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A Stochastic Approach to an Interpolation Problem with Applications to Hellinger Integrals and Arithmetic-Geometric Mean Relationship

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A class of interpolation formulas is defined as the convergent in the mean Abel-Goncharov series with the interpolation nodes viewed as certain random variables. It is shown that a special choice of the distribution of the random nodes leads to a particularly useful formula. As a first application of this formula, the expansion is obtained of the arithmetic-geometric mean difference for positive binary random variables in terms of certain central moments. As a second application, we obtain an expansion of the Hellinger integral.

1. Introduction

In this paper we consider the following interpolation problem for a function f defined on the interval $[0,1]$. Let the numeric values be given of the following functionals of f:

$$\begin{aligned}
\Delta_0(f) &= f(0) \\
\Delta_1(f) &= f(1) - f(0) \\
\Delta_2(f) &= f(1) - 2f(\frac{1}{2}) + f(0) \\
&\vdots
\end{aligned}$$

$$\Delta_n(f) \quad = \quad \sum_{k=0}^{n}(-1)^{n-k}\left(\begin{array}{c} n \\ k \end{array}\right) f(\frac{k}{n}) \tag{1}$$

Using these values, construct the interpolating polynomial $\mathrm{P}_n(x; f)$, $x \in [0,1]$, of degree n such that

$$\Delta_k(\mathrm{P}_n) = \Delta_k(f) \tag{2}$$

for $k = 0, \ldots, n$ (see [6]: the interpolation problem of section 1.5 reduces to our problem with the special choice of the interpolation nodes $x_{kn} = \frac{k}{n}$).

It is easily seen that the problem consists of constructing the polynomials $\mathrm{C}_m(x)$ of degree m for $m = 0, \ldots, n$, uniquely defined by the conditions

$$\Delta_k(\mathrm{C}_m) = \left\{ \begin{array}{ll} 0 & \text{if } k \neq m \\ 1 & \text{if } k = m \end{array} \right. \tag{3}$$

where $\Delta_k(\mathrm{C}_m) = 0$ for $k > m$ automatically, since $\Delta_n(f) = 0$ for every polynomial f of degree $m < n$ (this is a direct consequence of the identity (29)). Note that $\mathrm{C}_0(x) = 1$ and $\mathrm{C}_1(x) = x$. The desired interpolating polynomial $\mathrm{P}_n(x; f)$ is then expressed as follows:

$$\mathrm{P}_n(x; f) = \sum_{m=0}^{n} \mathrm{C}_m(x)\Delta_m(f). \tag{4}$$

Indeed, $\Delta_k(\cdot)$ is a linear functional so that Δ_k of the sum on the right hand side of (4) equals the sum $\Delta_k(\mathrm{C}_0)\Delta_0(f) + \ldots + \Delta_k(\mathrm{C}_n)\Delta_n(f)$. Hence the condition (3) implies the identity (2).

In Section 2 the various properties of the basic polynomials $\mathrm{C}_0, \mathrm{C}_1, \ldots$ are discussed. It is shown in particular that these polynomials are easily constructed by using the recurrence relationship (13). As is shown in Section 3 there is an interesting link between the polynomials $\mathrm{C}_0, \mathrm{C}_1, \ldots$ and the well-known Goncharov polynomials envolved in the classical Abel-Goncharov interpolation series (see Section 3 for a short description of the Abel-Goncharov interpolation problem; see [1], [6], and [11] for more details). Due to this link we will show that if the Abel-Goncharov series for a function f converges for some $x \in [0,1]$, then the interpolating polynomial $\mathrm{P}_n(x; f)$ given by (4) converges to $f(x)$ for the same value of x. Consider for instance the function $f(x) = \mathrm{e}^{ax}$, with a fixed constant a. In this case $\Delta_0(f) = 1$ and

$$\Delta_m(f) = (\mathrm{e}^{\frac{a}{m}} - 1)^m$$

for $m \geq 1$. It will be seen in section **??** that if $|a| < \log 2$, then

$$\mathrm{e}^{ax} = 1 + \sum_{m=1}^{\infty} \mathrm{C}_m(x)\,(\mathrm{e}^{\frac{a}{m}} - 1)^m \tag{5}$$

for each fixed $x \in [0,1]$.

In Section 5 we will show that the series in (5) converges uniformly in x and a (see formula (32)). The Sections 5 and 6 are devoted to applications of the last result.

