, they are duals of each other. The matroid M_G is known as the *graphic matroid* of G and $\mathcal{M}_G^*$ as its *cographic matroid*.

We consider here the case when $\mathcal{M} = \mathcal{M}_G^*$ is the cographic matroid of $G = ([n], E)$. Then, $\mathrm{CYC}(\mathcal{M}) = \mathrm{CUT}(G)$ is the cut polytope of G and $I\mathcal{M}$ is the *cut ideal* of G (denoted earlier by IG), thus defined as the vanishing ideal of all cut vectors in G.

So IG is an ideal in $\mathbb{R}E$, while IK_n is an ideal in $\mathbb{R}E_n$. One can easily verify that IG is the elimination ideal, $IK_n \cap \mathbb{R}E$, of IK_n with respect to E. By Theorem 3.4, we know that the (edge) binomials $x_e^2 - 1$ ($e \in E$) together with the binomials $1 - \mathbf{x}^C$ (C chordless circuit of G) generate the cut ideal IG. When $G = K_n$ is a complete graph, the only chordless circuits are the triangles so that, beside the edge binomials, it suffices to consider the binomials $1-x_{\{i,j\}}x_{\{i,k\}}x_{\{j,k\}}$ (or $x_{\{i,j\}}-x_{\{i,k\}}x_{\{j,k\}}$) for distinct $i,j,k \in [n]$.

When G is connected, there are 2^{n-1} distinct cuts in G (corresponding to the partitions of $[n]$ into two classes) and, when G has p connected components, there are 2^{n-p} cuts in G and thus $\dim \mathbb{R}E/IG = 2^{n-p}$.

The following notion of T-joins arises naturally when considering the equivalence relation (10). Given a set $T \subseteq [n]$, a set $F \subseteq E$ is called a *T-join* if $T = \{v \in [n] \mid \deg_F(v) \text{ is odd}\}$. For instance, the $\emptyset$-joins are the cycles of G and, for $T = \{s,t\}$, the minimum T-joins correspond to the shortest $s-t$ paths in G. If F is a T-join and F' is T'-join, then $F\Delta F'$ is a $(T\Delta T')$-join. In particular, $F \sim F'$, i.e. $F\Delta F' \in \mathcal{C}_G$, precisely when F, F' are both T-joins for the same $T \subseteq [n]$.

Thus the equivalence classes of $\sim$ correspond to the members of the set $\mathcal{T}_G := \{T \subseteq [n] \mid \exists T\text{-join in } G\}$ (which consists of the sets $T_1 \cup \ldots \cup T_p$, where each T_i is an even subset of V_i and $V_1, \ldots, V_p$ are the connected components of G). The set $\mathcal{F}$ (in (11)) consists of one T-join F_T for each $T \in \mathcal{T}_G$, and $\mathcal{F}_k = \{F_T \mid T \in \mathcal{T}_k\}$, after defining $\mathcal{T}_k$ as the set of all $T \in \mathcal{T}_G$ for which there exists a T-join of size at most k. Then the corresponding basis of $\mathbb{R}E/IG$ is $\mathcal{B} = \{\mathbf{x}^{F_T} + IG \mid T \in \mathcal{T}_G\}$, $\mathcal{B}_k = \{\mathbf{x}^{F_T} + IG \mid T \in \mathcal{T}_k\}$ and (4) holds.

For instance, $\mathcal{F}_1$ consists of all edges $e \in E$ together with the empty set. Hence the first order theta body $\mathrm{TH}_1(IG)$ consists of the vectors $\mathbf{y} \in \mathbb{R}^E$ for which there exists a positive semidefinite matrix X indexed by $E \cup \{\emptyset\}$ satisfying $y_e = X_{\emptyset,e}$ $(e \in E)$ and

$$\begin{array}{ll} \text{(i)} & X_{\emptyset,\emptyset} = X_{e,e} = 1 \ \text{ for all } e \in E, \\ \text{(ii)} & X_{e,f} = X_{\emptyset,g} \text{ if } \{e,f,g\} \text{ is a triangle in } G, \\ \text{(iii)} & X_{e,f} = X_{g,h} \text{ if } \{e,f,g,h\} \text{ is a circuit in } G. \end{array} \tag{16}$$

Remark 3.7. When $G = K_n$ is the complete graph, for any even $T \subseteq [n]$, the minimum cardinality of a T-join is $|T|/2$; just choose for F_T a set of $|T|/2$ disjoint edges (i.e. a perfect matching) on T. Hence the set $\mathcal{T}_k$ consists of all even $T \subseteq [n]$ with $|T| \leq 2k$. As an illustration, if we index the combinatorial moment matrices by $\mathcal{T}_k$, then the condition (13)(ii) reads:

$$X_{T_1,T_2} = X_{T_3,T_4} \quad \text{if } \ T_1\Delta T_2 = T_3\Delta T_4. \tag{17}$$

This observation will enable us to relate the theta body hierarchy to the semidefinite relaxations of the cut polytope considered in [15], cf. Section 3.3.

