Predicting a cyclic Poisson process

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Abstract We construct and investigate a $(1-\alpha)$-upper prediction bound for a future observation of a cyclic Poisson process using past data. A normal based confidence interval for our upper prediction bound is established. A comparison of the new prediction bound with a simpler nonparametric prediction bound is also given.

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1 Introduction and main results

Let X be a Poisson process on the real line $\mathbf{R}$ with (unknown) locally integrable intensity function λ. We assume that λ is periodic with (known) period $\tau > 0$ and is positive a.e. w.r.t. Lebesgue measure. We do not assume any parametric form of λ, except that it is periodic. For each point $s \in \mathbf{R}$ and all $k \in \mathbf{Z}$, we have

$$\lambda(s + k\tau) = \lambda(s), \tag{1.1}$$

where **Z** denotes the set of integers.

Suppose that, for some $\omega \in \Omega$, a single realization $X(\omega)$ of the Poisson process X defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with intensity function λ is observed, though only within a bounded interval $[-n, 0]$.

Our goal in this paper is to propose and investigate a $(1-\alpha)$-upper prediction bound for the time Z of the first event of the Poisson process X after the present time 0, using only a single realization $X(\omega)$ of the cyclic Poisson process X observed in the past, i.e. in an interval $[-n, 0]$. A much simpler but related prediction problem for the homogeneous Poisson process was investigated in Vit (1973).

It is well-known that, for any real number $z > 0$, the distribution function of Z is given by :

$$F_Z(z) \;=\; \mathbf{P}\left(Z \leq z\right) = 1 - \mathbf{P}\left(Z > z\right) = 1 - e^{-\Lambda(z)}, \tag{1.2}$$

with $\Lambda(z) = \int_0^z \lambda(s)ds$. Let $z_r = z - \tau[\frac{z}{\tau}]$ where for any real number x, $[x]$ denotes the largest integer that less than or equal to x. Then, for any $z > 0$ we have $z = \tau[\frac{z}{\tau}] + z_r$ with $0 < z_r < \tau$. Let $\theta = \tau^{-1}\int_0^\tau \lambda(s)ds$ be the global intensity of X. Then, for any $z > 0$, we can write

$$\Lambda(z) = \theta\tau[\frac{z}{\tau}] + \Lambda(z_r). \tag{1.3}$$

Since $\lambda(s) > 0$ a.e. we also have $\theta > 0$. This latter condition is equivalent to the requirement that, with **P**-probability one, $|X(\omega)| = \infty$, which is obviously a necessary assumption for obtaining our consistency results.

In view of (1.2) and (1.3) our probability model for Z is a semiparametric one, the nonparametric component is given by the function $\Lambda(z_r) = \int_0^{z_r} \lambda(s)ds$, $0 < z_r < \tau$, whereas the parametric component is described by θ (with known period τ).

Let $\hat{F}_{Z,n}(z)$ denote the empirical counterpart of $F_Z(z)$, using the available past data set at hand, i.e. $X(\omega) \cap [-n, 0]$, the Poisson process X observed in $[-n, 0]$, which is given by

$$\hat{F}_{Z,n}(z) = 1 - e^{-\hat{\Lambda}_n(z)} \tag{1.4}$$

with

$$\hat{\Lambda}_n(z) = \tau[\frac{z}{\tau}]\hat{\theta}_n + \hat{\Lambda}_n(z_r) \tag{1.5}$$

where

$$\hat{\theta}_n = \frac{X([-\tau n_\tau, 0])}{\tau n_\tau}, \tag{1.6}$$

$$\hat{\Lambda}_n(z_r) = \frac{1}{n_\tau} \sum_{k=1}^{n_\tau} X([-k\tau, z_r - k\tau]), \tag{1.7}$$

and $n_\tau = [\frac{n}{\tau}]$.

A $(1-\alpha)$-prediction interval for a future observation of X, i.e. the time of the first event after time 0, is given by $(0, \xi_{Z,1-\alpha})$, where $\xi_{Z,1-\alpha}$ is defined by

$$\xi_{Z,1-\alpha} = \inf\{z : F_Z(z) \geq 1-\alpha\}, \tag{1.8}$$

i.e. $\xi_{Z,1-\alpha} = F_Z^{-1}(1-\alpha)$, where F_Z^{-1} denotes the inverse of F_Z. In other words, $\xi_{Z,1-\alpha}$ is nothing but the (smallest) solution of

$$\mathbf{P}\left(Z \leq \xi_{Z,1-\alpha}\right) = 1-\alpha. \tag{1.9}$$

Since the distribution of Z is unknown, we replace equation (1.8) by its empirical counterpart, i.e. we define $\hat{\xi}_{Z,n,1-\alpha}$ by

$$\hat{\xi}_{Z,n,1-\alpha} = \hat{F}_{Z,n}^{-1}(1-\alpha). \tag{1.10}$$

As a simple consequence of (1.10) we have that

$$\hat{F}_{Z,n}(\hat{\xi}_{Z,n,1-\alpha}) = 1-\alpha + \mathcal{O}_p(n^{-1}),$$

as $n \to \infty$, which in turn easily reduces to the equation

$$\tau[\frac{\hat{\xi}_{Z,n,1-\alpha}}{\tau}]\hat{\theta}_n + \frac{1}{n_\tau}\sum_{k=1}^{n_\tau} X([-k\tau, \hat{\xi}_{Z,n,1-\alpha,r} - k\tau]) = \ln\left(\frac{1}{\alpha}\right) + \mathcal{O}_p\left(\frac{1}{n}\right), \tag{1.11}$$

as $n \to \infty$, where $\hat{\xi}_{Z,n,1-\alpha,r} = \hat{\xi}_{Z,n,1-\alpha} - \tau[\frac{\hat{\xi}_{Z,n,1-\alpha}}{\tau}]$. In other words, $\hat{\xi}_{Z,n,1-\alpha}$ given by (1.10) is nothing but the (smallest) solution of (1.11). Note that the non negative $\mathcal{O}_p(n^{-1})$ error term appearing in (1.11) is due to the fact that $\hat{F}_{Z,n}$ is discrete, a step function with jumps of size $\mathcal{O}_p(n^{-1})$ occuring at points $z = s_i + k\tau$ for positive integers k and events s_i which belong to our past data set $X(\omega) \cap [-n, 0]$.

The density of Z exists and is given by (cf. (1.2))

$$f_Z(z) = \frac{d}{dz}\left(F_Z(z)\right) = \lambda(z)e^{-\Lambda(z)}. \tag{1.12}$$

Clearly f_Z is unknown, but we can estimate f_Z at a given point z by

$$\hat{f}_{Z,n}(z) = \hat{\lambda}_{n,K}(z) e^{-\hat{\Lambda}_n(z)}, \tag{1.13}$$

where, for any $z > 0$, $\hat{\lambda}_{n,K}(z)$ is given by

$$\hat{\lambda}_{n,K}(z) = \frac{\tau}{n} \sum_{k=0}^{\infty} \frac{1}{h_n} \int_{-n}^{0} K\left(\frac{x - (z + k\tau)}{h_n}\right) X(dx), \tag{1.14}$$

which is the kernel-type estimator of the intensity function λ of X introduced in Helmers *et al.* (2003) and investigated also in Helmers *et al.* (2005). Here, h_n is a sequence of positive real numbers such that $h_n \downarrow 0$, as $n \to \infty$, and K denotes a kernel function $K : \mathbf{R} \to [0, \infty)$ satisfying the following properties: (K.1) K is a probability density function, (K.2) K is bounded, and (K.3) K has support in $[-1, 1]$.

The main result of this paper is the following theorem:

Theorem 1 *Suppose that λ is periodic and locally integrable. Let $\hat{\xi}_{Z,n,1-\alpha}$ given by (1.10), i.e. the smallest solution of (1.11).*

(i) (Consistency) We have

$$\mathbf{P}\left(Z \leq \hat{\xi}_{Z,n,1-\alpha}\right) \; \to \; 1 - \alpha, \tag{1.15}$$

as $n \to \infty$.

(ii) (Asymptotic Normality) We have

$$\frac{\sqrt{n_\tau} \lambda(\xi_{Z,1-\alpha})}{\sqrt{q(\xi_{Z,1-\alpha})}} \left(\hat{\xi}_{Z,n,1-\alpha} - \xi_{Z,1-\alpha}\right) \; \stackrel{d}{\to} \; N(0,1) \tag{1.16}$$

as $n \to \infty$, provided $\xi_{Z,1-\alpha}$ is a Lebesgue point of λ, where for any $z > 0$

$$q(z) = [\frac{z}{\tau}]^2 \tau\theta + (1 + 2[\frac{z}{\tau}])\Lambda(z_r) \tag{1.17}$$

with $z_r = z - \tau[\frac{z}{\tau}]$.

