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Packing Odd Paths

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Let s, t be vertices of a graph G, and let each edge e have a "capacity" $c(e) \in \mathbf{R}_+$. We prove a conjecture of Cook and Sebő that for every $k \in \mathbf{R}_+$, the following two statements are equivalent:

(i) there is a "fractional packing" of value k of the odd length $s-t$ paths, so that no edge is used more than its capacity;

(ii) for every subgraph H of G with $s, t \in V(H)$ in which there is no odd $s-t$ path,

$$\sum_{v \in V(H)} \sum (c(e) \colon e \in E(G) - E(H), \text{ and } e \text{ is incident with } v) \geqslant 2k.$$

1. INTRODUCTION

All graphs in this paper are finite and undirected, and may have loops or multiple edges. The vertex- and edge-sets of a graph G are denoted by $V(G)$ and $E(G)$. A *path* in a graph G is a non-null connected subgraph P with $|E(P)| = |V(P)| - 1$ and with no vertex of valency $\geqslant 3$. Thus, paths have no "repeated" vertices. The *ends* of a path are defined in the natural way, and a path with ends s, t is called an *$s-t$ path.* A path P is *odd* or *even* depending whether $|E(P)|$ is odd or even. The sets of non-negative real numbers, rationals and integers are denoted by $\mathbf{R}_+$, $\mathbf{Q}_+$, and $\mathbf{Z}_+$. If H is a subgraph of a graph G and $c \in \mathbf{R}_+^{E(G)}$ we denote $\sum (c(e) \colon e \in E(H))$ by $c(H)$.

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Let s, t be distinct vertices of a graph G, and let $c \in \mathbf{Q}_+^{E(G)}$. How can we determine the minimum of $c(P)$ taken over all odd $s-t$ paths P? Edmonds (see [2]) gave a polynomial algorithm for this, by reducing the problem to a minimum weight perfect matching problem, as follows. Take the disjoint union of two copies G_1, G_2 of G, and for each $v \in V(G)$ and $e \in E(G)$ let v_i, e_i denote the corresponding vertex or edge of $G_i (i=1, 2)$. For each $v \in V(G)$ add a new edge e_v say with ends v_1, v_2, and delete s_2 and t_2. Let H be the graph we obtain. For each $f \in E(H)$, let

$$d(f) = \begin{cases} c(e), & \text{if } e \in E(G) \text{ and } f = e_1 \text{ or } e_2, \\ 0, & \text{if } v \in V(G) \text{ and } f = e_v. \end{cases}$$

Then it is easy to see that the desired minimum of $c(P)$ over all odd $s-t$ paths of G equals the minimum of $d(F)$ taken over all perfect matchings F of H, and the latter is a well-solved problem from matching theory.

However, there remain some problems about odd $s-t$ paths which resist solution by this approach. In particular, let $\Pi \subseteq \mathbf{R}_+^{E(G)}$ be the polyhedron defined by $c \in \Pi$ if and only if $c \in \mathbf{R}_+^{E(G)}$ and $c(P) \geqslant 1$ for every odd $s-t$ path P of G. Edmonds' method gives us a polynomial time algorithm to test if an arbitrary $c \in \mathbf{Q}_+^{E(G)}$ belongs to Π, but it tells us little about the vertices of Π, as was observed by Grötschel [1].

Not all the vertices of Π need be integral. For instance, let G have six vertices s, t, u, v, w, x, and edges $su, sv, tu, tw, tx, uv, vw, wx$; then $(0, \frac{1}{2}, 0, 0, 0, \frac{1}{2}, \frac{1}{2}, 0)$ is a vertex of Π, with the obvious notation. But we shall show that in general, every vertex of Π is $(0, \frac{1}{2}, 1)$-valued, thereby proving an unpublished conjecture of Cook and Sebő.