Example 3.8. If G has no circuit of length 3 or 4, then $\mathrm{TH}_1(IG) = [-1,1]^E$, since the conditions (16)(ii)-(iii) are void. For instance, if G is a forest, then $\mathrm{TH}_1(IG) = [-1,1]^E = \mathrm{CUT}(G)$ and thus IG is TH_1-exact. On the other hand, if $G = C_n$ is a circuit of length $n \geq 5$, then $\mathrm{TH}_1(IG) = [-1,1]^E$ strictly contains the polytope $\mathrm{CUT}(C_n)$ (as $|E| = n$ and $\mathrm{CUT}(C_n)$ has only 2^{n-1} vertices). Thus IC_n is not TH_1-exact for $n \geq 5$.

Example 3.9. For $G = K_5$, IK_5 is not TH_1-exact. Indeed, the inequality $\sum_{e\in E_5} x_e + 2 \geq 0$ induces a facet of $\mathrm{CUT}(K_5)$ (cf. e.g. [6, Chapter 28.2]) and the linear form $\sum_{e\in E_5} x_e + 2$ takes three distinct values on the vertices of $\mathrm{CUT}(K_5)$ (namely, 0 on the facet, 12 on the trivial empty cut and 4 on the cut obtained by separating a vertex from all the others). Applying Theorem 2.2, we can conclude that IK_5 is not TH_1-exact.

In Section 3.3 below we will characterize the graphs whose cut ideals are TH_1-exact and we will determine the precise order k at which the cut ideal of a circuit is TH_k-exact in Section 5.

3.3. **Comparison with other SDP relaxations of the cut polytope.** We mention here the link between the theta bodies of the cut ideal IG and some other semidefinite relaxations of the cut polytope $\mathrm{CUT}(G)$. First note that the relaxation $\mathrm{TH}_1(IG)$ coincides with the edge-relaxation considered by Rendl and Wiegele (see [26]) and numerical experiments there indicates that it is often tighter than the basic semidefinite relaxation (7) of $\mathrm{CUT}(G)$.

Next we compare the theta bodies of IG with the relaxations $Q_t(G)$ of $\mathrm{CUT}(G)$ considered in [15][1]. For $t \in \mathbb{N}$, set $\mathcal{O}_t(n) := \{T \subseteq [n] \mid |T| \leq t \text{ and } |T| \equiv t \mod 2\}$. Then $Q_t(G)$ consists of the vectors $\mathbf{y} \in \mathbb{R}^E$ for which there exists a positive semidefinite matrix X indexed by $\mathcal{O}_t(n)$ satisfying (17), $X_{T,T} = 1$ ($T \in \mathcal{O}_t(n)$), and $y_{\{i,j\}} = X_{\emptyset,\{i,j\}}$ for t even (resp., $y_{\{i,j\}} = X_{\{i\},\{j\}}$ for t odd) for all edges $\{i,j\} \in E$. Therefore, for $t = 1$, $Q_1(G)$ coincides with the Goemans-Williamson SDP relaxation (7). Moreover, for even $t = 2k$, $Q_{2k}(K_n)$ coincides with the theta body $\mathrm{TH}_k(IK_n)$. (To see it use Remark 3.7.) The following chain of inclusions shows the link to the theta bodies:

$$\mathrm{CUT}(G) \subseteq Q_{2k}(G) = \pi_E(Q_{2k}(K_n)) = \pi_E(\mathrm{TH}_k(IK_n)) \subseteq \mathrm{TH}_k(IG) \tag{18}$$

(where the last inclusion follows using (2)). Therefore, the k-th theta body $\mathrm{TH}_k(IG)$ is in general a weaker relaxation than $Q_{2k}(G)$. For instance, for the 5-circuit, $\mathrm{CUT}(C_5) = Q_2(C_5)$ (see [15]) but $\mathrm{CUT}(C_5)$ is strictly contained in $\mathrm{TH}_1(IC_5) = [-1,1]^5$ (see Example 3.8).

On the other hand, the SDP relaxation $\mathrm{TH}_k(IG)$ can be much simpler and less costly to compute than $Q_{2k}(G)$, since its definition exploits the structure of G and thus often uses smaller matrices. Indeed, $Q_{2k}(G)$ is defined as the projection of $Q_{2k}(K_n)$, whose definition involves matrices indexed by all even sets $T \subseteq [n]$ of size at most $2k$, thus not depending on the structure of G. On the other hand, the matrices needed to define $\mathrm{TH}_k(IG)$ are indexed by the even sets $T \subseteq [n]$ of size at most $2k$ for which G has a T-join of size at most k. For instance, for $k = 1$, $\mathrm{TH}_1(IG)$ uses matrices of size $1 + |E|$, while $Q_2(G)$ needs matrices of size $1 + \binom{n}{2}$.