(iii) (Studentization) Let $\hat{\lambda}_{n,K}$ be the kernel-type estimator of λ given by (1.14), then we have

$$\frac{\sqrt{n_\tau} \hat{\lambda}_{n,K}(\hat{\xi}_{Z,n,1-\alpha})}{\sqrt{\hat{q}_n(\hat{\xi}_{Z,n,1-\alpha})}} \left(\hat{\xi}_{Z,n,1-\alpha} - \xi_{Z,1-\alpha}\right) \; \stackrel{d}{\to} \; N(0,1) \tag{1.18}$$

as $n \to \infty$, provided $\xi_{Z,1-\alpha}$ is a Lebesgue point of λ, where for any $z > 0$

$$\hat{q}_n(z) = [\frac{z}{\tau}]^2 \tau\hat{\theta}_n + (1 + 2[\frac{z}{\tau}])\hat{\Lambda}_n(z_r). \tag{1.19}$$

Note that, a point z is called a Lebesgue point of λ if $\lim_{h\downarrow 0}\frac{1}{2h}\int_{-h}^{h}|\lambda(z+x)-\lambda(z)|dx = 0$. This assumption is a rather mild one since the set of all Lebesgue point of λ is dense in $\mathbf{R}$, whenever λ is assumed to be locally integrable. The Lebesgue point assumption also occurs in Helmers *et al.* (2003) and Helmers *et al.* (2005).

It is easy to check (cf. (2.22) and (2.23)) that $q(\xi_{Z,1-\alpha})$ appearing in (1.16) reduces to $\Lambda(\xi_{Z,1-\alpha,r})$, with $\xi_{Z,1-\alpha,r} = \xi_{Z,1-\alpha} - \tau[\frac{\xi_{Z,1-\alpha}}{\tau}]$, whenever $\xi_{Z,1-\alpha} < \tau$ which happens if and only if $\theta\tau > \ln(1/\alpha)$ (cf. (2.24)). In other words

$$q(\xi_{Z,1-\alpha}) = \Lambda(\xi_{Z,1-\alpha,r}) = \Lambda(\xi_{Z,1-\alpha}) \iff \theta\tau > \ln(1/\alpha). \tag{1.20}$$

We note in passing that $q(\xi_{Z,1-\alpha}) = \Lambda(\xi_{Z,1-\alpha,r})$ also holds true in the case that θ is assumed to be known. To check this is an easy matter in view of (1.5); i.e. $\hat{\Lambda}_n(z)$ now reduces to $\tau[\frac{z}{\tau}]\theta + \hat{\Lambda}_n(z_r)$.

An important statistical application of (1.18) is that it enables one to construct a confidence interval for the $(1-\alpha)-$upper prediction bound $\xi_{Z,1-\alpha}$ as follows:

Corollary 1 *For any significance level p, $0 < p < 1$, a normal based confidence interval for $\xi_{Z,1-\alpha}$ with approximate coverage probability $1-p$ is given by*

$$I_n = \left(\hat{\xi}_{Z,n,1-\alpha} - \frac{\Phi^{-1}(1-\frac{p}{2})\sqrt{\hat{q}_n(\hat{\xi}_{Z,n,1-\alpha})}}{\sqrt{[\frac{n}{\tau}]}\ \hat{\lambda}_{n,K}(\hat{\xi}_{Z,n,1-\alpha})},\ \hat{\xi}_{Z,n,1-\alpha} + \frac{\Phi^{-1}(1-\frac{p}{2})\sqrt{\hat{q}_n(\hat{\xi}_{Z,n,1-\alpha})}}{\sqrt{[\frac{n}{\tau}]}\ \hat{\lambda}_{n,K}(\hat{\xi}_{Z,n,1-\alpha})}\right), \tag{1.21}$$

where Φ denote the distribution function of a standard normal r.v. and

$$\mathbf{P}\left(\xi_{Z,1-\alpha} \in I_n\right) = 1 - p + o(1), \tag{1.22}$$

as $n \to \infty$, provided $\xi_{Z,1-\alpha}$ is a Lebesgue point of λ, $\lambda(\xi_{Z,1-\alpha}) > 0$ and the period τ is known.

The upper prediction bound $\hat{\xi}_{Z,n,1-\alpha}$ can be viewed as an estimator of $\xi_{Z,1-\alpha}$ based on the semiparametric model (1.2). In contrast, a simple nonparametric estimator of $\xi_{Z,1-\alpha}$ is given by the sample quantile $\hat{\xi}^{NP}_{Z,N,1-\alpha}$, defined as

$$\hat{\xi}^{NP}_{Z,N,1-\alpha} = \hat{F}_N^{-1}(1-\alpha) \tag{1.23}$$

where for any $0 < s < 1$, $\hat{F}_N^{-1}(s) = \inf\{x : \hat{F}_N(x) \geq s\}$, and $\hat{F}_N$ denote the empirical distribution function (df) with random sample size N based on $Z_1, Z_2, \ldots, Z_N$, with

$$N = \sum_{i=1}^{n_\tau} \mathbf{I}(X([-(n_\tau - i + 1)\tau, -(n_\tau - i)\tau)) \geq 1) \tag{1.24}$$

where N has Binomial distribution with parameters n_τ and $1-e^{-\theta\tau}$. Note that for each i, $i=1,2,\ldots,n_\tau$, we have $\mathbf{P}(X([-(n_\tau-i+1)\tau,-(n_\tau-i)\tau))\geq 1)=1-e^{-\theta\tau}$, whereas the summands in (1.24) are i.i.d.

The $Z_i's$, $i=1,2,\ldots,N$, are the observed times to the first 'event' in $X(\omega)\cap[-n,0]$, starting at time $-(n_\tau-i+1)\tau$, $i=1,2,\ldots,n_\tau$, whenever well-defined. For instance, when $X([-n,0])=0$, i.e. the data set at hand is empty, the $Z_i's$ do not exist; i.e. $N=0$. If there is no 'event' of $X(\omega)$ in the interval $[-(n_\tau-i+1)\tau,-(n_\tau-i)\tau)$ but there is an 'event' in the next interval $[-(n_\tau-i)\tau,-(n_\tau-i-1)\tau)$, then we know that $\tau<Z_i<2\tau$. To obtain Z_{i+1} we observe the time to the next 'event' of $X(\omega)$ starting from time $-(n_\tau-i-1)\tau$. More generally, if $N=m$, $m=0,1,2,\ldots,n_\tau$, then precisely m waiting times, say $Z_1,Z_2,\ldots,Z_m$, are observed. Of course, the $Z_i's$ are i.i.d. with common df F_Z (cf. (1.2)), because of (1.1).

Using a well-known result for sample quantiles based on a sample with non random sample size (see, e.g., Reiss, 1989, p.109) and the fact that $\sqrt{N/(n_\tau(1-e^{-\theta\tau}))}\xrightarrow{p}1$, as $n\to\infty$, we have

$$\frac{\sqrt{n_\tau(1-e^{-\theta\tau})}f_Z(\xi_{Z,1-\alpha})}{\sqrt{\alpha(1-\alpha)}}\left(\hat{\xi}^{NP}_{Z,N,1-\alpha}-\xi_{Z,1-\alpha}\right)\xrightarrow{d}N(0,1) \tag{1.25}$$

as $n\to\infty$. So, the asymptotic variance of $\hat{\xi}^{NP}_{Z,N,1-\alpha}$ is equal to

$$\frac{\alpha(1-\alpha)}{n_\tau(1-e^{-\theta\tau})f_Z^2(\xi_{Z,1-\alpha})}, \tag{1.26}$$

provided $f_Z(\xi_{Z,1-\alpha})>0$.

Our prediction bound $\hat{\xi}_{Z,n,1-\alpha}$ uses the whole past data set $X(\omega)\cap[-n,0]$ at hand. So, in contrast to $\hat{\xi}^{NP}_{Z,N,1-\alpha}$, which based on a Binomial random sample of size N with mean $n_\tau(1-e^{-\theta\tau})$, our proposed prediction bound $\hat{\xi}_{Z,n,1-\alpha}$ is a function of $X([-n,0])$ data points - a Poisson random sample size with mean $\int_{-n}^0\lambda(s)ds\approx n_\tau\int_0^\tau\lambda(s)ds=n_\tau\,\theta\tau$. Since for any $\theta\tau>0$ we have $\theta\tau>(1-e^{-\theta\tau})$, we use, on the average, a bigger data set in constructing $\hat{\xi}_{Z,n,1-\alpha}$ compared with $\hat{\xi}^{NP}_{Z,N,1-\alpha}$. Comparing (1.26) with the asymptotic variance of $\hat{\xi}_{Z,n,1-\alpha}$ (cf. (1.16)) which is equal to

$$\frac{q(\xi_{Z,1-\alpha})}{n_\tau\lambda^2(\xi_{Z,1-\alpha})}=\frac{q(\xi_{Z,1-\alpha})e^{-2\Lambda(\xi_{Z,1-\alpha})}}{n_\tau f_Z^2(\xi_{Z,1-\alpha})}=\frac{q(\xi_{Z,1-\alpha})\alpha^2}{n_\tau f_Z^2(\xi_{Z,1-\alpha})}, \tag{1.27}$$

provided $\lambda(\xi_{Z,1-\alpha})>0$, one can check - cf. Theorem 2 below - that the variance in (1.27) is smaller than the variance in (1.26), as one would perhaps expect.