Let $s, t \in V(G)$ be distinct. A subgraph H of G is *odd-free* if $s, t \in V(H)$ and there is no odd $s-t$ path in H. A function $h \in \mathbf{Z}_+^{E(G)}$ is called a *slice* if there is an odd-free subgraph H such that for each $e \in E(G)$ with ends u, v,

$$h(e) = \begin{cases} 2, & \text{if } u, v \in V(H) \text{ and } e \notin E(H), \\ 1, & \text{if exactly one of } u, v \text{ belongs to } V(H), \\ 0, & \text{otherwise.} \end{cases}$$

In this situation h is called the *slice defined by H*. The following is our main result.

(1.1) *Let $s, t \in V(G)$ be distinct, let $k \in \mathbf{Z}_+$, and let $c \in \mathbf{Z}_+^{E(G)}$, such that $c(P)$ is even for every circuit P and for every $s-t$ path P. Then the following are equivalent*:

(i) *$c(P) \geqslant 2k$ for every odd $s-t$ path P of G*

(ii) *there are k slices $h_1, \ldots, h_k$ such that $h_1 + \cdots + h_k \leqslant c$.*

If h is the slice determined by an odd-free subgraph H, and P is an odd $s-t$ path, then since P is not a subgraph of H and $s, t \in V(H)$, there are at least two vertices of P in $V(H)$ incident with edges of P not in $E(H)$. Consequently, $h(P) \geqslant 2$ and so that (ii) implies (i) in (1.1) is obvious. We prove the converse implication in the next section.

From (1.1), it follows (by scaling) that an arbitrary $c \in \mathbf{Q}_{+}^{E(G)}$ dominates a convex combination of slices if and only if $c(P) \geqslant 2$ for every odd $s-t$ path. In particular, every vertex of 2Π (where Π is as defined earlier) dominates a convex combination of slices. Since each slice belongs to 2Π, it follows that every vertex of 2Π is a slice, and so every vertex of Π is $(0, \frac{1}{2}, 1)$-valued.

In Section 3 we discuss the "blocking" problem, that of packing odd $s-t$ paths.

2. The Main Proof

The goal of this section is to prove that (i) implies (ii) in (1.1). The method of proof is, given c and $k \geqslant 1$ satisfying (1.1)(i), we shall construct a slice h so that $c \geqslant h$ and so that $c-h$, $k-1$ still satisfy (1.1)(i); then, by induction on k, $c-h$ dominates the sum of $k-1$ slices, and so c dominates the sum of k slices, as required. First, we need the following. If $X \subseteq E(G)$, a subgraph P of G is *X-odd* if $|E(P) \cap X|$ is odd, and is *X-even* otherwise. If G, H are graphs we write $H \subseteq G$ to denote that H is a subgraph of G; and if H_1, H_2 are subgraphs of G, the subgraphs $H_1 \cup H_2$, $H_1 \cap H_2$ have the natural definition.

(2.1) *Let $s, t \in V(G)$ be distinct, let $X \subseteq E(G)$, let $c \in \mathbf{R}_{+}^{E(G)}$, and let P, Q be X-even $s-t$ paths of G. If $P \cup Q$ has an X-odd circuit then there is an X-odd $s-t$ path $R \subseteq P \cup Q$ such that $c(R) \leqslant \frac{1}{2}(c(P)+c(Q))$.*

Proof. By adding parallel edges to G and X we may assume that $E(P \cap Q) = \emptyset$. We define an *arc* of P to be a subpath of P with distinct ends both in $V(Q)$, and with no internal vertex in $V(Q)$. Thus each edge of P belongs to a unique arc. For each arc A of P its *fundamental circuit* is the unique circuit in $A \cup Q$. We say that A is a *special arc* if its fundamental circuit is X-odd. We define the arcs and special arcs of Q similarly.

(1) *For each X-odd circuit C of $P \cup Q$, $C \cap Q$ includes a special arc of Q.*

For let the arcs of Q included in $C \cap Q$ be $B_1, \ldots, B_n$, and let the fundamental circuit of B_i be C_i $(1 \leqslant i \leqslant n)$. Then the modulo 2 sum of

$E(C), E(C_1), \ldots, E(C_n)$ is a subset of $E(P)$ with an even number of edges incident with every vertex, and hence is empty. Since $|E(C) \cap X|$ is odd it follows that $|E(C_i) \cap X|$ is odd for some i, and hence B_i is special, as required.