In Section 6 we consider a binary random variable X which takes on either the value e^a ($|a| < \log 2$) with probability x or the value 1 with probability $1-x$. Then the geometric mean $\mathrm{e}^{\mathbb{E}\log \mathrm{X}}$ of X and the arithmetic mean $\mathbb{E}\mathrm{X}$ of X equal e^{ax} and $1+x(\mathrm{e}^a-1)$ respectively. By using (5) we will show that the difference between the geometric mean and the arithmetic mean of X can be expanded as follows:

$$\mathrm{e}^{\mathbb{E}\log \mathrm{X}} - \mathbb{E}\mathrm{X} = \sum_{n=2}^{\infty}\sum_{m=0}^{[\frac{n}{2}]} \alpha_{nm} \mathbb{E}\left[(\mathrm{X}^{\frac{1}{n}} - \mathbb{E}\mathrm{X}^{\frac{1}{n}})^m\right] \mathbb{E}\left[(\mathrm{X}^{\frac{1}{n}} - \mathbb{E}\mathrm{X}^{\frac{1}{n}})^{n-m}\right] \quad (6)$$

where the α's are universal constants, independent of the distribution of X.

In Section 7 we will give another application of the expansion (5). Let f and g be positive probability density functions on $\mathbb{R}$. For $n \in \mathbb{N}^+$ define

$$h_n(f,g) = \int_{-\infty}^{\infty} \left(f(t)^{\frac{1}{n}} - g(t)^{\frac{1}{n}}\right)^n \, dt. \quad (7)$$

In case n is even $h_n(f,g)^{\frac{1}{n}}$ is called the Hellinger distance of order n between f and g. By Newton's binomial formula

$$h_n(f,g) = \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} H_{\frac{m}{n}}(f,g) \quad (8)$$

where $H_x(f,g)$ is the Hellinger integral of order $x \in [0,1]$ defined by

$$H_x(f,g) = \int_{-\infty}^{\infty} f(t)^x g(t)^{1-x} \, dt \quad (9)$$

(see [10], Section 3.2). Using the expansion (5) we will show that if f and g satisfy (39), then the following relation (inverse to (8)) holds:

$$H_x(f,g) = 1 + \sum_{m=1}^{\infty} \mathrm{C}_m(x)\, h_m(f,g). \quad (10)$$

The convergence is uniform in x in the sense of (41).

2. Polynomials C_n

Recall that for nonnegative integers n and m so that $n \geq m$, the Stirling number of the second kind is defined by

$$\mathcal{S}_{nm} = \frac{1}{m!}\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} k^n \quad (11)$$

with the convention that $\mathcal{S}_{nn} = 1$ if $n \geq 0$, $\mathcal{S}_{n0} = 0$ if $n \geq 1$ and $\mathcal{S}_{nm} = 0$ if $m > n$ (see for example [8]). Apart from the last mentioned zero values, the Stirling numbers of the second kind are positive integers.

Hence by definition (1) the functional $\Delta_m(f)$ of the special function $f(x) = x^n$ equals the normalized Stirling number

$$\mathrm{D}_{nm} = \frac{m!}{m^n}\mathcal{S}_{nm}.$$

Moreover, the interpolating polynomial $\mathrm{P}_n(x; f)$ of the monomial $f(x) = x^n$ is the monomial itself, so that according to (4)

$$x^n = \sum_{m=0}^{n} \mathrm{D}_{nm}\, \mathrm{C}_m(x). \tag{12}$$

Thus starting from $\mathrm{C}_0(x) = 1$, we have the following recurrence relationship

$$\mathrm{D}_{nn}\mathrm{C}_n(x) = x^n - \sum_{m=0}^{n-1} \mathrm{D}_{nm}\, \mathrm{C}_m(x) \tag{13}$$

with $\mathrm{D}_{nn} = \frac{n!}{n^n}$. According to (13) the polynomial $\hat{\mathrm{C}}_n(x) := \frac{n!}{n^n}\mathrm{C}_n(x)$ is monic, i.e. a polynomial with unit leading coefficient. The first 9 solutions of equation (13) are

$$\begin{aligned}
\hat{\mathrm{C}}_0(x) &= 1 \\
\hat{\mathrm{C}}_1(x) &= x \\
\hat{\mathrm{C}}_2(x) &= x(x-1) \\
\hat{\mathrm{C}}_3(x) &= x(x-\frac{1}{2})(x-1) \\
\hat{\mathrm{C}}_4(x) &= x(x-\frac{1}{2})^2(x-1) \\
\hat{\mathrm{C}}_5(x) &= x(x-\frac{1}{3})(x-\frac{1}{2})(x-\frac{2}{3})(x-1) \\
\hat{\mathrm{C}}_6(x) &= x(x-\frac{1}{4})(x-\frac{1}{2})^2(x-\frac{3}{4})(x-1) \\
\hat{\mathrm{C}}_7(x) &= \hat{\mathrm{C}}_5(x)(x^2-x+\frac{8}{45}) \\
\hat{\mathrm{C}}_8(x) &= \hat{\mathrm{C}}_6(x)(x^2-x+\frac{25}{144})
\end{aligned}$$