Theorem 2 *Suppose that λ is periodic and locally integrable. If*

$$\theta\tau > \frac{\ln(1/\alpha)}{3}, \tag{1.28}$$

then for any $0 < \alpha < 1$, we have

$$\frac{q(\xi_{Z,1-\alpha})\alpha^2}{n_\tau f_Z^2(\xi_{Z,1-\alpha})} < \frac{\alpha(1-\alpha)}{n_\tau(1-e^{-\theta\tau})f_Z^2(\xi_{Z,1-\alpha})}, \tag{1.29}$$

provided $f_Z(\xi_{Z,1-\alpha}) > 0$.

Comparing the r.h.s. of (1.29) (cf. (1.26)) with the l.h.s. of (1.29) in the special case that (1.20) holds true, i.e. when $q(\xi_{Z,1-\alpha})$ reduces to $\Lambda(\xi_{Z,1-\alpha,r}) = \Lambda(\xi_{Z,1-\alpha}) = \ln(\alpha^{-1})$, a simple calculation shows that

$$\frac{\text{asymp. var } (\hat{\xi}^{NP}_{Z,n,1-\alpha})}{\text{asymp. var } (\hat{\xi}_{Z,n,1-\alpha})} = \frac{\alpha^{-1}-1}{\ln(\alpha^{-1})(1-e^{-\theta\tau})} \tag{1.30}$$

holds true, provided $\theta\tau > \ln(1/\alpha)$. Condition $\theta\tau > \ln(1/\alpha)$, when $\alpha = 0.05$ (0.10), is equivalent to assuming that, on the average, there are at least 2.9957 (2.3026) events of the process X in any interval of length τ. In particular this means, for instance, when $\alpha = 0.05$ (0.10), the ratio in (1.30) is bigger or equal to 6.6762 (4.3430), whenever $\theta\tau > 2.9957$ (2.3026).

To obtain a Studentized version of (1.25) (cf. Ho & Lee, 2005; Reiss, 1989) one need to estimate θ and $f_Z(\xi_{Z,1-\alpha})$ by $\hat{\theta}_n$ (cf. (1.6)) and a density estimate $\hat{f}_{Z,n}(\hat{\xi}^{NP}_{Z,N,1-\alpha})$, where $\hat{f}_{Z,n}$ (cf. (1.13)) denotes an appropriate density estimate of f. For any significance level p, $0 < p < 1$, a normal based confidence interval for $\xi_{Z,1-\alpha}$ with approximate coverage probability $1-p$ is given by

$$I_n^{NP} = \left(\hat{\xi}^{NP}_{Z,N,1-\alpha} - \frac{\Phi^{-1}(1-\frac{p}{2})\sqrt{\alpha(1-\alpha)}}{\sqrt{[\frac{n}{\tau}](1-e^{-\tau\hat{\theta}_n})}\hat{f}_{Z,n}(\hat{\xi}^{NP}_{Z,N,1-\alpha})}, \hat{\xi}^{NP}_{Z,N,1-\alpha} + \frac{\Phi^{-1}(1-\frac{p}{2})\sqrt{\alpha(1-\alpha)}}{\sqrt{[\frac{n}{\tau}](1-e^{-\tau\hat{\theta}_n})}\hat{f}_{Z,n}(\hat{\xi}^{NP}_{Z,N,1-\alpha})}\right)$$

where

$$\mathbf{P}\left(\xi_{Z,1-\alpha} \in I_n^{NP}\right) = 1-p+o(1), \tag{1.31}$$

as $n \to \infty$, provided $\xi_{Z,1-\alpha}$ is a Lebesgue point of λ, $\lambda(\xi_{Z,1-\alpha}) > 0$ and the period τ is known.

Ho and Lee (2005) recently obtained an iterated smoothed bootstrap-t method for setting confidence interval for quantiles like $\hat{\xi}^{NP}_{Z,N,1-\alpha}$ for a non random sample size n, with coverage error of order $n^{-58/57}$, i.e. the classical normal error $\mathcal{O}(n^{-1/2})$, which one would expect in (1.31), is replaced by a much smaller coverage error $\mathcal{O}(n^{-58/57})$ using an iterated smoothed bootstrap method to approximate the distribution of a Studentized sample quantile. The question remains whether we can obtain such much smaller coverage errors using bootstrap methods for (1.31) and (1.22) as well. The authors hope to pursue this matter elsewhere.

In certain cases of interest the intensity function λ is apriori known to be sufficiently smooth and one may estimate $\Lambda(z)$ by $\int_0^z \hat{\lambda}_{n,K}(s)ds$ instead of $\hat{\Lambda}_n(z)$, for any $z > 0$. In this set up, it might be of interest to construct a confidence region for the function $\Lambda(z)$, $z > 0$ (cf. (1.3)) using a kernel type estimator for λ, somewhat similar to the methodology used in Helmers *et al.* (2009).

To conclude this section we also want to refer to Helmers and Zitikis (1999) and Helmers and Mangku (2009) for some related statistical work on Poisson intensity functions.

2 Proof of Theorem 1 and relation (1.20)

First we prove part (i) of Theorem 1. To check this, we write the l.h.s. of (1.15) as

$$\mathbf{P}\left(Z \leq \xi_{Z,1-\alpha} + (\hat{\xi}_{Z,n,1-\alpha} - \xi_{Z,1-\alpha})\right) = \mathbf{P}\left(Z - (\hat{\xi}_{Z,n,1-\alpha} - \xi_{Z,1-\alpha}) \leq \xi_{Z,1-\alpha}\right)$$

Then, by (1.9), proving (1.15) is equivalent to showing that

$$\mathbf{P}\left(Z - (\hat{\xi}_{Z,n,1-\alpha} - \xi_{Z,1-\alpha}) \leq \xi_{Z,1-\alpha}\right) \;\rightarrow\; \mathbf{P}\left(Z \leq \xi_{Z,1-\alpha}\right), \tag{2.1}$$

as $n \to \infty$. To prove (2.1), it suffices to check

$$(\hat{\xi}_{Z,n,1-\alpha} - \xi_{Z,1-\alpha}) \;\xrightarrow{p}\; 0, \tag{2.2}$$

as $n \to \infty$. By (1.8) and (1.10), to verify (2.2), it suffices to show

$$\left(\inf\left\{x : \hat{F}_{Z,n}(x) \geq 1-\alpha\right\} - \inf\left\{x : F_Z(x) \geq 1-\alpha\right\}\right) \;\xrightarrow{p}\; 0, \tag{2.3}$$

as $n \to \infty$. By writing $\hat{F}_{Z,n}(x) = F_Z(x) + (\hat{F}_{Z,n}(x) - F_Z(x))$, proving (2.3) is equivalent to checking that

$$\left(\inf\left\{x : F_Z(x) + (\hat{F}_{Z,n}(x) - F_Z(x)) \geq 1-\alpha\right\} - \inf\left\{x : F_Z(x) \geq 1-\alpha\right\}\right) \;\xrightarrow{p}\; 0, \tag{2.4}$$

as $n \to \infty$. By part (i) of Proposition 1 (cf. section 4) and the fact that F_Z is continuous in a neighborhood of $\xi_{Z,1-\alpha}$, we obtain (2.4) . This completes the proof of part (i) of Theorem 1.

Next we prove part (ii) of Theorem 1. To verify this, by (1.8) and (1.10), we write the l.h.s. of (1.16) as

$$\frac{\sqrt{n_\tau}\lambda(\xi_{Z,1-\alpha})}{\sqrt{q(\xi_{Z,1-\alpha})}}\left(\inf\left\{x : \hat{F}_{Z,n}(x) \geq 1-\alpha\right\} - \inf\left\{x : F_Z(x) \geq 1-\alpha\right\}\right). \tag{2.5}$$

