Rational Systems in Control and System Theory

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Contents

Chapter 1
Introduction

In the last few decades, control and system theory has been enriched by results obtained by methods of commutative algebra and algebraic geometry. The algebraic geometric approach to system theory of linear dynamical systems is presented for example in [19, 35, 36]. Within the class of nonlinear systems, the systems with an obvious algebraic structure, such as polynomial and rational systems, are studied by means of algebraic geometry in [4, 5, 7, 10, 11] and others. Besides additional insight into control and system theory algebraic approaches provide, their importance and usefulness also lie in their connection to computational algebra and hence to procedures and algorithms for solving algebraic problems. These tools allow one to develop and implement new and/or more efficient constructive methods of control and system theory.

In this thesis we present an algebraic approach to realization theory and system identification for the classes of rational and Nash systems.

With the growing interest in life sciences, rational systems have become the focus of increased attention. In systems biology, in particular, rational systems are widely used as models of various phenomena such as gene expression, metabolic networks, and enzyme-catalyzed reactions, [62, 57]. They also appear in physics, engineering and economy. Moreover, as Bartosiewicz stated in [10], the theory of rational systems could be simpler and more powerful, once developed, than the theory of smooth systems. In the modeling framework originated by Savageau, one derives the models of metabolic networks which belong to the class of Nash systems. These systems lie between the classes of rational and analytic systems. Specifically, they provide an extension of rational systems which still allows for an algebraic structure.

Realization theory is one of the central topics of control and system theory. It deals with the characterization of all systems, within a certain class of systems - in our case rational and Nash systems, which have a given input-output behavior. Apart from the existence issues, the realization problem concerns properties of realizations such as canonicity and minimality, relations between different realizations of the same map, algorithms and procedures for constructing realization of desired properties. Furthermore, realization theory serves as a theoretical foundation for model reduction, system identification and control/observer design. It provides the tools for deriving better modeling techniques, more efficient reduction methods and new methods for the construction of controllers and observers within the considered class of systems.

The problems of system identification deal with modeling a phenomenon based on the observed measurements. This involves the selection of a model structure, experimental design, identifiability analysis, parameter estimation and evaluation methods. In this thesis we consider only the identifiability problem for the deterministic classes of polynomial and rational systems and for noise-free data.

Let us describe the contents of the thesis in more detail.

Chapter 2 In Chapter 2 we introduce basic terminology, notation and facts of commutative algebra and algebraic geometry used in the thesis.

Chapter 3 The framework to study rational systems, presented in Chapter 3, is adopted from [10] and from [11]. We motivate the study of rational systems by examples from systems biology and engineering. The notions of algebraic reachability and of algebraic/rational observability are introduced. For algebraic reachability of rational systems we provide a characterization in terms of polynomial ideals satisfying certain conditions. Both concepts, of reachability and of observability, are related to different notions of controllability, accessibility and observability of linear and nonlinear systems. Part of Chapter 3 which deals with reachability properties of rational systems is based on the paper

J. Němcová, *Algebraic reachability of rational systems*, in Proceedings of European Control Conference, Budapest, Hungary, 2009.

Chapter 4 The realization problem considered in Chapter 4 deals with determining initialized rational systems whose input-output behavior is the same as the one of a considered response map. We derive necessary and sufficient conditions for a response map to be realizable by a rational system. The characterization of the existence of rationally observable, canonical, and minimal rational realizations for a given response map is provided as well. We relate minimality of rational realizations to their rational observability, algebraic reachability and canonicity. The relations between birational equivalence of rational realizations and their canonicity and minimality properties are determined. Namely, we show that all canonical rational realizations of the same response map are birationally equivalent, and that birational equivalence preserves minimality of rational realizations. Chapter 4 is based on the following two papers:

J. Němcová and J.H. van Schuppen, *Realization theory for rational systems: The existence of rational realizations*, to appear in SIAM Journal on Control and Optimization.

J. Němcová and J.H. van Schuppen, *Realization theory for rational systems: Minimal rational realizations*, to appear in Acta Applicandae Mathematicae (DOI: 10.1007/s10440-009-9464-y).

Preliminary results are presented in

J. Němcová, *The realizations of response maps by rational systems*, in Proceedings of the 18th International Symposium on Mathematical Theory of Networks & Systems, Blacksburg, Virginia, USA, July 28 - August 1, 2008.

Chapter 5 By applying the results of [11] and of Chapter 4 we derive in Chapter 5 the characterization of structural and global identifiability of parametrizations of parametrized polynomial and parametrized rational systems. The corresponding method for checking identifiability is employed to investigate identifiability properties of systems modeling certain biological phenomena. Note that identifiability is a necessary condition for the uniqueness of parameter values determining a model fitting measurements. Without the existence of a unique solution to the parameter estimation problem it could happen that the methods for estimating parameters will never find the true values of the parameters. Therefore, verification of identifiability of a parametrization should precede estimation of numerical values of parameters, and thus formulation of a fully specified model of a phenomenon. The contents of Chapter 5 is presented in the papers

J. Němcová, *Structural identifiability of polynomial and rational systems*, submitted to Mathematical Biosciences.

J. Němcová, *Structural and global identifiability of parametrized rational systems*, in Proceedings of 15th IFAC Symposium on System Identification, Saint-Malo, France, 2009.

Chapter 6 In Chapter 6 we investigate realization theory of Nash systems. In particular, we introduce the class of Nash systems and then formulate and partially solve the realization problem for them. In analogy with the results of Chapter 4 we derive necessary and sufficient conditions for the existence of Nash realizations of a response map. Further, the concepts of semi-algebraic observability and semi-algebraic reachability of Nash realizations are defined and their relationship with minimality is explained. Nevertheless, there remain many open problems. Chapter 6 is based on the paper

J. Němcová, M. Petreczky, J.H. van Schuppen, *Realization theory of Nash systems*, to appear in the Proceedings of 48th IEEE Conference on Decision and Control, Shanghai, China, 2009.

Chapter 7 The thesis is concluded by providing directions for further research.

An overview of the results on realization theory and identifiability properties for the class of rational systems presented in the thesis and an overview of system identification problems concerning polynomial and rational systems can be found in

J. Němcová and J.H. van Schuppen, *Rational systems - realization and identification*, to appear in Festschrift tentatively titled *Three Decades of Progress in Systems and Control*, Springer, 2009.

J. Němcová and J.H. van Schuppen, *Tutorial on system identification of polynomial and of rational systems*, in Proceedings of 15th IFAC Symposium on System Identification, Saint-Malo, France, 2009.

Chapter 2
Preliminaries

In this chapter we recall the basic notions and terminology of commutative algebra and algebraic geometry used within subsequent chapters. We do not intend to provide an extensive overview of all (basic) facts and/or introduction to these fields. This is done in far more comprehensive way in [17, 25, 68, 70, 128, 130] and others.

2.1 Commutative algebra

By a polynomial p in finitely many indeterminates $X_1,\ldots,X_n$ with real coefficients we mean a finite formal sum

$$p = \sum_{\alpha_1,\ldots,\alpha_n \in \mathbb{N}\cup\{0\}} a_{\alpha_1,\ldots,\alpha_n} X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n},$$

where $a_{\alpha_1,\ldots,\alpha_n} \in \mathbb{R}$ are such that only finitely many of them are non-zero. We identify X_i^0 with the unit element 1 of $\mathbb{R}$, i.e. $X_i^0 = 1$, for all $i = 1,\ldots,n$. To emphasize the dependence of p on the indeterminates $X_1,\ldots,X_n$ we write $p(X_1,\ldots,X_n)$ instead of p. We denote the ring of all polynomials in n indeterminates with real coefficients by $\mathbb{R}[X_1,\ldots,X_n]$. Concerning the polynomials in one indeterminate the following theorem will be needed later.

Theorem 2.1. *Let R be a ring and let $S = R[X]$ be the ring of polynomials with coefficients in R and in one indeterminate. If R is an integral domain then S is also an integral domain.*

Proof. This statement is proven for example in [130, Vol.1, Ch. 1.16]. □

The same statement holds also for rings of polynomials in more indeterminates. In particular, since the field $\mathbb{R}$ is an integral domain, the algebra $\mathbb{R}[X_1,\ldots,X_n]$ is an integral domain. Therefore, we can define the field of quotients of $\mathbb{R}[X_1,\ldots,X_n]$ as the set of fractions $\{p/q \mid p,q \in \mathbb{R}[X_1,\ldots,X_n]; q \neq 0\}$. This field, which is a field extension of $\mathbb{R}$, is denoted by $\mathbb{R}(X_1,\dots,X_n)$ and we refer to its elements as rational functions. We use the notation $\mathscr{Q}(S)$ for the field of quotients of an integral domain S. Thus, for example, $\mathscr{Q}(\mathbb{R}[X_1,\dots,X_n]) = \mathbb{R}(X_1,\dots,X_n)$. We denote the quotient (factor) ring of a ring R modulo an ideal $I \subseteq R$ by R/I.

By a formal power series in n indeterminates over $\mathbb{R}$ we mean an infinite sequence $f = (f_0, f_1, \dots, f_q, \dots)$ of homogeneous polynomials of $\mathbb{R}[X_1,\dots,X_n]$ where $f_q = 0$ or f_q is of degree q, i.e. f_q is of the form $a_q X_1^{\alpha_1(q)} \cdots X_n^{\alpha_n(q)}$ with $\alpha_1(q),\dots,\alpha_n(q) \in \mathbb{N} \cup \{0\}$ and $a_q \in \mathbb{R}$. By considering addition and multiplication defined as $f+g = (f_0+g_0,\dots,f_q+g_q,\dots)$ and $fg = (f_0g_0,\dots,\sum_{i+j=q} f_ig_j,\dots)$, respectively, we derive that the set $\mathbb{R}[[X_1,\dots,X_n]]$ of all power series in n indeterminates over $\mathbb{R}$ is a ring. The elements of $\mathbb{R}[[X_1,\dots,X_n]]$ are denoted by $\sum_{q=0}^{+\infty} a_q X_1^{\alpha_1(q)} \cdots X_n^{\alpha_n(q)}$.

Theorem 2.2. *The ring of convergent formal power series over $\mathbb{R}$ in finitely many indeterminates is an integral domain.*

Proof. According to [130, Vol.2, Ch.7, Theorem 1] or [70, Ch. VI.3] the ring $\mathbb{R}[[X_1,\dots,X_n]]$ is an integral domain. Then, the ring of convergent formal power series is a subring of the integral domain $\mathbb{R}[[X_1,\dots,X_n]]$ which implies that it is also an integral domain. $\square$

Let F be a field extension of $\mathbb{R}$. We call the elements $\varphi_1,\dots,\varphi_s \in F$ *algebraically independent* over $\mathbb{R}$ if there does not exist a non-zero polynomial $p \in \mathbb{R}[X_1,\dots,X_s]$ such that $p(\varphi_1,\dots,\varphi_s) = 0$ in F. We denote by trdeg F the *transcendence degree* of F over $\mathbb{R}$ which is defined as the largest number of elements of F which are algebraically independent over $\mathbb{R}$. An arbitrary subset of F cardinality of which is trdeg F and which consists only of algebraically independent elements, is called a *transcendence basis* of F. For more details on transcendence degree, basis and extensions see [17, 24, 70, 83, 128].

Remark 2.3. We will use the term *transcendence degree* also when dealing with integral domains containing $\mathbb{R}$. In particular, if A is an integral domain (over $\mathbb{R}$) and if F is the field of fractions of A then we set trdeg A = trdeg F (over $\mathbb{R}$).

Definition 2.4. Let F be a subfield of a field G. An element $g \in G$ is said to be *algebraic* over F if there exist elements $f_0,\dots,f_j \in F$, $j \geq 1$, not all equal to zero, such that

$$f_0 + f_1 g + \cdots + f_j g^j = 0.$$

In the following three propositions we state properties of transcendence degree which can be found or derived from the statements in [24, 70, 130]. Proposition 2.5 can be derived as a consequence of [24, Ch. 6.2, Proposition 2] (the same statement as [24, Ch. 6.2, Proposition 2] can be found also in [130, Vol.1]). The statement [130, Vol.1, Ch. II, Theorem 28] says that the transcendence degree of an integral domain which is a homeomorphic image of another integral domain is lower than the transcendence degree of its preimage. Since a field is an integral domain but not the other way round, Proposition 2.7 is a weaker version of this theorem.

Proposition 2.5. *Let F be a subfield of a field G and let F and G be field extensions of $\mathbb{R}$. Then* trdeg $F \leq$ trdeg G *(over $\mathbb{R}$).*

Proof. By [70, Ch. X, Theorem 1] any two transcendence bases of the same field have the same cardinality. Moreover, by the same theorem, a transcendence basis can be chosen from a set of generators.

Since F is a subfield of a field G, we can assume that the set of generators of F is a subset of the set of generators of G. Hence, if a transcendence basis S_F of F is chosen as a subset of a set of generators of F, we can find a transcendence basis S_G of G such that $S_F \subseteq S_G$. Therefore, directly from the definition of transcendence degree, trdeg $F \leq$ trdeg G. □

Proposition 2.6. *Let F and G be field extensions of $\mathbb{R}$ such that F is a subfield of G and* trdeg $F =$ trdeg G*. If the elements of $G \setminus F$ are not algebraic over F then $F = G$.*

Proof. Let $\{f_1, \ldots, f_{\text{trdeg } F}\} = S_F \subset F$ be a transcendence basis of F. Since $F \subseteq G$ and trdeg $F =$ trdeg G we assume without loss of generality that $S_F = S_G$.

Let us assume by contradiction that there exists $g \in G$ such that $g \notin F$. Since S_G is a maximal algebraically independent set of G over $\mathbb{R}$, the set $S_G \cup \{g\}$ is algebraically dependent over $\mathbb{R}$. Therefore, there exists a non-zero polynomial p of $\mathbb{R}[X_1, \ldots, X_{\text{trdeg } F}, X_{\text{trdeg } F+1}]$ such that $p(f_1, \ldots, f_{\text{trdeg } F}, g) = 0$. The polynomial p can be rewritten in the form

$$p_0(f_1, \ldots, f_{\text{trdeg } F}) + p_1(f_1, \ldots, f_{\text{trdeg } F})g + \cdots + p_j(f_1, \ldots, f_{\text{trdeg } F})g^j = 0,$$

where $p_0, p_1, \ldots, p_j \in \mathbb{R}[X_1, \ldots, X_{\text{trdeg } F}]$ are such that

$$p_0(f_1, \ldots, f_{\text{trdeg } F}), \ldots, p_j(f_1, \ldots, f_{\text{trdeg } F}) \in F.$$

Hence, by Definition 2.4, g is algebraic over F. This contradicts the assumption that all elements of $G \setminus F$ are not algebraic over F. Therefore, $G \setminus F = \emptyset$. Since $F \subseteq G$, we conclude that $F = G$. □

Proposition 2.7. *Let F and G be field extensions of $\mathbb{R}$ such that there exists a field isomorphism $i : F \to G$, $G = i(F)$. Then* trdeg $F =$ trdeg G *(over $\mathbb{R}$).*

Proof. Let $S_F = \{f_1, \ldots, f_{\text{trdeg } F}\}$ be a transcendence basis of F. Since $f_1, \ldots, f_{\text{trdeg } F}$ are algebraically independent over $\mathbb{R}$, all non-zero $p \in \mathbb{R}[X_1, \ldots, X_{\text{trdeg } F}]$ satisfy the inequality $p(f_1, \ldots, f_{\text{trdeg } F}) \neq 0$. Because the isomorphism i preserves sums and products, $p(i(f_1), \ldots, i(f_{\text{trdeg } F})) = i(p(f_1, \ldots, f_{\text{trdeg } F})) \neq 0$ for every non-zero $p \in \mathbb{R}[X_1, \ldots, X_{\text{trdeg } F}]$. If the image $i(p(f_1, \ldots, f_{\text{trdeg } F}))$ of a non-zero element $p(f_1, \ldots, f_{\text{trdeg } F})$ of F was zero, it would contradict the fact that i is injective. Therefore, the set $i(S_F) = \{i(f_1), \ldots, i(f_{\text{trdeg } F})\}$ is a subset of a transcendence basis of G. Thus, trdeg $F \leq$ trdeg G.

The inequality trdeg $F \leq$ trdeg G can be proven in the same way, but by considering the inverse i^{-1} of $i : F \to G$. Let $S_G = \{g_1, \ldots, g_{\text{trdeg } G}\}$ be a transcendence basis of G. Then $p(g_1,\dots,g_{\text{trdeg }G}) \neq 0$ and consequently $i^{-1}(p(g_1,\dots,g_{\text{trdeg }G})) = p(i^{-1}(g_1),\dots,i^{-1}(g_{\text{trdeg }G})) \neq 0$ for all non-zero $p \in \mathbb{R}[X_1,\dots,X_{\text{trdeg }G}]$. Hence, the set $i^{-1}(S_G)$ is a subset of a transcendence basis of F and trdeg $F \geq$ trdeg G. □

The following proposition can be found in [83, Proposition 2.2.27].

Proposition 2.8. *Let A be an integral domain (over $\mathbb{R}$) such that* trdeg $A = n < +\infty$ *(over $\mathbb{R}$) and let I be a prime ideal of A. Then* trdeg $A/I = m \leq n$. *If $m = n$ then $I = (0)$.*

As the last useful fact of commutative algebra directly used in the subsequent chapters let us state the following theorem.

Theorem 2.9. *If F is a finitely generated field containing $\mathbb{R}$ then every subfield G of F containing $\mathbb{R}$ is finitely generated.*

Proof. This is a consequence of a more general theorem stating that if F is a finitely generated extension of a field K then every subextension G of F is finitely generated. See [17, Ch. V.14.7, Corollary 3] for the proof. □

2.2 Algebraic geometry

In this section we state the notions and basic facts of algebraic geometry used in the subsequent chapters. Since we work over real numbers, all facts presented in this section are stated for real numbers even if they hold more generally.

Definition 2.10. A *real affine variety* X is a subset of $\mathbb{R}^n$ of zero points of finitely many polynomials of $\mathbb{R}[X_1,\dots,X_n]$. Formally, $X = V(\{f_1,\dots,f_N\}) = \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid f_1(x_1,\dots,x_n) = \dots = f_N(x_1,\dots,x_n) = 0\}$ where $f_1,\dots,f_N \in \mathbb{R}[X_1,\dots,X_n]$ and $N, n < +\infty$.

We say that a variety is *irreducible* if it cannot be written as a union of two non-empty varieties which are its strict subvarieties. In the subsequent chapters we work with irreducible real affine varieties. Working with real varieties allows us to have a better geometric understanding of the state-spaces of rational systems and it is also sufficient for real-life applications.

On $\mathbb{R}^n$ we consider the *Zariski topology* which is given by defining the closed sets as real affine varieties. We refer to an open, closed, dense set in Zariski topology as to a Z-open, Z-closed, Z-dense set, respectively. The closure of a set $S \subseteq \mathbb{R}^n$ in the Zariski topology is denoted by $Z\text{-}cl(S)$. On a variety $X \subseteq \mathbb{R}^n$ the related topology is considered. We refer to it as to the Zariski topology on X. For more details on Zariski topology see for example [56, 68].

Definition 2.11. By a *polynomial* (*polynomial function*) on an irreducible real affine variety $X \subseteq \mathbb{R}^n$ we mean a map $p : X \to \mathbb{R}$ for which there exists a polynomial $q \in \mathbb{R}[X_1, \dots, X_n]$ such that $p = q$ on X. We denote by A the algebra of all polynomials on X.

Let I be the ideal of all polynomials of $\mathbb{R}[X_1, \dots, X_n]$ which are zero at every point of a variety $X \subseteq \mathbb{R}^n$. If $X = V(\{f_1, \dots, f_N\})$ then I is the ideal generated by the polynomials $f_1, \dots, f_N$. Let $q \in \mathbb{R}[X_1, \dots, X_n]$ be arbitrary. All polynomials of the set $q + I$ represent the same polynomial p on the variety X and p is independent of the choice of its representant of $q + I$. Thus, p is well-defined. The algebra A of all polynomials on X is then isomorphic to $\mathbb{R}[X_1, \dots, X_n]/I$, i.e. $A \cong \mathbb{R}[X_1, \dots, X_n]/I$.

From Hilbert Basis Theorem and from the fact that the ideals in the quotient ring $\mathbb{R}[X_1, \dots, X_n]/I$ are in a one-to-one correspondence with the ideals of $\mathbb{R}[X_1, \dots, X_n]$ containing I, every ideal in $\mathbb{R}[X_1, \dots, X_n]/I$ is finitely generated. Therefore A is a finitely generated algebra of polynomials, i.e. there exist $\varphi_1, \dots, \varphi_k \in A, k < +\infty$ such that $A = \mathbb{R}[\varphi_1, \dots, \varphi_k]$.

Proposition 2.12. *A real affine variety V is irreducible if and only if the ideal of polynomials which vanish on V is a prime ideal.*

Proof. This statement is proven for example in [25, Ch. 5.1, Proposition 4] or in [68, Proposition 1.10]. □

Let $X \subseteq \mathbb{R}^n$ be an irreducible real affine variety. Then the algebra A of all polynomials on X is an integral domain and one can define Q, the field of quotients of A. Again, the elements of Q do not depend on the choice of representants for polynomials of A. Further, generators of A can be considered generators of Q. The elements of Q are called *rational functions on* X. Even if $\varphi \in Q$ is not defined on all of X, we write $\varphi : X \to \mathbb{R}$. Note that rational functions on a variety X are defined on Z-dense open subsets of X.

Definition 2.13. Let X be an irreducible real affine variety and let Q denote the field of rational functions on X. We define the dimension of X as dim $X =$ trdeg Q (transcendence degree of Q over $\mathbb{R}$).

Note that trdeg Q also corresponds to the dimension of the rational vector fields on X considered as a vector space over Q, [49, Corollary to Theorem 6.1].

Definition 2.14. Let X be an irreducible real affine variety and let Q be the field of rational functions on X. A *rational vector field* f on X is an $\mathbb{R}$-linear map $f : Q \longrightarrow Q$ such that $f(\varphi \cdot \psi) = f(\varphi) \cdot \psi + \varphi \cdot f(\psi)$ for all $\varphi, \psi \in Q$.

A rational vector field f on $X \subseteq \mathbb{R}^n$ is defined at the point $x \in X$ if $f(O_x) \subseteq O_x$ where $O_x = \{\varphi \in Q \mid \varphi \text{ is defined at } x\}$. The set of all points at which a rational vector field f is defined is the set $D_f = \{x \in X \mid f(O_x) \subseteq O_x\}$. We can write f in the form $f = \sum_{i=1}^n f_i \frac{\partial}{\partial X_i}$ where $f_i \in Q$ for all $i = 1, \dots, n$. For any $x \in D_f$ the value of f at x is the vector $(f_1(x), \dots, f_n(x))$ which is a tangent vector to X at x. If $x \in D_f$ is a singular point then $f(x) = 0$.

Definition 2.15. The *trajectory of a rational vector field* f from a point $x_0 \in D_f$ is the map $x : [0,T) \to X$ such that for all $t \in [0,T)$ and for all $\varphi \in A$,

$$\frac{d}{dt}(\varphi \circ x)(t) = (f\varphi)(x(t)) \text{ and } x(0) = x_0.$$

Because a rational vector field f is an $\mathbb{R}$-linear map from Q to Q, it would be straightforward to consider in Definition 2.15 that $\varphi \in Q$. It is explained in [10] why it is sufficient to consider $\varphi \in A$ instead of $\varphi \in Q$.

Theorem 2.16. *For any rational vector field f and any point $x_0 \in D_f$ there exists an unique trajectory of f from x_0 defined on the maximal interval $[0,T)$ (T may be infinite).*

Proof. This theorem is stated in [10]. The proof follows the ideas of the proof of the same statement for polynomial vector fields given in [9]. □

In the rest of this section we will deal with polynomial and rational mappings between varieties. Let $X \subseteq \mathbb{R}^m$ and $X' \subseteq \mathbb{R}^n$ be irreducible real affine varieties. A *polynomial mapping* from X to X' is a function $\psi : X \to X'$ given as $\psi(x_1,\ldots,x_m) = (g_1(x_1,\ldots,x_m),\ \ldots, g_n(x_1,\ldots,x_m))$ for all $(x_1,\ldots,x_m) \in X$, where $g_i \in \mathbb{R}[X_1,\ldots,X_m]$ for all $i = 1,\ldots,n$. A *rational mapping* from X to X' is a function $\phi : X \to X'$ given as $\phi = (f_1,\ldots,f_n)$, where $f_i \in \mathbb{R}(X_1,\ldots,X_m)$, $i = 1,\ldots,n$ are such that ϕ is defined at some point of X, and if ϕ is defined at a point $(x_1,\ldots,x_m) \in X$ then $\phi(x_1,\ldots,x_m) \in X'$. Similarly as the rational functions on a variety, rational mappings between varieties do not have to be defined everywhere. They are defined on Z-dense subsets of respective varieties, [25, Ch. 5.5, Proposition 8].

The composition $\psi \circ \phi$ of two rational mappings $\phi : X \to X'$ and $\psi : X' \to X''$ is defined if there is a point $x \in X$ such that ϕ is defined at x and ψ is defined at $\phi(x)$. Let $\phi, \psi : X \to X'$ be rational mappings given as $\phi = \left(\frac{f_1}{g_1},\ldots,\frac{f_n}{g_n}\right)$, $\psi = \left(\frac{h_1}{k_1},\ldots,\frac{h_n}{k_n}\right)$ where $f_i, g_i, h_i, k_i \in \mathbb{R}[X_1,\ldots,X_m]$ for all $i = 1,\ldots,n$. If $f_i k_i - h_i g_i \in I_X$ for every $i = 1,\ldots,n$, where I_X is the ideal of polynomials of $\mathbb{R}[X_1,\ldots,X_m]$ vanishing on X, then $\phi = \psi$.

Definition 2.17. Let $X \subseteq \mathbb{R}^m$ and $X' \subseteq \mathbb{R}^n$ be irreducible real affine varieties. We say that X and X' are *isomorphic* if there exist polynomial mappings $\phi : X \to X'$ and $\psi : X' \to X$ such that $\phi \circ \psi = 1_{X'}$ and $\psi \circ \phi = 1_X$. We say that X and X' are *birationally equivalent* if there exist rational mappings $\phi : X \to X'$ and $\psi : X' \to X$ such that $\phi \circ \psi = 1_{X'}$ and $\psi \circ \phi = 1_X$.