It is not hard to prove that in fact all polynomials of odd degree exceeding 5, have the common factor $\hat{\mathrm{C}}_5(x)$, and all polynomials of even degree exceeding 6, have the common factor $\hat{\mathrm{C}}_6(x)$. Also the property

$$\mathrm{C}_n(x) = (-1)^n\mathrm{C}_n(1-x) \tag{14}$$

is preserved for all $n \geq 2$.

3. The Abel-Goncharov problem

The Abel-Goncharov interpolation problem is usually formulated for functions f of a complex variable z, analytic in a certain complex region, i.e. having all derivatives in this region. If the numeric value of the m^{th} derivative $f^{(m)}$, evaluated at a certain complex number a_m, is given for all $m = 0, 1, \ldots,$ then the following interpolation problem can be considered. Given the numbers $f^{(m)}(a_m)$, $m = 0, \ldots, n$, construct the interpolating polynomial $\mathrm{P}_n(z; f)$ of degree n such that

$$\mathrm{P}_n^{(k)}(a_k; f) = f^{(k)}(a_k) \tag{15}$$

for $k = 0, \ldots, n$.

It is easily seen that the interpolating polynomial $\mathrm{P}_n(z; f)$ can be presented in the following form:

$$\mathrm{P}_n(z; f) = \sum_{m=0}^{n} \mathrm{G}_m(z) \, \frac{f^{(m)}(a_m)}{m!} \tag{16}$$

where $\mathrm{G}_m(z)$ for $m = 0, \ldots, n$ are polynomials of degree m, the so-called Goncharov polynomials, determined by the conditions

$$\begin{array}{rcl} \mathrm{G}_m^{(k)}(a_k) & = & 0 \quad \text{if } k \neq m \\ \mathrm{G}_m^{(m)}(z) & = & m! \end{array} \tag{17}$$

Indeed, take all the required derivatives from both sides of (16) and evaluate them at $a_0, \ldots, a_n$, respectively. Due to the conditions (17) we get the identity (16).

The Goncharov polynomials can be constructed recurrently starting from $\mathrm{G}_0(z) = 1$. Indeed, apply (16) to the monomial $f(z) = z^n$ and take into account that in this case the interpolating polynomial $\mathrm{P}_n(z; f)$ coincides with the monomial itself. So, we get

$$z^n = \sum_{m=0}^{n} \mathrm{G}_m(z) \begin{pmatrix} n \\ m \end{pmatrix} a_m^{n-m} \tag{18}$$

i.e.

$$\mathrm{G}_n(z) = z^n - \sum_{m=0}^{n-1} \mathrm{G}_m(z) \begin{pmatrix} n \\ m \end{pmatrix} a_m^{n-m}. \tag{19}$$

For instance, $\mathrm{G}_1(z) = z - a_0$. For $n > 1$

$$\mathrm{G}_n(z) = n! \int_{a_0}^{z} \int_{a_1}^{z_1} \cdots \int_{a_{n-1}}^{z_{n-1}} dz_n \cdots dz_1 \tag{20}$$

which is directly verified by checking the property (17) of the integrals on the right hand side. Note that $\mathrm{G}_n(z)$ depends only on $a_0, \ldots, a_{n-1}$. If needed, we will exhibit this dependence by writing $\mathrm{G}_n(z) = \mathrm{G}_n(z; a_0, \ldots, a_{n-1})$.