By part (ii) of Proposition 1 we can write

$$\hat{F}_{Z,n}(x) = F_Z(x) + \frac{N(0,1)\sqrt{q(x)}}{\sqrt{n_\tau}e^{\Lambda(x)}} + o_p\left(\frac{1}{\sqrt{n}}\right), \tag{2.6}$$

as $n \to \infty$. By (2.3), we know from the proof of part (i) of Theorem 1 that

$$\left(\inf\left\{x : \hat{F}_{Z,n}(x) \geq 1-\alpha\right\} - \xi_{Z,1-\alpha}\right) = o_p(1),$$

as $n \to \infty$. Hence, to prove part (ii) of Theorem 1 we only need to consider x in a shrinking neighborhood of $\xi_{Z,1-\alpha}$, i.e. $|x - \xi_{Z,1-\alpha}| = o_p(1)$, as $n \to \infty$. Next we show that, for any x, $|x - \xi_{Z,1-\alpha}| = o_p(1)$, as $n \to \infty$, $N(0,1)\sqrt{q(x)}e^{-\Lambda(x)}/\sqrt{n_\tau}$ in (2.6) can be replaced by $N(0,1)\sqrt{q(\xi_{Z,1-\alpha})}e^{-\Lambda(\xi_{Z,1-\alpha})}/\sqrt{n_\tau}$. To verify this we have to show

$$\frac{1}{\sqrt{n_\tau}}\left(\sqrt{q(x)}\ e^{-\Lambda(x)} - \sqrt{q(\xi_{Z,1-\alpha})}\ e^{-\Lambda(\xi_{Z,1-\alpha})}\right) = o_p\left(\frac{1}{\sqrt{n}}\right), \tag{2.7}$$

which is equivalent to checking that

$$\left(\sqrt{q(x)}\ e^{-\Lambda(x)} - \sqrt{q(\xi_{Z,1-\alpha})}\ e^{-\Lambda(\xi_{Z,1-\alpha})}\right) = o_p(1), \tag{2.8}$$

as $n \to \infty$. To prove (2.8), we write the l.h.s. of (2.8) as

$$\sqrt{q(x)}\left(e^{-\Lambda(x)} - e^{-\Lambda(\xi_{Z,1-\alpha})}\right) + e^{-\Lambda(x)}\left(\sqrt{q(x)} - \sqrt{q(\xi_{Z,1-\alpha})}\right). \tag{2.9}$$

Since $|x - \xi_{Z,1-\alpha}| = o_p(1)$ as $n \to \infty$, a simple argument show that, the quantity in (2.9) is of order $o_p(1)$ as $n \to \infty$, provided

$$\Lambda(x) - \Lambda(\xi_{Z,1-\alpha}) = o_p(1), \tag{2.10}$$

as $n \to \infty$. To verify (2.10) we note that the l.h.s. of (2.10) is equal to

$$\begin{aligned} \Lambda(x) - \Lambda(\xi_{Z,1-\alpha}) &= \int_0^x \lambda(s)ds - \int_0^{\xi_{Z,1-\alpha}} \lambda(s)ds = \int_{\xi_{Z,1-\alpha}}^x \lambda(s)ds \\ = & \int_0^{(x-\xi_{Z,1-\alpha})} \lambda(s+\xi_{Z,1-\alpha})ds = \lambda(\xi_{Z,1-\alpha})(x-\xi_{Z,1-\alpha}) \\ & +(x-\xi_{Z,1-\alpha})\left(\frac{1}{(x-\xi_{Z,1-\alpha})}\int_0^{(x-\xi_{Z,1-\alpha})}(\lambda(s+\xi_{Z,1-\alpha}) - \lambda(\xi_{Z,1-\alpha}))ds\right). \end{aligned} \tag{2.11}$$

Since $\xi_{Z,1-\alpha}$ is a Lebesgue point of λ and $|x-\xi_{Z,1-\alpha}| = o_p(1)$, then the r.h.s. of (2.11) is $o_p(1)$, as $n \to \infty$. Hence we have (2.7).

Next, substituting (2.6) with $N(0,1)\sqrt{q(x)}e^{-\Lambda(x)}/\sqrt{n_\tau}$ replaced by $N(0,1)\sqrt{q(\xi_{Z,1-\alpha})}e^{-\Lambda(\xi_{Z,1-\alpha})}/\sqrt{n_\tau}$ into (2.5), we obtain that the l.h.s. of (1.16) is equal to

$$\begin{aligned}
& \frac{\sqrt{n_\tau}\lambda(\xi_{Z,1-\alpha})}{\sqrt{q(\xi_{Z,1-\alpha})}}\left(\inf\left\{x : F_Z(x) + \frac{N(0,1)\sqrt{q(\xi_{Z,1-\alpha})}}{\sqrt{n_\tau}\ e^{\Lambda(\xi_{Z,1-\alpha})}} + o_p\left(\frac{1}{\sqrt{n}}\right) \geq 1-\alpha\right\}\right. \\
& \left. - \inf\left\{x : F_Z(x) \geq 1-\alpha\right\}\right) \\
= & \frac{\sqrt{n_\tau}\lambda(\xi_{Z,1-\alpha})}{\sqrt{q(\xi_{Z,1-\alpha})}}\left(\inf\left\{x : F_Z(x) \geq 1-\alpha + \frac{N(0,1)\sqrt{q(\xi_{Z,1-\alpha})}}{\sqrt{n_\tau}\ e^{\Lambda(\xi_{Z,1-\alpha})}} + o_p\left(\frac{1}{\sqrt{n}}\right)\right\}\right. \\
& \left. - \inf\left\{x : F_Z(x) \geq 1-\alpha\right\}\right) \\
= & \frac{\sqrt{n_\tau}\lambda(\xi_{Z,1-\alpha})}{\sqrt{q(\xi_{Z,1-\alpha})}}\left(F_Z^{-1}\left(1-\alpha + \frac{N(0,1)\sqrt{q(\xi_{Z,1-\alpha})}}{\sqrt{n_\tau}\ e^{\Lambda(\xi_{Z,1-\alpha})}} + o_p\left(\frac{1}{\sqrt{n}}\right)\right) - F_Z^{-1}(1-\alpha)\right) \\
= & \frac{\sqrt{n_\tau}\lambda(\xi_{Z,1-\alpha})}{\sqrt{q(\xi_{Z,1-\alpha})}}\left(\frac{N(0,1)\sqrt{q(\xi_{Z,1-\alpha})}}{\sqrt{n_\tau}\ e^{\Lambda(\xi_{Z,1-\alpha})}} + o_p\left(\frac{1}{\sqrt{n}}\right)\right)\left(\frac{1}{f_Z(F_Z^{-1}(1-\alpha))} + o_p(1)\right) \\
= & \frac{\sqrt{n_\tau}\lambda(\xi_{Z,1-\alpha})}{\sqrt{q(\xi_{Z,1-\alpha})}}\left(\frac{N(0,1)\sqrt{q(\xi_{Z,1-\alpha})}e^{-\Lambda(\xi_{Z,1-\alpha})}}{\sqrt{n_\tau}f_Z(\xi_{Z,1-\alpha})}\right) + o_p(1) \\
= & N(0,1) + o_p(1), \qquad (2.12)
\end{aligned}$$

as $n \to \infty$, where for any $0 < s < 1$, $F_Z^{-1}(s) = \inf\{x : F_Z(x) \geq s\}$. This completes the proof of part (ii) of Theorem 1.

Next we prove part (iii) of Theorem 1. To check this, by (1.16), it suffices to show

$$\frac{\sqrt{q(\xi_{Z,1-\alpha})}}{\sqrt{\hat{q}_n(\hat{\xi}_{Z,n,1-\alpha})}} \xrightarrow{p} 1, \qquad (2.13)$$

and

$$\frac{\hat{\lambda}_{n,K}(\hat{\xi}_{Z,n,1-\alpha})}{\lambda(\xi_{Z,1-\alpha})} \xrightarrow{p} 1, \qquad (2.14)$$

as $n \to \infty$.

First we consider (2.13). By writing

$$\frac{\sqrt{q(\xi_{Z,1-\alpha})}}{\sqrt{\hat{q}_n(\hat{\xi}_{Z,n,1-\alpha})}} = \sqrt{\frac{q(\xi_{Z,1-\alpha})}{q(\xi_{Z,1-\alpha}) + (\hat{q}_n(\hat{\xi}_{Z,n,1-\alpha}) - q(\xi_{Z,1-\alpha}))}},$$

to prove (2.13), it suffices to check

$$\left(\hat{q}_n(\hat{\xi}_{Z,n,1-\alpha}) - q(\xi_{Z,1-\alpha})\right) \xrightarrow{p} 0, \qquad (2.15)$$

as $n \to \infty$. By (1.17) and (1.19), and since $(\hat{\xi}_{Z,n,1-\alpha} - \xi_{Z,1-\alpha}) = \mathcal{O}_p(n^{-1/2})$ (cf. (1.16)) and $(\hat{\theta}_n - \theta) = \mathcal{O}_p(n^{-1/2})$, as $n \to \infty$ (cf. (4.37)), a simple argument shows that, to prove (2.13), it suffices to verify

$$\left(\hat{\Lambda}_n(\hat{\xi}_{Z,n,1-\alpha,r}) - \Lambda(\xi_{Z,1-\alpha,r})\right) \xrightarrow{p} 0, \tag{2.16}$$

as $n \to \infty$. To verify (2.16), we write the l.h.s. of (2.16) as

$$\left(\hat{\Lambda}_n(\hat{\xi}_{Z,n,1-\alpha,r}) - \hat{\Lambda}_n(\xi_{Z,1-\alpha,r})\right) + \left(\hat{\Lambda}_n(\xi_{Z,1-\alpha,r}) - \Lambda(\xi_{Z,1-\alpha,r})\right). \tag{2.17}$$

By (4.33) with z replaced by $\xi_{Z,1-\alpha,r}$, we have the second term of (2.17) is of order $\mathcal{O}_p(n^{-1/2})$, as $n \to \infty$. Next we show the first term of (2.17) is of the same order, i.e.