Example 3.10. It was shown in [14] that $\mathrm{CUT}(K_n)$ is strictly contained in $Q_k(K_n)$ for $k < \lceil \frac{n}{2} \rceil - 1$. Therefore, $\mathrm{CUT}(K_n) \subset \mathrm{TH}_k(IK_n) = Q_{2k}(K_n)$ for

[1]For simplicity in the notation we shift the indices by 1 with respect to [15].

all $2k < \lceil \frac{n}{2} \rceil - 1$. This implies that IK_n is not TH_k-exact for $k \leq \lfloor \frac{n-1}{4} \rfloor$. However, it is known that $\mathrm{CUT}(K_n) = Q_{\lceil \frac{n}{2} \rceil}(K_n)$ when $n \leq 7$. Therefore, IK_5, IK_6 and IK_7 are all TH_2-exact.

For some graphs there is a special inclusion relationship between the theta bodies and the Q_t-hierarchy. We consider first graphs with bounded diameter.

Lemma 3.11. *Let G be a graph with diameter at most k, i.e., such that any two vertices can be joined by a path traversing at most k edges. Then* $\mathrm{TH}_k(IG) \subseteq Q_2(G)$.

Proof: It suffices to observe that the set $\mathcal{T}_k$ indexing the matrices in the definition of $\mathrm{TH}_k(IG)$ (which consists of the even sets $T \subseteq V$ for which there is a T-join of size at most k) contains all pairs of vertices. Thus $\mathcal{T}_k$ contains the set $\mathcal{O}_2(n)$ indexing the matrices in the definition of $Q_2(G)$. $\square$

Next we observe that $\mathrm{TH}_k(IG)$ refines the Goemans-Williamson relaxation (7) for graphs with radius k.

Lemma 3.12. *Let G be a graph with radius at most k, i.e., there exists a vertex that can be joined to any other vertex by a path traversing at most k edges. Then* $\mathrm{TH}_k(IG) \subseteq Q_1(G)$.

Proof: Say vertex 1 can be joined to all other vertices $i \in [n] \setminus \{1\}$ by a path of length at most k. Then the set $\mathcal{T}_k$ contains $\emptyset$, $\{i,j\}$ for all edges $ij \in E$, and all pairs $\{1,i\}$ for $i \in [n] \setminus \{1\}$. Let $\mathbf{y} \in \mathrm{TH}_k(IG)$, i.e. there exists a positive semidefinite matrix X indexed by $\mathcal{T}_k$ satisfying (17) and $y_e = X_{\emptyset,e}$ for $e \in E$. Consider the $n \times n$ matrix Y defined by $Y_{ii} = 1$ ($i \in [n]$), $Y_{1i} = X_{\emptyset,\{1,i\}}$ ($i \in [n] \setminus \{1\}$), and $Y_{ij} = X_{\{1,i\},\{1,j\}}$ ($i \neq j \in [n] \setminus \{1\}$). Then $Y \succeq 0$ (since Y coincides with the principal submatrix of X indexed by $\emptyset, \{1,2\}, \ldots, \{1,n\}$), $y_{\{i,j\}} = Y_{ij}$ for all $\{i,j\} \in E$ (using (17)). This shows $\mathbf{y} \in Q_1(G)$, concluding the proof. $\square$

In particular, as already noted in [26], $\mathrm{TH}_1(IG) \subseteq Q_1(G)$ if G contains a vertex adjacent to all other vertices. For an arbitrary graph G, let G^* be the graph obtained by adding edges to G so that one of its vertices is adjacent to all other vertices. Thus, $\mathrm{TH}_1(IG^*) \subseteq Q_1(G^*)$ by Lemma 3.12. Taking projections onto the edge set of G, the relaxation $\pi_E(\mathrm{TH}_1(IG^*))$ is contained in $\pi_E(Q_1(G^*)) = Q_1(G)$ (and in $\mathrm{TH}_1(IG)$).

3.4. **Application to circuits in graphs.** Let us consider briefly the case when $\mathcal{M} = \mathcal{M}_G$ is the graphic matroid of a graph $G = (V, E)$, i.e. $\mathcal{C} = \mathcal{C}_G$ is the collection of cycles of G and $\mathcal{C}^* = \mathcal{D}_G$ is its collection of cuts.

One can find a set $\mathcal{F}$ of representatives for the equivalence classes of (10) as follows. Namely, assume for simplicity that G is connected and let $E_0 \subseteq E$ be the edge set of a spanning tree in G. Then the collection $\mathcal{F} := \mathcal{P}(E \setminus E_0)$ is a set of distinct representatives for the classes of (10). Indeed, note first that no two distinct subsets F, F' of $E \setminus E_0$ are in relation by $\sim$, since each non-empty cut meets the tree E_0. Next, any subset $X \subseteq E_0$ determines a unique cut D_X for which $D_X \cap E_0 = X$, so that $X \sim X \Delta D_X$. Hence, for any set $Z \subseteq E$, write $Z = X \cup Y$ with $X \subseteq E_0$ and $Y \subseteq E \setminus E_0$; then $Z \sim X \Delta D_X \Delta Y$ is thus in the same equivalence class as a subset of $E \setminus E_0$.