The answer to the question whether the sequence of the interpolating polynomials $\mathrm{P}_0(z; f), \mathrm{P}_1(z; f), \ldots$ converges to $f(z)$ for a fixed z from a certain domain depends not only on the function f in question but also on the choice of the numbers $a_0, a_1, \ldots$ The usual setting of the convergence problem is as follows: the considerations are restricted to the domain $|z| \le 1$ (1 can be replaced by any other positive number) and a class of functions f is characterized for which the convergence takes place for all numbers $a_0, a_1, \ldots$ from the same domain. Evidently, this class of functions f is quite restricted (see [1], theorem 9.11.1 which tells us that it consists of entire functions of exponential type less than $\log 2$). On the other hand, certain special choices of the numbers $a_0, a_1, \ldots$ enlarge the convergence class considerably. For instance, put $a_n = a$ for all $n \ge 0$. Then $\mathrm{G}_n(z) = (z-a)^n$ for $n \ge 0$, so that the Abel-Goncharov series reduces to the usual Taylor expansion. If $a_n = a + nh$ with h not necessarily zero, then $\mathrm{G}_n(z) = (z-a)(z-a-nh)^{n-1}$ and we get the classical Abel interpolation series (see [1], Section 9.10). As is seen in the next section, it is useful for certain purposes to view $a_0, a_1, \ldots$ as a sequence of random variables.

We conclude this section by the following example of the Abel-Goncharov series. For a fixed complex number w let $f(z) = \mathrm{e}^{wz}$. According to [1], theorem 9.11.1, the Abel-Goncharov series for this function f converges absolutely if $|w| < \log 2$ (since in this case the function in question is of exponential type less than $\log 2$). Namely, for each $|z| \le 1$ and $|w| < \log 2$

$$\mathrm{e}^{zw} = \sum_{m=0}^{\infty} \mathrm{G}_m(z)\, \mathrm{e}^{a_m w}\, \frac{w^m}{m!}, \tag{21}$$

if $|a_m| \le 1$ for all $m \in \mathbb{N}$. Note that the proof of (21) is based on the following inequality: for each $n \in \mathbb{N}$ and each $|z| \le 1$

$$|\mathrm{G}_n(z)| \le n!\, (\log 2)^{-n} \tag{22}$$

if $a_0 = 0$ and $|a_m| \le 1$ for all $m \in \mathbb{N}^+$ (see [1], formula 9.11.9).

4. Link between C_n and G_n

Suppose that $a_0, a_1, \ldots$ is a sequence of independent random variables, such that for all possible outcomes of the random sequence $a_0, a_1, \ldots$ the Abel-Goncharov series

$$f(z) = \sum_{m=0}^{\infty} \mathrm{G}_m(z)\, \frac{f^{(m)}(a_m)}{m!} \tag{23}$$

is absolutely convergent for each $|z| \le 1$. Note that $\mathrm{G}_m(z) = \mathrm{G}_m(z; a_0, \ldots, a_{m-1})$ is a random variable, independent of $f^{(m)}(a_m)$. Since the expectation of the product of independent random variables equals the product of their expectations, we have

$$\mathbb{E}\{\mathrm{G}_m(z;a_0,\ldots,a_{m-1})f^{(m)}(a_m)\} = \mathbb{E}\mathrm{G}_m(z;a_0,\ldots,a_{m-1})\mathbb{E}f^{(m)}(a_m).$$

Using this simple fact, take the expectation of both sides in (23). The left hand side is non-random and therefore its expectation is $f(z)$ itself. On the other hand, the expectation of the sum in (23) equals the sum of the expectations, hence (23) implies

$$f(z) = \sum_{m=0}^{\infty} \frac{1}{m!}\, \mathbb{E}\mathrm{G}_m(z)\, \mathbb{E}f^{(m)}(a_m) \tag{24}$$

due to our assumption that the series in (23) converges absolutely. In particular (21) implies

$$\mathrm{e}^{wz} = \sum_{m=0}^{\infty} \frac{w^m}{m!}\, \mathbb{E}\mathrm{G}_m(z)\, \mathbb{E}\mathrm{e}^{a_m w}. \tag{25}$$

Thus if (23) holds for all possible outcomes of the random sequence $a_0, a_1, \ldots$ then also (24) holds, i.e. the convergence class of functions f for (24) includes (and is in fact wider than) that of the Abel-Goncharov series (23). The characterization of this class (which would heavily depend on the distribution of the random variables $a_0, a_1, \ldots$) is obviously beyond the scope of the present paper. In the remainder of this section we only discuss two special examples of the random sequence $a_0, a_1, \ldots$ in which the series (24) takes a particularly transparent form.