For each arc A of P, define $d(A) = c(A) + \frac{1}{2}c(Q)$, and for each arc A of Q, define $d(A) = c(A) + \frac{1}{2}c(P)$. Now there is an X-odd circuit in $P \cup Q$ by hypothesis, and so by (1) there is a special arc. Let A be a special arc of either P or Q, chosen with $d(A)$ minimal. From the symmetry we may assume that $A \subseteq P$. Let R be the $s-t$ path different from Q in $Q \cup A$, and let C be the fundamental circuit of A. Since C is X-odd and Q is X-even it follows that R is X-odd, and we claim it satisfies the theorem.

For by (1), there is a special arc B of Q with $B \subseteq C \cap Q$. Then $E(B) \cap E(R) = \emptyset$, and so

$$c(R) \leqslant c(Q) - c(B) + c(A).$$

But from the choice of A, $d(A) \leqslant d(B)$, that is

$$c(A) + \tfrac{1}{2}c(Q) \leqslant c(B) + \tfrac{1}{2}c(P).$$

It follows that $c(R) \leqslant \frac{1}{2}(c(P) + c(Q))$, as required. ∎

(2.1) has the following corollary.

(2.2) *Let $s, t \in V(G)$ be distinct, let $c \in \mathbf{R}_+^{E(G)}$, and let P, Q be $s-t$ paths. Let L be a path of G with ends u, v, such that $V(L \cap P) = \{u\}$ and $V(L \cap Q) = \{v\}$. Let e, f be edges of the subpaths of P between u and s, t respectively. Then there is an $s-t$ path $R \subseteq P \cup Q \cup L$ such that exactly one of e, f belongs to R, and $c(R) \leqslant \frac{1}{2}(c(P) + c(Q)) + c(L)$.*

Proof. Let G' be obtained from $P \cup Q \cup L$ by contracting all edges of L, thereby identifying the vertices of L into one new vertex w say. Now $u \neq s, t$ and $V(L \cap P) = \{u\}$, so $s, t \notin V(L)$. Since only one vertex of L belongs to $V(P)$, there is an $s-t$ path P' of G' with $E(P') = E(P)$. Define Q' similarly, and let $X = \{e, f\} \subseteq E(G')$. Now $P' \cup Q'$ has an X-odd circuit, since the closed walk formed by following P' from s to w and then Q' from w to s has exactly one edge in X. Moreover, P' and Q' are both X-even $s-t$ paths in G'. By (2.1) applied to G', P', Q', X, there is an X-odd $s-t$ path $R \subseteq P' \cup Q'$ with

$$c(R') \leqslant \tfrac{1}{2}(c(P') + c(Q')) = \tfrac{1}{2}(c(P) + c(Q)).$$

Since R' is X-odd, exactly one of e, f belong to $E(R')$. Let R be the $s-t$ path of G with $E(R') \subseteq E(R) \subseteq E(R') \cup E(L)$. Then

$$c(R) \leqslant c(R') + c(L) \leqslant \tfrac{1}{2}(c(P) + c(Q)) + c(L),$$

as required. ∎

Throughout the remainder of this section, s and t are distinct vertices of a graph G, $c \in \mathbf{Z}_+^{E(G)}$ is such that $c(P)$ is even for every circuit or $s-t$ path P, and $k \in \mathbf{Z}_+$ is such that $c(P) \geqslant 2k$ for every odd $s-t$ path P. We define J to be the subgraph of G formed by s, t and the union of all $s-t$ paths P with $c(P) < 2k$. Thus, J is connected unless $c(P) \geqslant 2k$ for every $s-t$ path.