Note however that the above set $\mathcal{F}$ may not consist of the minimum cardinality representatives. In fact, as observed in Remark 3.6, finding a minimum cardinality representative in each equivalence class amounts to solving a maximum weight cut problem, thus a hard problem. Nevertheless this collection $\mathcal{F}$ can be used to index truncated moment matrices (simply index the k-th order matrix by all $F \in \mathcal{F}$ with $|F| \leq k$). However, studying this SDP hierarchy is less relevant for optimization purposes since the linear inequality description of $\mathrm{CYC}(\mathcal{M}_G)$ is completely known (see Theorem 4.4 below), and one can find a maximum weight cycle in a graph in polynomial time (with algorithms for maximum T-joins; cf. [7]).

4. Matroids whose cycle ideals are TH_1-exact

4.1. **Matroid minors.** Let $\mathcal{M} = (E, \mathcal{C})$ be a binary matroid and $e \in E$. Set

$$\mathcal{C}\backslash e := \{C \in \mathcal{C} \mid e \notin C\}, \qquad \mathcal{C}/e := \{C \setminus \{e\} \mid C \in \mathcal{C}\}.$$

Then, $\mathcal{M}\backslash e := (E \setminus \{e\}, \mathcal{C}\backslash e)$ and $\mathcal{M}/e := (E \setminus \{e\}, \mathcal{C}/e)$ are again binary matroids; one says that $\mathcal{M}\backslash e$ is obtained by *deleting* e and $\mathcal{M}/e$ by *contracting* e. A *minor* of $\mathcal{M}$ is obtained by a sequence of deletions and contractions, thus of the form $\mathcal{M}\backslash X/Y$ for disjoint $X, Y \subseteq E$. In the language of binary spaces, $C\backslash e$ arises from $\mathcal{C}$ by taking the intersection with the hyperplane $x_e = 0$, while $\mathcal{C}/e$ arises by projecting $\mathcal{C}$ onto $\mathbb{R}^{E\setminus\{e\}}$.

Example 4.1. Let M_r denote the $r \times (2^r - 1)$ matrix whose columns are all non-zero 0/1 vectors of length r, and let $\mathcal{P}_r$ denote the binary matroid represented by M_r, called the binary projective space of dimension $r-1$. One can verify that $\mathcal{P}_r$ has 2^r cocycles; the non-empty cocycles have size 2^{r-1} and thus are cocircuits. Hence, $\mathrm{CYC}(\mathcal{P}_r^*)$ is a simplex and $I\mathcal{P}_r^*$ is TH_1-exact. When $n = 3$, $\mathcal{P}_3 =: F_7$ is called the *Fano matroid*. It will follow from Theorem 4.6 that IF_7 is also TH_1-exact.

Example 4.2. R_{10} is the binary matroid on 10 elements, represented by the matrix

$$\begin{array}{c}
\begin{array}{cccccccccc} 34 & 35 & 45 & 23 & 24 & 25 & 13 & 14 & 15 & 12 \end{array} \\
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1
\end{pmatrix}
\end{array},$$

where it is convenient to index the columns by the edge set E_5 of K_5. Then the cycles of R_{10} correspond to the even cycles of K_5, and the cocycles of

R_{10} to the cuts of K_5 and their complements. Note that R_{10} is isomorphic to its dual. Consider the inequality:

$$\sum_{e\in F} x_e - \sum_{e\in E_5\setminus F} x_e \geq -4, \tag{19}$$

where F consists of three edges adjacent to a common vertex (e.g. $F = \{12, 13, 14\}$). (Thus (19) is of the form (20), but with a shifted right hand side.) One can verify that (19) defines a facet of $\mathrm{CYC}(R_{10})$ and that the linear function in (19) takes three distinct values on the cycles of R_{10} (namely, 0, 4, and -4). Therefore, in view of Theorem 2.2, we can conclude that R_{10} is not TH_1-exact.

4.2. **The cycle polytope.** As each cycle and cocycle have an even intersection, the following inequalities are valid for the cycle polytope $\mathrm{CYC}(\mathcal{M})$:

$$\sum_{e\in F} x_e - \sum_{e\in D\setminus F} x_e \geq 2 - |D| \quad \text{for } D \in \mathcal{C}^*, F \subseteq D, |F| \text{ odd.} \tag{20}$$

Let $\mathrm{MET}(\mathcal{M})$ be the polyhedron in $\mathbb{R}^E$ defined by the inequalities (20) together with $-1 \leq x_e \leq 1$ ($e \in E$). We have $\mathrm{CYC}(\mathcal{M}) \subseteq \mathrm{MET}(\mathcal{M})$. In particular, $\mathrm{CYC}(\mathcal{M})$ is contained in the hyperplane $x_e = 1$ if e is a coloop of $\mathcal{M}$, and it is contained in the hyperplane $x_e - x_f = 0$ if e, f are coparallel. Thus we may assume without loss of generality that $\mathcal{M}$ has no coloops and no coparallel elements. We will use the following known results.