First, suppose that $a_0, a_1, \ldots$ are independent identically distributed random variables. If, besides, each a_m has a standard normal distribution, then

$$\mathbb{E}\mathrm{e}^{a_m w} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{xw - \frac{1}{2}x^2}\, dx = \mathrm{e}^{\frac{1}{2}w^2},$$

so that (25) reduces to

$$\mathrm{e}^{zw - \frac{1}{2}w^2} = \sum_{m=0}^{\infty} \mathbb{E}\mathrm{G}_m(z)\, \frac{w^m}{m!}$$

with the generating function of the Hermite polynomials on the left hand side. Hence in this special case the polynomials $\mathbb{E}\mathrm{G}_m(z)$ are the Hermite polynomials.

Consider now the triangular scheme of independent identically distributed random variables

$$\begin{array}{lllll}
a_{11} & & & & \\
a_{21} & a_{22} & & & \\
\vdots & & \ddots & & \\
a_{m1} & \cdots & \cdots & a_{mm} & \\
\vdots & & & & \ddots
\end{array}$$

Denote by a_m the arithmetic mean of the random variables in the m^{th} row, i.e.

$$a_m = \frac{1}{m} \sum_{k=1}^{m} a_{mk}$$

and put $a_0 \equiv 0$. Besides, suppose that all the random variables a_{mk} are uniformly distributed in the interval $[0,1]$. Using again the fact that the expectation of the product of independent random variables equals the product of their expectations, we get

$$\mathbb{E}\mathrm{e}^{a_m w} = \prod_{k=1}^{m} \mathbb{E}\mathrm{e}^{a_{mk}\frac{w}{m}} = \left(\int_0^1 \mathrm{e}^{x\frac{w}{m}}\, dx \right)^m = \left(\frac{\mathrm{e}^{\frac{w}{m}} - 1}{\frac{w}{m}} \right)^m .$$

So it follows that (25), restricted to real valued w and z (replace them by a real valued constant a and a real valued variable x), reduces to

$$\mathrm{e}^{ax} = 1 + \sum_{m=1}^{\infty} \frac{m^m}{m!} \mathbb{E}\mathrm{G}_m(x)\, (\mathrm{e}^{\frac{a}{m}} - 1)^m, \tag{26}$$

for $x \in [0,1]$ and $|a| < \log 2$. Compare (5) and (26) to conclude that these two series agree, provided

$$\mathrm{C}_n(x) = \frac{n^n}{n!} \mathbb{E}\mathrm{G}_n(x; a_0, \ldots, a_{n-1}). \tag{27}$$

In the remainder of this section we establish the last relation between the Goncharov polynomials G_n and the polynomials C_n of section **??**. Note meanwhile that (20) and (27) yield yet another characterization of the polynomials C_n:

$$\mathrm{C}_n(x) = n^n\, \mathbb{E} \int_0^x \int_{a_1}^{x_1} \cdots \int_{a_{n-1}}^{x_{n-1}} dx_n \cdots dx_1. \tag{28}$$

To prove (27), note first that the functionals Δ_n defined in (1) can be expressed for $n \geq 1$ as follows

$$\Delta_n(f) = \int_0^{\frac{1}{n}} \int_{t_0}^{t_0+\frac{1}{n}} \cdots \int_{t_{n-1}}^{t_{n-1}+\frac{1}{n}} f^{(n)}(t_{n-1})\, dt_{n-1} \cdots dt_0$$

so that

$$n^n \Delta_n(f) = \mathbb{E} f^{(n)}(a_n) = \int_0^1 \cdots \int_0^1 f^{(n)}\left(\frac{x_1 + \cdots + x_n}{n} \right) dx_1 \cdots dx_n. \tag{29}$$

If f is a polynomial of degree $m < n$, for instance, we get $\Delta_n(f) = 0$, as was noted in the introduction (see (3) and the remark following it).

Next recall that the normalized Stirling number D_{nm} equals $\Delta_m(f)$ when $f(x) = x^n$, and $\mathrm{D}_{nn} = \frac{n!}{n^n}$. Hence by (29)

$$\frac{\mathcal{S}_{nm}}{m^{n-m}} = \frac{\mathrm{D}_{nm}}{\mathrm{D}_{mm}} = \begin{pmatrix} n \\ m \end{pmatrix} \mathbb{E} a_m^{n-m}. \tag{30}$$

Finally, take expectations of both sides in (18), using (30) and the independence of G_m and a_m. We get (replace z by x)

$$x^n = \sum_{m=0}^{n} \frac{\mathrm{D}_{nm}}{\mathrm{D}_{mm}} \mathbb{E}\mathrm{G}_m(x).$$

But we also have (12), so that the identity (28) holds.