Because every polynomial mapping is a rational mapping, the set of irreducible varieties birationally equivalent to a given irreducible variety X contains, among all irreducible varieties isomorphic to X, many different non-isomorphic irreducible varieties. Hence, birational equivalence is weaker equivalence relation than isomorphism. The following theorems, which characterize birational equivalence of irreducible varieties, correspond to [25, Ch. 9.5, Corollary 7] and [25, Ch. 5.5, Theorem 10].

Theorem 2.18. *Let X and X' be irreducible real affine varieties which are birationally equivalent. Then* $\dim X = \dim X'$.

Theorem 2.19. *Let X and X' be irreducible real affine varieties and let Q and Q' be the fields of rational functions on X and X', respectively. The varieties X and X' are birationally equivalent if and only if there exists an isomorphism of the fields Q and Q' which is the identity on the constant functions $\mathbb{R} \subset Q$.*

2.3 Real algebraic geometry

In this section we follow the notation and terminology of [15]. However, we do not state all results in their full generality (over arbitrary real closed fields).

Recall that by $\mathbb{R}$ we denote the field of real numbers. A *real field* is a field which can be ordered (ordering of a field is a total order relation $\leq$ satisfying the conditions $x \leq y \Rightarrow x+z \leq y+z$ and $(0 \leq x, 0 \leq y) \Rightarrow 0 \leq xy$ for all elements x,y,z of the field). Equivalently, a field F is real if and only if for every $f_1,\ldots,f_n \in F$ such that $\sum_{i=1}^{n} f_i^2 = 0$ it holds that $f_1 = \ldots = f_n = 0$. This characterization implies the following lemma:

Lemma 2.20. *A prime ideal I of a commutative ring R is real if and only if the field of fractions of R/I is real.*

A real field which has no nontrivial real algebraic extension is a *real closed field*. Further, a *real closure* of an ordered field F is its algebraic extension which is real closed and whose ordering extends the ordering of F. A real closure of F is unique up to an F-isomorphism.

Definition 2.21. A subset $S \subseteq \mathbb{R}^n$ is called *semi-algebraic* if it is of the form

$$\bigcap_{i=1}^{d} \bigcup_{j=1}^{m_i} \{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid P_{i,j}(x_1,\ldots,x_n)\ \varepsilon_{i,j}\ 0\},$$

where $P_{i,j} \in \mathbb{R}[X_1,\ldots,X_n]$ and $\varepsilon_{i,j} \in \{<,=\}$ for all $i = 1,\ldots,d$ and $j = 1,\ldots,m_i$.

Note that a semi-algebraic subset of $\mathbb{R}^n$ can be obtained by finite unions and intersections of sets of the points of $\mathbb{R}^n$ which satisfy finitely many polynomial equalities and inequalities. The *dimension* of a semi-algebraic set S is given as the maximal length of chains of prime ideals of the ring of polynomial functions on S. One can derive the following characterization of the dimension of a semi-algebraic set:

Proposition 2.22. *For a semi-algebraic set $S \subseteq \mathbb{R}^n$ it holds that* $\dim\ (Z\text{-}cl(S)) = \dim\ S$.

We say that a semi-algebraic subset $S \subseteq \mathbb{R}^n$ is *semi-algebraically connected* if it cannot be written as a union of two disjoint closed semi-algebraic sets in S. Let $S \subseteq \mathbb{R}^n$ and $S' \subseteq \mathbb{R}^m$ be semi-algebraic sets. A map $f : S \to S'$ is a *semi-algebraic map* if its graph is a semi-algebraic set in $\mathbb{R}^{n+m}$. The following technical lemma on the existence of a partition of a semi-algebraic set is proven in [15, Lemma 2.6.3].

Lemma 2.23. *Let $S \subseteq \mathbb{R}^n$ be a semi-algebraic set and let $f : S \to \mathbb{R}$ be a semi-algebraic function. There exist semi-algebraic subsets $S_1, \ldots, S_m$ of S and polynomials $g_i \in \mathbb{R}[X_1, \ldots, X_n, Y]$, $i = 1, \ldots, m$ such that $S = \bigcup_{i=1}^m S_i$, $S_i \cap S_j = \emptyset$ for all $i \neq j \in \{1, \ldots, m\}$ and such that for all $x \in S_i$, $g_i(x, Y)$ is not identically zero and $g_i(x, f(x)) = 0$.*

A *Nash submanifold M* of $\mathbb{R}^n$ is a semi-algebraic set which has also an analytic manifold property. Namely, M is a Nash submanifold of $\mathbb{R}^n$ of dimension d if for every point $m \in M$ there exists a Nash diffeomorphism φ from an open semi-algebraic neighborhood Ω of the origin in $\mathbb{R}^n$ onto an open semi-algebraic neighborhood Ω' of m in $\mathbb{R}^n$ such that $\varphi(0) = m$ and $\varphi((\mathbb{R}^d \times \{0\}) \cap \Omega) = M \cap \Omega'$, where $\mathbb{R}^d \times \{0\} = \{(x_1, \ldots, x_d, 0, \ldots, 0) \in \mathbb{R}^n \mid (x_1, \ldots, x_d) \in \mathbb{R}^d\}$. Because a Nash submanifold M of $\mathbb{R}^n$ is a semi-algebraic subset of $\mathbb{R}^n$, the *dimension* of M coincides with the dimension of a real affine variety given as the Zariski closure of M in $\mathbb{R}^n$, see Proposition 2.22. Further, the Zariski closure of M has, by [15, Proposition 8.4.1], the following property:

Proposition 2.24. *Let $M \subseteq \mathbb{R}^n$ be a Nash submanifold which is semi-algebraically connected. Then Z-cl(M) is an irreducible real affine variety.*

Definition 2.25. By a *Nash function* on a Nash submanifold $M \subseteq \mathbb{R}^n$ we mean an analytic function from M to $\mathbb{R}$ which satisfies an algebraic equation. We denote the ring of Nash functions on M by $\mathscr{N}(M)$.

The following two statements are stated more generally (for arbitrary real closed field) in [15], see [15, Proposition 8.1.9] and [15, Theorem 8.6.4].

Proposition 2.26. *Let $M \subseteq \mathbb{R}^n$ be a Nash submanifold which is semi-algebraically connected, let U be a non-empty open subset of M, and let $f : M \to \mathbb{R}$ be a Nash function. If $f|_U = 0$ then $f = 0$.*

Theorem 2.27. *Let $M \subseteq \mathbb{R}^n$ be a Nash submanifold and let I be a prime ideal of $\mathscr{N}(M)$. Then the set $\mathscr{Z}_M(I) = \{m \in M \mid \forall f \in I : f(m) = 0\}$ is semi-algebraically connected.*

The set $\mathscr{Z}_M(f_1, \ldots, f_p)$, where $f_1, \ldots, f_p \in \mathscr{N}(M)$, is called a *Nash subset* of M. The fact that $\mathscr{Z}_M(I)$ is a Nash set for every ideal $I \subseteq \mathscr{N}(M)$ can be considered a weak form of the noetherian property of the ring $\mathscr{N}(M)$ stated in the following theorem, see [15, Theorem 8.7.18].

Theorem 2.28. *Let $M \subseteq \mathbb{R}^n$ be a Nash submanifold. Then the ring $\mathscr{N}(M)$ is noetherian (any ideal of $\mathscr{N}(M)$ has a finite system of generators).*

Chapter 3
Rational Systems

In this chapter we introduce the framework adopted from [10, 11] to study rational systems. Further, we deal with reachability and observability of rational systems. For the introduction of the terminology on algebraic/rational observability and on algebraic reachability of polynomial/rational systems and for other approaches see [4, 5, 10, 11] and others. A polynomial (rational) system is canonical if it is algebraically reachable and algebraically (rationally) observable. This concept of canonicity will play an important role in Chapter 4 for the characterization of minimal rational realizations.

Let us outline the contents of this chapter. The first section explains the motivation to study rational systems in greater detail. We provide two examples, one of biological and one of engineering relevance. In Section 3.2 the notion of rational systems is formalized. Section 3.3 and Section 3.4 deal with the concepts of algebraic reachability and algebraic/rational observability of rational systems, respectively. Both concepts are related to different concepts of controllability and observability of linear and nonlinear systems.

Within this chapter we use the notation and terminology introduced in Section 2.1 and in Section 2.2.

3.1 Motivation

Rational systems arise as models of phenomena in systems biology, engineering, physics and economy. Thus, they are useful for analyzing data and simulating phenomena in these fields. Concerning systems biology, rational systems model metabolic, signaling and genetic networks, see [62]. Metabolic networks describe material and energy flows of a cell, signaling networks describe how signals are conveyed from one location in a cell to another, and genetic networks describe the processes from the reading of DNA to the production of proteins. We derive a model of an enzyme-catalyzed reaction in Example 3.1. Since this reaction is the simplest example of an enzyme-catalyzed reaction, the way to model it is explained in many biological textbooks, for example in [62]. Further examples of rational systems in systems biology are presented in Chapter 5. For an application of rational systems in engineering see the rational system modeling the movement of a satellite in Earth orbit in Example 3.2. This example is adopted from [110] and [18].

Example 3.1. Consider a reaction

$$E+S \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} ES \xrightarrow{k_2} E+P$$

where a substrate S is irreversibly transformed into a product P by the catalytic influence of an enzyme E. The substrate S and the enzyme E form an intermediate complex ES which then dissolves into the product P and the enzyme E, or to the substrate S and the enzyme E.

Let S, P, E, ES denote the concentrations of the respective chemical species. The change of these concentrations in time follows the dynamics of the equations

$$\begin{aligned}
\dot{S} &= -k_1 E \cdot S + k_{-1} ES, \\
\dot{ES} &= k_1 E \cdot S - (k_{-1}+k_2) ES, \\
\dot{E} &= -k_1 E \cdot S + (k_{-1}+k_2) ES, \\
\dot{P} &= k_2 ES,
\end{aligned}$$

where k_1, k_{-1}, k_2 are real numbers called parameters. Under additional assumptions on ES, namely $\dot{ES} = 0$, one derives that $\dot{E} = 0$ (by summing up the equations for $\dot{ES}$ and $\dot{E}$ and by substituting $\dot{ES} = 0$ to the resulting equation). Therefore, one assumes that the total concentration $E_{total} = E + ES$ of the enzyme E (in free form or involved in the complex ES) is constant. Then, the concentration ES is given as $\frac{E_{total} S}{S + \frac{k_{-1}+k_2}{k_1}}$ and consequently the equation for the change of the concentration P takes the form

$$\dot{P} = \frac{k_2 E_{total} S}{S + \frac{k_{-1}+k_2}{k_1}} = \frac{v_{max} S}{S + K_m}.$$

This type of kinetics is usually referred as Michaelis-Menten kinetics. The value of the parameter v_{max} specifies the maximal reaction rate which can be reached for a large substrate concentration. The Michaelis constant K_m equals the substrate concentration which yields a half-maximal reaction rate.

The derived nonlinear system with the dynamics

$$\begin{aligned}
\dot{S} &= -k_1 \left(E_{total} - \frac{E_{total} S}{S + K_m} \right) S + k_{-1} \frac{E_{total} S}{S + K_m}, \\
\dot{P} &= \frac{v_{max} S}{S + K_m},
\end{aligned}$$

with the output function $y = P$ and the concentration E_{total} considered as an input u is a rational system.

Example 3.2. For the communication satellites in Earth orbit it is important to stay in the same position with respect to the devices on the surface for which they reflect electromagnetic signals the devices emit. Let us assume for simplicity that a satellite is moving only in the equator plane of the Earth. Its position at time t is given by the polar coordinates $(r(t), \theta(t))$, where $r(\cdot)$ refers to the distance from the Earth center and $\theta(\cdot)$ denotes the angular distance of the satellite from the zero meridian. Further, the satellite is equipped with jets for maneuvering which can generate forces F_r and F_θ in the radial and tangential direction, respectively. Then the movement of the satellite is described by the equations

$$\begin{aligned}\ddot{r}(t) &= r(t)\dot{\theta}(t)^2 - \frac{gm_E}{r(t)^2} + \frac{F_r(t)}{m_S},\\ \ddot{\theta}(t) &= -2\frac{\dot{r}(t)\dot{\theta}(t)}{r(t)} + \frac{F_\theta(t)}{m_S r(t)},\end{aligned}$$

derived by Newton's laws. Here m_E and m_S denote the mass of the Earth and of the satellite, respectively, g stands for the gravitational constant.

The goal for the satellite's navigation is to keep the satellite on a geostationary orbit so that it does not spend extra energy on maneuvering. Thus, the satellite is supposed to fly at the same altitude r_0, without using its jets ($F_r(\cdot) = F_\theta(\cdot) = 0$), above the same point at the equator which is determined by the angle θ_0 (angular distance from the zero meridian). Since the Earth rotates with the angular velocity Ω, the satellite's polar coordinates at time t are supposed to be $(r_0, \theta_0 + \Omega t)$.

Let us substitute relations $r(t) = r_0, \theta(t) = \theta_0 + \Omega t, F_r(t) = F_\theta(t) = 0$, which characterize the desired trajectory of the satellite, to the equations which describe the movement of the satellite. We obtain the altitude $r_0 = \sqrt[3]{\frac{gm_E}{\Omega^2}}$ at which the satellite should fly. Further, let us introduce new variables $x_1(t) = r(t) - r_0$, $x_2(t) = \dot{r}(t)$, $x_3(t) = \theta(t) - (\theta_0 + \Omega t)$, $x_4(t) = \dot{\theta}(t) - \Omega$. They correspond to the deviation of the satellite from the desired altitude (x_1), from the desired velocity for the altitude level change (x_2), from the desired angle (angular distance from the zero meridian, x_3), and from the desired angular velocity (x_4). By rewriting the equations for the satellite's movement using these new variables we derive the following system of the first order ordinary differential equations:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t),\\ \dot{x}_2(t) &= (x_1(t) + r_0)(x_4(t) + \Omega)^2 - \frac{gm_E}{(x_1(t) + r_0)^2} + \frac{F_r(t)}{m_S},\\ \dot{x}_3(t) &= x_4(t),\\ \dot{x}_4(t) &= -\frac{2x_2(t)(x_4(t) + \Omega)}{x_1(t) + r_0} + \frac{F_\theta(t)}{m_S(x_1(t) + r_0)}.\end{aligned} \tag{3.1}$$

Note that the right-hand sides of the equations above are rational functions in the state variables x_1, x_2, x_3, x_4. As the output function $y(\cdot)$ one can consider the deviation of the satellite from the desired angle since it can be measured from the surface. Thus, $y(t) = x_3(t)$. The forces F_r and F_θ are considered the inputs to the system.

The nonlinear control system whose dynamics is specified by (3.1) with the output function $y(t) = x_3(t)$ and the inputs $u = \begin{pmatrix} F_r \\ F_\theta \end{pmatrix}$ is an example of a rational system.

3.2 Framework

This section introduces the concepts of polynomial and rational systems as they are developed in [10, 11] and the relevant references therein. This approach generalizes the common approach of nonlinear control theory from the geometric viewpoint since the considered state-spaces are not necessarily smooth affine varieties. Polynomial/rational systems are considered control systems on irreducible real affine varieties with the dynamics defined by polynomial/rational vector fields and with output functions having polynomial/rational components. The inputs are taken to be piecewise-constant functions with the values in an *input-space* U which is an arbitrary subset of $\mathbb{R}^m$. We take $\mathbb{R}^r$ as the *output-space*. The formal definition of a polynomial/rational system is given in the following definition.

Definition 3.3. A *polynomial/rational system* Σ with an input-space U and an output-space $\mathbb{R}^r$ is a triple (X, f, h) where

(i) the state-space X is an irreducible real affine variety,
(ii) $f = \{f_\alpha \mid \alpha \in U\}$ is a family of polynomial/rational vector fields on X,
(iii) $h : X \to \mathbb{R}^r$ is an output map with polynomial/rational components, i.e. for polynomial systems $h_i \in A$ for all $i = 1, \dots, r$ where A is the algebra of all polynomials on the variety X and for rational systems $h_i \in Q$ for all $i = 1, \dots, r$ where Q is the field of all rational functions on the variety X.

Definition 3.4. An *initialized polynomial/rational system* Σ with an input-space U and an output-space $\mathbb{R}^r$ is a quadruple (X, f, h, x_0) where

(i) (X, f, h) is a polynomial/rational system,
(ii) $x_0 \in X$ is an initial state such that all $h_i, i = 1, \dots, r$ and at least one of $f_\alpha, \alpha \in U$ are defined at x_0.

Example 3.5. Let us consider the rational system derived in Example 3.1 as the model of an enzyme-catalyzed irreversible reaction. We formulate this system in the framework of Definition 3.3. The state-space is considered to be $X = \mathbb{R}^2$. The input-space $U = [0, \infty)$ is given by all possible values of the total concentration of the enzyme E. Then, the dynamics is specified by the family f of vector fields $f_\alpha = \left(-k_1(\alpha - \frac{\alpha S}{S+K_m})S + k_{-1}\frac{\alpha S}{S+K_m}\right)\frac{\partial}{\partial S} + \frac{v_{max}S}{S+K_m}\frac{\partial}{\partial P}$ for $\alpha \in U$. The output function is $h(S, P) = P$.

Consider a rational system $\Sigma = (X, f = \{f_\alpha \mid \alpha \in U\}, h)$. The states at which all components of the output function h are defined and at which at least one of the rational vector fields $f_\alpha, \alpha \in U$ is defined is a Z-dense open subset of X, see [10]. If U is a finite set then the set of points at which all components of h and all vector fields $f_\alpha, \alpha \in U$ are defined is also a Z-dense open subset of X. For polynomial systems all components of h and all vector fields $f_\alpha, \alpha \in U$ are defined at any $x_0 \in X$.

As the *space of input functions* we consider the set $\mathscr{U}_{pc}$ of piecewise-constant functions $u : [0, T_u] \to U$ where $T_u \geq 0$ depends on u. Every input $u \in \mathscr{U}_{pc}$ can be represented as $u = (\alpha_1, t_1)(\alpha_2, t_2) \cdots (\alpha_{n_u}, t_{n_u})$ where $n_u \in \mathbb{N}$, and where $\alpha_i \in U$, $t_i \in [0, \infty]$ for $i = 1, \ldots, n_u$. For such representation it holds that $u(0) = \alpha_1$ and $u(t) = \alpha_{i+1}$ for $t \in (\sum_{j=0}^{i} t_j, \sum_{j=0}^{i+1} t_j]$, where $t_0 = 0$ and $i = 0, 1, \ldots, n_u - 1$. Then $T_u = \sum_{j=1}^{n_u} t_j$. Note that an input $u \in \mathscr{U}_{pc}$ can be represented in different ways as a sequence of tuples (α, t) with $\alpha \in U, t \in [0, \infty]$. For example, $u = (\alpha_1, t_1)(\alpha_1, t_2)(\alpha_2, 0)(\alpha_3, t_3) = (\alpha_1, t_1 + t_2)(\alpha_3, t_3)$. We consider all these representations equivalent. The concatenation $(u)(v)$ of inputs $u = (\alpha_1, t_1) \cdots (\alpha_n, t_n), v = (\beta_1, s_1) \cdots (\beta_k, s_k) \in \mathscr{U}_{pc}$ is the input $(u)(v) = (\alpha_1, t_1) \cdots (\alpha_n, t_n)(\beta_1, s_1) \cdots (\beta_k, s_k) \in \mathscr{U}_{pc}$. The restriction of an input u to a shorter time-domain $[0, t], t < T_u$ is denoted by $u_{[0,t]}$. The empty input e is such an input that $T_e = 0$.

Let $\Sigma = (X, f, h, x_0)$ be an initialized polynomial/rational system. Consider a constant input $u = (\alpha, T_u) \in \mathscr{U}_{pc}$. The *trajectory* of Σ corresponding to the constant input u is the trajectory of a vector field f_α from the initial state $x_0 \in X$, i.e. it is the map $x(\cdot; x_0, u) : [0, T_u] \to X$ such that $\frac{d}{dt}(\varphi \circ x)(t; x_0, u) = (f_\alpha \varphi)(x(t; x_0, u))$ and $x(0) = x_0$ for $t \in [0, T_u]$ and for all $\varphi \in A$, see Definition 2.15. The existence and uniqueness of trajectories for polynomial and rational vector fields are treated in [11, Theorem 1] and in Theorem 2.16, respectively. The trajectory of Σ corresponding to the empty input e equals the initial state x_0, i.e. $x(0; x_0, e) = x_0$. Let us consider an input $u = (\alpha_1, t_1) \cdots (\alpha_{n_u}, t_{n_u}) \in \mathscr{U}_{pc}$. By the *trajectory* of Σ corresponding to the piecewise-constant input u we mean the continuous map $x(\cdot; x_0, u) : [0, T_u] \to X$ such that $x(0; x_0, u) = x_0$ and $x(t; x_0, u) = x_{\alpha_i}(t - \sum_{j=0}^{i-1} t_j)$ for $t \in [\sum_{j=0}^{i-1} t_j, \sum_{j=0}^{i} t_j]$, $i = 1, \ldots, n_u$ where $x_{\alpha_i} : [0, t_i] \to X$ is a trajectory of a vector field f_{α_i} from the initial state $x(\sum_{j=0}^{i-1} t_j; x_0, u) = x_{\alpha_{i-1}}(t_{i-1})$ for $i = 2, \ldots, n_u$, and from the initial state x_0 for $i = 1$.

Let $u \in \mathscr{U}_{pc}$ be an input defined on a time-domain $[0, T_u]$. It is possible that the trajectory of a polynomial/rational system corresponding to u blows up before the time T_u, or that a rational system is steered by the input u to the state where the dynamics determined by this input is not defined.

Definition 3.6. Let $\Sigma = (X, f, h, x_0)$ be an initialized polynomial/rational system. We define the set $\mathscr{U}_{pc}(\Sigma)$ of *admissible inputs for the system* Σ as a subset of the set of piecewise-constant inputs $\mathscr{U}_{pc}$ for which there exist a trajectory of Σ.

Note that for every $u \in \mathscr{U}_{pc}(\Sigma)$ and for every $t \in [0, T_u]$ the input $u_{[0,t]} \in \mathscr{U}_{pc}(\Sigma)$. It is possible that the set of admissible inputs $\mathscr{U}_{pc}(\Sigma)$ contains only the empty input e. The set of all $x_0 \in X$ such that $\mathscr{U}_{pc}(\Sigma) \setminus \{e\} \neq \emptyset$ is a Z-dense open subset of X and we denote it by X_Σ.

Let us denote the set of all maps from $\mathscr{U}_{pc}(\Sigma)$ to $\mathbb{R}$ by $(\mathscr{U}_{pc}(\Sigma) \to \mathbb{R})$.

Definition 3.7. Let $\Sigma = (X, f, h, x_0)$ be an initialized polynomial/rational system and let A denote the algebra of polynomials on X. We define the *input-to-state map* $\tau : \mathscr{U}_{pc}(\Sigma) \to X$ as the map $\tau(u) = x(T_u; x_0, u)$ for $u \in \mathscr{U}_{pc}(\Sigma)$. The map $\tau^* : A \to (\mathscr{U}_{pc}(\Sigma) \to \mathbb{R})$ defined as $\tau^*(\varphi) = \varphi \circ \tau$ for all $\varphi \in A$ is called a *dual input-to-state map*.

For an initialized polynomial/rational system $\Sigma = (X, f, h, x_0)$ with the algebra $A = \mathbb{R}[\varphi_1, \dots, \varphi_k]$ of polynomials on X, the map $\tau^* : A \to (\mathscr{U}_{pc}(\Sigma) \to \mathbb{R})$ is a homomorphism and $\tau^*(\mathbb{R}[\varphi_1, \dots, \varphi_k]) = \mathbb{R}[\tau^*\varphi_1, \dots, \tau^*\varphi_k]$. Then, the map $\widehat{\tau^*} : A/\mathrm{Ker}\ \tau^* \to \mathbb{R}[\tau^*\varphi_1, \dots, \tau^*\varphi_k]$, defined as $\widehat{\tau^*}([\varphi]) = \tau^*\varphi$ for every $\varphi \in A$, is an isomorphism.

3.3 Algebraic reachability

A system is called reachable from an initial state if all states in its state-space can be reached by applying a suitable input to the system. Because rational functions on an irreducible variety $X \subseteq \mathbb{R}^n$ are defined on Z-dense subsets of X, the natural way of defining algebraic reachability for rational systems is to say that a rational system is algebraically reachable from an initial state if the set of all states which can be reached by applying admissible inputs to the system from the initial state is a Z-dense subset of the state-space. This means, since the state-spaces of rational systems are irreducible varieties, that the smallest variety containing this set is already the state-space of the considered system.

Definition 3.8. An initialized rational system $\Sigma = (X, f, h, x_0)$ is said to be *algebraically reachable (from x_0)* if its reachable set

$$\mathscr{R}(x_0) = \{x(T_u; x_0, u) \in X \mid u \in \mathscr{U}_{pc}(\Sigma)\}$$

is Z-dense in X. A rational system $\Sigma = (X, f, h)$ is said to be *algebraically reachable* if it is algebraically reachable from any $x_0 \in X_\Sigma$.

If a rational system $\Sigma = (X, f, h)$ is algebraically reachable from an initial state $x_0 \in X_\Sigma$ then its reachable set $\mathscr{R}(x_0)$ is sufficiently large to distinguish the elements of the algebra A of all polynomials on X and the elements of the field Q of all rational functions on X. Thus, the equality $\varphi_1 = \varphi_2$ on $\mathscr{R}(x_0)$ of two polynomials of A implies that $\varphi_1 = \varphi_2$ on X. The same holds if $\varphi_1, \varphi_2 \in Q$. Note that $\mathscr{R}(x_0) \subseteq X$.