Lemma 4.3. [1, Corollary 4.21] *Let $\mathcal{M}$ be a binary matroid with no F_7^* minor. The inequality (20) defines a facet of* $\mathrm{CYC}(\mathcal{M})$ *if and only if D is a chordless cocircuit of $\mathcal{M}$.*

Theorem 4.4. [1, Theorem 4.22] *For a binary matroid $\mathcal{M}$,* $\mathrm{CYC}(\mathcal{M}) = \mathrm{MET}(\mathcal{M})$ *if and only if $\mathcal{M}$ has no F_7^*, R_{10} or $\mathcal{M}_{K_5}^*$ minors.*

Recall that $I\mathcal{M}$ is TH_1-exact if $\mathrm{CYC}(\mathcal{M}) = \mathrm{TH}_1(I\mathcal{M})$.

Lemma 4.5. *Assume $\mathcal{M}$ has no F_7^* minor. If $I\mathcal{M}$ is* TH_1*-exact then $\mathcal{M}$ does not have any chordless cocircuit of length at least five.*

Proof: Suppose $D = \{e_1, \ldots, e_k\}$ is a chordless cocircuit of $\mathcal{M}$ with $k = |D| \geq 5$. By Lemma 4.3, the inequality

$$x_{e_1} - x_{e_2} - \cdots - x_{e_k} \geq 2 - k$$

defines a facet of $\mathrm{CYC}(\mathcal{M})$. We now use the following claim [1, Lemma 4.17]: For each even subset $F \subseteq D$, there exists a cycle $C \in \mathcal{C}$ for which $C \cap D = F$. Thus we can find three cycles whose intersections with D are respectively $\emptyset$, $\{e_2, e_3\}$ and $\{e_2, e_3, e_4, e_5\}$. Then the linear form $x_{e_1} - x_{e_2} - \cdots - x_{e_k}$ evaluated at each of these three cycles takes the values $2-k, 6-k, 10-k$. In view of Theorem 2.2 we can thus conclude that $I\mathcal{M}$ is not TH_1-exact. $\square$

Theorem 4.6. *Assume $\mathcal{M}$ has no F_7^*, R_{10} or $\mathcal{M}_{K_5}^*$ minors. Then $I\mathcal{M}$ is* TH_1*-exact if and only if $\mathcal{M}$ does not have any chordless cocircuit of length at least 5.*

Proof: Lemma 4.5 gives the 'only if' part. For the 'if' part, it suffices to verify that, if D is a cocircuit of length at most 4 and F is an odd subset of D, then the linear form $\sum_{e\in F} x_e - \sum_{e\in D\setminus F} x_e$ takes two values when evaluated at cycles of $\mathcal{M}$, and then to apply Theorems 4.4 and 2.2. □

Corollary 4.7. *The cycle ideal of a graphic matroid $\mathcal{M}_G$ is* TH_1*-exact if and only if G has no chordless cut of size at least 5.*

Proof: Directly from Theorem 4.6 since graphic matroids do not have F_7^*, R_{10} or $\mathcal{M}_{K_5}^*$ minors. □

Lemma 4.8. *If $I\mathcal{M}$ is* TH_k*-exact, then the cycle ideal of any deletion minor of $\mathcal{M}$ is also* TH_k*-exact.*

Proof: Say $\mathcal{M}' = \mathcal{M}\backslash e_1$ is a deletion minor of $\mathcal{M}$, where $E = \{e_1, \ldots, e_m\}$ and $E' = E \setminus \{e_1\}$. Take $\mathbf{x}' \in \mathrm{TH}_k(I\mathcal{M}')$; we show that $\mathbf{x}' \in \mathrm{CYC}(\mathcal{M}')$. For this extend $\mathbf{x}'$ to $\mathbf{x} \in \mathbb{R}^E$ by setting $x_{e_1} := 1$. We verify that $\mathbf{x} \in \mathrm{TH}_k(I\mathcal{M})$.

For this consider a linear polynomial $f \in \mathbb{R}E$ of the form $f = s + q$ where s is a sos of degree at most $2k$ and $q \in I\mathcal{M}$. Define the polynomials $f', s', q' \in \mathbb{R}E'$ by $f'(x_{e_2}, \ldots, x_{e_m}) = f(1, x_{e_2}, \ldots, x_{e_m})$; similarly for q', s'. Obviously s' is sos with degree at most $2k$. Since q vanishes on $\{\chi^C : C \in \mathcal{C}\}$, it vanishes on all χ^C, $C \in \mathcal{C}$, with $x_{e_1} = 1$. This last fact is equivalent to saying that q' vanishes on $\{\chi^C : C \in \mathcal{C}'\}$. Therefore, f' is k-sos modulo $I\mathcal{M}'$ and so $f'(\mathbf{x}') \geq 0$ as $\mathbf{x}' \in \mathrm{TH}_k(I\mathcal{M}')$. In particular, $f(\mathbf{x}) = f'(\mathbf{x}') \geq 0$ and $\mathbf{x} \in \mathrm{TH}_k(I\mathcal{M}) = \mathrm{CYC}(\mathcal{M})$.