5. Uniform convergence

We have already shown in the previous section that the expansion (5) holds for each $x \in [0,1]$ and $|a| < \log 2$, which means that the remainder term

$$R_n(x,a) = \mathrm{e}^{ax} - 1 - \sum_{m=1}^{n} \mathrm{C}_m(x)\, (\mathrm{e}^{\frac{a}{m}} - 1)^m, \tag{31}$$

evaluated at such x and a, tends to zero as $n \to \infty$. In this section we will show that this convergence is uniform in the sense that for a fixed $\rho \in (0, \log 2)$

$$\sup_{x\in[0,1],a\in[-\rho,\rho]} |R_n(x,a)| \to 0 \quad \text{as} \;\; n \to \infty. \tag{32}$$

In fact we will show that for each $x \in [0,1]$ and $a \in [-\rho, \rho]$

$$|R_n(x,a)| \le \frac{2\,\theta^{n+1}}{1-\theta} \tag{33}$$

with $\theta = \frac{\rho}{\log 2}$, which implies (32) since $\theta \in (0,1)$.

To verify (33) note first that for each $n \in \mathbb{N}$ and $x \in [0,1]$

$$|\mathrm{C}_n(x)| \le n^n\, (\log 2)^{-n} \tag{34}$$

due to (22) and (27). Next, apply (29) to $f(x) = \mathrm{e}^{ax}$. Since in this case $f^{(n)}(x) = a^n \mathrm{e}^{ax}$, $\Delta_0(f) = 1$ and $\Delta_n(f) = (\mathrm{e}^{\frac{a}{n}} - 1)^n$, $n \ge 1$, we get

$$|n^n\, (\mathrm{e}^{\frac{a}{n}} - 1)^n| \le \max(1, \mathrm{e}^a)\, |a|^n \le 2\,\rho^n. \tag{35}$$

Now, since (5) holds for each fixed $x \in [0,1]$ and $|a| < \log 2$, the remainder term (31) can be written in the form

$$R_n(x,a) = \sum_{m=n+1}^{\infty} \mathrm{C}_m(x)\, (\mathrm{e}^{\frac{a}{m}} - 1)^m.$$

Thus by (34) and (35) we get (33):

$$|R_n(x,a)| \le \sum_{m=n+1}^{\infty} |\mathrm{C}_m(x)|\, |(\mathrm{e}^{\frac{a}{m}} - 1)^m| \le 2 \sum_{m=n+1}^{\infty} \theta^m \le \frac{2\,\theta^{n+1}}{1-\theta}.$$

6. Arithmetic-Geometric mean relationship

Suppose that the random variable X takes on either the value e^a with probability x or the value 1 with probability $1-x$, where a is an arbitrary non-zero constant. Note that any binary random variable X can be presented in this form by suitable normalization, if needed.

By definition, the arithmetic and geometric means of X are

$$\mathbb{E}X = xe^a + 1 - x = 1 + x(e^a - 1)$$

and

$$e^{\mathbb{E}\log X} = e^{ax},$$

respectively. Therefore if $|a| < \log 2$, then by (5)

$$e^{\mathbb{E}\log X} = \mathbb{E}X + \sum_{n=2}^{\infty} C_n(x)\,(e^{\frac{a}{n}} - 1)^n \tag{36}$$

since $C_0(x) = 1$ and $C_1(x) = x$, as we already know.

Compare (6) and (36) term by term. It is then seen that the following equality has to be satisfied for $n \geq 2$

$$\sum_{m=0}^{[\frac{n}{2}]} \alpha_{nm}\, \mathbb{E}(X^{\frac{1}{n}} - \mathbb{E}X^{\frac{1}{n}})^m\, \mathbb{E}(X^{\frac{1}{n}} - \mathbb{E}X^{\frac{1}{n}})^{n-m} = C_n(x)\,(e^{\frac{a}{n}} - 1)^n \tag{37}$$

with a suitable choice of the constants α_{nm}. For $n = 2$, for instance, the left hand side equals

$$\alpha_{20}\, \mathbb{E}(X^{\frac{1}{2}} - \mathbb{E}X^{\frac{1}{2}})^2$$

with

$$\begin{aligned} \mathbb{E}(X^{\frac{1}{2}} - \mathbb{E}X^{\frac{1}{2}})^2 &= x(e^{\frac{a}{2}} - xe^{\frac{a}{2}} - 1 + x)^2 + (1-x)(1 - xe^{\frac{a}{2}} - 1 + x)^2 \\ &= x(1-x)(e^{\frac{a}{2}} - 1)^2, \end{aligned}$$

so that $\alpha_{20} = -2$ since $C_2(x) = 2x(x-1)$.