Lemma 3.9. *Let $\Sigma = (X, f, h, x_0)$ be an initialized rational system and let the map τ^* be determined by the system Σ as in Definition 3.7. Then*

$$Z\text{-}cl(\mathscr{R}(x_0)) = \{x \in X \mid \varphi(x) = 0 \textit{ for all } \varphi \in \mathrm{Ker}\ \tau^*\}.$$

Proof. Because $\tau^*(\varphi) = \varphi \circ \tau$ for all $\varphi \in A$, where τ is defined as $\tau(u) = x(T_u; x_0, u)$ for $u \in \mathscr{U}_{pc}(\Sigma)$, we derive the following equalities:

$$\begin{aligned}\text{Ker } \tau^* &= \{\varphi \in A \mid \tau^*(\varphi) = \varphi \circ \tau = 0 \text{ on } \mathscr{U}_{pc}(\Sigma)\} \\ &= \{\varphi \in A \mid \varphi(x(T_u; x_0, u)) = 0 \text{ for all } u \in \mathscr{U}_{pc}(\Sigma)\} \\ &= \{\varphi \in A \mid \varphi = 0 \text{ on } \mathscr{R}(x_0)\}.\end{aligned}$$

Ker τ^* is the largest ideal of A containing all polynomials of A vanishing on $\mathscr{R}(x_0)$. Hence, Ker τ^* determines the smallest variety in X containing $\mathscr{R}(x_0)$. Therefore, $Z\text{-}cl(\mathscr{R}(x_0)) = \{x \in X \mid \varphi(x) = 0 \text{ for all } \varphi \in \text{Ker } \tau^*\}$. □

Consider an initialized rational system $\Sigma = (X, f, h, x_0)$. Further, let I be the ideal of A determining the variety $Z\text{-}cl(\mathscr{R}(x_0))$ and let (0) denote the zero ideal of A. Because $Z\text{-}cl(\mathscr{R}(x_0))$ contains at least the point x_0, the variety $Z\text{-}cl(\mathscr{R}(x_0))$ cannot be determined by the ideal $I = A$. Otherwise $Z\text{-}cl(\mathscr{R}(x_0))$ would have to be an empty set contradicting the fact that $x_0 \in Z\text{-}cl(\mathscr{R}(x_0))$. If the ideal I is such that $(0) \subsetneq I \subsetneq A$ then the variety $Z\text{-}cl(\mathscr{R}(x_0))$ satisfies the inclusions $\emptyset \subsetneq Z\text{-}cl(\mathscr{R}(x_0)) \subsetneq X$. In this case the rational system Σ is not algebraically reachable from x_0. On the other hand, if $I = (0)$ then $Z\text{-}cl(\mathscr{R}(x_0)) = X$ and thus the rational system Σ is algebraically reachable from x_0. The following theorem summarizes this characterization of algebraic reachability.

Theorem 3.10. *The following statements are equivalent:*

(i) an initialized rational system $\Sigma = (X, f, h, x_0)$ is algebraically reachable from x_0,
(ii) $Z\text{-}cl(\mathscr{R}(x_0)) = X$,
(iii) the only ideal $I \subseteq A$ such that

$$Z\text{-}cl(\mathscr{R}(x_0)) \subseteq \{x \in X \mid \varphi(x) = 0 \textit{ for all } \varphi \in I\}$$

is the ideal $I = (0)$,
(iv) Ker $\tau^* = (0)$*,*
(v) τ^* is injective.

Proof. The equivalence $(i) \Leftrightarrow (ii)$ follows from the definition of algebraic reachability. The equivalence $(iv) \Leftrightarrow (v)$ is obvious. Next we prove the chain of implications $(ii) \Rightarrow (iv) \Rightarrow (iii) \Rightarrow (ii)$.

$(ii) \Rightarrow (iv)$ By Lemma 3.9, $Z\text{-}cl(\mathscr{R}(x_0)) = \{x \in X \mid \varphi(x) = 0 \text{ for all } \varphi \in \text{Ker } \tau^*\}$. From (ii), $Z\text{-}cl(\mathscr{R}(x_0)) = X$. Then it follows that $X = \{x \in X \mid \varphi(x) = 0 \text{ for all } \varphi \in \text{Ker } \tau^*\}$, and consequently every polynomial $\varphi \in$ Ker τ^* is identically zero on X. Therefore Ker τ^* is the zero ideal of A, i.e. Ker $\tau^* = (0)$.

$(iv) \Rightarrow (iii)$ Let Ker $\tau^* = (0)$. We prove that $I = (0) \subseteq A$ if and only if I is an ideal of A such that

$$Z\text{-}cl(\mathscr{R}(x_0)) \subseteq \{x \in X \mid \varphi(x) = 0 \text{ for all } \varphi \in I\}. \tag{3.2}$$

If $I = (0)$ then $I = \text{Ker } \tau^* = (0)$, and, by Lemma 3.9, $Z\text{-}cl(\mathscr{R}(x_0)) = \{x \in X \mid \varphi(x) = 0 \text{ for all } \varphi \in I\}$. Next, let $I \subseteq A$ be an ideal satisfying (3.2). Because $Z\text{-}cl(\mathscr{R}(x_0)) = \{x \in X \mid \varphi(x) = 0 \text{ for all } \varphi \in \text{Ker } \tau^*\}$ and because Ker $\tau^* = (0)$, it follows that

$Z\text{-}cl(\mathscr{R}(x_0)) = X$. Further, from (3.2), $X \subseteq \{x \in X \mid \varphi(x) = 0 \text{ for all } \varphi \in I\}$ which implies that $I = (0)$.

Note that the implication $(iv) \Rightarrow (iii)$ can be proven also in the following way. Let us assume by contradiction that Ker $\tau^* = (0)$ and that there exists an ideal $I \subseteq A$ satisfying (3.2) such that $I \neq (0)$. Because Ker $\tau^* = (0) \subsetneq I$, it holds that $\{x \in X \mid \varphi(x) = 0 \text{ for all } \varphi \in I\} \subsetneq \{x \in X \mid \varphi(x) = 0 \text{ for all } \varphi \in \text{Ker } \tau^*\} = Z\text{-}cl(\mathscr{R}(x_0))$ which contradicts (3.2). Thus, the ideal $I = (0)$ is the only ideal satisfying (3.2).

$(iii) \Rightarrow (ii)$ Let I be an ideal of A determining the variety $Z\text{-}cl(\mathscr{R}(x_0))$. Then $Z\text{-}cl(\mathscr{R}(x_0)) = \{x \in X \mid \varphi(x) = 0 \text{ for all } \varphi \in I\}$. From (iii), $I = (0)$ and $Z\text{-}cl(\mathscr{R}(x_0)) = \{x \in X \mid \varphi(x) = 0 \text{ for } \varphi = 0 \text{ on } X\} = X$. □

Proposition 3.11. *Let $\Sigma = (X, f, h, x_0)$ be an initialized rational system with the algebra $A = \mathbb{R}[\varphi_1, \ldots, \varphi_k]$ of polynomials on X. We assume that Σ is such that the algebra $\mathbb{R}[\tau^*\varphi_1, \ldots, \tau^*\varphi_k]$ determined by the dual input-to-state map $\tau^* : A \to (\mathscr{U}_{pc}(\Sigma) \to \mathbb{R})$ is an integral domain. Then,*

(i) Ker τ^* *is a prime ideal, and*
(ii) $Z\text{-}cl(\mathscr{R}(x_0)) \subseteq X$ is an irreducible variety.

Proof. (i) Because $\mathbb{R}[\tau^*\varphi_1, \ldots, \tau^*\varphi_k]$ is an integral domain, and because the map $\widehat{\tau^*} : A/\text{Ker } \tau^* \to \mathbb{R}[\tau^*\varphi_1, \ldots, \tau^*\varphi_k]$ is an isomorphism, we obtain that Ker τ^* is a prime ideal.

(ii) Obviously, $Z\text{-}cl(\mathscr{R}(x_0))$ is a variety. From Proposition 2.12, $Z\text{-}cl(\mathscr{R}(x_0))$ is irreducible if and only if the ideal of polynomials on X which vanish on $Z\text{-}cl(\mathscr{R}(x_0))$ is a prime ideal. By Lemma 3.9, the ideal of polynomials on X vanishing on $Z\text{-}cl(\mathscr{R}(x_0))$ equals Ker τ^*. Since, from (i), Ker τ^* is a prime ideal, $Z\text{-}cl(\mathscr{R}(x_0))$ is an irreducible variety. □

Note that the statements (i) and (ii) of the proposition above are equivalent and that this proposition is independent of the choice of generators $\varphi_1, \ldots, \varphi_k$ of the algebra A of polynomials on X.

With respect to the computational issues, it is useful to have a characterization of algebraic reachability of rational systems in terms of rational vector fields corresponding to a rational system. For polynomial systems such characterization is provided by Bartosiewicz in [11]. We recall this characterization in the following proposition.

Proposition 3.12. *An initialized polynomial system $\Sigma = (X, f, h, x_0)$ is algebraically reachable from x_0 if and only if there are no ideals $(0) \neq I \subseteq A$ such that*

(i) $\varphi(x_0) = 0$ for every $\varphi \in I$, and
(ii) $f_\alpha I \subseteq I$ for all $\alpha \in U$.

We obtain the corresponding result for initialized rational systems $\Sigma = (X, f = \{f_\alpha \mid \alpha \in U\}, h, x_0)$ by finding a set $X' \subseteq X$ such that $x_0 \in X'$ and all vector fields $f_\alpha, \alpha \in U$ are tangent to X'. The proof of Theorem 3.13 is analogous to the Bartosiewicz's proof of Proposition 3.12.

Theorem 3.13. *If an initialized rational system $\Sigma = (X, f, h, x_0)$ is algebraically reachable then there are no ideals $(0) \neq I \subseteq A$ such that for every $\varphi \in I$ and for every $\alpha \in U$ it holds that $\varphi(x_0) = 0$ and $\{p \mid f_\alpha \varphi = p/q,$ and $p, q \in A\} \subseteq I$.*

If the set $\mathscr{U}_{pc}(\Sigma)$ of admissible inputs for Σ is such that for every $u \in \mathscr{U}_{pc}(\Sigma)$ and for every $\alpha \in U$ there exists $\varepsilon > 0$ such that $(u)(\alpha, \varepsilon) \in \mathscr{U}_{pc}(\Sigma)$ then also the converse implication holds.

Proof. ($\Rightarrow$) Consider an initialized rational system $\Sigma = (X, f, h, x_0)$. Suppose that there exists an ideal $(0) \neq I \subseteq A$ such that for every $\varphi \in I$ and for every $\alpha \in U$ it holds that $\varphi(x_0) = 0$ and $\{p \mid p/q = f_\alpha \varphi; p, q \in A\} \subseteq I$. For such an ideal I we define

$$X' = \{x \in X \mid \varphi(x) = 0 \text{ for all } \varphi \in I\}.$$

Then $x_0 \in X'$ and for every $\alpha \in U, \varphi \in I$ it holds that $f_\alpha \varphi = 0$ on X'. Therefore the rational vector fields f_α are tangent to X' which implies that the trajectories of Σ starting at $x_0 \in X'$ remain in X'. Hence, $\mathscr{R}(x_0) \subseteq X'$. Because X' is a variety containing $\mathscr{R}(x_0)$ and because the ideal I determining X' is such that $I \neq (0)$ we get that $Z\text{-}cl(\mathscr{R}(x_0)) \subseteq X' \subsetneq X$. This implies that Σ is not algebraically reachable from x_0.

($\Leftarrow$) Consider an initialized rational system $\Sigma = (X, f, h, x_0)$ which is not algebraically reachable. Hence, $Z\text{-}cl(\mathscr{R}(x_0)) = X' \subsetneq X$. Further, let the set $\mathscr{U}_{pc}(\Sigma)$ satisfy the assumption from the theorem. Let us define

$$I = \{\varphi \in A \mid \varphi(x) = 0 \text{ for every } x \in X'\}.$$

Because $x_0 \in X'$ and $X' \subsetneq X$, the ideal I satisfies $(0) \subsetneq I \subsetneq A$. It also follows that $\varphi(x_0) = 0$ for every $\varphi \in I$. We will prove that $\{p \mid p/q = f_\alpha \varphi; p, q \in A\} \subseteq I$ for every $\alpha \in U$ and for every $\varphi \in I$.

From the definition of a trajectory of a rational system and from the assumption on $\mathscr{U}_{pc}(\Sigma)$, it holds that

$$((f_\alpha \varphi) \circ \tau)(u) = \frac{d}{ds}(\varphi \circ \tau)((u)(\alpha, s))|_{s=0+}$$

for all $\varphi \in A$, $u \in \mathscr{U}_{pc}(\Sigma)$, and $\alpha \in U$. Consider an arbitrary $\varphi \in I$ and an arbitrary $\alpha \in U$. Then $\varphi \circ \tau = 0$ on $\mathscr{U}_{pc}(\Sigma)$, and for any $u \in \mathscr{U}_{pc}(\Sigma)$ it follows that $\frac{d}{ds}(\varphi \circ \tau)((u)(\alpha, s))|_{s=0+} = \frac{d}{ds}0 = 0$. Consequently, $((f_\alpha \varphi) \circ \tau)(u) = 0$ for all $u \in \mathscr{U}_{pc}(\Sigma)$ and thus $f_\alpha \varphi = 0$ on $\mathscr{R}(x_0)$. Because $f_\alpha \varphi = \frac{p}{q}$ where $p, q \in A$ and $q \neq 0$, we get that $p = 0$ on $\mathscr{R}(x_0)$. Hence, $p \in I$. $\square$

The assumption of Theorem 3.13 on the set $\mathscr{U}_{pc}(\Sigma)$ of admissible inputs for $\Sigma = (X, f, h, x_0)$ implies that all rational vector fields $f_\alpha \in f$ are defined at x_0.

Corollary 3.14. *Consider a rational system $\Sigma = (X, f, h)$ such that $\mathscr{U}_{pc}(\Sigma)$ satisfies the assumption from Theorem 3.13 for every $x_0 \in X_\Sigma$. Then Σ is algebraically reachable if and only if for any $x_0 \in X_\Sigma$ there are no ideals $(0) \neq I \subseteq A$ such that for every $\varphi \in I$ and for every $\alpha \in U$ it holds that $\varphi(x_0) = 0$ and $\{p \mid p/q = f_\alpha \varphi; p, q \in A\} \subseteq I$.*

Different methods for checking algebraic reachability are demonstrated in Example 3.15, Example 3.17 and in the examples presented in Chapter 5.

Example 3.15. Let $\Sigma = (X, f = \{f_\alpha \mid \alpha \in U\}, h, x_0)$ be an initialized rational system with the input-space $U = [0,\infty)$ and the output-space $\mathbb{R}$ such that $X = \mathbb{R}$, $f_\alpha = \frac{\alpha}{1+x}\frac{\partial}{\partial x}$ for $\alpha \in U$, $h(x) = x$, and $x_0 = 0 \in X \setminus \{-1\}$. The admissible inputs $\mathscr{U}_{pc}(\Sigma)$ for Σ are the piecewise-constant functions u with the values in U which do not steer the system Σ to the state -1 before the time T_u. Thus, $\mathscr{U}_{pc}(\Sigma) = \mathscr{U}_{pc}$.

Consider an input $u = (\alpha_1, t_1)\cdots(\alpha_k, t_k) \in \mathscr{U}_{pc}(\Sigma)$. The trajectory of Σ corresponding to u is the map $x(\cdot; x_0, u) : [0, \sum_{i=1}^{k} t_i] \to \mathbb{R}$ such that

$$
\begin{aligned}
x(t;x_0,u) &= -1 + \sqrt{1+2\alpha_1 t}, \text{ for } t \in [0,t_1],\\
x(t;x_0,u) &= -1 + \sqrt{1+2\alpha_1 t_1 + 2\alpha_2(t-t_1)}, \text{ for } t \in [t_1, t_1+t_2],\\
&\vdots\\
x(t;x_0,u) &= -1 + \sqrt{1+2\sum_{i=1}^{k-1}\alpha_i t_i + 2\alpha_k\Big(t-\sum_{i=1}^{k-1} t_i\Big)}, \text{ for } t \in \Big[\sum_{i=1}^{k-1} t_i, \sum_{i=1}^{k} t_i\Big].
\end{aligned}
$$

Consider a closed interval $J = [x_0, x_0+\delta] = [0,\delta]$ with $\delta > 0$. Let $x_1 \in J$ be arbitrary. There exists $u = (\alpha, t) \in \mathscr{U}_{pc}$ such that $x_1 = -1 + \sqrt{1+2\alpha t}$. Therefore $J \subseteq \mathscr{R}(x_0)$ and, because a closed subset of $\mathbb{R}$ with a non-empty interior is Z-dense in $\mathbb{R}$, it follows that $\mathbb{R} = Z\text{-}cl(J) \subseteq Z\text{-}cl(\mathscr{R}(x_0)) \subseteq \mathbb{R}$. This implies that $Z\text{-}cl(\mathscr{R}(x_0)) = \mathbb{R}$. Therefore, Σ is algebraically reachable from x_0.

Below we prove that the condition of Theorem 3.13 implied by algebraic reachability of Σ is satisfied. Namely, since the algebra A of real polynomials on $\mathbb{R}$ equals $\mathbb{R}[X]$, we prove by contradiction that there are no ideals $(0) \neq I \subseteq \mathbb{R}[X]$ such that for every $\varphi \in I$ and for every $\alpha \in \mathbb{R}$ it holds that $\varphi(x_0) = 0$ and $\{p \mid f_\alpha \varphi = \frac{p}{q}; p, q \in \mathbb{R}[X]\} \subseteq I$. Let us consider such an ideal $(0) \neq I \subseteq \mathbb{R}[X]$. For every $\varphi \in I$ there is $\varphi_{(x-x_0)} \in \mathbb{R}[X]$ such that $\varphi = (x-x_0)^{i_\varphi}\varphi_{(x-x_0)}$ where $i_\varphi \in \mathbb{N}$ and $\varphi_{(x-x_0)}(x_0) \neq 0$. For arbitrary $\alpha \in \mathbb{R}$ and arbitrary non-zero $\varphi \in I$ we get

$$
f_\alpha \varphi(x) = \frac{\alpha}{1+x} i_\varphi (x-x_0)^{i_\varphi - 1}\varphi_{(x-x_0)}(x) + \frac{\alpha}{1+x}(x-x_0)^{i_\varphi}\frac{\partial}{\partial x}\varphi_{(x-x_0)}(x).
$$

This implies that $p_1(x) = \alpha i_\varphi (x-x_0)^{i_\varphi-1}\varphi_{(x-x_0)}(x) + \alpha (x-x_0)^{i_\varphi}\frac{\partial}{\partial x}\varphi_{(x-x_0)}(x)$ is an element of I. By applying the vector field f_α to p_1 we obtain that $f_\alpha p_1(x) = \frac{1}{1+x}p_2(x)$ where $p_2 \in I$ is of the form

$$
\begin{aligned}
p_2(x) = \alpha^2 i_\varphi(i_\varphi - 1)(x-x_0)^{i_\varphi-2}\varphi_{(x-x_0)}(x) + {} & \text{terms containing the powers}\\
& \text{of } (x-x_0) \text{ with the exponents higher than } i_\varphi - 2.
\end{aligned}
$$

In the same way we continue applying the vector field f_α to the polynomials $p_j, j > 2$ and deriving the polynomials p_{j+1} as $f_\alpha p_j = \frac{1}{1+x}p_{j+1}$. After i_φ steps we obtain the equality

$$p_{i_\varphi}(x) = \text{sum of terms containing non-zero powers of } (x-x_0) \text{ (these terms are zero at } x_0\text{), and the term } \alpha^{i_\varphi}(i_\varphi)!\varphi_{(x-x_0)}(x) \text{ which is non-zero at } x_0.$$

Therefore $p_{i_\varphi} \notin I$ and consequently $\varphi \notin I$. Because φ was an arbitrary non-zero polynomial of I, it follows that $I = (0)$ which contradicts the assumption $I \neq (0)$.

3.3.1 Relations with other concepts of controllability

Since piecewise-constant inputs approximate the effect of arbitrary controls (see for example [98]), to relate the concept of algebraic reachability to other concepts of controllability for the classes of linear and nonlinear systems it is sufficient to consider these classes of systems only with the inputs of $\mathscr{U}_{pc}$.

3.3.1.1 Linear systems

The concepts of reachability, controllability, and null controllability for linear systems (see [110]) are equivalent and we refer to them as controllability. Consider a linear system Σ given as

$$\begin{aligned} \dot{x} &= Ax + Bu,\ x(0) = x_0, \\ y &= Cx + Du, \end{aligned}$$

with the state-space $X = \mathbb{R}^n$, A, B, C, D real matrices of suitable dimensions, and the inputs u from $\mathscr{U}_{pc}$. Then Σ is *controllable* if for every $x_0 \in X$ the reachable set $\mathscr{R}(x_0)$ is such that $\mathscr{R}(x_0) = X$.

Therefore, if a linear system Σ is controllable then Σ is algebraically reachable (from any $x_0 \in X$). Note also that $\mathscr{R}(x_0)$ is a linear subspace of $X = \mathbb{R}^n$ and consequently $Z\text{-}cl(\mathscr{R}(x_0)) = \mathscr{R}(x_0)$ is an irreducible variety in $\mathbb{R}^n$. Hence, if Σ is not controllable then Σ is not algebraically reachable, and even Σ is not algebraically reachable from any $x_0 \in X$. We summarize these remarks in the following theorem.

Theorem 3.16. *Consider a linear system Σ. Then the following statements are equivalent:*

(i) Σ is controllable,
(ii) Σ is algebraically reachable,
(iii) Σ is algebraically reachable from any initial state.

3.3.1.2 Polynomial systems

Since polynomial systems are a subclass of rational systems, we can define algebraic reachability of polynomial systems in the same way as it is done for rational systems in Definition 3.8. Let us compare this concept of algebraic reachability of polynomial systems with the concepts of controllability introduced for continuous-time polynomial systems in [4, 5, 11].

Algebraic reachability of initialized polynomial systems is characterized by Bartosiewicz in [11, Proposition 2], see Proposition 3.12, by the non-existence of ideals satisfying certain conditions. Let $\Sigma = (X, f, h, x_0)$ be an initialized polynomial system. Because polynomial vector fields of f are defined at all points of the state-space X and because X_Σ equals X, the conditions from Proposition 3.12 are the same as the conditions from Theorem 3.13 for algebraic reachability of an initialized rational system when the system is actually polynomial.

Since the state-spaces X of polynomial systems we consider are algebraic varieties, the property of reachability (from an initial state) on X in [4] corresponds to our notion of algebraic reachability (from an initial state). Because any smooth algebraic variety $X \subseteq \mathbb{R}^n$ is a submanifold of $\mathbb{R}^n$, the accessibility property on a smooth variety X according to [4] implies algebraic reachability. As the example from [4, pg. 886-887] for α irrational and the state-space considered to be $V(\{f_1, f_2\})$ (see that example for more details) shows, the converse implication, i.e. algebraic reachability implies accessibility, does not hold. The strong accessibility property introduced in [5] for polynomial systems with the state-spaces being $\mathbb{R}^n$ implies their algebraic reachability.

3.3.1.3 Nonlinear systems

To compare the concept of algebraic reachability of rational systems with the concepts of controllability introduced for nonlinear systems we follow the theory of nonlinear systems presented in [79] using the Lie-theoretic approach (for local accessibility and local strong accessibility), and [105] (for global properties such as controllability, accessibility and strong accessibility).

Consider a smooth affine nonlinear control system Σ on a smooth manifold X given as

$$\dot{x} = f(x) + \sum_{j=1}^{m} g_j(x)\alpha_j, \ \alpha = (\alpha_1, \dots, \alpha_m) \in U \subseteq \mathbb{R}^m.$$

The reachable set from $x_0 \in X$ at time $T > 0$, with the trajectories staying in an open neighborhood V of x_0, is the set

$$\mathscr{R}^V(x_0, T) = \{x \in X \mid \exists\, u \in \mathscr{U}_{pc}(\Sigma) : x(0; x_0, u) = x_0,\ x(t; x_0, u) \in V \text{ for } 0 \leq t \leq T, \\ \text{and } x(T; x_0, u) = x\}.$$

The reachable set from $x_0 \in X$ till time $T > 0$ is denoted by

$$\mathscr{R}^V_T(x_0) = \bigcup_{0 \leq \tau \leq T} \mathscr{R}^V(x_0, \tau).$$

The system Σ is called *locally accessible from* x_0 if for any neighborhood V of x_0 and for any time $T > 0$ the set $\mathscr{R}_T^V(x_0)$ contains a non-empty open set. If this condition holds for every $x_0 \in X$, Σ is called *locally accessible*. Further, we say that Σ is *locally strongly accessible from* x_0 if for any neighborhood V of x_0 and for any $T > 0$ sufficiently small the set $\mathscr{R}^V(x_0, T)$ contains a non-empty open set.

Therefore, for a rational system Σ with the state-space being a smooth irreducible variety $X \subseteq \mathbb{R}^n$ such that the components $h_i, i = 1,\dots,r$ and $f_{\alpha,i}, i = 1,\dots,n, \alpha \in U$ (determined by the rational vector fields $f_\alpha = \sum_{i=1,\dots,n} f_{\alpha,i} \frac{\partial}{\partial x_i}, \alpha \in U$) can be considered smooth functions on X, it holds that if Σ is locally accessible then Σ is also algebraically reachable. If Σ is locally accessible from an initial state x_0 or locally strongly accessible from x_0 then it is also algebraically reachable from x_0. All three implications follow from the fact that a non-empty open set in an irreducible real affine variety X is Z-dense open in X. In general, none of the converse implications holds as the following example demonstrates.

Example 3.17. Consider a rational system Σ, with the state-space $X = \mathbb{R}^2$, the dynamics of which is given as

$$\begin{aligned} \dot{x}_1 &= \tfrac{1}{1+x_2^2},\ x_1(0) = x_1^0 \in \mathbb{R}, \\ \dot{x}_2 &= 1, \qquad x_2(0) = x_2^0 \in \mathbb{R}. \end{aligned}$$

The trajectory of Σ from the initial state (x_1^0, x_2^0) is the map $x : [0,\infty] \to \mathbb{R}^2$ such that

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \arctan(t + x_2^0) + x_1^0 - \arctan x_2^0 \\ t + x_2^0 \end{pmatrix}.$$

Therefore, the reachable set $\mathscr{R}((x_1^0, x_2^0))$ of the system Σ equals the set $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = \arctan x_2 + x_1^0 - \arctan x_2^0; x_1 \geq x_1^0; x_2 \geq x_2^0\}$. Because the smallest variety in $\mathbb{R}^2$ containing $\mathscr{R}((x_1^0, x_2^0))$ is $\mathbb{R}^2$, the system Σ is algebraically reachable from (x_1^0, x_2^0). Since this holds for any initial state $(x_1^0, x_2^0) \in \mathbb{R}^2$, Σ is algebraically reachable. Further, because the sets $\mathscr{R}_T^V((x_1^0, x_2^0))$ and $\mathscr{R}^V((x_1^0, x_2^0), T)$ are subsets of $\mathscr{R}((x_1^0, x_2^0))$ and thus they do not contain an open set in $\mathbb{R}^2$ for any open neighborhood V of (x_1^0, x_2^0) and any time $T > 0$, the system Σ is neither locally strongly accessible from (x_1^0, x_2^0) nor locally accessible from (x_1^0, x_2^0), and consequently it is not locally accessible.