Thus $\mathbf{x}$ is a convex combination of ±1-incidence vectors of cycles of $\mathcal{M}$; as $x_{e_1} = 1$ no cycle in the combination uses e_1, which thus gives a decomposition of $\mathbf{x}'$ as a convex combination of cycles of $\mathcal{M}'$. □

Remark 4.9. On the other hand, the property of being TH_1-exact is *not* preserved under taking contraction minors. Indeed, every binary matroid can be realized as a contraction minor of some dual binary projective space $\mathcal{P}_r^*$ (see [11]). Now we observed in Example 4.1 that the cycle ideal of $\mathcal{P}_r^*$ is TH_1-exact, while $I\mathcal{M}$ is not always TH_1-exact.

See Section 5 for examples of cographic matroids whose cycle ideal is TH_2-exact while they have a contraction minor whose cycle ideal is not TH_k-exact for large k (this is the case for wheels, cf. Corollary 5.10).

We now characterize the TH_1-exact cographic matroids. We begin with a lemma relating graph and matroid minors involving K_5.

Lemma 4.10. *The cographic matroid $\mathcal{M}_G^*$ of a graph G has a $\mathcal{M}_{K_5}^*$ minor if and only if K_5 is a contraction minor of G.*

Proof: The 'if part' is obvious since if K_5 is a contraction minor of G, then $\mathcal{M}^*_{K_5}$ is a deletion minor of $\mathcal{M}^*_G$. Conversely assume that $\mathcal{M}^*_{K_5}$ is a minor of $\mathcal{M}^*_G$. By Whitney's 2-isomorphism theorem (cf. [21]), K_5 is 2-isomorphic to a minor H of G; but then H must be isomorphic to K_5 as the the only graph 2-isomorphic to K_5 is K_5 itself. Hence K_5 is a minor of G, which implies that K_5 is also a contraction minor of G. $\square$

Corollary 4.11. *The cycle ideal of a cographic matroid $\mathcal{M}^*_G$ is* TH_1*-exact if and only if M^*_G has no $M^*_{K_5}$ minor and no chordless cocircuit of length at least 5.*

Proof: Note that $\mathcal{M}^*_G$ contains no F_7^* or R_{10} minor. Hence in view of Theorem 4.6, it suffices to show that if $\mathcal{M}^*_G$ is TH_1-exact then $\mathcal{M}^*_G$ has no $M^*_{K_5}$ minor. So assume that $\mathcal{M}^*_G$ is TH_1-exact. As $\mathcal{M}^*_{K_5}$ is not TH_1-exact (cf. Example 3.9), Lemma 4.8 implies that $\mathcal{M}^*_{K_5}$ is not a deletion minor of $\mathcal{M}^*_G$. Hence K_5 is not a contraction minor of G which, by Lemma 4.10, implies that $\mathcal{M}^*_{K_5}$ is not a minor of $\mathcal{M}^*_G$. $\square$

Reformulating this last result we arrive at a characterization of 'cut-perfect' graphs, answering Problem 8.4 in [19].

Corollary 4.12. *The cut ideal of a graph G is* TH_1*-exact if and only if G has no K_5 minor and no chordless circuit of length at least 5.*

In [24, Theorem 3.2], Sullivant obtains the same characterization for *compressed* cut polytopes; namely he proves that $\mathrm{CUT}(G)$ is compressed if and only if G has no K_5 minor and no chordless cycles of length at least 5. See [10, Section 4] for comments on the connection between compressed polytopes and TH_1-exactness.

5. The theta bodies for cut ideals of circuits

In this section we determine the exact order k for which the cut ideal IC_n of a circuit C_n with n edges is TH_k-exact. We also obtain some results on graphs whose cut ideal is TH_2-exact. We begin with a result determining when the inequalities (20) associated to circuits of G are valid for $\mathrm{TH}_k(IG)$.

Theorem 5.1. *Let C be a circuit of a graph G, let $e \in C$, and let k be an integer such that $4k \geq |C|$. Then the inequality*

$$x_e - \sum_{f \in C \setminus \{e\}} x_f \geq 2 - |C| \tag{21}$$

is valid for $\mathrm{TH}_k(IG)$.

The proof uses the following preliminary results. For convenience, for a graph $G = (V, E)$, let $\mathcal{S}_k$ denote the set of polynomials $f \in \mathbb{R}E$ that are k-sos modulo the cut ideal IG.