(2.3) *Every $s-t$ path of J is even.*

Proof. We may assume that J is connected. Define A (respectively B) to be the set of all $v \in V(J)$ such that there is an $s-t$ path P with $c(P) < 2k$ and $v \in V(P)$, where the subpath of P between s and v is odd (respectively, even). Since J is connected it follows that $A \cup B = V(J)$. Moreover, for every edge e of J with ends u, v say, e belongs to some $s-t$ path P with $c(P) < 2k$, and so one of u, v belongs to A and the other to B. We claim that $A \cap B = \emptyset$. For suppose not; then there are $s-t$ paths P, Q with $c(P)$, $c(Q) < 2k$, such that $P \cup Q$ has an odd closed walk and hence an odd circuit. Then P, Q are even, since $c(R) \geqslant 2k$ for every odd $s-t$ path R. By (2.1) (with $X = E(G)$) there is an odd $s-t$ path $R \subseteq P \cup Q$ such that $c(R) \leqslant \frac{1}{2}(c(P) + c(Q)) < 2k$, a contradiction. Thus $A \cap B = \emptyset$. It follows that (A, B) is a 2-colouring of J, and so J is bipartite. Since J is connected there is an $s-t$ path P with $c(P) < 2k$, and hence P is even. But since J is bipartite, all $s-t$ paths in J have the same parity as P, and the result follows. ▮

(2.4) *If $k \geqslant 1$ and L is a path of G with distinct ends, both in $V(J)$, and with no edge or internal vertex in J, then $c(L) \geqslant 2$.*

Proof. Let L have ends u, v. If $\{u, v\} = \{s, t\}$ then L is an $s-t$ path and $L \nsubseteq J$, and so $c(L) \geqslant 2k \geqslant 2$ as required. We may assume then that $u \neq s, t$. Consequently $V(J) \neq \{s, t\}$ and so J is connected. Let P be an $s-t$ path with $u \in V(P)$ and with $c(P) < 2k$. Since J is connected, there is also an $s-t$ path Q with $v \in V(Q)$ and with $c(Q) < 2k$. Since $c(P)$, $c(Q)$ are even it follows that $c(P)$, $c(Q) \leqslant 2k-2$. Suppose first that $u \in V(Q)$. Let Q' be the $s-t$ path in $Q \cup L$ different from Q. Since $Q' \nsubseteq J$ it follows that $c(Q') \geqslant 2k$; but $c(Q') \leqslant c(Q) + c(L)$ and $c(Q) \leqslant 2k-2$, and so $c(L) \geqslant 2$ as required. We may assume then that $u \notin V(Q)$, and similarly that $v \notin V(P)$.

Let e, f be the edges of P incident with u. By (2.2) there is an $s-t$ path $R \subseteq P \cup Q \cup L$ such that exactly one of e, f belongs to R, and $c(R) \leqslant \frac{1}{2}(c(P) + c(Q)) + c(L)$. Since exactly one of e, f belong to R and $u \notin V(Q)$, it follows that u has valency 1 in $R \cap J$, and since $u \neq s, t$ we deduce that $R \nsubseteq J$. Consequently $c(R) \geqslant 2k$. Since $c(P)$, $c(Q) \leqslant 2k-2$ we have

$$2k \leqslant c(R) \leqslant \tfrac{1}{2}(c(P) + c(Q)) + c(L) \leqslant 2k - 2 + c(L)$$

and so $c(L) \geqslant 2$ as required. ▮

Let G' be the subgraph of G with $V(G') = V(G)$ and

$$E(G') = E(J) \cup \{e \in E(G): c(e) = 0\}.$$

Let H be the union of the (one or two) components of G' that intersect $\{s, t\}$. Then $J \subseteq H$, and for every $v \in V(H)$ there exist $u \in V(J)$ and a $u-v$ path L such that $c(e) = 0$ for all $e \in E(L)$ and $V(L \cap J) = \{u\}$.

(2.5) *If $k \geqslant 1$, H is odd-free.*

Proof. Suppose that $P \subseteq H$ is an odd $s-t$ path. Since $P \nsubseteq J$ by (2.3), there is a subpath L of P with distinct ends both in $V(J)$ and with no edge or internal vertex in J; and consequently $c(e) = 0$ for all $e \in E(L)$. This contradicts (2.4), and so there is no such P, as required. ∎

Let h be the slice defined by H. Let $c' = c - h$.