If a rational system Σ satisfies the conditions from [105] stated there for nonlinear systems (see Section 4 of that paper for details) then, by using the terminology of [105], Σ is algebraically reachable if it is controllable (all states can be reached) or if it has accessibility (the set of all reachable states contains an open set) or strong accessibility (the set of all states reachable at time $T > 0$ contains an open set) property. The same relations hold also for initialized rational systems with the respective properties (algebraically reachable from an initial state, controllable from an initial state, accessibility or strong accessibility property from an initial state). Since there are cases when accessibility implies strong accessibility, and controllability implies strong accessibility, it could also happen that algebraic reachability implies some of these properties for a subclass of rational systems. To establish these relations further research is needed.

3.3.2 Further notions

The accessibility notions for rational systems can be introduced in an analogy with the corresponding notions for nonlinear systems. One can define for example algebraic accessibility (from an initial state) and strong algebraic accessibility (from an initial state) for rational systems as follows:

Definition 3.18. A rational system $\Sigma = (X, f, h)$ is *algebraically accessible from an initial state* $x_0 \in X_\Sigma$ if for every time $T > 0$ the set of states which can be reached from x_0 by applying admissible inputs u with time-domains $[0, T_u], T_u < T$ is Z-dense in X. If this holds for any $x_0 \in X_\Sigma$ then Σ is called *algebraically accessible*.

A rational system $\Sigma = (X, f, h)$ is *strongly algebraically accessible from an initial state* $x_0 \in X_\Sigma$ if there exists $T > 0$ such that the set of states which can be reached from x_0 by applying admissible inputs u with time-domains $[0, T]$ is Z-dense in X. If this holds for any $x_0 \in X_\Sigma$ then Σ is called *strongly algebraically accessible*.

Both these properties (from an initial state) imply algebraic reachability (from an initial state). Further research is needed to characterize these properties in terms of rational vector fields of a system and polynomial or rational functions on a state-space of a system.

Since algebro-geometric and Lie-theoretic approaches for studying nonlinear systems are related (see for example [49]), the study of relations between algebraic concepts of controllability and the "usual" concepts of controllability in more detail is possible. These relations can result in computationally feasible methods for checking controllability properties. This follows from the fact that many algorithms and tools for dealing with polynomial ideals and field extensions (which are the building stones of the considered algebraic conditions) are already available, see for example [13, 76] and the references therein.

3.4 Rational observability

The algebra A of all polynomials on a variety X is a system of functions on X which distinguishes the points of X (if $a \neq b$ are two different points of X then there exists $p \in A$ such that $p(a) \neq p(b)$). The smallest system of functions distinguishing the points of X and containing all rational functions on X is the field Q of all rational functions on X.

Because observability of a system means that the initial states of the system are distinguished by the input-output maps given by this system, one way of defining observability for a rational system $\Sigma = (X, f, h)$ is to characterize it by the property that F distinguishes the points of the state-space X, where F is the subfield of the field of rational functions on X which is generated by all components of the output function h and which is closed with respect to the vector fields of f defining the dynamics of Σ.

Definition 3.19. Let $\Sigma = (X, f = \{f_\alpha \mid \alpha \in U\}, h)$ be a rational system and let Q denote the field of rational functions on X. The *observation algebra* $A_{obs}(\Sigma)$ of Σ is the smallest subalgebra of the field Q which contains all components $h_i, i = 1, \ldots, r$ of h, and which is closed with respect to the derivations given by the rational vector fields $f_\alpha, \alpha \in U$. The *observation field* $Q_{obs}(\Sigma)$ of Σ is the field of quotients of $A_{obs}(\Sigma)$.

Recall that since X is an irreducible variety, the algebra A of polynomials on X is an integral domain. Further, as the field of fractions of an integral domain is also an integral domain, the field Q of rational functions on X is an integral domain. Because the observation algebra $A_{obs}(\Sigma)$ of Σ is a subalgebra of Q, it is an integral domain, too. Therefore, the observation field $Q_{obs}(\Sigma)$ is well-defined. Note that $Q_{obs}(\Sigma)$ is also closed with respect to the derivations given by the rational vector fields f_α for all $\alpha \in U$. The following proposition deals with finite generatedness of the observation fields of rational systems, it is proven in [10, Proposition 1].

Proposition 3.20. *For a rational system Σ, $Q_{obs}(\Sigma)$ is a finitely generated field extension of $\mathbb{R}$, i.e. there exist $\varphi_1, \ldots, \varphi_k \in Q_{obs}(\Sigma)$ such that $Q_{obs}(\Sigma) = \mathbb{R}(\varphi_1, \ldots, \varphi_k)$.*

Definition 3.21. Let $\Sigma = (X, f = \{f_\alpha \mid \alpha \in U\}, h)$ be a rational system. Let A denote the algebra of polynomials on X and let Q denote the field of rational functions on X. The system Σ is called *algebraically observable* if $A_{obs}(\Sigma) = A$ and *rationally observable* if $Q_{obs}(\Sigma) = Q$.

If a polynomial/rational system is algebraically observable then it is also rationally observable. However, there exist polynomial and rational systems which are rationally observable but not algebraically observable, see Example 3.22. Therefore, algebraic observability of a rational system implies its rational observability but not the other way round.

The procedure for checking algebraic and rational observability follows directly from Definition 3.21. It is demonstrated in the following example and in the examples presented in Chapter 5.

Example 3.22. Let $\Sigma = (X, f = \{f_\alpha \mid \alpha \in \mathbb{R}\}, h)$ be a polynomial system given as $X = \mathbb{R}, f_\alpha = \alpha x^2 \frac{\partial}{\partial x}$ for $\alpha \in \mathbb{R}, h = x^2$. By simple calculation we derive that

$$A_{obs}(\Sigma) = \mathbb{R}[X^2, X^3, X^4, \ldots] \subsetneq \mathbb{R}[X] = A.$$

Therefore, Σ is not algebraically observable. On the other hand, for the observation field of Σ it holds that $Q_{obs}(\Sigma) = \mathbb{R}(X) = Q$ and thus the system Σ is rationally observable.

3.4.1 Linear systems

Let Σ be a linear system with the state-space $X = \mathbb{R}^n$ given as

$$\begin{aligned} \dot{x} &= Ax + Bu,\ x(0) = x_0, \\ y &= Cx + Du, \end{aligned} \tag{3.3}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{r\times n}, D \in \mathbb{R}^{r\times m}$. We assume that the inputs u are piecewise-constant functions with the values in $U \subseteq \mathbb{R}^m$. Because observability of linear systems does not depend on the inputs, to study observability of the system Σ it is sufficient to study observability of the linear system Σ_0 given as

$$\begin{aligned} \dot{x} &= Ax,\ x(0) = x_0, \\ y &= Cx. \end{aligned} \tag{3.4}$$

Theorem 3.23. *Consider linear systems Σ and Σ_0 of the form (3.3) and (3.4), respectively. The following statements are equivalent:*

(i) Σ_0 is observable,
(ii) Σ is observable,
(iii) the observability rank condition rank $\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = n$ *is satisfied.*

Theorem 3.24. *Consider a linear system Σ of the form (3.3) and the corresponding linear system Σ_0 determined by (3.4). The following statements are equivalent:*

(i) Σ is observable,
(ii) Σ_0 is algebraically observable,
(iii) Σ_0 is rationally observable.

Proof. By Theorem 3.23, Σ is observable if and only if Σ_0 is observable. Let us formulate the system Σ_0 in the framework introduced in Section 3.2. Thus, Σ_0 is a linear system defined on the state-space $X = \mathbb{R}^n$. The dynamics of Σ_0 is given by the vector field $f = Ax\frac{\partial}{\partial x} = \sum_{i=1}^n A_i x \frac{\partial}{\partial x_i}$ on X, where A_i denotes the i-th row of the matrix A. The output function of Σ_0 is the map $h(x) = Cx = \begin{pmatrix} C_1 x \\ \vdots \\ C_r x \end{pmatrix}$, where C_i denotes the i-th row of the matrix C. By applying the vector field f to the components of the output map h we derive that

$$\begin{aligned} f(C_j x) &= \sum_{i=1}^n A_i x \frac{\partial}{\partial x_i}(C_j x) = \sum_{i=1}^n A_i x C_{j,i} = C_j A x \\ &\vdots \end{aligned}$$

$$\underbrace{f\cdots f}_{k-times}(C_jx) = f^k(C_jx) = C_jA^kx$$

for $j = 1,\ldots,r$ and $k \in \mathbb{N}$. Therefore, the observation algebra $A_{obs}(\Sigma_0)$ equals the algebra $\mathbb{R}[\{C_jx, f^k(C_jx) \mid j = 1,\ldots,r; k \in \mathbb{N}\}] = \mathbb{R}[\{C_jx, C_jA^kx \mid j = 1,\ldots,r; k \in \mathbb{N}\}]$. Then, Σ_0 is algebraically observable if $A_{obs}(\Sigma_0) = \mathbb{R}[\{C_jx, C_jA^kx \mid j = 1,\ldots,r; k \in \mathbb{N}\}] = \mathbb{R}[x_1,\ldots,x_n] = A$, and it is rationally observable if $Q_{obs}(\Sigma_0) = \mathscr{Q}(A_{obs}(\Sigma_0)) = \mathbb{R}(x_1,\ldots,x_n) = Q$.

$(i) \Leftrightarrow (ii)$ We show that the observability rank condition (see Theorem 3.23) for the system Σ is satisfied if and only if $A_{obs}(\Sigma_0) = \mathbb{R}[\{C_jx, C_jA^kx \mid j = 1,\ldots,r; k \in \mathbb{N}\}] = \mathbb{R}[x_1,\ldots,x_n] = A$. The Cayley-Hamilton theorem implies that for proving the equality $A_{obs}(\Sigma_0) = A$ it is sufficient to prove that $\mathbb{R}[\{C_jx, C_jAx,\ldots,C_jA^{n-1}x \mid j = 1,\ldots,r\}] = \mathbb{R}[x_1,\ldots,x_n]$.

Note that $x_i \in \mathbb{R}[\{C_jx, C_jAx,\ldots,C_jA^{n-1}x \mid j = 1,\ldots,r\}]$ for $i = 1,\ldots,n$ if and only if $x_i \in <\{C_jx, C_jAx,\ldots,C_jA^{n-1}x \mid j = 1,\ldots,r\}>$, where $<\{a_1,\ldots,a_s\}>$ denotes the linear vector space over $\mathbb{R}$ generated by the elements $a_1,\ldots,a_s$. Indeed, if $x_i \in \mathbb{R}[\{C_jx, C_jAx,\ldots,C_jA^{n-1}x \mid j = 1,\ldots,r\}]$ then there exists a polynomial $p_i \in \mathbb{R}[X_1,\ldots,X_{r(n-1)}]$ such that $x_i = p_i(C_1x,\ldots,C_rA^{n-1}x)$. Therefore, $x_i = \sum_{j=1,\ldots,r,\ k=0,\ldots,n-1,\ l\in\mathbb{N}\cup\{0\}} a_{j,k,l}(C_jA^kx)^l$ with finitely many non-zero coefficients $a_{j,k,l} \in \mathbb{R}$. Because the degree of every monomial of $(C_jA^kx)^l$ equals l and because x_i is a monomial of degree 1, it follows that $a_{j,k,l} = 0$ for every $l \in (\mathbb{N} \cup \{0\}) \setminus \{1\}$. Thus, $x_i = \sum_{j=1,\ldots,r,\ k=0,\ldots,n-1} a_{j,k,1}C_jA^kx$ which implies that x_i belongs to $<\{C_jx, C_jAx,\ldots,C_jA^{n-1}x \mid j = 1,\ldots,r\}>$. The converse implication is obvious.

Consider a vector space V over $\mathbb{R}$ with the basis $x_1,\ldots,x_n$. Let $g : V \to V$ be a linear map defined as $g(z) = Mz$ where $M = \begin{pmatrix} C \\ \vdots \\ CA^{n-1} \end{pmatrix} \in \mathbb{R}^{r(n-1)\times n}$. Then g is surjective if and only if $x_i \in g(V) = <\{C_jx, C_jAx,\ldots,C_jA^{n-1}x \mid j = 1,\ldots,r\}>$ for every $i = 1,\ldots,n$, i.e. if and only if the system Σ_0 is algebraically observable. Furthermore, g is surjective if and only if the rank of the matrix M equals n, i.e. if and only if the observability rank condition for Σ is satisfied.

$(ii) \Leftrightarrow (iii)$ Because algebraic observability implies rational observability, it holds that $A_{obs}(\Sigma_0) = A$ implies $Q_{obs}(\Sigma_0) = Q$. To complete the proof we prove the converse implication. If $Q_{obs}(\Sigma_0) = Q$ then $\mathbb{R}(\{C_jx, C_jA^kx \mid j = 1,\ldots,r; k \in \mathbb{N}\}) = \mathbb{R}(x_1,\ldots,x_n)$. Since all monomials of C_jx and C_jA^kx, for $j = 1,\ldots,r, k \in \mathbb{N}$, are of degree 1, it follows that by taking the quotients of the elements of $A_{obs}(\Sigma_0)$ we do not introduce any polynomial which would not be an element of $A_{obs}(\Sigma_0)$. Hence, since $\mathscr{Q}(A_{obs}(\Sigma_0)) = Q_{obs}(\Sigma_0) = Q = \mathscr{Q}(A)$, we conclude that $A_{obs}(\Sigma_0) = A$. □

3.4.2 Nonlinear systems

Consider a rational system $\Sigma = (X, f = \{f_\alpha \mid \alpha \in U\}, h)$ and two points $x_1 \neq x_2 \in X$ such that all components of h and at least one of f_α, $\alpha \in U$ are defined at x_1 and x_2. Let $\mathscr{U}$ denote the set of inputs of $\mathscr{U}_{pc}$ which are admissible for both systems $\Sigma_1 = (X, f, h, x_1)$ and $\Sigma_2 = (X, f, h, x_2)$, i.e. $\mathscr{U} = \mathscr{U}_{pc}(\Sigma_1) \cap \mathscr{U}_{pc}(\Sigma_2)$. We say that x_1 and x_2 are *indistinguishable* if $h(x(T_u; x_1, u)) = h(x(T_u; x_2, u))$ for all $u \in \mathscr{U}$.

Definition 3.25. A rational system Σ is *observable* if it has no indistinguishable states.

In [11, Proposition 3] it is proven that algebraically observable polynomial systems are observable. Let us assume that a rational system $\Sigma = (X, f, h)$ is such that all initialized rational systems $\Sigma_0 = (X, f, h, x_0)$, where $x_0 \in X_\Sigma$, have the same admissible inputs $\mathscr{U} = \mathscr{U}_{pc}(\Sigma_0)$. Then, in the same way as in [11, Proposition 3], one can conclude that algebraically observable rational system Σ is observable. Moreover, one can prove that rational observability of such Σ implies its observability.

There are many observability concepts for nonlinear systems, [79, 98, 55, 40, 50] and others. In [12] the relations between several of these concepts are reviewed. Let us point out that algebraic observability in differential-algebraic setting [30, 42, 40, 41] has a different meaning than algebraic observability introduced in Definition 3.21. We leave the comparison of algebraic and rational observability defined in Definition 3.21 and other nonlinear notions of observability for future research.

Chapter 4
Realization Theory

The focus of this chapter is realization theory for the class of rational systems. We deal with the existence of rational realizations for a given response map and with the existence and the characterization of rational realizations which are canonical and/or minimal. Further, we discuss the development of the procedures and algorithms for the construction of rational realizations which have desired properties.

Within this chapter we use the notation and terminology introduced in Section 2.1 and in Section 2.2.

4.1 Introduction

Realization theory deals with the problem of determining a dynamical system, within a certain class of systems, and an initial state of this system such that it corresponds to an a priori given input-output or response map. The correspondence is given by the equality between the input-output behaviors of the system and of the map. Such a system is then called a realization of the considered map. The behavioral approach to realization theory, which is more general, is not addressed in this thesis. That approach considers any relation between observed variables (without specifying inputs and outputs) and asks for a realization as a system from a certain class of systems.

Another goal of realization theory is to characterize realizations of a given map which have certain properties. One wants to find the conditions under which the systems realizing the considered map are observable, controllable, or minimal. The relations between realizations having these properties are of interest since they lead to several applications of realization theory. Because controllability appears to be often an equivalent condition for the existence of a control law which achieves a particular control objective and because observability is an equivalent condition for the existence of an observer of a system, the realization theory is useful in control and observer synthesis. Realization theory finds its application also in system identification because the study of minimal realizations and equivalence relations between them provides a tool for studying the parametrizations of systems.

Realization theory within system theory originates in Kalman's paper [60] where he deals with realizations of finite-dimensional linear systems. There is also prior work on realizations of automata. The generalization of the results of realization theory from linear to nonlinear systems goes via bilinear systems to smooth and to analytic systems. For the realizations of bilinear systems see for example [27]. There are three approaches described in [58] to solve the realization problem for nonlinear continuous-time systems. See the references therein for Jakubczyk's approach, the approach by formal power series in non-commuting variables, and the Volterra series approach.

4.1.1 Realization theory of polynomial and of rational systems

Polynomial and rational systems are special classes of nonlinear systems admitting a more refined algebraic structure. Realization theory of discrete-time polynomial systems was formulated by Sontag in [96]. Later, in [124], Wang and Sontag published their results on realization theory of polynomial and rational continuous-time systems based on the approach of formal power series in non-commuting variables and on the relation of two characterizations of observation spaces. In [125] they generalize [124] to analytic realization theory and they relate it to the differential-geometric approach. In [126] the relation between orders of input/output equations and minimal dimensions of realizations is explored for both analytic and algebraic input/output equations. Another extension of [124] to the analytic case can be found in [122]. Further generalizations of [122] are presented in [123].

An algebraic-geometric approach to realization theory of polynomial continuous-time systems, motivated by the results of Jakubczyk in [59] for nonlinear realizations, is introduced by Bartosiewicz in [8, 11]. Furthermore, in [10], he introduces the concept of rational systems but he does not solve the realization problem for this class of systems. Nevertheless, the problem of immersion of smooth systems into rational systems treated in [10] is similar to the problem of rational realizations. Because the approach to realization theory of rational systems presented in this chapter is based on [8, 10, 11], there is an analogy between the results of this chapter and Bartosiewicz's results presented in [10, 11].

Compared to the realization theory of rational systems developed by Wang and Sontag in [124], the approach presented in this chapter is different. We apply an algebraic-geometric approach rather than the techniques based on formal power series. We solve the same problem of the existence of rational realizations (compare Theorem 5.2 in [124] and Theorem 4.19 in this chapter). In addition we deal with the questions of rational observability and algebraic reachability of rational realizations which are not treated in [124]. Another difference is that the realizations within the class of rational systems we consider do not have to be affine in the inputs as assumed by Wang and Sontag. This is motivated by the planned application of realization theory to biochemical systems where the inputs may enter in a rational way which is not affine.

Because of the possible biological applications of polynomial and rational systems, it is necessary to study these systems with positivity constraints. The first step, motivated by biochemical reaction networks, in developing a realization theory of rational positive systems is done in [115]. Rational positive realizations are not considered in this chapter.

4.1.2 Minimality of realizations

Minimal linear realizations of a given response map are defined as linear systems realizing this map for which the state-space dimension is minimal over all such linear systems. For linear realizations, the property of being minimal in this sense is equivalent to the property of being observable and controllable. Many papers deal with generalizing the same concept of minimality to nonlinear realizations, see for example [58, 59, 102, 103, 126, 94, 90]. The history of the development of concepts of minimality is sketched in [97].

Minimal realizations within the class of polynomial discrete-time systems were firstly defined in [96] by Sontag as minimal-dimensional realizations, i.e. as polynomial discrete-time realizations having the state-spaces of minimal dimension within all such realizations. The dimension of a state-space X is defined as the transcendence degree of the algebra of polynomial functions on X. In [11], Bartosiewicz generalizes discrete-time polynomial case to continuous-time case. He defines the so-called algebraically minimal polynomial realizations by using the same concept Sontag uses for minimal-dimensionality. Bartosiewicz proves that algebraically minimal polynomial realizations are algebraically observable and algebraically controllable.

Apart from the papers [80, 81], where we developed the results presented in this chapter, we are aware only of two papers, [10] and [124], concerning rational realizations. In [124] the problem of minimality of rational realizations is not considered. There is a relationship between the definition of minimal rational realizations we use in [80, 81] and thus also in this chapter and the concept introduced in [10] of minimal dimensions of rational systems to which a smooth system can be immersed. We define minimality for rational realizations as minimal-dimensionality. Then, in analogy with [10, 11] we derive that a minimal rational realization of a given response map is a rational realization of that map whose state-space dimension equals the transcendence degree of the observation field of the map realized by the system.

4.1.3 Outline of the chapter

The concept of response maps considered for rational realizations is introduced in Section 4.2. The realization problem for rational systems is stated formally in Section 4.3. It consists of three subproblems. The first subproblem is studied in Section 4.4 where sufficient and necessary conditions for a response map to be realizable by a rational system are provided. The second subproblem, considered in Sections 4.5, 4.6, 4.7, deals with the existence and characterization of canonical and minimal rational realizations. In Section 4.5 we recall the notions of rationally observable and algebraically reachable rational realizations. Further, we prove that the existence of a rational realization, of a rationally observable rational realization, and of a canonical (both rationally observable and algebraically reachable) rational realization are equivalent properties of a response map. The minimality of rational realizations is introduced and related to canonicity of rational realizations in Section 4.6. The existence of minimal rational realizations is proven equivalent to the existence of rational realizations. Section 4.7 deals with isomorphic relations of canonical and minimal rational realizations. The existence and implementations of the algorithms for checking the properties of rational realizations, for computing rational realizations, and for transforming rational realizations to canonical or minimal rational realizations, is the last subproblem of the realization problem considered in this chapter. This problem is discussed in Section 4.8 where we sketch the directions for further research concerning realization theory of rational systems.

4.2 Response maps

The realization problem for rational systems deals with determining initialized rational systems (possibly having some additional properties) such that their input-output behavior is the same as the one of the considered input-output or response map. We work with response maps rather than with input-output maps since it is more convenient from the technical viewpoint. Input-output maps are considered to be the maps between the spaces of input and output functions (functions of time) mapping an input to an output. We call a map which describes the outputs immediately after applying finite parts of the inputs a response map. See for example [96] for a more detailed explanation.

Let $U \subseteq \mathbb{R}^m$ be an input-space. Next we define the sets of admissible inputs for rational systems with values in U. These sets of inputs are the sets on which the response maps studied with respect to the realization problem for rational systems are defined.

Definition 4.1. A set $\widetilde{\mathscr{U}_{pc}} \subseteq \mathscr{U}_{pc}$ of input functions with the values in an input-space $U \subseteq \mathbb{R}^m$ is called a *set of admissible inputs* if:

(i) $\forall u \in \widetilde{\mathscr{U}_{pc}}\ \forall t \in [0, T_u] : u_{[0,t]} \in \widetilde{\mathscr{U}_{pc}}$,

(ii) $\forall u \in \widetilde{\mathscr{U}_{pc}}\ \forall \alpha \in U\ \exists t > 0 : (u)(\alpha, t) \in \widetilde{\mathscr{U}_{pc}}$,
(iii) $\forall u = (\alpha_1, t_1) \cdots (\alpha_k, t_k) \in \widetilde{\mathscr{U}_{pc}}\ \exists \delta > 0\ \forall \overline{t_i} \in [0, t_i + \delta], i = 1, \ldots, k :$

$$\overline{u} = (\alpha_1, \overline{t_1}) \cdots (\alpha_k, \overline{t_k}) \in \widetilde{\mathscr{U}_{pc}}.$$

Definition 4.2. Consider a set $\widetilde{\mathscr{U}_{pc}}$ of admissible inputs with the values in $U \subseteq \mathbb{R}^m$. Let $u \in \widetilde{\mathscr{U}_{pc}}$. We denote the derivation of a real function $\varphi : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}$ at the switching time T_u of the input $(u)(\alpha, t) \in \widetilde{\mathscr{U}_{pc}}$, where $t > 0$ is sufficiently small and $\alpha \in U$, as

$$(D_\alpha \varphi)(u) = \frac{d}{dt} \varphi((u)(\alpha, t))|_{t=0+}.$$

Note that, by Definition 4.1, $\widetilde{\mathscr{U}_{pc}}$ is a suitable domain for real functions on $\widetilde{\mathscr{U}_{pc}}$ to be D_α-differentiable. Consider a real function $\varphi : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}$ and an input $u = (\alpha_1, t_1) \cdots (\alpha_k, t_k) \in \widetilde{\mathscr{U}_{pc}}$. We define the function $\widehat{\varphi}_u(t) = \varphi(u_{[0,t]})$ for $t \in [0, T_u]$. If $t \in [\sum_{i=0}^{j} t_i, \sum_{i=0}^{j+1} t_i]$, $j = 0, \ldots k-1$ then $\widehat{\varphi}_u(t) = \varphi((\alpha_1, t_1) \cdots (\alpha_j, t - \sum_{i=0}^{j} t_i))$. The derivation $(D_\alpha \varphi)(u)$ is well-defined if the function $\widehat{\varphi}_{(u)(\alpha,t)}(\hat{t}) = \varphi((u)(\alpha, \hat{t} - T_u))$, $\hat{t} \in [T_u, T_u + t]$ is differentiable at T_u+.

We say that the map $\varphi : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}$ is *smooth* if the derivations $D_{\alpha_1} \cdots D_{\alpha_i} \varphi$ are well-defined on $\widetilde{\mathscr{U}_{pc}}$ for every $i \in \mathbb{N}$ and every $\alpha_j \in U$, $j = 1, \ldots, i$. To simplify the notation, the derivation $D_{\alpha_1} \cdots D_{\alpha_i} \varphi$ can be rewritten as $D_\alpha \varphi$ where α is the multi-index $\alpha = (\alpha_1, \ldots, \alpha_i)$.

Definition 4.3. Consider a set $\widetilde{\mathscr{U}_{pc}}$ of admissible inputs with the values in $U \subseteq \mathbb{R}^m$. We say that a function $\varphi : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}$ is *analytic at the switching times* of the inputs of $\widetilde{\mathscr{U}_{pc}}$ if for every input $u = (u_1, t_1) \cdots (u_k, t_k) \in \widetilde{\mathscr{U}_{pc}}$ the function

$$\varphi_{u_1, \ldots, u_k}(t_1, \ldots, t_k) = \varphi((u_1, t_1) \cdots (u_k, t_k))$$

is analytic, i.e. we can write $\varphi_{u_1, \ldots, u_k}$ in the form of convergent formal power series in k indeterminates $t_1, \ldots, t_k$. We denote the set of real functions $\varphi : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}$ which are analytic at the switching times of the inputs of $\widetilde{\mathscr{U}_{pc}}$ by $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$. We refer to the elements of $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ as to the *analytic functions on* $\widetilde{\mathscr{U}_{pc}}$.