$$\left(\hat{\Lambda}_n(\hat{\xi}_{Z,n,1-\alpha,r}) - \hat{\Lambda}_n(\xi_{Z,1-\alpha,r})\right) = \mathcal{O}_p\left(\frac{1}{\sqrt{n}}\right), \tag{2.18}$$

as $n \to \infty$. To verify (2.18), note that by (1.16), we have $\hat{\xi}_{Z,n,1-\alpha} = \xi_{Z,1-\alpha} + \mathcal{O}_p(n^{-1/2})$, which also implies $\hat{\xi}_{Z,n,1-\alpha,r} = \xi_{Z,1-\alpha,r} + \mathcal{O}_p(n^{-1/2})$, as $n \to \infty$. The l.h.s. of (2.18) can be written as

$$\begin{aligned}
&\frac{1}{n_\tau}\sum_{k=1}^{n_\tau} X([-k\tau, \hat{\xi}_{Z,n,1-\alpha,r} - k\tau]) - \frac{1}{n_\tau}\sum_{k=1}^{n_\tau} X([-k\tau, \xi_{Z,1-\alpha,r} - k\tau]) \\
&= \frac{1}{n_\tau}\sum_{k=1}^{n_\tau} \left\{X([-k\tau, \xi_{Z,1-\alpha,r} + \mathcal{O}_p(n^{-1/2}) - k\tau]) - X([-k\tau, \xi_{Z,1-\alpha,r} - k\tau])\right\} \\
&= \mathcal{O}_p\left(\frac{1}{\sqrt{n}}\right),
\end{aligned} \tag{2.19}$$

as $n \to \infty$, since clearly $X([\xi_{Z,1-\alpha,r} - k\tau, \xi_{Z,1-\alpha,r} - k\tau + \mathcal{O}_p(n^{-1/2})]) = \mathcal{O}_p(n^{-1/2})$ uniformly in k, because λ is periodic and $\xi_{Z,1-\alpha}$ is a Lebesgue point of λ. Hence we have (2.18). Therefore, we obtain (2.16).

Next we prove (2.14). By writing the l.h.s. of (2.14) as

$$\left(\frac{\lambda(\xi_{Z,1-\alpha}) + (\hat{\lambda}_{n,K}(\hat{\xi}_{Z,n,1-\alpha}) - \lambda(\xi_{Z,1-\alpha}))}{\lambda(\xi_{Z,1-\alpha})}\right),$$

to prove (2.14), it suffices to check

$$\left(\hat{\lambda}_{n,K}(\hat{\xi}_{Z,n,1-\alpha}) - \lambda(\xi_{Z,1-\alpha})\right) \xrightarrow{p} 0, \tag{2.20}$$

as $n \to \infty$. To verify (2.20), we write the l.h.s. of (2.20) as

$$\left(\hat{\lambda}_{n,K}(\hat{\xi}_{Z,n,1-\alpha}) - \hat{\lambda}_{n,K}(\xi_{Z,1-\alpha})\right) + \left(\hat{\lambda}_{n,K}(\xi_{Z,1-\alpha}) - \lambda(\xi_{Z,1-\alpha})\right). \tag{2.21}$$

Since $\xi_{Z,1-\alpha}$ is a Lebesgue point of λ, by Theorem 2.1. of Helmers *et al.* (2003) for the case τ is known, we have the second term of (2.21) is $o_p(1)$, as $n \to \infty$. By a similar argument as the one used to prove (2.18), we also obtain the first term of (2.21) is $o_p(1)$, as $n \to \infty$. Hence we have (2.20). Therefore, we obtain (2.14). This completes the proof of Theorem 1.

Proof of (1.20):

To begin with, first we show

$$\xi_{Z,1-\alpha} < \tau \ \text{ if and only if } \ \theta\tau > \ln\left(\frac{1}{\alpha}\right). \tag{2.22}$$

To verify (2.22) we argue as follows. By (1.9) we have $\mathbf{P}\left(Z > \xi_{Z,1-\alpha}\right) = \alpha$. Note also that

$$\xi_{Z,1-\alpha} < \tau \ \ <==> \ \ \mathbf{P}\left(Z > \tau\right) < \mathbf{P}\left(Z > \xi_{Z,1-\alpha}\right) \ \ <==> \ \ \mathbf{P}\left(Z > \tau\right) < \alpha. \tag{2.23}$$

Since $\mathbf{P}(Z > \tau) = e^{-\theta\tau}$, the statement in (2.23) is equivalent to

$$e^{-\theta\tau} < \alpha \ \ <==> \ \ \theta\tau > \ln\left(\frac{1}{\alpha}\right). \tag{2.24}$$

Combining (2.23) and (2.24) we obtain (2.22). By the l.h.s. of (2.22) we have $[\frac{\xi_{Z,1-\alpha}}{\tau}] = 0$. Substituting $[\frac{\xi_{Z,1-\alpha}}{\tau}] = 0$ into $q(\xi_{Z,1-\alpha})$ we obtain the l.h.s. of (1.20). This completes the proof of (1.20).

3 Proof of Theorem 2

Since $\Lambda(\xi_{Z,1-\alpha,r}) = \Lambda(\xi_{Z,1-\alpha}) - [\frac{\xi_{Z,1-\alpha}}{\tau}]\theta\tau$, $q(\xi_{Z,1-\alpha})$ can also be written as

$$q(\xi_{Z,1-\alpha}) = (1 + 2[\frac{\xi_{Z,1-\alpha}}{\tau}])\Lambda(\xi_{Z,1-\alpha}) - (1 + [\frac{\xi_{Z,1-\alpha}}{\tau}])[\frac{\xi_{Z,1-\alpha}}{\tau}]\tau\theta,$$

instead of (1.17). Then, proving (1.29) is equivalent to checking that

$$\{(1 + 2[\frac{\xi_{Z,1-\alpha}}{\tau}])\Lambda(\xi_{Z,1-\alpha}) - (1 + [\frac{\xi_{Z,1-\alpha}}{\tau}])[\frac{\xi_{Z,1-\alpha}}{\tau}]\tau\theta\} < \frac{(1-\alpha)}{\alpha(1 - e^{-\theta\tau})}. \tag{3.1}$$

To prove (3.1), we split up condition (1.28) into three cases, namely, case (i) $\theta\tau > \ln(1/\alpha)$, case (ii) $\ln(1/\alpha)/2 < \theta\tau \leq \ln(1/\alpha)$ and case (iii) $\ln(1/\alpha)/3 < \theta\tau \leq \ln(1/\alpha)/2$.

First we consider case (i). In this case, by (2.22), we have $[\frac{\xi_{Z,1-\alpha}}{\tau}] = 0$. Since $(1 - e^{-\theta\tau}) < 1$, proving (3.1) in this case, it suffices to check

$$\Lambda(\xi_{Z,1-\alpha,r}) < \frac{(1-\alpha)}{\alpha}. \tag{3.2}$$

By noting that $\Lambda(\xi_{Z,1-\alpha,r}) \leq \Lambda(\xi_{Z,1-\alpha}) = -\ln(\alpha)$, to prove (3.2), it suffices to check

$$-\ln(\alpha) < \frac{(1-\alpha)}{\alpha} \quad <==> \quad \ln(\alpha) + \frac{1}{\alpha} - 1 > 0.$$

Let $h(\alpha) = \ln(\alpha) + 1/\alpha - 1$. We have to show, for all $0 < \alpha < 1$, $h(\alpha) > 0$. To do this, note that $h(1) = 0$ and $h'(\alpha) = \alpha^{-1}(1 - \alpha^{-1})$. Since $0 < \alpha < 1$, we have $\alpha^{-1} > 0$ and $(1 - \alpha^{-1}) < 0$, which implies $h'(\alpha) < 0$ for all $0 < \alpha < 1$. Hence, $h(\alpha)$ is monotone decreasing to 0 in interval $(0, 1)$, which implies $h(\alpha) > 0$ for all $0 < \alpha < 1$. Therefore we obtain (1.29).

Next we consider case (ii). By a similar argument as the proof of (2.22), we have

$$\frac{\ln(1/\alpha)}{2} < \theta\tau \leq \ln(\frac{1}{\alpha}) <==> \theta\tau \leq \ln(\frac{1}{\alpha}) < 2\theta\tau \text{ if and only if } \tau \leq \xi_{Z,1-\alpha} < 2\tau. \quad (3.3)$$

By (3.3) we have $[\frac{\xi_{Z,1-\alpha}}{\tau}] = 1$. Since $\theta\tau \leq \ln(1/\alpha)$, we have $(1-\alpha)/(1-e^{-\theta\tau}) \geq 1$. Then to prove (3.1) in this case, it suffices to check

$$\{3\Lambda(\xi_{Z,1-\alpha}) - 2\tau\theta\} < \frac{1}{\alpha}. \quad (3.4)$$

By noting that $\Lambda(\xi_{Z,1-\alpha}) = \ln(1/\alpha)$ and $2\theta\tau > \ln(1/\alpha)$ (cf. (3.3)), to prove (3.4), it suffices to verify

$$\{3\ln(1/\alpha) - \ln(1/\alpha)\} < \frac{1}{\alpha} <==> \alpha(\ln(1/\alpha)) < \frac{1}{2}. \quad (3.5)$$

Since the maximum value of $\alpha(\ln(1/\alpha))$ is e^{-1} (when $\alpha = e^{-1}$) which is less than 1/2, we have (3.5).