In general, all the central moments for $n \geq 2$ required in (37) are easily determined:

$$\mathbb{E}(X^{\frac{1}{n}} - \mathbb{E}X^{\frac{1}{n}})^m = [x(1-x)^m + (-1)^m(1-x)x^m](e^{\frac{a}{n}} - 1)^m,$$

with 1 on the right hand side if $m = 0$, and 0 if $m = 1$. Hence the equality (37) is reduced to

$$C_n(x) = \sum_{m=0}^{[\frac{n}{2}]} \alpha_{nm}\, \psi_{nm}(x) \tag{38}$$

for $n \geq 2$, with the polynomials

$$\psi_{n0}(x) = x(1-x)^n + (-1)^n(1-x)x^n$$

which are of degree n, $\psi_{n1}(x) \equiv 0$, and for $2 \le m \le [\frac{n}{2}]$

$$\psi_{nm}(x) = [x(1-x)^m + (-1)^m(1-x)x^m][x(1-x)^{n-m} +$$
$$(-1)^{n-m}(1-x)x^{n-m}],$$

which are of degree n. Note that the polynomials $\psi_{nm}(x)$ so defined possess the same symmetry property (14) as $\mathrm{C}_n(x)$. Therefore the polynomial identity (38) reduces to a system of linear equations with unique roots α_{nm}. This is not hard to verify, however we do not wish to enter here in required technical details. Instead, we present explicitly the first few terms of the expansion (6):

$$\begin{aligned}
\mathrm{e}^{\mathbb{E}\log \mathrm{X}} &= \mathbb{E}\mathrm{X} - \frac{2^2}{2!}\mathbb{E}_{22}(\mathrm{X}) + \frac{3^3}{3!}\frac{1}{2}\mathbb{E}_{33}(\mathrm{X}) \\
&\quad - \frac{4^4}{4!}\left(\frac{1}{2}\right)^2\left[\mathbb{E}_{44}(\mathrm{X}) - [\mathbb{E}_{42}(\mathrm{X})]^2\right] \\
&\quad + \frac{5^5}{5!}\frac{1}{2}\frac{1}{3}\frac{2}{3}\left[\mathbb{E}_{55}(\mathrm{X}) - \frac{5}{2}\mathbb{E}_{52}(\mathrm{X})\mathbb{E}_{53}(\mathrm{X})\right] \\
&\quad - \frac{6^6}{6!}\left(\frac{1}{2}\right)^2\frac{1}{4}\frac{3}{4}\left[\mathbb{E}_{66}(\mathrm{X}) - \mathbb{E}_{62}(\mathrm{X})\mathbb{E}_{64}(\mathrm{X}) - \frac{10}{3}[\mathbb{E}_{63}(\mathrm{X})]^2\right] + \ldots
\end{aligned}$$

with $\mathbb{E}_{nm}(\mathrm{X}) = \mathbb{E}[(\mathrm{X}^{\frac{1}{n}} - \mathbb{E}\mathrm{X}^{\frac{1}{n}})^m]$.

We have shown that the expansion (6) with the constants α_{nm} determined by (38) is identical to the expansion (5), and thus in view of (32) the convergence of the remainder term in (6) is uniform in x and a.

7. Hellinger Integrals

In this section we will give another application of the expansion (5), namely we will prove formula (10). Suppose that f and g are positive probability density functions which satisfy the following condition: there exists a $\tau \in (1, 2)$ so that

$$\frac{1}{\tau} \le \frac{f(t)}{g(t)} \le \tau \tag{39}$$

for all $t \in \mathbb{R}$. If, for instance, f and g are densities of Cauchy's distribution

$$f(t) = \frac{1}{\pi}\,\frac{\sigma}{1+(\sigma t)^2} \qquad \text{and} \qquad g(t) = \frac{1}{\pi}\,\frac{1}{1+t^2},$$

then (39) is satisfied if and only if the parameter $\sigma \in (\frac{1}{2}, 2)$. In another example of Laplace's densities

$$f(t) = \frac{1}{2}\,\mathrm{e}^{-|t-\theta|} \qquad \text{and} \qquad g(t) = \frac{1}{2}\,\mathrm{e}^{-|t|}$$

(39) is satisfied if and only if $|\theta| < \log 2$.