Note that the analytic functions on $\widetilde{\mathscr{U}_{pc}}$ are also smooth with respect to D_α derivations. Let φ be an analytic function on $\widetilde{\mathscr{U}_{pc}}$. Then for every $(u)(\alpha, 0)(v) \in \widetilde{\mathscr{U}_{pc}}$ it holds that

$$\varphi((u)(\alpha, 0)(v)) = \varphi((u)(v))).$$

Theorem 4.4. *Let $\widetilde{\mathscr{U}_{pc}}$ be a set of admissible inputs with the values in $U \subseteq \mathbb{R}^m$. The ring $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ of analytic functions on $\widetilde{\mathscr{U}_{pc}}$ is an integral domain.*

Proof. To prove that $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ is an integral domain we prove that for $f, g \in \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ it holds that if $fg = 0$ on $\widetilde{\mathscr{U}_{pc}}$ then either $f = 0$ on $\widetilde{\mathscr{U}_{pc}}$ or $g = 0$

on $\widetilde{\mathscr{U}_{pc}}$. Consider $f,g \in \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ such that $fg = 0$. Then $fg(u) = 0$ for every $u \in \widetilde{\mathscr{U}_{pc}}$. Let $u \in \widetilde{\mathscr{U}_{pc}}$ be an arbitrary input. Because $f(u), g(u) \in \mathbb{R}$ and because $\mathbb{R}$ is an integral domain, the equality $fg(u) = f(u)g(u) = 0$ implies that either $f(u) = 0$ or $g(u) = 0$. It still needs to be proven that either f or g stay zero for all inputs $u \in \widetilde{\mathscr{U}_{pc}}$, otherwise neither of them would be zero on $\widetilde{\mathscr{U}_{pc}}$.

Hence, to complete the proof we have to prove that there do not exist $u,v \in \widetilde{\mathscr{U}_{pc}}$ such that $f(u) = g(v) = 0$ and $f(v), g(u) \neq 0$. Let us assume by contradiction that there exist such $u = (\alpha_1,t_1^u)\cdots(\alpha_k,t_k^u), v = (\beta_1,t_1^v)\cdots(\beta_l,t_l^v) \in \widetilde{\mathscr{U}_{pc}}$. Since $u,v \in \widetilde{\mathscr{U}_{pc}}$, we derive from Definition 4.1(ii) that

$$\exists\, t_i^{v'} \in [0,t_i^v], i = 1,\ldots,l : w = (\alpha_1,t_1^u)\cdots(\alpha_k,t_k^u)(\beta_1,t_1^{v'})\cdots(\beta_l,t_l^{v'}) \in \widetilde{\mathscr{U}_{pc}}.$$

From Definition 4.1(iii), there is $\delta > 0$ such that $w' = (\alpha_1,t_1^{u'})\cdots(\alpha_k,t_k^{u'})(\beta_1,t_1^{v''})\cdots \cdots(\beta_l,t_l^{v''}) \in \widetilde{\mathscr{U}_{pc}}$ for all $t_i^{u'} \in [0,t_i^u+\delta)$, $i = 1,\ldots,k$ and for all $t_j^{v''} \in [0,t_j^{v'}+\delta)$, $j = 1,\ldots,l$. Because $fg = 0$ on $\widetilde{\mathscr{U}_{pc}}$ it follows that $0 = fg(w') = f(w')g(w')$ and thus

$$0 = f_{\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_l}(t_1^{u'},\ldots,t_k^{u'},t_1^{v''},\ldots,t_k^{v''})g_{\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_l}(t_1^{u'},\ldots,t_k^{u'},t_1^{v''},\ldots,t_k^{v''})$$

for $t_i^{u'} \in [0,t_i^u+\delta),\ t_j^{v''} \in [0,t_j^{v'}+\delta),\ i = 1,\ldots,k,\ j = 1,\ldots,l$. Hence,

$$f_{\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_l} g_{\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_l} = 0 \ \text{ on } \prod_{i=1,\ldots,k}[0,t_i^u+\delta) \times \prod_{j=1,\ldots,l}[0,t_i^{v'}+\delta). \quad (4.1)$$

Because $f,g \in \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ it holds that $f_{\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_l}$ and $g_{\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_l}$ are convergent formal power series in $k+l$ indeterminates for all $\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_l \in U$, and $k,l \in \mathbb{N}$. Since $\prod_{i=1,\ldots,k}[0,t_i^u+\delta) \times \prod_{j=1,\ldots,l}[0,t_i^{v'}+\delta)$ is an open connected set in $\mathbb{R}^{k+l}$, we derive from Theorem 2.2 and from (4.1) that either

$$f_{\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_l} = 0 \ \text{ on } \prod_{i=1,\ldots,k}[0,t_i^u+\delta) \times \prod_{j=1,\ldots,l}[0,t_j^{v'}+\delta), \quad (4.2)$$

or

$$g_{\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_l} = 0 \ \text{ on } \prod_{i=1,\ldots,k}[0,t_i^u+\delta) \times \prod_{j=1,\ldots,l}[0,t_j^{v'}+\delta). \quad (4.3)$$

By assuming that either (4.2) or (4.3) holds we come to a contradiction which completes the proof.

Let us assume that (4.2) holds. Therefore, for $\tau_i \in [0,t_i^u+\delta),\ \tau_j' \in [0,t_j^{v'}+\delta),\ i = 1,\ldots,k,\ j = 1,\ldots,l$, we obtain that $f_{\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_l}(\tau_1,\ldots,\tau_k,\tau_1',\ldots,\tau_l') = 0$. Because $f_{\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_l}(0,\ldots,0,\tau_1',\ldots,\tau_l') = f_{\beta_1,\ldots,\beta_l}(\tau_1',\ldots,\tau_l')$, we derive that $f_{\beta_1,\ldots,\beta_l}(\tau_1',\ldots,\tau_l') = 0$ for $\tau_j' \in [0,t_j^{v'}+\delta),\ j = 1,\ldots,l$. Further, because $v \in \widetilde{\mathscr{U}_{pc}}$, from Definition 4.1(iii) it follows that

$$\exists\, \varepsilon > 0\ \forall\, t_j \in [0,t_j^v+\varepsilon),\ j = 1,\ldots,l : (\beta_1,t_1)\cdots(\beta_l,t_l) \in \widetilde{\mathscr{U}_{pc}}.$$

Thus, $f_{\beta_1,\dots,\beta_l}$ can be represented as a convergent formal power series in l indeterminates with a convergence domain containing $\prod_{j=1,\dots,l}[0,t_j^v+\varepsilon)$. Since $f_{\beta_1,\dots,\beta_l}=0$ on $\prod_{j=1,\dots,l}[0,t_j^{v'}+\delta)$, then $f_{\beta_1,\dots,\beta_l}=0$ also on $\prod_{j=1,\dots,l}[0,t_j^v+\varepsilon)$. Therefore $f_{\beta_1,\dots,\beta_l}(t_1^v,\dots,t_l^v)=f(v)=0$ which contradicts the assumption $f(v)\neq 0$.

Let us assume that (4.3) holds. Then $g_{\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_l}(\tau_1,\dots,\tau_k,\tau_1',\dots,\tau_l')=0$ for all $\tau_i\in[0,t_i^u+\delta),\ \tau_j'\in[0,t_j^{v'}+\delta),\ i=1,\dots,k,\ j=1,\dots,l$, and thus especially for $\tau_i=t_i^u,\ i=1,\dots,k$, and $\tau_j'=0,\ j=1,\dots,l$. Therefore,

$$0=g_{\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_l}(t_1,\dots,t_k,0,\dots,0)=g_{\alpha_1,\dots,\alpha_k}(t_1,\dots,t_k)=g(u)$$

which contradicts the assumption $g(u)\neq 0$. □

Remark 4.5. Because $\mathscr{A}(\widetilde{\mathscr{U}_{pc}}\to\mathbb{R})$ for a set $\widetilde{\mathscr{U}_{pc}}$ of admissible inputs is an integral domain, we can define the field $\mathscr{Q}(\widetilde{\mathscr{U}_{pc}}\to\mathbb{R})$ of the quotients of elements of $\mathscr{A}(\widetilde{\mathscr{U}_{pc}}\to\mathbb{R})$.

Definition 4.6. Let $\widetilde{\mathscr{U}_{pc}}$ be a set of admissible inputs. A map $p:\widetilde{\mathscr{U}_{pc}}\to\mathbb{R}^r$ is called a *response map* if its components $p_i:\widetilde{\mathscr{U}_{pc}}\to\mathbb{R}$, $i=1,\dots,r$ are such that $p_i\in\mathscr{A}(\widetilde{\mathscr{U}_{pc}}\to\mathbb{R})$.

We imposed extra assumptions on response maps in the definition above because to solve the problem of realization of a response map by a rational system we use the objects such as observation algebra and observation field of a response map. Those assumptions are necessary for well-definedness of these objects. The following definition of observation algebra and observation field of a response map is an analogy to the definition of observation algebra and observation field of a rational system, see Definition 3.19.

Definition 4.7. Let $\widetilde{\mathscr{U}_{pc}}$ be a set of admissible inputs with the values in U, and let $p:\widetilde{\mathscr{U}_{pc}}\to\mathbb{R}^r$ be a response map. The *observation algebra* $A_{obs}(p)$ of p is the smallest subalgebra of the algebra $\mathscr{A}(\widetilde{\mathscr{U}_{pc}}\to\mathbb{R})$ which contains the components $p_i, i=1,\dots,r$ of p, and which is closed with respect to the derivations D_α for all $\alpha\in U$. The *observation field* $Q_{obs}(p)$ of p is the field of quotients of $A_{obs}(p)$.

Note that the observation field $Q_{obs}(p)$ of p is well-defined only if $A_{obs}(p)$ is an integral domain. This is the case for response maps because $A_{obs}(p)$ is a subalgebra of $\mathscr{A}(\widetilde{\mathscr{U}_{pc}}\to\mathbb{R})$ for a set $\widetilde{\mathscr{U}_{pc}}$ of admissible inputs which is an integral domain, see Theorem 4.4. For well-definedness of the observation algebra of p it is sufficient to assume that the components of p are smooth (with respect to D_α derivations).

4.3 Problem formulation

An initialized rational system which for each input generates the same output as a response map p is called a *rational realization of* p. Equivalently we can say that a rational system *realizes* p. Then the *realization problem* for rational systems consists of 1) determining a rational realization of a response map, 2) determining and characterizing such rational realizations which are canonical and/or minimal, 3) providing algorithms for computing rational realizations, checking their properties, and for transforming rational realizations to canonical and/or minimal rational realizations. Formally we state the existence part of the realization problem for rational systems as follows:

Problem 4.8. Let $\widetilde{\mathscr{U}_{pc}}$ be a set of admissible inputs with the values in $U \subseteq \mathbb{R}^m$. Consider a response map $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$. The existence part of the *realization problem* for rational systems consists of determining an initialized rational system $\Sigma = (X, f, h, x_0)$ such that

$$\widetilde{\mathscr{U}_{pc}} \subseteq \mathscr{U}_{pc}(\Sigma) \ \text{ and } \ p(u) = h(x(T_u; x_0, u)) \ \text{ for all } \ u \in \widetilde{\mathscr{U}_{pc}}.$$

4.4 Existence of rational realizations

In this section we provide sufficient and necessary conditions for a response map to be realizable by a rational system. The realizability of response maps by polynomial systems is treated in [11, Theorem 2]. The proof of that theorem and the proof of Proposition 4.17, which deals with sufficient conditions for rational realizability, have the same structure. Recall that the main result of this section stated in Theorem 4.19 corresponds to the results stated in [124, Theorem 5.2].

Definition 4.9. Consider a set $\widetilde{\mathscr{U}_{pc}}$ of admissible inputs and a rational realization $\Sigma = (X, f, h, x_0)$ of a response map $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$. We define the *input-to-state map* $\tau : \widetilde{\mathscr{U}_{pc}} \to X$ as the map $\tau(u) = x(T_u; x_0, u)$ for $u \in \widetilde{\mathscr{U}_{pc}}$. The *dual input-to-state map* τ^* determined by τ is defined as $\tau^* : A \to \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ such that $\tau^*(\varphi) = \varphi \circ \tau$ for all $\varphi \in A$, where A denotes the algebra of polynomials on X.

Remark 4.10. Note that this definition corresponds to Definition 3.7. It refines the notion of input-to-state maps and dual input-to-state maps for the sets $\widetilde{\mathscr{U}_{pc}}$ of admissible inputs.

We state some properties of the map τ^* in Proposition 4.11. The proof of this proposition is omitted because it directly follows from the definition of τ^*.

Proposition 4.11. *Let $\Sigma = (X, f, h, x_0)$ be a rational realization of a response map $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$ where $\widetilde{\mathscr{U}_{pc}}$ is a set of admissible inputs. Let A be the algebra of polynomials on X, and let $\varphi_1, \dots, \varphi_k \in A$, $k < +\infty$ be such that $A = \mathbb{R}[\varphi_1, \dots, \varphi_k]$. Then the map $\tau^* : A \to \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ defined in Definition 4.9 is a homomorphism and $\tau^*(\mathbb{R}[\varphi_1, \dots, \varphi_k]) = \mathbb{R}[\tau^*\varphi_1, \dots, \tau^*\varphi_k]$. Moreover, the map $\widehat{\tau^*} : A/\mathrm{Ker}\ \tau^* \to \mathbb{R}[\tau^*\varphi_1, \dots, \tau^*\varphi_k]$, defined as $\widehat{\tau^*}([\varphi]) = \tau^*\varphi$ for every $\varphi \in A$, is an isomorphism.*

The map $\widehat{\tau^*}$ can be extended to an isomorphism of the fields $\mathcal{Q}(A/\mathrm{Ker}\ \tau^*)$ and $\mathbb{R}(\tau^*\varphi_1, \dots, \tau^*\varphi_k)$.

The following lemma can be found in [11] stated for polynomial systems.

Lemma 4.12. *Let $\Sigma = (X, f, h, x_0)$ be a rational realization of a response map $p : \widetilde{\mathcal{U}_{pc}} \to \mathbb{R}^r$ where $\widetilde{\mathcal{U}_{pc}}$ is a set of admissible inputs and let $\tau : \widetilde{\mathcal{U}_{pc}} \to X$ be as in Definition 4.9. Then for any $\varphi \in A$, where A is the algebra of polynomials on X, and for any $\alpha \in U$, where U is the set of all values of the inputs of $\widetilde{\mathcal{U}_{pc}}$, it holds that $D_\alpha(\varphi \circ \tau) = (f_\alpha \varphi) \circ \tau$.*

Proof. Let $u \in \widetilde{\mathcal{U}_{pc}}$ and $\alpha \in U$ be arbitrary. Because $\widetilde{\mathcal{U}_{pc}} \subseteq \mathcal{U}_{pc}(\Sigma)$, the trajectories of the rational system Σ corresponding to the input $(u)(\alpha, s)$ with sufficiently small $s > 0$ and corresponding to all restrictions of $(u)(\alpha, s)$ to shorter time-domains are well-defined. Definition 4.2 implies that for arbitrary $\varphi \in A$

$$D_\alpha(\varphi \circ \tau)(u) = \frac{d}{ds}(\varphi \circ \tau)((u)(\alpha, s))|_{s=0+} = \frac{d}{ds}\varphi(\tau((u)(\alpha, s)))|_{s=0+}.$$

By the definition of the input-to-state map τ, $\tau((u)(\alpha, s)) = x(T_u + s; x_0, (u)(\alpha, s))$. Then, from Definition 2.15,

$$D_\alpha(\varphi \circ \tau)(u) = (f_\alpha \varphi)(x(T_u + s; x_0, (u)(\alpha, s)))|_{s=0+}.$$

By the continuity of the rational function $f_\alpha \varphi$ along the trajectory of Σ determined by the input $(u)(\alpha, s)$ and by the properties of trajectory, we get that $D_\alpha(\varphi \circ \tau)(u) = (f_\alpha \varphi)(\tau(u))$. □

Proposition 4.13. *Let $p : \widetilde{\mathcal{U}_{pc}} \to \mathbb{R}^r$ be a response map realizable by a rational system $\Sigma = (X, f, h, x_0)$. Let $\tau : \widetilde{\mathcal{U}_{pc}} \to X$ be as in Definition 4.9. Then the map $\tau^*_{ext} : A_{obs}(\Sigma) \to A_{obs}(p)$ defined as $\tau^*_{ext}\varphi = \varphi \circ \tau$ for every $\varphi \in A_{obs}(\Sigma)$ is a well-defined surjective homomorphism, i.e. $\tau^*_{ext}(A_{obs}(\Sigma)) = A_{obs}(p)$.*

Proof. Note that τ^*_{ext} is defined in the same way as τ^* but on a different domain, see Definition 4.9. Obviously, τ^*_{ext} is a homomorphism. We prove that τ^*_{ext} is well-defined and that it is surjective.

By Definition 3.19 and by Definition 4.7, the observation algebras $A_{obs}(\Sigma)$ and $A_{obs}(p)$ are generated by h_i, $f_{\alpha_1} \cdots f_{\alpha_j} h_i$ and p_i, $D_{\alpha_1} \cdots D_{\alpha_j} p_i$, respectively, such that $j \in \mathbb{N}$, $\alpha_1, \dots, \alpha_j \in U$, and $i = 1, \dots, r$. Since τ^*_{ext} is a homomorphism, we show that $\tau^*_{ext}(A_{obs}(\Sigma)) = A_{obs}(p)$ by proving that the generators of $A_{obs}(\Sigma)$ and $A_{obs}(p)$ are mapped to each other by τ^*_{ext}. To show that τ^*_{ext} is well-defined it is sufficient to prove that τ^*_{ext} is well-defined for the generators of the algebra $A_{obs}(\Sigma)$.

Since Σ is a rational realization of p, we know that $p_i = h_i \circ \tau$ for $i = 1, \dots, r$ and that p is well-defined on $\widetilde{\mathcal{U}_{pc}}$. Because $h_i \in A_{obs}(\Sigma)$, $i = 1, \dots, r$, it follows that $p_i = \tau^*_{ext} h_i$ for $i = 1, \dots, r$ which implies that τ^*_{ext} is well-defined at h_i for all $i = 1, \dots, r$.

Let $i \in \{1, \ldots, r\}$ be arbitrary and let $h_{i,num}, h_{i,den} \in A$ be such that $h_i = \frac{h_{i,num}}{h_{i,den}}$. For a rational vector field $f_\alpha \in f$, it holds that

$$(f_\alpha h_i) \circ \tau = (f_\alpha \frac{h_{i,num}}{h_{i,den}}) \circ \tau = \frac{(f_\alpha h_{i,num} \circ \tau)(h_{i,den} \circ \tau) - (f_\alpha h_{i,den} \circ \tau)(h_{i,num} \circ \tau)}{(h_{i,den} \circ \tau)^2},$$

and further, by Lemma 4.12, that

$$(f_\alpha h_i) \circ \tau = \frac{D_\alpha(h_{i,num} \circ \tau)(h_{i,den} \circ \tau) - D_\alpha(h_{i,den} \circ \tau)(h_{i,num} \circ \tau)}{(h_{i,den} \circ \tau)^2}.$$

Therefore, $\tau^*_{ext}(f_\alpha h_i) = (f_\alpha h_i) \circ \tau = D_\alpha(\frac{h_{i,num}}{h_{i,den}} \circ \tau) = D_\alpha(h_i \circ \tau) = D_\alpha(p_i)$. As p is a response map, the derivations D_α of p are well-defined. Consequently τ^*_{ext} is well-defined at $f_\alpha h_i \in A_{obs}(\Sigma) \subseteq Q$ for all $\alpha \in U$ and for all $i = 1, \ldots, r$. $\square$

The following proposition states necessary conditions for a response map to be realizable by a rational system.

Proposition 4.14 (Necessity). *Let $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$ be a response map realizable by a rational system $\Sigma = (X, f, h, x_0)$. Let $\tau^*_{ext} : A_{obs}(\Sigma) \to A_{obs}(p)$ be as in Proposition 4.13. Then*

(i) $Q_{obs}(p) = \widehat{\tau}^*(\mathscr{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}))$, where $\widehat{\tau}^*$ is the extension of the isomorphism $\widehat{\tau_i^*} : A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext} \to A_{obs}(p)$ which is derived from the map $\tau^*_{ext} : A_{obs}(\Sigma) \to A_{obs}(p)$,
(ii) $Q_{obs}(p)$ is a finite field extension of $\mathbb{R}$.

Proof. (i) According to Proposition 4.13, the map $\tau^*_{ext} : A_{obs}(\Sigma) \to A_{obs}(p)$ is a surjective homomorphism which is not necessarily injective. Then the map

$$\widehat{\tau_i^*} : A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext} \to A_{obs}(p)$$

defined as $\widehat{\tau_i^*}([\varphi]) = \tau^*_{ext}(\varphi)$ for every $\varphi \in A_{obs}(\Sigma)$ is an isomorphism. Since the algebras $A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}$ and $A_{obs}(p)$ are integral domains, we can construct the fields of fractions of $A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}$ and of $A_{obs}(p)$. By extending the isomorphism $\widehat{\tau_i^*}$ from the algebra $A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}$ to the field $\mathscr{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext})$ we derive the isomorphism $\widehat{\tau}^* : \mathscr{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}) \to Q_{obs}(p)$. Therefore $Q_{obs}(p) = \widehat{\tau}^*(\mathscr{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}))$.

(ii) The field $\mathscr{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext})$ is a field isomorphic to a subfield F of $Q_{obs}(\Sigma)$. Since $F \subseteq Q_{obs}(\Sigma) \subseteq Q$ and Q is a finite field extension of $\mathbb{R}$, we get by Theorem 2.9 that F is finitely generated. Because a field isomorphic to a finitely generated field is finitely generated, there exist $\varphi_1, \ldots, \varphi_k \in \mathscr{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext})$ such that

$$\mathscr{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}) = \mathbb{R}(\varphi_1, \ldots, \varphi_k). \tag{4.4}$$

From (i), $Q_{obs}(p) = \widehat{\tau}^*(\mathscr{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}))$ where $\widehat{\tau}^*$ is an isomorphism. Then, by (4.4), $Q_{obs}(p) = \widehat{\tau}^*(\mathbb{R}(\varphi_1, \ldots, \varphi_k)) = \mathbb{R}(\widehat{\tau}^* \varphi_1, \ldots, \widehat{\tau}^* \varphi_k)$. Thus, $Q_{obs}(p)$ is a finite field extension of $\mathbb{R}$. $\square$

Remark 4.15. Because $\widehat{\tau_i^*}$ is an isomorphism and because $A_{obs}(\Sigma)$ and $A_{obs}(p)$ are integral domains, Ker τ_{ext}^* is a prime ideal of $A_{obs}(\Sigma)$.

In the following proposition we prove that the generators of the observation field $Q_{obs}(p)$ of a response map p can be chosen from $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$. This implies that Proposition 4.14(ii) can be equivalently stated as: "If there exists a rational realization of a response map p then $Q_{obs}(p)$ is finitely generated by the elements of $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$."

Proposition 4.16. *Let $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$ be a response map. The observation field $Q_{obs}(p)$ is a finite field extension of $\mathbb{R}$ if and only if it is finitely generated by the elements of $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$, i.e. there exist finitely many $\varphi_1, \ldots, \varphi_k \in \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ such that $Q_{obs}(p) = \mathbb{R}(\varphi_1, \ldots, \varphi_k)$.*

Proof. ($\Leftarrow$) Let $Q_{obs}(p)$ be finitely generated by the elements of $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$. Then it is obviously a finite field extension of $\mathbb{R}$.

($\Rightarrow$) Let $Q_{obs}(p)$ be a finite field extension of $\mathbb{R}$. There exist $\varphi_1, \ldots, \varphi_k \in Q_{obs}(p)$ such that $Q_{obs}(p) = \mathbb{R}(\varphi_1, \ldots, \varphi_k)$. As $\varphi_i \in Q_{obs}(p)$, $i = 1, \ldots, k$ we know that $\varphi_i = \frac{\varphi_{i,num}}{\varphi_{i,den}}$ where $\varphi_{i,num}, \varphi_{i,den} \in A_{obs}(p)$ for $i = 1, \ldots, k$. Let us define the field

$$F = \mathbb{R}(\varphi_{1,num}, \varphi_{1,den}, \ldots, \varphi_{k,num}, \varphi_{k,den}).$$

Because $\varphi_{i,num}, \varphi_{i,den} \in A_{obs}(p)$ for $i = 1, \ldots, k$, and because $A_{obs}(p)$ is a subalgebra of $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$, it follows that

$$\varphi_{i,num}, \varphi_{i,den} \in \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R}) \text{ for } i = 1, \ldots, k, \tag{4.5}$$

and

$$\mathbb{R}[\varphi_{1,num}, \varphi_{1,den}, \ldots, \varphi_{k,num}, \varphi_{k,den}] \subseteq A_{obs}(p). \tag{4.6}$$

According to (4.5), the field F is generated by elements of $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$. From the definition of F it is obvious that $F \supseteq Q_{obs}(p)$. By taking the quotients in (4.6) we derive that $F \subseteq Q_{obs}(p)$. Therefore $Q_{obs}(p) = F$ and thus the field $Q_{obs}(p)$ is finitely generated by the elements of $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$. $\square$

The following proposition specifies sufficient conditions for a response map to be realizable by a rational system.

Proposition 4.17 (Sufficiency). *Let $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$ be a response map. If there exists a field $F \subseteq \mathscr{Q}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ such that*

(i) F is finitely generated by the elements of $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$,
(ii) F is closed with respect to D_α derivations, i.e.

$$\forall i \in \mathbb{N}\, \forall \alpha_j \in U,\, j = 1, \ldots, i :\; D_{\alpha_1} \cdots D_{\alpha_i} F \subseteq F,$$

(iii) $Q_{obs}(p) \subseteq F$,

then p has a rational realization.

Proof. Consider a response map $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$. We assume that there exists a field $F \subseteq \mathscr{Q}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ satisfying $(i)-(iii)$.