(2.6) *If $k \geqslant 1$ then $c'(e) \geqslant 0$ for every edge e of G.*

Proof. Since $c(e) \geqslant 0$ we may assume that $h(e) \geqslant 1$. Hence $e \notin E(H)$, and at least one end of e belongs to $V(H)$. From the definition of H, $c(e) > 0$, and so we may assume that $h(e) = 2$, and both ends u, v of e belong to $V(H)$. Let P, Q be minimal paths of H from $V(J)$ to u, v respectively. Then $c(f) = 0$ for every edge f of $P \cup Q$. If $V(P \cap Q) = \varnothing$, let L be the path formed by P, Q, and e. By (2.4), $c(L) \geqslant 2$, and so $c(e) \geqslant 2$, and hence $c'(e) \geqslant 0$, as required. On the other hand, if $V(P \cap Q) \neq \varnothing$ there is a circuit C of G with $e \in E(C)$, such that $c(f) = 0$ for every edge $f \neq e$ of C. Since $c(C)$ is even by hypothesis it follows that $c(e)$ is even, and so $c(e) \geqslant 2$; and hence again $c'(e) \geqslant 0$, as required. ∎

(2.7) *If $k \geqslant 1$ then $c'(P) \geqslant 2k-2$ for every $s-t$ path P with $P \nsubseteq H$.*

Proof. If possible, choose an $s-t$ path P with $c'(P) < 2k-2$ and $P \nsubseteq H$, with $P \cup H$ minimal. Since $P \nsubseteq H$ it follows that $P \cap H$ has at least two components, one containing s and the other t. If it has exactly two components then $h(P) = 2$ from the definition of h, and so $c'(P) = c(P) - 2$, and $c(P) < 2k$; yet, $P \nsubseteq J$ since $P \nsubseteq H$, a contradiction. Consequently, $P \cap H$ has at least three components. Let D be one of them with $s, t \notin V(D)$. Let L be a minimal path of H between $V(D)$ and $V(J)$ (thus, if $V(D \cap J) \neq \varnothing$ then $E(L) = \varnothing$). Let L have ends $u \in V(D)$ and $v \in V(J)$. Then $V(L \cap D) = \{u\}$ and $V(L \cap J) = \{v\}$, and $c(e) = 0$ for all $e \in E(L)$.

Suppose that $V(L \cap P) \neq \{u\}$, and let L' be a minimal subpath of L between u and $V(P) - V(D)$, with ends u, w say. Let P' be the $s-t$ path in $P \cup L'$ different from P. Then

$$c'(P') \leqslant c'(P) + c'(L') = c'(P) < 2k-2.$$

Moreover, since $P \cap H$ has at least three components and L' meets only two of them, it follows that $P' \cap H$ has at least two components, and so $P' \not\subseteq H$. But u, w are in different components of $P \cap H$, and so the subpath of P between them is not included in H. Since no edge of this subpath belongs to P' it follows that $P' \cup H$ is a proper subgraph of $P \cup H$, contrary to the choice of P.

We deduce that $V(L \cap P) = \{u\}$. Since $s, t \notin V(D)$, there are edges e, f of the subpaths of P between u and s, t, respectively, such that $e, f \notin E(H)$. Let Q be an $s-t$ path with $v \in V(Q)$ and $c(Q) \leqslant 2k-2$. By (2.2) applied to P, Q, L, c', there is an $s-t$ path $R \subseteq P \cup Q \cup L$ such that exactly one of e, f (say e) belongs to R, and

$$c'(R) \leqslant \tfrac{1}{2}(c'(P) + c'(Q)) + c'(L).$$

Now $c'(P) < 2k-2$, $c'(Q) = c(Q) \leqslant 2k-2$, and $c'(L) = 0$, and so $c'(R) < 2k-2$. But $R \not\subseteq H$ since $e \in E(R)$, and $R \cup H$ is a proper subgraph of $P \cup H$ since $f \notin E(R)$. This contradicts the choice of P. Consequently there is no such P, and the result follows. ∎