Next we consider case (iii). Similarly to (3.3), we now have

$$\frac{\ln(1/\alpha)}{3} < \theta\tau \leq \frac{\ln(1/\alpha)}{2} <==> 2\theta\tau \leq \ln(\frac{1}{\alpha}) < 3\theta\tau \text{ iff } 2\tau \leq \xi_{Z,1-\alpha} < 3\tau. \quad (3.6)$$

By (3.6) we have $[\frac{\xi_{Z,1-\alpha}}{\tau}] = 2$. Next to prove (3.1) in this case, it suffices to check

$$\{5\Lambda(\xi_{Z,1-\alpha}) - 6\tau\theta\} < \frac{(1-\alpha)}{\alpha(1-e^{-\theta\tau})}. \quad (3.7)$$

Since $\theta\tau \leq \ln(1/\alpha)/2$, we have $(1-e^{-\theta\tau}) \leq (1-\alpha^{1/2})$. By noting that $\Lambda(\xi_{Z,1-\alpha}) = \ln(1/\alpha)$ and $\theta\tau > \ln(1/\alpha)/3$ (cf. (3.6)), to prove (3.7), it suffices to verify

$$\{5\ln(1/\alpha) - 2\ln(1/\alpha)\} < \frac{(1-\alpha)}{\alpha(1-\alpha^{1/2})}. <==> \frac{(1-\alpha)}{3(1-\alpha^{1/2})} + \alpha\ln(\alpha) > 0. \quad (3.8)$$

Define

$$f_3(\alpha) = \frac{(1-\alpha)}{3(1-\alpha^{1/2})} + \alpha \ln(\alpha) = \frac{1}{3} + \frac{\sqrt{\alpha}}{3} + \alpha \ln(\alpha).$$

It remains to show that $f_3(\alpha) > 0$ for all $0 < \alpha < 1$. To verify this, first note that $f_3'(\alpha) = 1/(6\sqrt{\alpha}) + \ln(\alpha) + 1$ and $f_3''(\alpha) = -1/(12\alpha^{3/2}) + 1/\alpha$. Since the first derivative f_3' is monotone increasing on $(0,1)$ with $f_3'(0) = -\infty$ and $f_3'(1) = 7/6$, the function f_3' is equal to zero for exactly one value of α, namely 0.266351... Because $f_3''(0.266351) = 3.148215 > 0$, we can conclude that $f_3(0.266351) = 0.152998$ is the minimum value of f_3 on $(0,1)$. Hence $f_3(\alpha) > 0$ for all $0 < \alpha < 1$. This completes the proof of Theorem 2.

4 Some asymptotics

In this section we investigate the asymptotic behaviour of $\hat{F}_{Z,n}$ (cf. (1.4)), our estimator of F_Z.

Proposition 1 *Suppose that λ is periodic and locally integrable.*

(i) (Consistency) For any $z > 0$ we have

$$\hat{F}_{Z,n}(z) \xrightarrow{p} F_Z(z), \tag{4.1}$$

as $n \to \infty$.

(ii) (Asymptotic normality) For any $z > 0$ we have

$$\frac{\sqrt{n_\tau} e^{\Lambda(z)}}{\sqrt{q(z)}} \left(\hat{F}_{Z,n}(z) - F_Z(z) \right) \xrightarrow{d} N(0,1) \tag{4.2}$$

as $n \to \infty$, where $N(0,1)$ denotes a standard normal random variable and $q(z)$ is given by (1.17).

(iii) (Studentization) For any $z > 0$ we have

$$\frac{\sqrt{n_\tau} e^{\hat{\Lambda}_n(z)}}{\sqrt{\hat{q}_n(z)}} \left(\hat{F}_{Z,n}(z) - F_Z(z) \right) \xrightarrow{d} N(0,1) \tag{4.3}$$

as $n \to \infty$, where $\hat{q}_n(z)$ is given by (1.19).

The error of the normal approximation in (4.2) is easily seen to be of the classical order $n^{-1/2}$. A correction term of Edgeworth type, correcting not only for bias and skewness but also for the lattice character of the Poisson distribution, can in principle be established using a general result on Edgeworth expansions for lattice distributions due to Kolassa & McCullagh (1990). We also refer to (4.17) for a simple explicit bias correction term to $\hat{F}_{Z,n}(z)$ of order $n^{-1/2}$.

Next we prove Proposition 1. To check Proposition 1 we need the following lemmas.

Lemma 1 *Suppose that λ is periodic and locally integrable. Then for any $z > 0$ we have*

$$\mathbf{E}\hat{\Lambda}_n(z) = \Lambda(z), \tag{4.4}$$

$$Var\left(\hat{\Lambda}_n(z)\right) = \frac{q(z)}{n_\tau}, \tag{4.5}$$

where $q(z)$ is given by (1.17), and

$$\frac{\sqrt{n_\tau}}{\sqrt{q(z)}}\left(\hat{\Lambda}_n(z) - \Lambda(z)\right) \xrightarrow{d} N(0,1) \tag{4.6}$$

as $n \to \infty$.

Note also that, since $0 \le \Lambda(z_r) \le \tau\theta$, from (4.5), we have that, for any $z > 0$, $Var(\hat{\Lambda}_n(z)) = \mathcal{O}(n^{-1})$, as $n \to \infty$.

Proof: Define $\Lambda^c(z_r) = \int_{z_r}^{\tau} \lambda(s)ds$. Then $\Lambda(z_r) + \Lambda^c(z_r) = \theta\tau$, so that for any $z > 0$, we can write

$$\Lambda(z) = (1 + [\frac{z}{\tau}])\Lambda(z_r) + [\frac{z}{\tau}]\Lambda^c(z_r) \tag{4.7}$$

instead of (1.3). An estimator of $\Lambda^c(z_r)$ is given by

$$\hat{\Lambda}_n^c(z_r) = \frac{1}{n_\tau}\sum_{k=1}^{n_\tau} X((z_r - k\tau, \tau - k\tau)). \tag{4.8}$$

Note that $\hat{\Lambda}_n(z_r)$ and $\hat{\Lambda}_n^c(z_r)$ are independent and $\hat{\Lambda}_n(z_r) + \hat{\Lambda}_n^c(z_r) = \tau\hat{\theta}_n$. Hence, for any $z > 0$, we can write $\hat{\Lambda}_n(z)$ in (1.5) as

$$\hat{\Lambda}_n(z) = (1 + [\frac{z}{\tau}])\hat{\Lambda}_n(z_r) + [\frac{z}{\tau}]\hat{\Lambda}_n^c(z_r). \tag{4.9}$$

$\mathbf{E}\hat{\Lambda}_n(z_r)$ can be computed as follows.

$$\begin{aligned}\mathbf{E}\hat{\Lambda}_n(z_r) &= \frac{1}{n_\tau}\sum_{k=1}^{n_\tau}\mathbf{E}X([-k\tau, z_r - k\tau]) = \frac{1}{n_\tau}\sum_{k=1}^{n_\tau}\int_{-k\tau}^{z_r - k\tau}\lambda(x)dx = \frac{1}{n_\tau}\sum_{k=1}^{n_\tau}\int_0^{z_r}\lambda(x)dx \\ &= \Lambda(z_r).\end{aligned} \tag{4.10}$$

Similarly we have $\mathbf{E}\hat{\Lambda}_n^c(z_r) = \Lambda^c(z_r)$. Replacing the r.v.'s on the r.h.s. of (4.9) with their expectations, we obtain the r.h.s. of (4.7). Hence we have (4.4).

Next we prove (4.5). Since $\hat{\Lambda}_n(z_r)$ and $\hat{\Lambda}_n^c(z_r)$ are independent, by (4.9), we have

$$Var(\hat{\Lambda}_n(z)) = (1 + [\frac{z}{\tau}])^2 Var(\hat{\Lambda}_n(z_r)) + [\frac{z}{\tau}]^2 Var(\hat{\Lambda}_n^c(z_r)). \tag{4.11}$$

For any $0 < z_r < \tau$ and any pair of integers (k, j), with $k \neq j$, we have that $X([-k\tau, z_r - k\tau])$ and $X([-j\tau, z_r - j\tau])$ are independent. Then, by a similar calculation as the one in (4.10), we obtain

$$Var\left(\hat{\Lambda}_n(z_r)\right) \;=\; \frac{1}{n_\tau^2}\sum_{k=1}^{n_\tau} Var(X([-k\tau, z_r - k\tau]) = \frac{\Lambda(z_r)}{n_\tau}. \tag{4.12}$$

Similarly we also have $Var(\hat{\Lambda}_n^c(z_r)) = \Lambda^c(z_r)/n_\tau$. Substituting these variances into the r.h.s. of (4.11) we obtain

$$Var(\hat{\Lambda}_n(z)) = \frac{(1 + [\frac{z}{\tau}])^2\Lambda(z_r) + [\frac{z}{\tau}]^2\Lambda^c(z_r)}{n_\tau}. \tag{4.13}$$

Since $\Lambda^c(z_r) = \theta\tau - \Lambda(z_r)$, we have

$$(1 + [\frac{z}{\tau}])^2\Lambda(z_r) + [\frac{z}{\tau}]^2\Lambda^c(z_r) = [\frac{z}{\tau}]^2\tau\theta + (1 + 2[\frac{z}{\tau}])\Lambda(z_r) = q(z) \tag{4.14}$$

(cf. (1.17)). Substituting (4.14) into the r.h.s. of (4.13), we obtain (4.5).