Let $\varphi_1,\ldots,\varphi_k : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}$ be the elements of $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ such that $F = \mathbb{R}(\varphi_1,\ldots,\varphi_k)$. Because F is closed with respect to D_α derivations, for all $\alpha = (\alpha_1,\ldots,\alpha_j) \in U^j$, $j \in \mathbb{N}$, and for all φ_i, $i = 1,\ldots,k$, there exists $v_i^\alpha \in \mathbb{R}(X_1,\ldots,X_k)$ such that

$$D_\alpha \varphi_i = v_i^\alpha(\varphi_1,\ldots,\varphi_k). \tag{4.7}$$

Since $Q_{obs}(p) \subseteq F$, the components of the map $p = (p_1,\ldots,p_r)$ are the elements of F and thus for every p_j, $j = 1,\ldots,r$ there exists $w_j \in \mathbb{R}(X_1,\ldots,X_k)$ such that

$$p_j = w_j(\varphi_1,\ldots,\varphi_k). \tag{4.8}$$

We prove that a rational system $\Sigma = (X, f, h, x_0)$ where

$$\begin{aligned}
&X = \mathbb{R}^k,\\
&f_\alpha = \sum_{i=1}^{k} v_i^\alpha \frac{\partial}{\partial X_i},\ \alpha \in U,\\
&h_j(X_1,\ldots,X_k) = w_j(X_1,\ldots,X_k),\ j = 1\ldots r,\\
&x_0 = (\varphi_1(e),\ldots,\varphi_k(e)),\ \text{where } e \text{ is the empty input,}
\end{aligned}$$

is a rational system realizing p. Let us define $\Psi(t) = (\varphi_1,\ldots,\varphi_k)(u_{[0,t]})$ for $u \in \widetilde{\mathscr{U}_{pc}}$, $t \in [0,T_u]$. It is well-defined because $\varphi_i \in \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$, $i = 1,\ldots,k$ are defined for every $u \in \widetilde{\mathscr{U}_{pc}}$ and because $u_{[0,t]} \in \widetilde{\mathscr{U}_{pc}}$ if $u \in \widetilde{\mathscr{U}_{pc}}$ and $t \in [0,T_u]$. In particular, $\Psi(0) = (\varphi_1,\ldots,\varphi_k)(u_{[0,0]})$ is well-defined because $u_{[0,0]} = e \in \widetilde{\mathscr{U}_{pc}}$ and the functions $\varphi_1,\ldots,\varphi_k \in \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ are defined at e. Consider a constant input $u = (\alpha, T_u) \in \widetilde{\mathscr{U}_{pc}}$. Note that according to Definition 4.1(i),(ii) there exists $\varepsilon > 0$ such that for every $t, \tau \geq 0$ such that $t + \tau \in [0, T_u + \varepsilon]$ it holds that $u' = (\alpha, t+\tau) \in \widetilde{\mathscr{U}_{pc}}$. If $t + \tau \leq T_u$ then $u' = u_{[0,t+\tau]}$, and if $T_u < t+\tau$ then $u = u'_{[0,T_u]}$. In both cases we refer to the corresponding inputs as to the inputs $u_{[0,t+\tau]}$, $t+\tau \in [0, T_u+\varepsilon]$. Then,

$$\begin{aligned}
\Psi(0) &= (\varphi_1,\ldots,\varphi_k)(u_{[0,0]}) = (\varphi_1(e),\ldots,\varphi_k(e)) = x_0 \text{ , and}\\
\frac{d}{dt}\Psi(t) &= \frac{d}{d\tau}\Psi(t+\tau)|_{\tau=0+} = \frac{d}{d\tau}(\varphi_1(u_{[0,t+\tau]}),\ldots,\varphi_k(u_{[0,t+\tau]}))|_{\tau=0+}.
\end{aligned}$$

Because $(u_{[0,t]})(\alpha,\tau) = u_{[0,t+\tau]}$ and $D_\alpha \varphi(u_{[0,t]}) = \frac{d}{d\tau}\varphi((u_{[0,t]})(\alpha,\tau))|_{\tau=0+}$ for all $\varphi \in \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$, the derivation $\frac{d}{dt}\Psi(t)$ equals $(D_\alpha\varphi_1(u_{[0,t]}),\ldots,D_\alpha\varphi_k(u_{[0,t]}))$. Furthermore, from (4.7), $\frac{d}{dt}\Psi(t) = (v_1^\alpha(\varphi_1,\ldots,\varphi_k)(u_{[0,t]}),\ldots,v_k^\alpha(\varphi_1,\ldots,\varphi_k)(u_{[0,t]}))$. Finally, as $\Psi(t) = (\varphi_1,\ldots,\varphi_k)(u_{[0,t]})$, we get that

$$\frac{d}{dt}\Psi(t) = (v_1^\alpha(\Psi(t)),\ldots,v_k^\alpha(\Psi(t))) \text{ , and } \Psi(0) = x_0.$$

The definition of f_α and the first equation derived above imply that for all $\varphi \in A$

$$(f_\alpha \varphi)(\Psi)(t) = \left(\sum_{i=1}^{k} v_i^\alpha \frac{\partial}{\partial X_i}\varphi\right)(\Psi(t)) = \sum_{i=1}^{k} v_i^\alpha(\Psi(t))\frac{\partial \varphi}{\partial X_i}(\Psi(t))$$

$$= \sum_{i=1}^{k} \frac{\partial \varphi}{\partial X_i}(\Psi(t)) v_i^\alpha(\Psi(t)) = \sum_{i=1}^{k} \frac{\partial \varphi}{\partial X_i}(\Psi(t))\left(\frac{d\Psi}{dt}(t)\right)_i = \frac{d}{dt}(\varphi \circ \Psi)(t).$$

From Definition 2.15 it follows that Ψ is the trajectory of f_α from x_0. Hence, $\Psi(t) = x(t;x_0,u_{[0,t]})$ for a constant input $u \in \widetilde{\mathscr{U}_{pc}}, t \in [0,T_u]$. Let us consider a piecewise-constant input $u = (\alpha_1,t_1)\cdots(\alpha_j,t_j) \in \widetilde{\mathscr{U}_{pc}}$. For $t \in [0,t_1]$ we already know that $\Psi(t) = x(t;x_0,u_{[0,t]})$. If we consider $t \in [t_1,t_1+t_2]$ instead of $t \in [0,t_1]$, we derive that $\Psi(t_1) = x(t_1;x_0,u_{[0,t_1]})$ and $\frac{d}{dt}\Psi(t) = (v_1^{\alpha_2}(\Psi(t)),\ldots,v_k^{\alpha_2}(\Psi(t)))$ with the same reasoning as before. Thus, $\Psi(t) = x(t-t_1;\Psi(t_1),u_{[t_1,t]}) = x(t-t_1;x(t_1;x_0,u_{[0,t_1]}),u_{[t_1,t]})$ for $t \in [t_1,t_1+t_2]$. In the analogous way we study the cases for $t \in [t_1+t_2,t_1+t_2+t_3],\ldots,t \in [t_1+\cdots+t_{i-1},t_1+\cdots+t_i],\ldots$. Finally, $\Psi(t) = x(t;x_0,u_{[0,t]})$ for an arbitrary $u \in \widetilde{\mathscr{U}_{pc}}$ and $t \in [0,T_u]$. Thus the trajectories of Σ are described by Ψ.

To prove that the rational system Σ is a realization of the response map p, we have to prove that $p(u) = h(x(T_u;x_0,u))$ for every $u \in \widetilde{\mathscr{U}_{pc}}$. Consider an arbitrary $u \in \widetilde{\mathscr{U}_{pc}}$. According to (4.8),

$$p(u) = (p_1,\ldots,p_r)(u) = (w_1(\varphi_1,\ldots,\varphi_k),\ldots,w_r(\varphi_1,\ldots,\varphi_k))(u).$$

Further, by the definitions of h_j, $j = 1,\ldots,r$, and Ψ, it follows that

$$p(u) = (h_1(\varphi_1,\ldots,\varphi_k),\ldots,h_r(\varphi_1,\ldots,\varphi_k))(u) = (h_1(\Psi(T_u)),\ldots,h_r(\Psi(T_u))).$$

Finally, since $\Psi(T_u) = x(T_u;x_0,u)$ for $u \in \widetilde{\mathscr{U}_{pc}}$, we derive that $p(u) = h(x(T_u;x_0,u))$ for $u \in \widetilde{\mathscr{U}_{pc}}$. $\square$

Remark 4.18. Proposition 4.17 can be stated as an equivalence. The proof of the other implication is the same as the sufficiency part of the proof of Theorem 4.19.

The main theorem of this section solving the problem of the existence of rational realizations for response maps is based on the three propositions above.

Theorem 4.19 (Existence of rational realizations). *A response map $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$ has a rational realization if and only if $Q_{obs}(p)$ is a finite field extension of $\mathbb{R}$.*

Proof. ($\Rightarrow$) See Proposition 4.14(ii) for this statement and the proof.
($\Leftarrow$) Since $Q_{obs}(p)$ is a finite field extension of $\mathbb{R}$, it follows from Proposition 4.16 and Definition 4.7 that $Q_{obs}(p)$ satisfies the conditions $(i)-(iii)$ of Proposition 4.17. By following the steps of the proof of Proposition 4.17 for $F = Q_{obs}(p)$, we construct a rational realization of p. $\square$

We illustrate the procedure given in the proof of Proposition 4.17 to compute a rational realization of a response map on the following example. The example is motivated by an example from [11].

Example 4.20. Let $\widetilde{\mathscr{U}_{pc}}$ be a set of admissible inputs given as the set of all piecewise-constant functions on $\mathbb{R}$ with the values in $\mathbb{R}$. We consider a map $p:\widetilde{\mathscr{U}_{pc}}\to\mathbb{R}$ defined as $p(u)=\exp(\int_0^{T_u}\frac{u(s)}{(1+s)^2}ds)$. By $u(s)$ we denote the value of an input $u\in\widetilde{\mathscr{U}_{pc}}$ at a time $s\in[0,T_u]$.

To determine a rational realization of p we first compute D_α derivations of p. Consider $\alpha_1,\alpha_2\in\mathbb{R}$ and $u\in\widetilde{\mathscr{U}_{pc}}$. For sufficiently small $t_1,t_2>0$ it holds that $(u)(\alpha_1,t_1)(\alpha_2,t_2)\in\widetilde{\mathscr{U}_{pc}}$. Then

$$
\begin{aligned}
&(D_{\alpha_1}p)(u)=\\
&=\frac{d}{d\tau}p((u)(\alpha_1,\tau))|_{\tau=0+}\\
&=\left[\frac{d}{d\tau}\exp\left(\int_0^{T_u}\frac{u(s)}{(1+s)^2}ds+\int_{T_u}^{T_u+\tau}\frac{\alpha_1}{(1+s)^2}ds\right)\right]_{\tau=0+}\\
&=0+p(u)\alpha_1\left[\exp\left(\int_{T_u}^{T_u+\tau}\frac{\alpha_1}{(1+s)^2}ds\right)\right]_{\tau=0+}\left[\frac{d}{d\tau}\int_{T_u}^{T_u+\tau}\frac{1}{(1+s)^2}ds\right]_{\tau=0+}\\
&=\alpha_1 p(u)\frac{1}{(1+T_u)^2},
\end{aligned}
$$

$$
\begin{aligned}
&(D_{\alpha_2}D_{\alpha_1}p)(u)=\\
&=D_{\alpha_2}(\alpha_1 p(u)\frac{1}{(1+T_u)^2})=\alpha_1\frac{1}{(1+T_u)^2}D_{\alpha_2}p(u)+\alpha_1 p(u)D_{\alpha_2}\frac{1}{(1+T_u)^2}\\
&=\frac{\alpha_1\alpha_2}{(1+T_u)^4}p(u)+\alpha_1 p(u)\left[\frac{d}{d\tau}\frac{1}{(1+T_u+\tau)^2}\right]_{\tau=0+}\\
&=\alpha_1\alpha_2\frac{1}{(1+T_u)^4}p(u)+\alpha_1 p(u)\frac{-2}{(1+T_u)^3}.
\end{aligned}
$$

We can compute the derivations $(D_{\alpha_i}\cdots D_{\alpha_1}p)(u)$ for any $i\in\mathbb{N}$, $\alpha_j\in\mathbb{R}$, $j\in 1,\dots,i$, and $u\in\widetilde{\mathscr{U}_{pc}}$. If we define $\varphi_1(u)=p(u)$ and $\varphi_2(u)=1+T_u$ then for any $i\in\mathbb{N}$ and $\alpha_j\in\mathbb{R}$, $j\in 1,\dots,i$ it holds that $(D_{\alpha_i}\cdots D_{\alpha_1}p)(u)\in\mathbb{R}(\varphi_1(u),\varphi_2(u))$. Therefore, by Definition 4.7, $Q_{obs}(p)\subseteq\mathbb{R}(\varphi_1,\varphi_2)$ and consequently, by Theorem 2.9, $Q_{obs}(p)$ is finitely generated. Hence, from Theorem 4.19, we know that there exists a rational realization of p.

We construct an initialized rational system $\Sigma=(X,f,h,x_0)$ which realizes p by following the steps of the proof of Proposition 4.17. Let us consider the field $F=\mathbb{R}(\varphi_1,\varphi_2)$. It is finitely generated, contains $Q_{obs}(p)$, and is closed with respect to D_α derivations. The number of generators of F equals 2 which implies that the state-space X can be taken as $\mathbb{R}^2$. To determine a family of rational vector fields $f=\{f_\alpha\mid\alpha\in\mathbb{R}\}$ we compute

$$v_1^\alpha(\varphi_1,\varphi_2) = D_\alpha\varphi_1 = D_\alpha p = \alpha\varphi_1\frac{1}{\varphi_2^2} \quad \text{and} \quad v_2^\alpha(\varphi_1,\varphi_2) = D_\alpha\varphi_2 = 1.$$

Since $D_\alpha\varphi_2(u) = \left[\frac{d}{d\tau}\varphi_2((u)(\alpha,\tau))\right]_{\tau=0+} = \left[\frac{d}{d\tau}(1+T_u+\tau)\right]_{\tau=0+} = [1]_{\tau=0+} = 1$ for any $u \in \widetilde{\mathscr{U}_{pc}}$, the equality $D_\alpha\varphi_2 = 1$ holds. The output map h is determined by a map w so that $w(\varphi_1,\varphi_2) = p = \varphi_1$. The initial point x_0 is given as $x_0 = (\varphi_1(e),\varphi_2(e))$ where e is the empty input. Finally, the rational realization $\Sigma = (X,f,h,x_0)$ of p is given as

$$\begin{aligned}
&X = \mathbb{R}^2,\\
&f_\alpha = v_1^\alpha(X_1,X_2)\frac{\partial}{\partial X_1} + v_2^\alpha(X_1,X_2)\frac{\partial}{\partial X_2} = \alpha\frac{X_1}{X_2^2}\frac{\partial}{\partial X_1} + \frac{\partial}{\partial X_2},\ \alpha \in \mathbb{R},\\
&h(X_1,X_2) = w(X_1,X_2) = X_1,\\
&x_0 = (\varphi_1(e),\varphi_2(e)) = (1,1).
\end{aligned}$$

4.5 Canonical rational realizations

The definitions of algebraic reachability and rational observability of rational realizations are based on [10, Definitions 3 and 4]. By rational observability of rational realizations of a response map we mean rational observability of rational systems defined in Definition 3.21. The definition of algebraic reachability of rational realizations slightly differs from the definition of algebraic reachability of rational systems (Definition 3.8) since we stress the inputs which play a role in the realization process. In this section we derive the characterization of the existence of rationally observable and algebraically reachable rational realizations of a given response map.

Definition 4.21. Let $\Sigma = (X,f,h,x_0)$ be a rational realization of a response map $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$ where $\widetilde{\mathscr{U}_{pc}}$ is a set of admissible inputs. Then Σ is said to be *algebraically reachable (from the initial state x_0)* if the reachable set

$$\mathscr{R}(x_0) = \{x(T_u;x_0,u) \in X \mid u \in \widetilde{\mathscr{U}_{pc}} \subseteq \mathscr{U}_{pc}(\Sigma)\}$$

is Z-dense in X.

Proposition 4.22. *Let $\Sigma = (X,f,h,x_0)$ be a rational realization of a response map $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$. Then the closure Z-cl$(\mathscr{R}(x_0))$ of the reachable set $\mathscr{R}(x_0)$ in Zariski topology on X is an irreducible variety.*

Proof. The Zariski closure of the reachable set $\mathscr{R}(x_0)$ is the smallest variety in X containing $\mathscr{R}(x_0)$. Hence, it is given as $Z\text{-}cl(\mathscr{R}(x_0)) = \{x \in X \mid \varphi(x) = 0 \text{ for all } \varphi \in A \text{ such that } \varphi = 0 \text{ on } \mathscr{R}(x_0)\}$. By considering the input-to-state map $\tau : \widetilde{\mathscr{U}_{pc}} \to X$ and the dual input-to-state map $\tau^* : A \to \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ defined in Definition 4.9, we derive that

$$\begin{aligned} Z\text{-}cl(\mathscr{R}(x_0)) &= \{x \in X \mid \varphi(x) = 0 \text{ for all } \varphi \in A \text{ such that } \varphi \circ \tau = 0 \text{ on } \widetilde{\mathscr{U}_{pc}}\} \\ &= \{x \in X \mid \varphi(x) = 0 \text{ for all } \varphi \in A \text{ such that } \tau^* \varphi = 0\} \\ &= \{x \in X \mid \varphi(x) = 0 \text{ for all } \varphi \in \text{Ker } \tau^*\}. \end{aligned}$$

Because τ^* is a homomorphism and thus $\tau^*(A)$ is a subalgebra of $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$, and because $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ is an integral domain, $\tau^*(A)$ is an integral domain. From Proposition 4.11, the map $\widehat{\tau^*} : A/\text{Ker } \tau^* \to \tau^*(A)$ is an isomorphism. This implies, since $\tau^*(A)$ is an integral domain, that Ker τ^* is a prime ideal.

Then, according to Proposition 2.12, the variety $Z\text{-}cl(\mathscr{R}(x_0))$ is irreducible.

□

Note that the proof of Proposition 4.22 corresponds to the combination of the proofs of Lemma 3.9 and Proposition 3.11.

Definition 4.23. We call a rational realization of a response map *canonical* if it is both rationally observable and algebraically reachable.

The following theorem deals with the existence of canonical rational realizations of a response map. It corresponds to [11, Theorem 3] characterizing the existence of minimal polynomial realizations. The proofs are analogous.

Theorem 4.24. *Let $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$ be a response map. The following statements are equivalent:*

(i) p has a rational realization which is rationally observable,
(ii) p has a canonical rational realization,
(iii) $Q_{obs}(p)$ is a finite field extension of $\mathbb{R}$.

Proof. $(i) \Rightarrow (ii)$ Let $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$ be a response map and let $\Sigma = (X, f, h, x_0)$ be its rational realization which is rationally observable. We denote the Zariski closure of the reachable set of Σ by X', i.e. $X' = Z\text{-}cl(\mathscr{R}(x_0))$. If $X = X'$ then Σ is algebraically reachable from x_0 and thus canonical. Let us assume that $X' \neq X$.

The variety X' is an irreducible real affine variety, see Proposition 4.22. Therefore X' can be considered a state-space of a rational system. Let $I \subseteq A$ be the ideal of polynomials which vanish on X'. The quotient ring A/I can be identified with the algebra A' of polynomials on X'. We denote the corresponding bijection by $\Psi : A/I \to A'$. This is a one-to-one and onto mapping which preserves sums and products. If we consider $\varphi' \in A'$ and $\varphi \in A$ such that $\Psi([\varphi]) = \varphi'$, it holds that $\varphi\restriction_{X'} = \varphi'$. More details can be found in [25, Chapter 5.2]. The algebra A' of polynomials on X' is finitely generated. Thus, there exist $\varphi'_1, \ldots, \varphi'_k \in A'$ such that $A' = \mathbb{R}[\varphi'_1, \ldots, \varphi'_k]$. Further, A' is an integral domain and we can define the field $Q' = \mathbb{R}(\varphi'_1, \ldots, \varphi'_k)$ of rational functions on X'.

Let us derive a canonical rational realization $\Sigma' = (X', f', h', x'_0)$ of p from Σ in the following way. We define the initial state x'_0 of Σ' as $x'_0 = x_0 \in Z\text{-}cl(\mathscr{R}(x_0)) = X'$. The output function h' of Σ' is defined as

$$h' = (h'_1, \ldots, h'_r) = \left(\frac{h'_{1,num}}{h'_{1,den}}, \ldots, \frac{h'_{r,num}}{h'_{r,den}} \right) = \left(\frac{\Psi([h_{1,num}])}{\Psi([h_{1,den}])}, \ldots, \frac{\Psi([h_{r,num}])}{\Psi([h_{r,den}])} \right)$$
$$= \left(\frac{h_{1,num}\restriction_{X'}}{h_{1,den}\restriction_{X'}}, \ldots, \frac{h_{r,num}\restriction_{X'}}{h_{r,den}\restriction_{X'}} \right) = (h_1\restriction_{X'}, \ldots, h_r\restriction_{X'}) = h\restriction_{X'},$$

where $h'_{i,num}, h'_{i,den} \in A'$ and $h_{i,num}, h_{i,den} \in A$ for $i = 1, \ldots, r$ are such that $h'_i = \frac{h'_{i,num}}{h'_{i,den}}$ and $h_i = \frac{h_{i,num}}{h_{i,den}}$. The output function h' is defined on a Z-dense subset of X' because X' is irreducible and because $h_{i,den}(x_0) \neq 0$ for $i = 1, \ldots, r$ and thus $h_{i,den} \notin I$ for $i = 1, \ldots, r$. We define the rational vector fields $f' = \{f'_\alpha : Q' \to Q' \mid \alpha \in U\}$ by relating them to the rational vector fields $f = \{f_\alpha : Q \to Q \mid \alpha \in U\}$ as

$$f'_\alpha \frac{\Psi([q_{num}])}{\Psi([q_{den}])} = \frac{\Psi([(f_\alpha q)_{num}])}{\Psi([(f_\alpha q)_{den}])}$$

for $q = \frac{q_{num}}{q_{den}} \in Q$ where $q_{num}, q_{den} \in A$ and $q_{den} \notin I$. Note that we assumed that $(f_\alpha q)_{num}, (f_\alpha q)_{den} \in A$ are such that $f_\alpha q = \frac{(f_\alpha q)_{num}}{(f_\alpha q)_{den}}$ for the considered $q \in Q$. Recall that for $\varphi \in A$ such that $\Psi([\varphi]) = \varphi'$ it holds that $\varphi\restriction_{X'} = \varphi'$. Then $f'_\alpha \frac{\Psi([q_{num}])}{\Psi([q_{den}])} = \frac{(f_\alpha q)_{num}\restriction_{X'}}{(f_\alpha q)_{den}\restriction_{X'}} = (f_\alpha q)\restriction_{X'}$ for $q = \frac{q_{num}}{q_{den}} \in Q$. Consequently, if $q' = \frac{\Psi([q_{num}])}{\Psi([q_{den}])}$, $q_{den} \notin I$ then

$$f'_\alpha q' = (f_\alpha q)\restriction_{X'} .$$

Let $u = (\alpha, T_u) \in \widetilde{\mathscr{U}_{pc}}$ be an arbitrary constant input from $\widetilde{\mathscr{U}_{pc}}$. Consider the trajectory $x'(\cdot; x'_0, u)$ of Σ' and the trajectory $x(\cdot; x_0, u)$ of Σ. Then $\frac{d}{dt}(\varphi' \circ x')(t; x'_0, u) = (f'_\alpha \varphi')(x'(t; x'_0, u))$ and $\varphi'(x'(0)) = x'_0$ for all $t \in [0, T_u]$ and for all $\varphi' \in A'$, and $\frac{d}{dt}(\varphi \circ x)(t; x_0, u) = (f_\alpha \varphi)(x(t; x_0, u))$ and $\varphi(x(0)) = x_0$ for all $t \in [0, T_u]$ and for all $\varphi \in A$. Since $X' = Z\text{-}cl(\mathscr{R}(x_0))$, both trajectories $x(\cdot; x_0, u)$ and $x'(\cdot; x'_0, u)$ stay in X'. Let $\varphi' \in A'$ and let $\varphi \in A$ be such that $\varphi' = \Psi([\varphi])$. Because $f'_\alpha \Psi([\varphi]) = (f_\alpha \varphi)\restriction_{X'}$, $x_0 = x'_0$, and because $\varphi = \varphi'$ on X', it follows that $(f_\alpha \varphi)\restriction_{X'} (x'(t; x'_0, u')) = (f_\alpha \varphi)(x'(t; x'_0, u')) = \frac{d}{dt}(\varphi \circ x')(t; x'_0, u')$, and $\varphi(x'(0)) = x'_0$. Therefore, by Theorem 2.16, the trajectories of Σ and Σ' are the same. Because the reachable sets of both systems coincide and because $X' = Z\text{-}cl(\mathscr{R}(x_0))$, the rational system $\Sigma' = (X', f', h', x'_0)$ is algebraically reachable.

The well-definedness of the rational vector fields $f'_\alpha, \alpha \in U$ follows from the fact that $(f_\alpha \frac{q_{num}}{q_{den}})\restriction_{X'}$ is independent of the choice of representatives $q_{num}, q_{den} \in A$ of the classes $[q_{num}], [q_{den}] \in A/I$. Let us consider arbitrary $\varphi_1, \varphi_2 \in I$. The polynomials φ_1, φ_2 are identically zero on X' and therefore $(q_{num} + \varphi_1)\restriction_{X'} = q_{num}\restriction_{X'}$ and $(q_{den} + \varphi_2)\restriction_{X'} = q_{den}\restriction_{X'}$. Because $\varphi_1 = 0$ on X', φ_1 is identically zero on the trajectories of Σ' and therefore $\varphi_1 \circ \tau = 0$ on $\widetilde{\mathscr{U}_{pc}}$. Further, $D_\alpha(\varphi_1 \circ \tau) = 0$ on $\widetilde{\mathscr{U}_{pc}}$ and, by Lemma 4.12, $(f_\alpha \varphi_1) \circ \tau = 0$ on $\widetilde{\mathscr{U}_{pc}}$. Because Σ' is algebraically reachable, the equality $f_\alpha \varphi_1 = 0$ on $\mathscr{R}(x'_0)$ implies that $f_\alpha \varphi_1 = 0$ on X'. In the same way we derive that $f_\alpha \varphi_2 = 0$ on X'. Finally, as f_α is an $\mathbb{R}$-linear map, $\left(f_\alpha \frac{q_{num}+\varphi_1}{q_{den}+\varphi_2}\right)\restriction_{X'} =$

$\left(\frac{(f_\alpha q_{num}+f_\alpha\varphi_1)(q_{den}+\varphi_2)-(f_\alpha q_{den}+f_\alpha\varphi_2)(q_{num}+\varphi_1)}{(q_{den}+\varphi_2)^2}\right)\restriction_{X'}=\left(\frac{(f_\alpha q_{num})q_{den}-(f_\alpha q_{den})q_{num}}{q^2_{den}}\right)\restriction_{X'}=\left(f_\alpha\frac{q_{num}}{q_{den}}\right)\restriction_{X'}$.