Proof of (1.1). We prove that (i) implies (ii) by induction on k. We may assume that $k \geqslant 1$, for if $k = 0$ the result is trivial. Define h, c' as earlier in this section. Then $c' \in \mathbf{Z}_+^{E(G)}$ by (2.6), and if P is a circuit or $s-t$ path of G, then $c'(P)$ is even, because $c(P)$ is even by hypothesis and $h(P)$ is even because h is a slice. For every odd $s-t$ path P, $P \not\subseteq H$ by (2.5), and so $c'(P) \geqslant 2k-2$ by (2.7). From the inductive hypothesis, there are $k-1$ slices $h_1, ..., h_{k-1}$ such that $h_1 + \cdots + h_{k-1} \leqslant c'$. But then $h + h_1 + \cdots + h_{k-1} \leqslant c$, as required. ∎

3. Path Packing

By standard linear programming duality techniques (for instance the theory of blocking polyhedra), (1.1) implies the following, which was conjectured in private communication by Cook and Sebő.

(3.1) *Let $s, t \in V(G)$ be distinct, let $c \in \mathbf{R}_+^{E(G)}$, and let $k \in \mathbf{R}_+$. Then the following are equivalent*:

(i) *for each odd $s-t$ path P there exists $q(P) \in \mathbf{R}_+$, so that $\sum_P q(P) = k$ and $\sum (q(P) : E(P) \ni e) \leqslant c(e)$ for each edge e*;

(ii) *$\sum (h(e)\, c(e) : e \in E(G)) \geqslant 2k$ for every slice h.*

Now (1.1) yields that, for suitably nice functions c, there is an integral packing of slices, but (3.1) only yields fractional packings of odd paths. It is natural to ask if there is an integral strengthening of (3.1). There are several ways in which this might be formulated, but what seems to us the most natural way is false. To see this, let G be the simple graph with seven vertices, s, t, u, v, w, x, y and with edges su, sv, uv, uw, ux, vw, vx, wx, xy, wy, yt. Let $c(e)=1$ for every edge e, except that $c(e)=2$ if $e=yt$; and let $k=2$. Then the function c is "Eulerian," meaning that it is integer-valued and for each vertex v, the sum of $c(e)$ over all edges e incident with v is even. Yet there is a unique function q satisfying (3.1)(i), and it is not integer-valued.

Incidentally, we do not know whether there is such a counterexample which can be drawn in the plane with s and t both on the infinite region.

A variation: what about even $s-t$ paths instead of odd? There are corresponding versions of (1.1) and (3.1) for even $s-t$ paths (using "even-free" instead of odd-free graphs H to define slices). These may easily be derived from (1.1) and (3.1) by adding a new vertex s' adjacent only to s, giving the new edge capacity zero (for (1.1)) or infinity (for (3.1)) and applying the corresponding odd paths theorem to s', t in this enlarged graph.

But finally, there is a more suprising extension of (3.1) to even $s-t$ paths. Let $s, t \in V(G)$ be distinct, and let $c \in \mathbf{R}_+^{E(G)}$. For $k_1, k_2 \in \mathbf{R}_+$, we say that (k_1, k_2) is *feasible* if for each $s-t$ path P there exists $q(P) \in \mathbf{R}_+$, so that

$$\sum (q(P): P \text{ odd}) = k_1,$$

$$\sum (q(P): P \text{ even}) = k_2,$$

$$\sum (q(P): E(P) \ni e) \leqslant c(e) \qquad (e \in E(G)).$$

(3.2) *(k_1, k_2) is feasible if and only if $(k_1, 0)$, $(0, k_2)$ are feasible and (k_1', k_2') is feasible for some $k_1', k_2' \geqslant 0$ with $k_1' + k_2' = k_1 + k_2$.*

The last condition here merely asserts that there is a flow of value $k_1 + k_2$ from s to t, so that the flow in any edge e is at most $c(e)$. (3.2) is easily deduced from (3.1) by adding two new vertices r, s' and three new edges rs, rs', ss' to G, with capacity k_1, k_2, and infinity, respectively, and applying (3.1) to r, t.

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