We will now prove that under the condition (39) the remainder term in (5)

$$R_n(x) = H_x(f,g) - 1 - \sum_{m=1}^{n} \mathrm{C}_m(x)\, h_m(f,g) \tag{40}$$

vanishes as $n \to \infty$ in the sense that

$$\sup_{x\in[0,1]} |R_n(x)| \to 0. \tag{41}$$

Indeed, put $a = \log\left(\frac{f(t)}{g(t)}\right)$ in (31) to conclude by (32) that

$$\sup_{x\in[0,1],t\in\mathbb{R}} |u(t,x) - u_n(t,x)| \to 0 \quad \text{as} \quad n \to \infty \tag{42}$$

with $u(t,x) = \left(\frac{f(t)}{g(t)}\right)^x$ and $u_n(t,x) = 1 + \sum\limits_{m=1}^{n} \mathrm{C}_m(x) \left(\left(\frac{f(t)}{g(t)}\right)^{\frac{1}{m}} - 1\right)^m$. This yields the desired result (41), since

$$R_n(x) = \int_{-\infty}^{\infty} [u(t,x) - u_n(t,x)]\, g(t)\, dt$$

and

$$\sup_{x\in[0,1]} \left| \int_{-\infty}^{\infty} [u(t,x) - u_n(t,x)]\, g(t)\, dt \right| \le \sup_{x\in[0,1],t\in\mathbb{R}} |u(t,x) - u_n(t,x)| \to 0$$

as $n \to \infty$.

8. Concluding remarks

As was mentioned in the introduction, our interpolation problem is a special case of Gelfond's interpolation problem in [6], Section 1.5. In fact the considerations of the present paper can be extended to Gelfond's general setting without many difficulties. However such extension would require some extra technicalities at the expense of the clarity of the exposition. Taking into account the introductory nature of the present paper we deliberately have chosen for the concrete setting with the special interpolation nodes $\frac{k}{n}$, and focused our attention to deriving formula (5) with interesting applications in Sections 6sh and 7.

Gelfond's method for describing the convergence class of functions is similar to ours: he also relates his problem to the corresponding Abel-Goncharov interpolation problem, derives the inequality (22) and consequently (34). However, the arguments used in [8], section 1.5, are quite complicated (at certain instances even not quite clear). In this paper we attempt to simplify the considerations: for the inequality (22) we refer to [1], theorem 9.11.1, where (22) is derived by simple arguments, and then directly infer (34) by noting that the polynomials C_n on the left hand side are certain expectations of the Goncharov polynomials G_n. (In fact, the last remark applies to the whole class of Gelfond's basic polynomials, and not only to the special polynomials C_n by taking appropriate expectations, but as was said earlier we do not dwell upon this here). It is perhaps worthwhile to mention here the possibility of using theorem 9.11.3 in [1], instead of theorem 9.11.1, which would allow for the wider range $|a| < 2\log(2+\sqrt{3})$ for the constant a in (5), and for $\rho \in (0, 2\log(2+\sqrt{3}))$ in (32). As is well known, the last bound cannot be essentially improved (see [1], p.173).

Regarding the Goncharov polynomials G_n, this means that the upper bound in (22) cannot be essentially improved. Hence, by the present method of deducing (34) from (22), we can not estimate C_n much better than (34). But close examination of the explicit expressions of $\hat{\mathrm{C}}_n$, presented at the end of Section 8 shows that this upper bound is quite unnatural, and that the first 9 polynomials satisfy the inequality $|\hat{\mathrm{C}}_n(x)| \leq 1$ if $x \in [0,1]$. By considerations quite different from that of the present paper, we will prove in the forthcoming report [3] that the last inequality holds for all $n \in \mathbb{N}$. Consequently it will be shown that (32) holds for each $\rho > 0$. In this manner we will improve upon results in Sections 6 and 7, allowing the constants a in Section 6 and τ in Section 7 to be any positive number.

In conclusion few remarks concerning the applications discussed in Sections 6 and 7. The arithmetic-geometric mean relationship is a classical subject (see [2] or [9], however the expansion (6) seems new. It is in fact valid not only for binary random variables, as is stated in the introduction (see [4]). As for the expansion (10), it might be useful in the context of [5] or [7]. The expansion up to the second term, for instance, plays a central rôle in proving the important functional central limit theorem in [7], p. 554.

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