Next we check (4.6). By (4.7), (4.9), (4.12) and the line after (4.12), we can write

$$\begin{aligned}
\sqrt{n_\tau}\left(\hat{\Lambda}_n(z) - \Lambda(z)\right) \;&=\; \sqrt{n_\tau}\left(1 + [\frac{z}{\tau}]\right)\left(\hat{\Lambda}_n(z_r) - \Lambda(z_r)\right) + \sqrt{n_\tau}[\frac{z}{\tau}]\left(\hat{\Lambda}_n^c(z_r) - \Lambda^c(z_r)\right) \\
&=\; \sqrt{\Lambda(z_r)}(1 + [\frac{z}{\tau}])\left(\frac{\sum_{k=1}^{n_\tau} X([-k\tau, z_r - k\tau]) - n_\tau\Lambda(z_r)}{\sqrt{n_\tau\Lambda(z_r)}}\right) \\
&\quad + \sqrt{\Lambda^c(z_r)}[\frac{z}{\tau}]\left(\frac{\sum_{k=1}^{n_\tau} X((z_r - k\tau, \tau - k\tau)) - n_\tau\Lambda^c(z_r)}{\sqrt{n_\tau\Lambda^c(z_r)}}\right).
\end{aligned} \tag{4.15}$$

Since $\sum_{k=1}^{n_\tau} X([-k\tau, z_r - k\tau])$ is a Poisson random variable with mean $n_\tau\Lambda(z_r) \to \infty$, as $n \to \infty$, then using normal approximation for Poisson random variables, we obtain the r.v. in the first term on the r.h.s. of (4.15) converges in distribution to $N(0, (1 + [\frac{z}{\tau}])^2\Lambda(z_r))$, as $n \to \infty$. Similarly, we also have that the r.v. in the second term on the r.h.s. of (4.15) converges in distribution to $N(0, [\frac{z}{\tau}]^2\Lambda^c(z_r))$, as $n \to \infty$. Note also that, these two normal r.v.'s are independent. Hence, by noting that sum of two independent normal r.v.'s is another normal r.v., we obtain

$$\sqrt{n_\tau}\left(\hat{\Lambda}_n(z) - \Lambda(z)\right) \xrightarrow{d} N(0, (1 + [\frac{z}{\tau}])^2\Lambda(z_r) + [\frac{z}{\tau}]^2\Lambda^c(z_r)) \tag{4.16}$$

as $n \to \infty$. Substituting (4.14) into the r.h.s. of (4.16) and then multiplying both sides by $(q(z))^{-1/2}$, we obtain (4.6). This completes the proof of Lemma 1.

Lemma 2 *Suppose that λ is periodic and locally integrable. Then for any $z > 0$ we have*

$$\mathbf{E}\left(\hat{F}_{Z,n}(z)\right) = F_Z(z) - \frac{q(z)e^{-\Lambda(z)}}{2n_\tau} + \mathcal{O}\left(\frac{1}{n^2}\right), \tag{4.17}$$

and

$$Var\left(\hat{F}_{Z,n}(z)\right) = \frac{q(z)e^{-2\Lambda(z)}}{n_\tau} + \mathcal{O}\left(\frac{1}{n^2}\right), \tag{4.18}$$

as $n \to \infty$.

Proof: First we check (4.17). By (4.9) and noting that $\hat{\Lambda}_n(z_r)$ and $\hat{\Lambda}_n^c(z_r)$ are independent, we can compute $\mathbf{E}(\hat{F}_{Z,n}(z))$ as follows.

$$\begin{aligned} \mathbf{E}\left(\hat{F}_{Z,n}(z)\right) &= 1 - \mathbf{E}e^{-\hat{\Lambda}_n(z)} = 1 - \mathbf{E}e^{-(1+[\frac{z}{\tau}])\hat{\Lambda}_n(z_r)-[\frac{z}{\tau}]\hat{\Lambda}_n^c(z_r)} \\ &= 1 - \mathbf{E}e^{-(1+[\frac{z}{\tau}])\hat{\Lambda}_n(z_r)}\mathbf{E}e^{-[\frac{z}{\tau}]\hat{\Lambda}_n^c(z_r)}. \end{aligned} \tag{4.19}$$

To compute the expectation on the r.h.s. of (4.19), we use the moment generating function of Poisson r.v., i.e., if Y is a Poisson r.v. with mean μ then $\mathbf{E}\exp(tY) = \exp(\mu(e^t - 1))$. Since $\sum_{k=1}^{n_\tau} X([-k\tau, z_r - k\tau])$ is a Poisson random variable with mean $n_\tau\Lambda(z_r)$, the moment generating function for Poisson r.v., with $t = -(1 + [\frac{z}{\tau}])/n_\tau$, yields

$$\begin{aligned} \mathbf{E}e^{-(1+[\frac{z}{\tau}])\hat{\Lambda}_n(z_r)} &= \mathbf{E}\exp\left(-\frac{(1+[\frac{z}{\tau}])}{n_\tau}\sum_{k=1}^{n_\tau} X([-k\tau, z_r - k\tau])\right) \\ &= \exp\left(n_\tau\Lambda(z_r)\left(e^{-(1+[\frac{z}{\tau}])/n_\tau} - 1\right)\right) \end{aligned} \tag{4.20}$$

By Taylor we have

$$e^{-(1+[\frac{z}{\tau}])/n_\tau} = 1 - \frac{(1+[\frac{z}{\tau}])}{n_\tau} + \frac{(1+[\frac{z}{\tau}])^2}{2n_\tau^2} + \mathcal{O}\left(\frac{1}{n^3}\right) \tag{4.21}$$

as $n \to \infty$. Substituting (4.21) into the r.h.s. of (4.20), we obtain

$$\begin{aligned} \mathbf{E}e^{-(1+[\frac{z}{\tau}])\hat{\Lambda}_n(z_r)} &= \exp\left(n_\tau\Lambda(z_r)\left(-\frac{(1+[\frac{z}{\tau}])}{n_\tau} + \frac{(1+[\frac{z}{\tau}])^2}{2n_\tau^2} + \mathcal{O}\left(\frac{1}{n^3}\right)\right)\right) \\ &= \exp\left(-(1+[\frac{z}{\tau}])\Lambda(z_r)\right)\ \exp\left(\frac{(1+[\frac{z}{\tau}])^2}{2n_\tau}\Lambda(z_r)\right)\ \exp\left(\mathcal{O}\left(\frac{1}{n^2}\right)\right), \end{aligned} \tag{4.22}$$

as $n \to \infty$. By Taylor, we can simplify the r.h.s. of (4.22) to obtain

$$\begin{aligned} \mathbf{E}e^{-(1+[\frac{z}{\tau}])\hat{\Lambda}_n(z_r)} &= e^{-(1+[\frac{z}{\tau}])\Lambda(z_r)}\left(1 + \frac{(1+[\frac{z}{\tau}])^2}{2n_\tau}\Lambda(z_r) + \mathcal{O}\left(\frac{1}{n^2}\right)\right)\left(1 + \mathcal{O}\left(\frac{1}{n^2}\right)\right) \\ &= e^{-(1+[\frac{z}{\tau}])\Lambda(z_r)} + \frac{(1+[\frac{z}{\tau}])^2\Lambda(z_r)e^{-(1+[\frac{z}{\tau}])\Lambda(z_r)}}{2n_\tau} + \mathcal{O}\left(\frac{1}{n^2}\right), \end{aligned} \tag{4.23}$$

as $n \to \infty$. Similarly we have

$$\mathbf{E}e^{-[\frac{z}{\tau}]\hat{\Lambda}_n^c(z_r)} = e^{-[\frac{z}{\tau}]\Lambda^c(z_r)} + \frac{[\frac{z}{\tau}]^2\Lambda^c(z_r)e^{-[\frac{z}{\tau}]\Lambda^c(z_r)}}{2n_\tau} + \mathcal{O}\left(\frac{1}{n^2}\right), \qquad (4.24)$$

as $n \to \infty$. Substituting (4.23) and (4.24) into (4.19) and by (4.14), we obtain (4.17).