Let $u \in \widetilde{\mathscr{U}_{pc}}$ be arbitrary. Then

$$\begin{aligned}
p(u) &= h(x(T_u;x_0,u)), \text{ because } \Sigma \text{ realizes } p,\\
&= (h\restriction_{X'})(x(T_u;x_0,u)), \text{ because } Z\text{-}cl(\mathscr{R}(x_0)) = X',\\
&= (h\restriction_{X'})(x'(T_u;x'_0,u)), \text{ from the equalities of the trajectories of } \Sigma \text{ and } \Sigma',\\
&= h'(x'(T_u;x'_0,u)), \text{ by the definition of } h'.
\end{aligned}$$

Thus, the system Σ' is a rational realization of p. We have already proven that Σ' is algebraically reachable. Below we show that Σ' is also rationally observable which completes the proof of the existence of a canonical rational realization of p.

The observation algebra $A_{obs}(\Sigma')$ is the smallest algebra containing the elements $h'_i, f'_\alpha h'_i$ for all $i = 1,\dots,r$ and $\alpha = (\alpha_1,\dots,\alpha_k) \in U^k, k \in \mathbb{N}$. As $h'_i = h_i\restriction_{X'}$ and $f'_\alpha h'_i = f_\alpha h_i\restriction_{X'}$ for $i = 1,\dots,k$ and for $\alpha = (\alpha_1,\dots,\alpha_k) \in U^k, k \in \mathbb{N}$, we get that $\varphi = \frac{\varphi_{num}}{\varphi_{den}} \in A_{obs}(\Sigma)$, where $\varphi_{num},\varphi_{den} \in A$, if and only if $\varphi' = \frac{\Psi([\varphi_{num}])}{\Psi([\varphi_{den}])} \in A_{obs}(\Sigma')$. Because $A_{obs}(\Sigma)$ and $A_{obs}(\Sigma')$ are integral domains, $\varphi = \frac{\varphi_{num}}{\varphi_{den}} \in Q_{obs}(\Sigma)$, $\varphi_{den} \notin I$ if and only if $\varphi' = \frac{\Psi([\varphi_{num}])}{\Psi([\varphi_{den}])} \in Q_{obs}(\Sigma')$. Therefore, from rational observability of Σ and from the relation $\mathscr{Q}(\Psi(A/I)) \cong \mathscr{Q}(A')$, we derive that the system Σ' is rationally observable.

$(ii) \Rightarrow (iii)$ It follows from Theorem 4.19.

$(iii) \Rightarrow (i)$ Let p be a response map such that $Q_{obs}(p)$ is a finite field extension of $\mathbb{R}$. From Proposition 4.16, $Q_{obs}(p)$ is finitely generated by the elements of $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$. Let $Q_{obs}(p) = \mathbb{R}(\varphi_1,\dots,\varphi_k)$ where $\varphi_i \in \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R}), i = 1,\dots,k$. By the definition of the observation field, $Q_{obs}(p)$ is closed with respect to D_α derivations for $\alpha = (\alpha_1,\dots,\alpha_i) \in U^i, i \in \mathbb{N}$.

The field $Q_{obs}(p)$ fulfills the conditions $(i)-(iii)$ of Proposition 4.17. By following the proof of Proposition 4.17, with $F = Q_{obs}(p)$, we construct a rational realization $\Sigma = (X,f,h,x_0)$ of p as

$$\begin{aligned}
X &= \mathbb{R}^k,\\
f_\alpha &= \sum_{i=1}^{k} v_i^\alpha(X_1,\dots,X_k)\frac{\partial}{\partial X_i},\ \alpha \in U,\\
h_j(X_1,\dots,X_k) &= w_j(X_1,\dots,X_k),\ \ j = 1\dots r,\\
x_0 &= (\varphi_1(e),\dots,\varphi_k(e)),
\end{aligned}$$

where w_j and v_i^α are determined by the equalities $p_j = w_j(\varphi_1,\dots,\varphi_k)$, $j = 1,\dots,r$ and $D_\alpha\varphi_i = v_i^\alpha(\varphi_1,\dots,\varphi_k)$, $i = 1,\dots,k$, $\alpha \in U$, respectively.

Because $X = \mathbb{R}^k$, the field $Q = \mathbb{R}(X_1,\dots,X_k)$ is the field of rational functions on X. Note that to consider h_j and $f_\alpha h_j$, $j = 1,\dots,r$ is the same as to consider p_j and $D_\alpha p_j$ but in different coordinates. Therefore, $Q_{obs}(\Sigma)$, as the field of quotients of the smallest subalgebra of Q containing all h_j, $f_\alpha h_j$ where $j = 1,\ldots,r$, $\alpha = (\alpha_1,\ldots,\alpha_k) \in U^k$, $k \in \mathbb{N}$, equals $\mathbb{R}(X_1,\ldots,X_k)$ in analogy to the relation $Q_{obs}(p) = \mathbb{R}(\varphi_1,\ldots,\varphi_k)$. That means that $Q_{obs}(\Sigma) = \mathbb{R}(X_1,\ldots,X_k) = Q$ which proves the rational observability of Σ. $\square$

Corollary 4.25. *Let p be a response map. According to Theorem 4.19 and Theorem 4.24 the following statements are equivalent:*

(i) p is realizable by a rational system,
(ii) p has a rational realization which is rationally observable,
(iii) p has a rational realization which is canonical.

4.6 Minimal rational realizations

The state-spaces of rational systems we consider are irreducible real affine varieties. Hence, we define the dimension of a rational system as the dimension of its state-space which equals the transcendence degree of the field of rational functions defined on the state-space, see Definition 2.13.

Definition 4.26. A rational realization of a response map is called *minimal* if its state-space dimension is minimal, i.e. if there does not exist a rational realization of the same map such that its state-space has a strictly lower dimension.

In this section we provide a characterization of minimal rational realizations and their existence. We also relate minimality of rational realizations to their rational observability and algebraic reachability.

Lemma 4.27. *Let $\widetilde{\mathscr{U}_{pc}}$ be a set of admissible inputs and let $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$ be a response map. For all rational realizations Σ of p it holds that*

$$\text{trdeg } Q_{obs}(p) \leq \text{trdeg } Q_{obs}(\Sigma).$$

Proof. From Proposition 4.14(i) and Proposition 2.7 it follows that trdeg $Q_{obs}(p) =$ trdeg $\mathscr{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext})$. Further,

$$\text{trdeg } \mathscr{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}) \leq \text{trdeg } \mathscr{Q}(A_{obs}(\Sigma)) = \text{trdeg } Q_{obs}(\Sigma)$$

which implies that trdeg $Q_{obs}(p) \leq$ trdeg $Q_{obs}(\Sigma)$. $\square$

Proposition 4.28. *Let Σ be a canonical rational realization of a response map p. Then Σ is a minimal rational realization of p.*

Proof. A canonical rational realization $\Sigma = (X, f, h, x_0)$ of p is, by the definition of canonicity, algebraically reachable. Hence, according to Definition 4.21, $X = Z\text{-}cl(\mathscr{R}(x_0))$. If $\varphi = \frac{\varphi_{num}}{\varphi_{den}} \in A_{obs}(\Sigma)$, where $\varphi_{num}, \varphi_{den} \in A$, $\varphi_{den} \neq 0$, is such that $\varphi = 0$ on $\mathscr{R}(x_0)$ then $\varphi_{num} = 0$ on $\mathscr{R}(x_0)$ and moreover $\varphi_{num} = 0$ on X. Otherwise $Z\text{-}cl(\mathscr{R}(x_0)) \subsetneq X$. Consider the map $\tau_{ext}^* : A_{obs}(\Sigma) \to A_{obs}(p)$ defined in Proposition 4.13 as $\tau_{ext}^*(\varphi) = \varphi \circ \tau$ for all $\varphi \in A_{obs}(\Sigma)$. It holds that

$$\text{Ker } \tau_{ext}^* = \{\varphi \in A_{obs}(\Sigma) \mid \varphi = 0 \text{ on } \mathscr{R}(x_0)\} = \{\varphi \in A_{obs}(\Sigma) \mid \varphi = 0 \text{ on } X\},$$

and thus Ker τ_{ext}^* is the zero ideal in $A_{obs}(\Sigma)$. Consequently, $A_{obs}(\Sigma)/\text{Ker } \tau_{ext}^* \cong A_{obs}(\Sigma)$, and furthermore $\mathscr{Q}(A_{obs}(\Sigma)/\text{Ker } \tau_{ext}^*) \cong Q_{obs}(\Sigma)$. Then, by Proposition 2.7,

$$\text{trdeg } \mathscr{Q}(A_{obs}(\Sigma)/\text{Ker } \tau_{ext}^*) = \text{trdeg } Q_{obs}(\Sigma).$$

As the map $\widehat{\tau}^* : \mathscr{Q}(A_{obs}(\Sigma)/\text{Ker } \tau_{ext}^*) \to Q_{obs}(p)$ defined in Proposition 4.14(i) is an isomorphism, it follows from Proposition 2.7 that

$$\text{trdeg } Q_{obs}(p) = \text{trdeg } \mathscr{Q}(A_{obs}(\Sigma)/\text{Ker } \tau_{ext}^*).$$

By the last two equalities, trdeg $Q_{obs}(p) =$ trdeg $Q_{obs}(\Sigma)$. Because Σ is canonical, it is rationally observable and thus, by Definition 3.21, $Q_{obs}(\Sigma) = Q$ and trdeg $Q_{obs}(\Sigma) =$ trdeg Q. Hence, trdeg $Q_{obs}(p) =$ trdeg Q and finally, by the definition of the dimension of an irreducible variety,

$$\dim X = \text{trdeg } Q = \text{trdeg } Q_{obs}(p).$$

Then, according to Lemma 4.27, the system Σ is a minimal rational realization of the response map p. □

The following theorem states sufficient and necessary condition for the existence of a minimal rational realization for a given response map.

Theorem 4.29 (Existence of minimal realizations). *A response map p has a minimal rational realization if and only if it has a rational realization.*

Proof. ($\Rightarrow$) It is obvious.
($\Leftarrow$) This statement follows directly from Corollary 4.25 and Proposition 4.28. □

Theorem 4.30 (Characterization of minimality). *A rational realization* $\Sigma = (X, f, h, x_0)$ *of a response map p is minimal if and only if* $\dim X = \text{trdeg } Q_{obs}(p)$.

Proof. ($\Rightarrow$) Let $\Sigma = (X, f, h, x_0)$ be a minimal rational realization of a response map p. Then Theorem 4.29 and Corollary 4.25 imply that there exists a canonical rational realization $\Sigma' = (X', f', h', x_0')$ of p. From the proof of Proposition 4.28 it follows that $\dim X' =$ trdeg $Q_{obs}(p)$. Further, from Lemma 4.27 or Proposition 4.28,

Σ' is a minimal rational realization of p. Since all minimal realizations of the same response map have the same dimension, we derive that dim $X =$ trdeg $Q_{obs}(p)$.

($\Leftarrow$) Let $\Sigma = (X, f, h, x_0)$ be a rational realization of a response map p such that dim $X =$ trdeg $Q_{obs}(p)$. It follows from Lemma 4.27 that Σ is a minimal rational realization of p. □

Proposition 4.28 states that canonicity is a sufficient condition for rational realizations to be minimal. The proposition below generalizes that statement by proving that it is not necessary to assume rational observability. In particular, we substitute a weaker condition for rational observability.

Proposition 4.31. *Let Σ be a rational realization of a response map p. If Σ is algebraically reachable and such that the elements of $Q \setminus Q_{obs}(\Sigma)$ are algebraic over $Q_{obs}(\Sigma)$ then Σ is a minimal rational realization of p.*

Proof. If $Q \setminus Q_{obs}(\Sigma) = \emptyset$ then Σ is rationally observable and the proposition follows from Proposition 4.28. Let us assume that $Q_{obs}(\Sigma) \subsetneqq Q$. Because the elements of $Q \setminus Q_{obs}(\Sigma)$ are algebraic over $Q_{obs}(\Sigma)$,

$$\text{trdeg } Q_{obs}(\Sigma) = \text{trdeg } Q.$$

From algebraic reachability of Σ, in the same way as in the proof of Proposition 4.28, it follows that

$$\text{trdeg } Q_{obs}(\Sigma) = \text{trdeg } \mathscr{Q}(A_{obs}(\Sigma)/\text{Ker } \tau_{ext}^{*}) = \text{trdeg } Q_{obs}(p).$$

Therefore trdeg $Q_{obs}(p) =$ trdeg $Q =$ dim X and thus the rational realization Σ of p is minimal. □

Further we consider the problem of determining whether algebraic reachability and/or rational observability of rational realizations are necessary conditions for rational realizations to be minimal. The next proposition specifies this necessity relationship for rational observability.

Proposition 4.32. *If Σ is a minimal rational realization of a response map p such that the elements of $Q \setminus Q_{obs}(\Sigma)$ are not algebraic over $Q_{obs}(\Sigma)$ then Σ is rationally observable.*

Proof. Let Σ be as in the proposition. From minimality of Σ it follows that trdeg $Q =$ trdeg $Q_{obs}(p)$, see Theorem 4.30. Further, by Proposition 4.14(i) and by Proposition 2.7, trdeg $\mathscr{Q}(A_{obs}(\Sigma)/\text{Ker } \tau_{ext}^{*}) =$ trdeg Q. Because $Q_{obs}(\Sigma)$ is a subfield of Q, it follows from Proposition 2.5 that trdeg $Q_{obs}(\Sigma) \leq$ trdeg Q. Consequently we obtain that

$$\text{trdeg } Q = \text{trdeg } \mathscr{Q}(A_{obs}(\Sigma)/\text{Ker } \tau_{ext}^{*}) \leq \text{trdeg } Q_{obs}(\Sigma) \leq \text{trdeg } Q.$$

Hence, trdeg $Q =$ trdeg $Q_{obs}(\Sigma)$. By the assumption that the elements of $Q \setminus Q_{obs}(\Sigma)$ are not algebraic over $Q_{obs}(\Sigma)$ and by Proposition 2.6, $Q = Q_{obs}(\Sigma)$ which proves rational observability of Σ. □

The facts that a rational realization Σ is minimal and that the elements of $Q \setminus Q_{obs}(\Sigma)$ are algebraic over $Q_{obs}(\Sigma)$ do not provide sufficient information to determine rational observability of Σ. See the proposition below for the proof.

Proposition 4.33. *If Σ is a rational realization of a response map p such that it is not rationally observable and such that the elements of $Q \setminus Q_{obs}(\Sigma)$ are not algebraic over $Q_{obs}(\Sigma)$ then Σ is not minimal.*

Proof. To prove that a rational realization Σ of p satisfying the assumptions of the proposition is not minimal it is sufficient to prove, by Theorem 4.30, that trdeg $Q_{obs}(p) <$ trdeg Q.

Since Σ is not rationally observable, $Q_{obs}(\Sigma) \subsetneq Q$. Moreover, as the elements of $Q \setminus Q_{obs}(\Sigma)$ are not algebraic over $Q_{obs}(\Sigma)$, trdeg $Q_{obs}(\Sigma) <$ trdeg Q.

Because trdeg $\mathscr{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}) \leq$ trdeg $Q_{obs}(\Sigma)$ and because by Proposition 4.14(i) there exists an isomorphism $\widehat{\tau^*} : \mathscr{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}) \to Q_{obs}(p)$, it follows by Proposition 2.7 that

$$\text{trdeg } Q_{obs}(p) = \text{trdeg } \mathscr{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}) \leq \text{trdeg } Q_{obs}(\Sigma).$$

Therefore, trdeg $Q_{obs}(p) \leq$ trdeg $Q_{obs}(\Sigma) <$ trdeg Q which completes the proof.

□

The problem of determining whether algebraic reachability is necessary for rational realizations to be minimal is considered in the following proposition.

Proposition 4.34. *Let Σ be a minimal rational realization of a response map p. If the elements of $Q \setminus Q_{obs}(\Sigma)$ are not algebraic over $Q_{obs}(\Sigma)$ then the rational realization Σ is algebraically reachable.*

Proof. Let $\Sigma = (X, f, h, x_0)$ be a minimal rational realization of a response map p such that the elements of $Q \setminus Q_{obs}(\Sigma)$ are not algebraic over $Q_{obs}(\Sigma)$. To prove that Σ is algebraically reachable we show that $Z\text{-}cl(\mathscr{R}(x_0)) \supseteq X$.

From Theorem 4.30, trdeg $Q =$ trdeg $Q_{obs}(p)$. This implies, by Proposition 4.14(i) and by Proposition 2.7, that trdeg $Q =$ trdeg $\mathscr{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext})$. Because Σ is according to Proposition 4.32 rationally observable, thus $Q = Q_{obs}(\Sigma)$, we derive that

$$\text{trdeg } Q_{obs}(\Sigma) = \text{trdeg } \mathscr{Q}(A_{obs}(\Sigma)) = \text{trdeg } \mathscr{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}).$$

According to Remark 2.3, trdeg $\mathscr{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}) =$ trdeg $(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext})$ and trdeg $Q_{obs}(\Sigma) =$ trdeg $A_{obs}(\Sigma)$. Therefore,

$$\text{trdeg } A_{obs}(\Sigma) = \text{trdeg } (A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}) < +\infty.$$

The finiteness of the transcendence degrees above follows from Proposition 3.20. Recall from Remark 4.15 that Ker τ^*_{ext} is a prime ideal in $A_{obs}(\Sigma)$. Then the equality above implies, by Proposition 2.8, that Ker $\tau^*_{ext} = (0) \subseteq A_{obs}(\Sigma)$.

Next we prove that

$$\forall 0 \neq \varphi \in A \; \exists u \in \widetilde{\mathscr{U}_{pc}} : \; \varphi(x(T_u; x_0, u)) \neq 0 \tag{4.9}$$

which is equivalent to $\forall 0 \neq \varphi \in A : \; \varphi \neq 0$ on $\mathscr{R}(x_0)$. Let us assume by contradiction that (4.9) is not true. Let $0 \neq \varphi \in A$ be such that $\varphi(x(T_u; x_0, u)) = 0$ for all $u \in \widetilde{\mathscr{U}_{pc}}$. As $Q = Q_{obs}(\Sigma)$, there exist $0 \neq \varphi_{num}, \varphi_{den} \in A_{obs}(\Sigma)$ such that $\varphi = \frac{\varphi_{num}}{\varphi_{den}}$. Furthermore, $\varphi_{num} = \frac{n_{num}}{n_{den}}$ and $\varphi_{den} = \frac{d_{num}}{d_{den}}$ for $0 \neq n_{num}, n_{den}, d_{num}, d_{den} \in A$. Therefore $\varphi \, n_{den} \, d_{num} = n_{num} \, d_{den} \in A$ which implies that

$$\begin{aligned} &\forall u \in \widetilde{\mathscr{U}_{pc}} : \\ &\qquad \varphi(x(T_u; x_0, u)) \, n_{den}(x(T_u; x_0, u)) \, d_{num}(x(T_u; x_0, u)) \\ &\qquad\qquad = n_{num}(x(T_u; x_0, u)) \, d_{den}(x(T_u; x_0, u)). \end{aligned} \tag{4.10}$$

Because $0 \neq \varphi_{num} \in A_{obs}(\Sigma)$ and because Ker $\tau_{ext}^* = (0)$, $\tau_{ext}^*(\varphi_{num}) \neq 0$. Hence, there exists $u \in \widetilde{\mathscr{U}_{pc}}$ such that $\varphi_{num}(x(T_u; x_0, u)) \neq 0$, and consequently

$$\exists u \in \widetilde{\mathscr{U}_{pc}} : \; n_{num}(x(T_u; x_0, u)) \neq 0.$$

Because $\tau_{ext}^*(\varphi_{den})$ is well-defined, it implies that

$$\forall u \in \widetilde{\mathscr{U}_{pc}} : \; d_{den}(x(T_u; x_0, u)) \neq 0.$$

According to the last two statements above, there exists an input $u \in \widetilde{\mathscr{U}_{pc}}$ such that $n_{num}(x(T_u; x_0, u)) \, d_{den}(x(T_u; x_0, u)) \neq 0$. Hence, by (4.10),

$$\exists u \in \widetilde{\mathscr{U}_{pc}} : \; \varphi(x(T_u; x_0, u)) \, n_{den}(x(T_u; x_0, u)) \, d_{num}(x(T_u; x_0, u)) \neq 0.$$

This contradicts the assumption that $\varphi(x(T_u; x_0, u)) = 0$ for all $u \in \widetilde{\mathscr{U}_{pc}}$. Therefore (4.9) is valid. Thus,

$$\forall \varphi \in A : \varphi = 0 \text{ on } \mathscr{R}(x_0) \Rightarrow \varphi = 0 \text{ on } X.$$

Consequently it follows that

$$I_1 = \{\varphi \in A \mid \varphi = 0 \text{ on } \mathscr{R}(x_0)\} \subseteq \{\varphi \in A \mid \varphi = 0 \text{ on } X\} = I_2,$$

and therefore $\{x \in X \mid \varphi(x) = 0 \text{ for all } \varphi \in I_1\} \supseteq \{x \in X \mid \varphi(x) = 0 \text{ for all } \varphi \in I_2\}$ which means that $Z\text{-}cl(\mathscr{R}(x_0)) \supseteq X$. □

Remark 4.35. From the proof of Proposition 4.34 we obtain the following statement: "If Σ is a minimal rational realization of a response map p and if Σ is rationally observable then Σ is also algebraically reachable."

Theorem 4.36 (Relation between canonicity and minimality). *Let Σ be a rational realization of a response map p such that the elements of $Q \setminus Q_{obs}(\Sigma)$ are not algebraic over $Q_{obs}(\Sigma)$. Then Σ is canonical if and only if Σ is minimal.*

Proof. This follows directly from the Propositions 4.32, 4.34, and 4.28. □

Example 4.37. In Example 4.20 we computed a rational realization $\Sigma = (X, f, h, x_0)$ for the response map $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}$ given as $p(u) = \exp(\int_0^{T_u} \frac{u(s)}{(1+s)^2} ds)$, where $\widetilde{\mathscr{U}_{pc}}$ is the set of all piecewise-constant inputs $u : \mathbb{R} \to \mathbb{R}$. Recall that the derived rational system $\Sigma = (X, f = \{f_\alpha \mid \alpha \in \mathbb{R}\}, h, x_0)$ realizing p is of the form:

$$\begin{aligned} X &= \mathbb{R}^2, \\ f_\alpha &= \alpha \frac{X_1}{X_2^2} \frac{\partial}{\partial X_1} + \frac{\partial}{\partial X_2}, \alpha \in \mathbb{R} \\ h(X_1, X_2) &= X_1, \\ x_0 &= (1,1). \end{aligned}$$

We show that this rational realization of p is canonical, and thus, according to Proposition 4.28, minimal.

As $X_1, X_2 \in Q_{obs}(\Sigma) \subseteq \mathbb{R}(X_1, X_2)$, the observation field $Q_{obs}(\Sigma)$ equals $\mathbb{R}(X_1, X_2) = Q$. Then, Σ is rationally observable.

Since $x_1(t) = \exp(\frac{\alpha t}{t+1})$, $x_2(t) = 1 + t$ for $t \in [0, T_u]$ are describing the trajectories of Σ for a constant input $u = (\alpha, T_u)$ with $\alpha \in \mathbb{R}$, we derive that the reachable set $\mathscr{R}_{const}(x_0)$ of Σ corresponding only to constant inputs is given as $\mathscr{R}_{const}(x_0) = \{(a,b) \in \mathbb{R}^2 \mid a > 0; b > 1\} \cup \{(1,1)\}$. The system Σ can be steered from the initial state $x_0 = (1,1)$ to the state $(a,b) \in \mathscr{R}_{const}(x_0)$ by applying the constant input with the value $\alpha = \frac{b}{b-1} \log a$ till the time $b-1$. The smallest irreducible real affine variety containing $\mathscr{R}_{const}(x_0)$ is $\mathbb{R}^2$ because the varieties in $\mathbb{R}^2$ are only $\mathbb{R}^2$, finite set of points, union of an algebraic plane curve and a finite set of points. Consequently, because $\mathbb{R}^2 = Z\text{-}cl(\mathscr{R}_{const}(x_0)) \subseteq Z\text{-}cl(\mathscr{R}(x_0))$, it follows that $Z\text{-}cl(\mathscr{R}(x_0)) = \mathbb{R}^2$. Thus Σ is algebraically reachable.

4.7 Birational equivalence of rational realizations

The equivalence relations of minimal rational realizations and of canonical rational realizations are the topic of this section. We prove that every rational realization of a response map which is birationally equivalent to a minimal rational realization of the same map is itself minimal. Further, we show that canonical rational realizations of the same response map are unique up to a birational equivalence. Therefore, minimal rational realizations are all birationally equivalent if they are canonical. Thus, for example, if the assumptions of Theorem 4.36 are satisfied then all minimal rational realizations of the same response map are birationally equivalent.

In [10, Definition 8] Bartosiewicz introduces the concept of isomorphic rational systems. Because rational realizations are initialized rational systems, and because in [10, Definition 8] only rational systems without specified initial states are considered, we slightly modify Bartosiewicz's definition to define isomorphic rational realizations.

Definition 4.38. Let $\Sigma = (X, f, h, x_0)$ and $\Sigma' = (X', f', h', x_0')$ be rational realizations of the same response map p with the same input-space U and the same output-space $\mathbb{R}^r$. We say that Σ and Σ' are *isomorphic* if

(i) the state-spaces X and X', which are irreducible real affine varieties, are birationally equivalent (with the corresponding rational mappings $\phi : X \to X'$, $\psi : X' \to X$),
(ii) $h'\phi = h$,
(iii) $f_\alpha(\varphi \circ \phi) = (f'_\alpha \varphi) \circ \phi$ for all $\varphi \in Q'$, $\alpha \in U$,
(iv) ϕ is defined at x_0, and $\phi(x_0) = x_0'$.

Definition 4.39. Let $\Sigma = (X, f, h, x_0)$ and $\Sigma' = (X', f', h', x_0')$ be rational realizations of the same response map p with the same input-space U and the same output-space $\mathbb{R}^r$. We say that Σ and Σ' are *birationally equivalent* if there exists a field isomorphism $i : Q' \to Q$ such that

(i) i is the identity on the constant functions $\mathbb{R} \subset Q'$,
(ii) $i(h') = h$ (componentwise),
(iii) $f_\alpha(i(\varphi)) = i(f'_\alpha \varphi)$ for all $\varphi \in Q'$, $\alpha \in U$,
(iv) $(i(\varphi))(x_0) = \varphi(x_0')$ for all $\varphi \in Q'$ such that φ is defined at x_0' and $i(\varphi)$ is defined at x_0.