Lemma 5.2. *For a graph G, let $F_1, F_2, F_3, F_4 \subseteq E$ with $|F_i| \leq k$ and such that $F_1 \Delta F_2 \Delta F_3 \Delta F_4$ is a cycle of G. Then $2 + \mathbf{x}^{F_1} - \mathbf{x}^{F_2} - \mathbf{x}^{F_3} - \mathbf{x}^{F_4} \in \mathcal{S}_k$.*

Proof: We use the following fact: As $C := F_1 \Delta F_2 \Delta F_3 \Delta F_4$ is a cycle, $1 - \mathbf{x}^C \in IG$ by Theorem 3.4, and thus $1 \equiv \mathbf{x}^C \equiv \mathbf{x}^{F_1}\mathbf{x}^{F_2}\mathbf{x}^{F_3}\mathbf{x}^{F_4}$ modulo IG. This implies that $\mathbf{x}^{F_i}\mathbf{x}^{F_j} \equiv \mathbf{x}^{F_k}\mathbf{x}^{F_l}$ for $\{i,j,k,l\} = \{1,2,3,4\}$. Now, one can easily verify that $(2+\mathbf{x}^{F_1} - \mathbf{x}^{F_2} - \mathbf{x}^{F_3} - \mathbf{x}^{F_4})^2 \equiv 4(2+\mathbf{x}^{F_1} - \mathbf{x}^{F_2} - \mathbf{x}^{F_3} - \mathbf{x}^{F_4})$ modulo IG, which gives the result. □

Lemma 5.3. *For a graph G, let $A, B \subseteq E$ with $|A|, |B|, |A\Delta B| \le k$. Then $1 + \mathbf{x}^A - \mathbf{x}^B - \mathbf{x}^{A\Delta B} \in \mathcal{S}_k$.*

Proof: We have $(1 + \mathbf{x}^A - \mathbf{x}^B - \mathbf{x}^{A\Delta B})^2 \equiv 4 + 2(\mathbf{x}^A - \mathbf{x}^B - \mathbf{x}^{A\Delta B}) + 2(-\mathbf{x}^A\mathbf{x}^B - \mathbf{x}^A\mathbf{x}^{A\Delta B} + \mathbf{x}^B\mathbf{x}^{A\Delta B}) \equiv 4(1 + \mathbf{x}^A - \mathbf{x}^B - \mathbf{x}^{A\Delta B})$ modulo IG. □

Lemma 5.4. *For a graph G, let $F \subseteq E$, $e \in F$, and $k \ge |F|$. Then:*

$$\begin{array}{ll} (i) & k - 1 + x_e - \sum_{f \in F\setminus\{e\}} x_f - \mathbf{x}^F \in \mathcal{S}_k, \\ (ii) & k - 1 - \sum_{f\in F} x_f + \mathbf{x}^F \in \mathcal{S}_k. \end{array} \tag{22}$$

Proof: It suffices to show the result for $k = |F|$. We show (i) using induction on $k \ge 2$. (The proof for (ii) is analogous.) For $k = 2$, $F = \{e, f\}$, we have $1 + x_e - x_f - x_e x_f \in \mathcal{S}_2$ by Lemma 5.3. Consider now $|F| = k \ge 3$ and let $g \in F \setminus \{e\}$. By the induction assumption applied to the set $F \setminus \{g\}$, we have:

$$k - 2 + x_e - \sum_{f \in F\setminus\{e,g\}} x_f - \mathbf{x}^{F\setminus\{g\}} \in \mathcal{S}_{k-1} \subseteq \mathcal{S}_k.$$

Applying Lemma 5.3 to the sets $F \setminus \{g\}$, $\{g\}$ and F, we obtain

$$1 + \mathbf{x}^{F\setminus\{g\}} - x_g - \mathbf{x}^F \in \mathcal{S}_k.$$

Summing up the above two relations yield the desired relation (22)(i). □

Proof: (*of Theorem 5.1*) Let C be a circuit in G with $|C| \le 4k$, i.e. $k \ge m := \lceil |C|/4 \rceil$. Let F denote the edge set of C and let $e \in F$. We show that the linear polynomial $f_C := x_e - \sum_{f\in F\setminus\{e\}} x_f + |C| - 2$ is k-sos modulo IG. For this we consider a partition of F into four sets $F_1, \ldots, F_4$ with $|F_i| \le m \le k$ for $i = 1, \ldots, 4$; say $e \in F_1$. Applying Lemma 5.2, we obtain that

$$2 + \mathbf{x}^{F_1} - \mathbf{x}^{F_2} - \mathbf{x}^{F_3} - \mathbf{x}^{F_4} \in \mathcal{S}_k.$$

Next, applying the condition (22)(i) to F_1 we obtain

$$|F_1| - 1 + x_e - \sum_{f\in F_1\setminus\{e\}} x_f - \mathbf{x}^{F_1} \in \mathcal{S}_k,$$

and applying the condition (22)(ii) to F_i yields

$$|F_i| - 1 - \sum_{f \in F_i} x_f + \mathbf{x}^{F_i} \in \mathcal{S}_k \quad \forall i = 2,3,4.$$