Next we verify (4.18).

$$Var\left(\hat{F}_{Z,n}(z)\right) \;=\; Var\left(e^{-\hat{\Lambda}_n(z)}\right) = \mathbf{E}\left(e^{-\hat{\Lambda}_n(z)}\right)^2 - \left(\mathbf{E}e^{-\hat{\Lambda}_n(z)}\right)^2. \qquad (4.25)$$

The first expectation on the r.h.s. of (4.25) can be computed as follows

$$\mathbf{E}\left(e^{-\hat{\Lambda}_n(z)}\right)^2 = \mathbf{E}\left(e^{-2\hat{\Lambda}_n(z)}\right) = \mathbf{E}e^{-2(1+[\frac{z}{\tau}])\hat{\Lambda}_n(z_r)}\ \mathbf{E}e^{-2[\frac{z}{\tau}]\hat{\Lambda}_n^c(z_r)}. \qquad (4.26)$$

By a similar calculation as the one in (4.20) - (4.23), with $-(1+[\frac{z}{\tau}])/n_\tau$ now replaced by $-2(1+[\frac{z}{\tau}])/n_\tau$, we obtain

$$\mathbf{E}e^{-2(1+[\frac{z}{\tau}])\hat{\Lambda}_n(z_r)} = e^{-2(1+[\frac{z}{\tau}])\Lambda(z_r)} + \frac{2(1+[\frac{z}{\tau}])^2\Lambda(z_r)e^{-2(1+[\frac{z}{\tau}])\Lambda(z_r)}}{n_\tau} + \mathcal{O}\left(\frac{1}{n^2}\right), \qquad (4.27)$$

as $n \to \infty$. Similarly we have

$$\mathbf{E}e^{-2[\frac{z}{\tau}]\hat{\Lambda}_n^c(z_r)} = e^{-2[\frac{z}{\tau}]\Lambda^c(z_r)} + \frac{2[\frac{z}{\tau}]^2\Lambda^c(z_r)e^{-2[\frac{z}{\tau}]\Lambda^c(z_r)}}{n_\tau} + \mathcal{O}\left(\frac{1}{n^2}\right), \qquad (4.28)$$

as $n \to \infty$. Substituting (4.27) and (4.28) into the r.h.s. of (4.26), and by (4.14), we obtain

$$\mathbf{E}\left(e^{-\hat{\Lambda}_n(z)}\right)^2 = e^{-2\Lambda(z)} + \frac{2q(z)\ e^{-2\Lambda(z)}}{n_\tau} + \mathcal{O}\left(\frac{1}{n^2}\right), \qquad (4.29)$$

as $n \to \infty$. From (4.17) we obtain

$$\left(\mathbf{E}e^{-\hat{\Lambda}_n(z)}\right)^2 = \left(1 - \mathbf{E}\hat{F}_{Z,n}(z)\right)^2 = e^{-2\Lambda(z)} + \frac{q(z)\ e^{-2\Lambda(z)}}{n_\tau} + \mathcal{O}\left(\frac{1}{n^2}\right), \qquad (4.30)$$

as $n \to \infty$. Substituting (4.29) and (4.30) into the r.h.s. of (4.25), we obtain (4.18). This completes the proof of Lemma 2.

Proof of Proposition 1

By Lemma 2, i.e. $\mathbf{E}(\hat{F}_{Z,n}(z) - F_Z(z)) = \mathcal{O}(n^{-1})$ and $Var(\hat{F}_{Z,n}(z)) = \mathcal{O}(n^{-1})$, as $n \to \infty$, Chebychev inequality yields part (i) of Proposition 1.

To prove part (ii) of Proposition 1, we argue as follows. First we write the l.h.s. of (4.2) as follows

$$\frac{\sqrt{n_\tau}e^{\Lambda(z)}}{\sqrt{q(z)}}\left(e^{-\Lambda(z)} - e^{-\hat{\Lambda}_n(z)}\right) = \frac{\sqrt{n_\tau}}{\sqrt{q(z)}}\left(1 - e^{-(\hat{\Lambda}_n(z)-\Lambda(z))}\right). \tag{4.31}$$

By Taylor we can write

$$\begin{aligned} e^{-(\hat{\Lambda}_n(z)-\Lambda(z))} &= 1 - \left(\hat{\Lambda}_n(z) - \Lambda(z)\right) + \frac{1}{2!}\left(\hat{\Lambda}_n(z) - \Lambda(z)\right)^2 \\ &\quad -\frac{1}{3!}\left(\hat{\Lambda}_n(z) - \Lambda(z)\right)^3 + \dots \end{aligned} \tag{4.32}$$

By (4.6) we have

$$\left(\hat{\Lambda}_n(z) - \Lambda(z)\right) = \frac{\sqrt{q(z)}}{\sqrt{n_\tau}}N(0,1) + o_p\left(\frac{1}{\sqrt{n}}\right), \tag{4.33}$$

as $n \to \infty$. Combining (4.32) and (4.33), we have

$$\left(1 - e^{-(\hat{\Lambda}_n(z)-\Lambda(z))}\right) = \frac{\sqrt{q(z)}}{\sqrt{n_\tau}}N(0,1) + o_p\left(\frac{1}{\sqrt{n}}\right), \tag{4.34}$$

as $n \to \infty$. Substituting (4.34) into r.h.s. of (4.31), we obtain

$$\frac{\sqrt{n_\tau}e^{\Lambda(z)}}{\sqrt{q(z)}}\left(\hat{F}_{Z,n}(z) - F_Z(z)\right) = N(0,1) + o_p(1), \tag{4.35}$$

as $n \to \infty$. Hence we have (4.2). This completes the proof of part(ii) of Proposition 1.

Next we prove part (iii) of Proposition 1. By part(ii) of Proposition 1, to verify part (iii) of Proposition 1, it suffices to check, for any $z > 0$,

$$\sqrt{\frac{q(z)}{\hat{q}_n(z)}}\; e^{(\hat{\Lambda}_n(z)-\Lambda(z))} \xrightarrow{p} 1, \tag{4.36}$$

as $n \to \infty$. By (4.6) we have $(\hat{\Lambda}_n(z) - \Lambda(z)) = \mathcal{O}_p(n^{-1/2})$, as $n \to \infty$. By (4.33) we have $(\hat{\Lambda}_n(z_r) - \Lambda(z_r)) = \mathcal{O}_p(n^{-1/2})$ and similarly $(\hat{\Lambda}^c_n(z_r) - \Lambda^c(z_r)) = \mathcal{O}_p(n^{-1/2})$, as $n \to \infty$. This implies

$$(\hat{\theta}_n - \theta) = \mathcal{O}_p(n^{-1/2}), \tag{4.37}$$

as $n \to \infty$. Hence we can write $\hat{q}_n(z) = q(z) + \mathcal{O}_p(n^{-1/2})$, as $n \to \infty$. Then the l.h.s. of (4.36) can be smplified as follows

$$\begin{aligned} &\sqrt{\frac{q(z)}{q(z) + \mathcal{O}_p(n^{-1/2})}}\; e^{\mathcal{O}_p(n^{-1/2})} = \sqrt{\frac{q(z)}{q(z)(1 + \mathcal{O}_p(n^{-1/2}))}}\; e^{\mathcal{O}_p(n^{-1/2})} \\ &= \sqrt{1 + \mathcal{O}_p\left(\frac{1}{\sqrt{n}}\right)}\left(1 + \mathcal{O}_p\left(\frac{1}{\sqrt{n}}\right)\right) = 1 + \mathcal{O}_p\left(\frac{1}{\sqrt{n}}\right), \end{aligned} \tag{4.38}$$

as $n \to \infty$. Hence we obtain (4.36). This completes the proof of Proposition 1.

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References

Helmers, R. and Zitikis, R. (1999). On estimation of Poisson intensity functions, *Annals of the Institute of Statistical Mathematics* **51**, 2, 265-280.

Helmers, R., Mangku, I W. and Zitikis, R. (2003). Consistent estimation of the intensity function of a cyclic Poisson process. *Journal of Multivariate Analysis.* **84**, 19-39.

Helmers, R., Mangku, I W. and Zitikis, R. (2005). Statistical properties of a kernel-type estimator of the intensity function of a cyclic Poisson process. *Journal of Multivariate Analysis.* **92**, 1-23.

Helmers, R., Wang, Q. and Zitikis, R. (2009). Confidence regions for the intensity function of a cyclic Poisson process. *Stat.Infer. Stoch. Process,* 12(1), 21-36.

Helmers, R. and Mangku, I W. (2009). Estimating the intensity of a cyclic Poisson process in the presence of linear trend. *Annals of the Institute of Statistical Mathematics*, **61**(3), 599-628.

Ho, Y. H. S. and Lee, S. M. S. (2005). Iterated smoothed bootstrap confidence intervals for population quantiles. *The Annals of Statistics.* **33**, 437-462.

Kolassa, J. E. and McCullagh, P. (1990). Edgeworth series for lattice distributions. *The Annals of Statistics.* **18**, 981-985.

Reiss, R. D. (1989). *Approximate Distributions of Order Statistics, with Applications to Nonparametric Statistics.* Springer, New York.

Vit, P. (1973). Interval prediction for a Poisson process. *Biometrika.* **60**, 667-668.

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