Summing up the above relations yields the desired result, namely f_C is k-sos modulo IG and thus $f_C \ge 0$ is valid for $\mathrm{TH}_k(IG)$. □

Corollary 5.5. *For the circuit C_n of length n, the equality* $\mathrm{TH}_k(IC_n) = \mathrm{CUT}(C_n)$ *holds for $n \leq 4k$.*

Proof: Consider the circuit $C_n = ([n], E)$ with $n \leq 4k$. By Theorem 4.4, the complete linear description of $\mathrm{CUT}(C_n)$ is provided by the inequalities (i) $\sum_{e\in F} x_e - \sum_{e\in E\setminus F} x_e \geq 2 - n$ where F is any odd subset of E, and (ii) $-1 \leq x_e \leq 1$ for all $e \in E$. Thus in order to show $\mathrm{TH}_k(IC_n) = \mathrm{CUT}(C_n)$, it suffices to show that the inequalities (i),(ii) are all valid for $\mathrm{TH}_k(IC_n)$. This is obvious for (ii). Using the well-known switching symmetries of the cut polytope (cf. [1], [6]), it suffices to show the desired property for the inequalities (i) with $|F| = 1$. But this result has just been shown in Theorem 5.1. □

Lemma 5.6. *If $n \geq 4k + 1$, then* $\mathrm{TH}_k(IC_n) = [-1, 1]^E$.

Proof: In view of Remark 3.5, it suffices to observe that the constraints (13) defining the theta body $\mathrm{TH}_k(IC_n)$ reduce to the constraints (13)(i) and (14). Let $\mathcal{F}_k$ be the set indexing the combinatorial moment matrices in the definition of $\mathrm{TH}_k(IC_n)$, where we can assume that each $F_i \in \mathcal{F}_k$ has cardinality at most k. Now consider a constraint of type (13)(ii). Since $F_1, \ldots, F_4 \in \mathcal{F}_k$ have size at most k and $\Delta_i F_i$ is a cycle of C_n, this cycle must be the empty set since $|\Delta_i F_i| \leq 4k < n$. Therefore we have a constraint of type (14). □

Corollary 5.7. *The smallest order k at which IC_n is TH_k-exact is $k = \lceil n/4 \rceil$.*

Proof: Directly from Theorem 5.1 and Lemma 5.6. □

Remark 5.8. One can verify that the linear form $x_e - \sum_{f\in C_n\setminus\{e\}} x_f$ takes $\lfloor (n + 1)/2\rfloor$ distinct values at the cut vectors of the circuit C_n. By (3), this permits to conclude that IC_n is TH_k-exact for $k = \lfloor (n + 1)/2\rfloor - 1$. This value is however larger than the order $\lceil n/4\rceil$ shown in Corollary 5.7 (for $n \geq 6$). Thus the reverse implication of (3) does not hold.

Corollary 5.9. *If the graph G has no K_5 minor and no chordless circuit of length at least 9, then its cut ideal IG is* TH_2*-exact.*

Proof: Direct application of Theorems 4.4 and 5.1. □

Note that the reverse implication in Corollary 5.9 does not hold. We will see below (in Corollary 5.10) that the cut ideal of a wheel is TH_2-exact, but a wheel can contain a chordless circuit of arbitrary length.

While we could characterize the graphs whose cut ideal is TH_1-exact, it is an open problem to characterize the graphs whose cut ideal is TH_2-exact. We conclude this section with several observations about these graphs.

Corollary 5.10. *If the graph G has no K_5 minor and has diameter at most 2 then its cut ideal IG is* TH_2*-exact.*

Proof: As G has diameter at most 2, Lemma 3.11 gives the inclusion $\mathrm{TH}_2(IG) \subseteq Q_2(G)$. It was shown in [15] that if G has no K_5 minor then $Q_2(G) = \mathrm{CUT}(G)$. □

A *wheel* of length n is a graph consisting of a circuit of length n with an additional vertex adjacent to all vertices on the circuit. As wheels have no K_5 minor and their diameter is 2, their cut ideal is TH_2-exact. Hence, within graphs with no K_5 minors, the cut ideal is TH_2-exact for the following two classes: graphs with diameter at most 2 and graphs with no chordless circuit of size at least 9. Note that there is no containment between these two classes; e.g. wheels of length $n \geq 9$ have diameter 2 but contain a circuit of length n, and C_8 has diameter larger than 2.

The following further graphs have a TH_2-exact cut ideal: K_5, K_6, K_7 (and probably K_8 too, as conjectured in [15]). Finally, if the cut ideal of a graph G is TH_2-exact, then the same holds for the cut ideal of any *contraction* minor H of G; in particular, C_9 is not a contraction minor of G.

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