Note that Definition 4.38 and Definition 4.39 are equivalent. In the proof of Theorem 2.19 it is shown that a field isomorphism $i : Q' \to Q$ from Definition 4.39 can be chosen as $i = \phi^*$ which is defined as $i(\varphi) = \varphi \circ \phi$ for all $\varphi \in Q'$ where ϕ is a rational mapping from Definition 4.38. From the same theorem we also obtain that if $i : Q' \to Q$ is a field isomorphism then there is a rational mapping $\phi : X \to X'$ such that $i = \phi^*$. Therefore, the conditions (ii) and (iii) of Definition 4.39 are only rewritten conditions (ii) and (iii) of Definition 4.38, respectively. Further, according to Theorem 2.19, Definition 4.39(i) corresponds to Definition 4.38(i). The rational function $i(\varphi) = \phi^*(\varphi) = \varphi \circ \phi$ is defined at x_0 if and only if $\varphi \circ \phi$ is defined at x_0. Hence, $i(\varphi)$ is defined at x_0 if and only if φ is defined at $\phi(x_0)$ and ϕ is defined at x_0. Because $\varphi(x_0') = (i(\varphi))(x_0) = \varphi(\phi(x_0))$ for all $\varphi \in Q'$ defined at x_0', $\phi(x_0) \in X'$, and because the rational functions Q' on X' distinguish the points of X', it follows that $\phi(x_0) = x_0'$. Therefore the (iv) conditions of Definition 4.38 and of Definition 4.39 are equivalent.

Theorem 4.40 (Minimality and birational equivalence). *Let Σ be a minimal rational realization of a response map p. If a rational realization Σ' of p with the same input- and output-space as Σ is birationally equivalent to Σ then Σ' is minimal.*

Proof. Let Σ and Σ' be as in the proposition. According to Definition 4.39(i) and Theorem 2.19, the state-spaces X and X' of Σ and Σ', respectively, are birationally equivalent. The minimality of Σ, Theorem 4.30 and Theorem 2.18 imply that $\dim X' = \dim X = \operatorname{trdeg} Q_{obs}(p)$. Therefore Σ' is minimal. $\square$

Theorem 4.41 (Canonicity and birational equivalence). *Let Σ and Σ' be canonical rational realizations of the same response map p with the same input-space U and the same output-space $\mathbb{R}^r$. Then Σ and Σ' are birationally equivalent.*

Proof. Let $\Sigma = (X, f, h, x_0)$ and $\Sigma' = (X', f', h', x_0')$ be canonical rational realizations of the same response map p with the same input-space U and the same output-space $\mathbb{R}^r$. By Proposition 4.14, the maps $\widehat{\tau^*} : \mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}) \to Q_{obs}(p)$ and $\widehat{\tau'^*} : \mathcal{Q}(A_{obs}(\Sigma')/\text{Ker } \tau'^*_{ext}) \to Q_{obs}(p)$ are field isomorphisms.

Consider a map $\Psi : A_{obs}(\Sigma) \to A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}$ defined as $\Psi(\varphi) = [\varphi]$ for all $\varphi \in A_{obs}(\Sigma)$. It is a surjective homomorphism. Because Σ is algebraically reachable, it follows from the proof of Proposition 4.28 that Ker τ^*_{ext} is a zero ideal in $A_{obs}(\Sigma)$. Therefore the map Ψ is also injective, and thus Ψ is an isomorphism. We extend the isomorphism $\Psi : A_{obs}(\Sigma) \to A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}$ to the field isomorphism which we denote, by the abuse of notation, as $\Psi : Q_{obs}(\Sigma) \to \mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext})$. Since $A_{obs}(\Sigma)$ and $A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}$ are integral domains, the fields $Q_{obs}(\Sigma) = \mathcal{Q}(A_{obs}(\Sigma))$ and $\mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext})$ are well-defined. Because Σ is rationally observable, the domain $Q_{obs}(\Sigma)$ of the field isomorphism Ψ equals Q. Thus, $\Psi : Q \to \mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext})$.

In the same way we derive the field isomorphism $\Psi' : Q' \to \mathcal{Q}(A_{obs}(\Sigma')/\text{Ker } \tau'^*_{ext})$ for the rational system Σ'.

Consider a field isomorphism $i : Q' \to Q$ defined as $i = \Psi^{-1} \circ \widehat{\tau^*}^{-1} \circ \widehat{\tau'^*} \circ \Psi'$. We show that i satisfies the conditions of Definition 4.39 which proves that the rational realizations Σ and Σ' are birationally equivalent.

Since all isomorphisms, $\Psi', \widehat{\tau'^*}, \widehat{\tau^*}^{-1}, \Psi^{-1}$ are the identities on constant functions $\mathbb{R}$, the field isomorphism i is the identity on $\mathbb{R} \subset Q'$, and thus i satisfies Definition 4.39(i).

Because $\Psi'(h') = [h']$, and because $\tau'^*_{ext}(h') = p$ from Proposition 4.13 and thus $\widehat{\tau'^*}([h']) = p$ from Proposition 4.14, it follows that

$$i(h') = (\Psi^{-1} \circ \widehat{\tau^*}^{-1} \circ \widehat{\tau'^*} \circ \Psi')(h') = \Psi^{-1}(\widehat{\tau^*}^{-1}(\widehat{\tau'^*}([h']))) = \Psi^{-1}(\widehat{\tau^*}^{-1}(p)).$$

Since Ψ and $\widehat{\tau^*}$ are field isomorphisms such that $\Psi(h) = [h]$ and $\widehat{\tau^*}([h]) = p$, for their inverses Ψ^{-1} and $\widehat{\tau^*}^{-1}$ it holds that $\Psi^{-1}([h]) = h$ and $\widehat{\tau^*}^{-1}(p) = [h]$. Then,

$$i(h') = \Psi^{-1}(\widehat{\tau^*}^{-1}(p)) = \Psi^{-1}([h]) = h, \tag{4.11}$$

and the field isomorphism i satisfies Definition 4.39(ii).

Since $Q' = Q_{obs}(\Sigma') = \mathcal{Q}(A_{obs}(\Sigma'))$, every element $\varphi \in Q'$ can be written as a rational combination of finitely many elements of the set $\{h', f'_\alpha h' \mid \alpha = (\alpha_1, \ldots, \alpha_i) \in U^i; i \in \mathbb{N}\}$. Consider an arbitrary $\varphi \in Q'$. There exist $i \in \mathbb{N}$, and $\varphi_{num}, \varphi_{den} \in \mathbb{R}[X_1, \ldots, X_i]$ such that $\varphi = \frac{\varphi_{num}(f'_{\alpha_1}h', \ldots, f'_{\alpha_i}h')}{\varphi_{den}(f'_{\alpha_1}h', \ldots, f'_{\alpha_i}h')}$ where $\alpha_j = (\alpha_j^1, \ldots, \alpha_j^{k_j}) \in U^{k_j}$ for all $j = 1, \ldots, i$ and for $k_j \in \mathbb{N} \cup \{0\}$ (if $k_j = 0$ then $f'_{\alpha_j}h' = h'$), and $\varphi_{den}(f'_{\alpha_1}h', \ldots, f'_{\alpha_i}h') \neq 0$. According to the proofs of Proposition 4.13 and Proposition 4.14, the field isomorphism $\widehat{\tau'^*} \circ \Psi' : Q' \to Q_{obs}(p)$ satisfies for any $\alpha = (\alpha_1, \ldots, \alpha_i) \in U^i$, $i \in \mathbb{N}$ the equality $(\widehat{\tau'^*} \circ \Psi')(f'_\alpha h') = D_\alpha p$. Hence, for $\varphi = \frac{\varphi_{num}(f'_{\alpha_1}h',\dots,f'_{\alpha_i}h')}{\varphi_{den}(f'_{\alpha_1}h',\dots,f'_{\alpha_i}h')} \in Q'$ it holds that

$$(\widehat{\tau'^*} \circ \Psi')(\varphi) = \frac{\varphi_{num}(D_{\alpha_1}p,\dots,D_{\alpha_i}p)}{\varphi_{den}(D_{\alpha_1}p,\dots,D_{\alpha_i}p)}.$$

By the same token, the field isomorphism $\Psi^{-1} \circ \widehat{\tau^*}^{-1} : Q_{obs}(p) \to Q$ satisfies for any $\alpha = (\alpha_1,\dots,\alpha_i) \in U^i$, $i \in \mathbb{N}$ the equality $(\Psi^{-1} \circ \widehat{\tau^*}^{-1})(D_\alpha p) = f_\alpha h$, and consequently

$$(\Psi^{-1} \circ \widehat{\tau^*}^{-1}) \left(\frac{\varphi_{num}(D_{\alpha_1}p,\dots,D_{\alpha_i}p)}{\varphi_{den}(D_{\alpha_1}p,\dots,D_{\alpha_i}p)} \right) = \frac{\varphi_{num}(f_{\alpha_1}h,\dots,f_{\alpha_i}h)}{\varphi_{den}(f_{\alpha_1}h,\dots,f_{\alpha_i}h)}.$$

Therefore, for any $\alpha \in U$,

$$\begin{aligned}
f_\alpha(i(\varphi)) &= f_\alpha \left((\Psi^{-1} \circ \widehat{\tau^*}^{-1} \circ \widehat{\tau'^*} \circ \Psi') \left(\frac{\varphi_{num}(f'_{\alpha_1}h',\dots,f'_{\alpha_i}h')}{\varphi_{den}(f'_{\alpha_1}h',\dots,f'_{\alpha_i}h')} \right) \right) \\
&= f_\alpha \left((\Psi^{-1} \circ \widehat{\tau^*}^{-1}) \left((\widehat{\tau'^*} \circ \Psi') \left(\frac{\varphi_{num}(f'_{\alpha_1}h',\dots,f'_{\alpha_i}h')}{\varphi_{den}(f'_{\alpha_1}h',\dots,f'_{\alpha_i}h')} \right) \right) \right) \\
&= f_\alpha \left((\Psi^{-1} \circ \widehat{\tau^*}^{-1}) \left(\frac{\varphi_{num}(D_{\alpha_1}p,\dots,D_{\alpha_i}p)}{\varphi_{den}(D_{\alpha_1}p,\dots,D_{\alpha_i}p)} \right) \right) \\
&= f_\alpha \left(\frac{\varphi_{num}(f_{\alpha_1}h,\dots,f_{\alpha_i}h)}{\varphi_{den}(f_{\alpha_1}h,\dots,f_{\alpha_i}h)} \right) \\
&= (\Psi^{-1} \circ \widehat{\tau^*}^{-1}) \left(D_\alpha \frac{\varphi_{num}(D_{\alpha_1}p,\dots,D_{\alpha_i}p)}{\varphi_{den}(D_{\alpha_1}p,\dots,D_{\alpha_i}p)} \right) \\
&= (\Psi^{-1} \circ \widehat{\tau^*}^{-1} \circ \widehat{\tau'^*} \circ \Psi') \left(f_\alpha \frac{\varphi_{num}(f'_{\alpha_1}h',\dots,f'_{\alpha_i}h')}{\varphi_{den}(f'_{\alpha_1}h',\dots,f'_{\alpha_i}h')} \right) = i(f_\alpha \varphi),
\end{aligned} \tag{4.12}$$

and the field isomorphism i satisfies Definition 4.39(iii).

Recall from Problem 4.8 that the sets $\mathscr{U}_{pc}(\Sigma)$ and $\mathscr{U}_{pc}(\Sigma')$ of admissible inputs for rational realizations Σ and Σ', respectively, both contain a set $\widetilde{\mathscr{U}_{pc}}$ of admissible inputs with the values in U. Therefore, h, f_α are defined at x_0, and h', f'_α are defined at x'_0, for all $\alpha \in U$. Because Σ and Σ' are rationally observable, to prove that $(i(\varphi))(x_0) = \varphi(x'_0)$ for all $\varphi \in Q'$ such that φ is defined at x'_0 and $i(\varphi)$ is defined at x_0 it is sufficient to prove that $(i(\varphi))(x_0) = \varphi(x'_0)$ for all $\varphi \in \{h', f'_\alpha h' \mid \alpha = (\alpha_1,\dots,\alpha_i) \in U^i; i \in \mathbb{N}\}$. From (4.11), $i(h') = h$, and then $(i(h'))(x_0) = h(x_0)$. Since Σ and Σ' realize the same map, $h(x_0) = h'(x'_0)$. Therefore, $(i(h'))(x_0) = h'(x'_0)$. Consider any $\alpha = (\alpha_1,\dots,\alpha_i) \in U^i$ with $i \in \mathbb{N}$. From (4.11) and (4.12), $i(f'_\alpha h') = f_\alpha(i(h')) = f_\alpha h$. Because Σ and Σ' realize the same map, it follows that $(f'_\alpha h')(x'_0) = (f_\alpha h)(x_0)$. Therefore, $(i(f'_\alpha h'))(x_0) = (f_\alpha h)(x_0) = (f'_\alpha h')(x'_0)$ which implies that i satisfies Definition 4.39(iv).

Finally it follows that the canonical rational realizations Σ and Σ' are birationally equivalent. □

Example 4.42. Consider the rational realization Σ of the map p derived in Example 4.20 and studied further in Example 4.37. Hence, $\Sigma = (X, f, h, x_0)$ where $X = \mathbb{R}^2$, $f = \{f_\alpha \mid \alpha \in \mathbb{R}\}$ with $f_\alpha = \alpha \frac{x_1}{x_2^2} \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$, $h = x_1$, and $x_0 = (1,1)$. In this example we derive a rational realization $\Sigma' = (X', f' = \{f'_\alpha \mid \alpha \in \mathbb{R}\}, h', x'_0)$ of the same map p which is birationally equivalent to Σ.

We define the state-space of Σ' as the unit sphere in $\mathbb{R}^3$, i.e. $X' = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 - 1 = 0\}$. The irreducible varieties X and X' are birationally equivalent since the stereographic projection $\Psi : X' \to X$ given as $\Psi(x,y,z) = (\frac{x}{1-z}, \frac{y}{1-z})$ and its inverse $\Phi : X \to X'$ given as $\Phi(x_1,x_2) = (\frac{2x_1}{x_1^2+x_2^2+1}, \frac{2x_2}{x_1^2+x_2^2+1}, \frac{x_1^2+x_2^2-1}{x_1^2+x_2^2+1})$ are the rational mappings such that $\Psi \circ \Phi = 1_X$ and $\Phi \circ \Psi = 1_{X'}$ on a Z-dense subsets of X and X', respectively. We derive the rational vector fields $f'_\alpha, \alpha \in \mathbb{R}$ on X' from the rational vector fields $f_\alpha, \alpha \in \mathbb{R}$ on X by following Definition 4.38. In particular, from Definition 4.38(iii) we obtain that

$$\begin{aligned} f_\alpha\left(\varphi\left(\frac{2x_1}{x_1^2+x_2^2+1}, \frac{2x_2}{x_1^2+x_2^2+1}, \frac{x_1^2+x_2^2-1}{x_1^2+x_2^2+1}\right)\right) &= \left(f'_\alpha \varphi(x,y,z)\right) \circ \Phi \\ &= \left(f'_{\alpha,1}\frac{\partial\varphi(x,y,z)}{\partial x} + f'_{\alpha,2}\frac{\partial\varphi(x,y,z)}{\partial y} + f'_{\alpha,3}\frac{\partial\varphi(x,y,z)}{\partial z}\right) \circ \Phi \quad (4.13) \end{aligned}$$

for all $\varphi \in Q'$, $\alpha \in \mathbb{R}$. Note that $f'_\alpha = f'_{\alpha,1}\frac{\partial}{\partial x} + f'_{\alpha,2}\frac{\partial}{\partial y} + f'_{\alpha,3}\frac{\partial}{\partial z}$. By considering the polynomial $\varphi(x,y,z) = x$ in (4.13) we derive that

$$f'_{\alpha,1} \circ \Phi = f_\alpha\left(\frac{2x_1}{x_1^2+x_2^2+1}\right) = \frac{2\alpha x_1(x_2^2 - x_1^2 + 1) - 4x_1x_2^3}{x_2^2(x_1^2+x_2^2+1)^2}$$

and thus

$$f'_{\alpha,1} = (1-z)^2\frac{(1-z)2\alpha x(y^2 - x^2 + (1-z)^2) - 4xy^3}{y^2(x^2+y^2+(1-z)^2)^2}.$$

Further, by considering the polynomials $\varphi(x,y,z) = y$ and $\varphi(x,y,z) = z$ in (4.13) we derive that $f'_{\alpha,2} \circ \Phi = \frac{2x_2(x_1^2-x_2^2+1)-4\alpha x_1^2}{x_2(x_1^2+x_2^2+1)^2}$ and $f'_{\alpha,3} \circ \Phi = \frac{4}{(x_1^2+x_2^2+1)^2}\left(\frac{\alpha x_1^2}{x_2^2} + x_2\right)$, respectively. Therefore,

$$f'_{\alpha,2} = (1-z)^2\frac{2y(x^2 - y^2 + (1-z)^2) - 4\alpha x^2(1-z)}{y(x^2+y^2+(1-z)^2)^2}$$

and

$$f'_{\alpha,3} = \frac{4(1-z)^4}{(x^2+y^2+(1-z)^2)^2}\left(\frac{\alpha x^2}{y^2} + \frac{y}{1-z}\right).$$

To sum up, the vector fields $f'_\alpha = (1-z)^2\frac{(1-z)2\alpha x(y^2-x^2+(1-z)^2)-4xy^3}{y^2(x^2+y^2+(1-z)^2)^2}\frac{\partial}{\partial x} + (1-z)^2\frac{2y(x^2-y^2+(1-z)^2)-4\alpha x^2(1-z)}{y(x^2+y^2+(1-z)^2)^2}\frac{\partial}{\partial y} + \frac{4(1-z)^4}{(x^2+y^2+(1-z)^2)^2}\left(\frac{\alpha x^2}{y^2} + \frac{y}{1-z}\right)\frac{\partial}{\partial z}$ for $\alpha \in \mathbb{R}$ are the rational vector fields on X' defining the dynamics of Σ'. Finally, from Definition 4.38(ii),(iv), the output function is $h'(x,y,z) = \frac{x}{1-z}$, and the initial state is $x_0' = (2/3, 2/3, 1/3)$.

4.8 Conclusions

In the preceding sections we provided sufficient and necessary conditions for a response map to be realizable by a rational system. We proved that for a given response map the problems of the existence of a rational, rationally observable rational, canonical rational, and minimal rational realization are equivalent. Canonical rational realizations were shown to be unique up to birational equivalence. Further, we proved that every rational realization of a response map which is birationally equivalent to a minimal rational realization of the same map is itself minimal. We derived relations between algebraic reachability, rational observability, and minimality of rational realizations. In particular, we proved that canonical realizations are minimal and that, under certain algebraic condition, minimal realizations are canonical, and realizations which are not rationally observable are not minimal. By assuming the negation of this algebraic condition, we showed that rational realizations which are algebraically reachable are also minimal. These results can be applied to problems of model reduction, system identification, and control and observer design.

4.8.1 Algorithms

To apply the results of realization theory of rational systems to systems biology and to engineering we need procedures to check the properties of rational realizations and to transform rational realizations to minimal rational realizations. For example, in systems biology one may be provided with a rational system realizing a response map which models concentration of glucose in a cell. It is not obvious whether this realization is algebraically reachable and/or rationally observable because of the modeling assumptions. However, it is important to be able to decide whether a system has these properties since they imply minimality of the realization and thus, arguably, allow for easier computations.

The algebraic framework we use is useful from the computational point of view. It allows us to formulate the procedures to check rational observability, algebraic reachability, and minimality of rational realizations, and the procedures for the construction of rational realizations having these properties in algebraic terms. This implies the possibility of using available computer algebra packages to solve the problems of realization theory for rational systems, even though the algorithms themselves have not been implemented yet. For more details on computational algebra and existing computer algebra systems, see for example [44, 13, 25, 26, 101, 45, 127, 71, 106, 31, 16, 75, 29, 107]. Many computer algebra packages can be found also in Maple, Mathematica, and Matlab.

The procedures for checking properties of rational realizations can be based on the verification of the definitions of the corresponding properties or on the propositions which characterize those properties. By following Definition 3.21 and Definition 3.19 we derive a *procedure for checking rational observability* of a rational system $\Sigma = (X, f, h, x_0)$ as follows:

1. Compute the observation algebra $A_{obs}(\Sigma)$ of a rational system Σ.
2. Compute the observation field $Q_{obs}(\Sigma)$ as a field of fractions of $A_{obs}(\Sigma)$.
3. Let B be a set of generators of the algebra of polynomials on X, i.e. a set of generators of $\mathbb{R}[X_1, \ldots, X_n]/I$ where I is the ideal of polynomials vanishing on X. Check whether $B \subseteq Q_{obs}(\Sigma)$.
4. If $B \subseteq Q_{obs}(\Sigma)$ then $Q_{obs}(\Sigma) = Q$, where Q is the field of rational functions on X, and the rational system is rationally observable. If $B \supsetneq Q_{obs}(\Sigma)$ then the system is not rationally observable.

The fact that the observation field $Q_{obs}(\Sigma)$ is finitely generated, see Proposition 3.20, and that a set B of generators of the algebra of polynomials on X is a finite set could simplify the computations for the second and the third step of the procedure. The third step of the procedure above could be executed element-wise. The algorithms for checking whether an element of B (and therefore an element of the field Q of rational functions on X) is also an element of the field $Q_{obs}(\Sigma)$ are described in [76] and [77].

According to Definition 4.21, a *procedure for checking algebraic reachability* of a rational realization $\Sigma = (X, f, h, x_0)$ of a response map $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$ has to consist of the steps for the construction of the set of points of X which are reachable from x_0 by applying only admissible inputs $\widetilde{\mathscr{U}_{pc}} \subseteq \mathscr{U}_{pc}(\Sigma)$ to the system Σ, and of the steps for checking whether this set is Z-dense in X. We can also derive a procedure for checking algebraic reachability of a rational realization from Proposition 4.34. This procedure needs to check whether the realization is minimal and whether the field Q of rational functions on X is such that the elements of $Q \setminus Q_{obs}(\Sigma)$ are not algebraic over $Q_{obs}(\Sigma)$.

We can base a *procedure for checking minimality* of a rational realization $\Sigma = (X, f, h, x_0)$ of a given response map p, for example, on Theorem 4.30 or on Proposition 4.28. In the latter case the procedure consists only of checking algebraic reachability and rational observability of Σ. In the former case we obtain the following steps of the procedure:

1. Compute dim X of the irreducible real affine variety X as the degree of the affine Hilbert polynomial of the corresponding ideal (ideal generated by the polynomials defining the variety X).
2. Compute the observation field $Q_{obs}(p)$ of the map p.
3. Compute the transcendence degree of the field $Q_{obs}(p)$.
4. If $\dim X = \operatorname{trdeg} Q_{obs}(p)$ then the rational realization Σ of p is minimal. Otherwise Σ is not minimal.

The first step of the procedure above is already implemented, for example, in Maple (see the command “HilbertDimension”). See also [127] and the references therein for computing the dimension of a variety. The algorithms for computing the transcendence degree of field extensions of a field are presented in [76] and (of not necessarily purely transcendental field extensions) in [77]. There are other algorithms for solving the same problem which can be found in the references therein. These algorithms can be used for computing the transcendence degree of an observation field of a response map since an observation field is a field extension of $\mathbb{R}$.

The proof of Proposition 4.17 provides a *procedure for constructing a rational realization* of a response map. The proof of Theorem 4.24 provides a *procedure for constructing a canonical rational realization*, and thus, by Proposition 4.28, a *procedure for constructing a minimal rational realization*, from a rationally observable rational realization. Moreover it shows that the procedure for constructing a rational realization from Proposition 4.17 gives as a result a rationally observable rational realization if we consider as a field F from Proposition 4.17 the observation field of a response map to be realized.

To specify details of these procedures and/or to develop new procedures further research is needed.

There are also other approaches to realization theory of nonlinear systems which provide the possibility of developing efficient procedures and algorithms to solve the realization problems. One of them is the differential-algebraic approach. See for example [39, 41, 40, 73] and the references therein for the application of differential algebra in control and system theory. An introduction to differential algebra, developed by Ritt [89] and by Kolchin [63], is provided by [61].

4.8.2 Further research

We restricted our attention only to the rational realizations with their state-spaces defined as irreducible real affine varieties. The generalization of the framework and of the corresponding approach to the realization theory for rational systems with the state-spaces defined as reducible varieties might be possible. Further research in this direction is left for the future. Further, due to the applications in real-life problems of biology and engineering, we have chosen to work with the field $\mathbb{R}$ of real numbers. Because algebraic geometry is not limited only to the field of real numbers, our results could be generalized to different fields. From the computational point of view, computable fields such as the field of rational numbers could be of interest.

Recall that the proofs of the existence of a rational realization for a given response map and of a rational realization which is even rationally observable are constructive. In both cases, the state-spaces of constructed rational realizations are taken to be $\mathbb{R}^n$ for $n \in \mathbb{N}$ which depends on the given response map. Hence, for any response map the observation field of which has a finite transcendence degree n there is a rational realization and a rational realization which is rationally observable with the state-space $\mathbb{R}^n$. The question arises whether this is true also for algebraically reachable, canonical, and minimal rational realizations. Moreover, smoothness, rationality, and other geometric properties of the possible state-spaces of rational realizations are of interest. The rationality of varieties which are the state-spaces of polynomial systems is considered in [41].

From Theorem 2.19, two irreducible real affine varieties are birationally equivalent if and only if their corresponding fields of rational functions are isomorphic. Therefore, better insight to the characterization of birational equivalence classes of rational realizations can be given by the study of field isomorphisms. The birational equivalence classes of rational realizations and rationality of their state-spaces are related. According to [52] every birational equivalence class of varieties over a field of characteristics 0, e.g. $\mathbb{R}$, contains at least one smooth projective variety. If one of these projective varieties equals a projective n-space then all state-spaces of rational realizations from the considered birational equivalence class are rational. For more details on rational varieties and their birational equivalence see [128, 64, 65, 2, 72, 1], for more details on field isomorphisms see [108, 82].

Chapter 5
System Identification

Since analysis and simulation of various phenomena usually require the availability of their fully specified models, one needs to be able to estimate unknown parameter values of the models. In this chapter we deal with identifiability of parametrizations which is the property of one-to-one correspondence of parameter values and the outputs of the models. We derive necessary and sufficient conditions for the parametrizations of polynomial and rational systems to be structurally or globally identifiable. The results are applied to investigate the identifiability properties of the systems modeling certain biological phenomena.

Within this chapter we use the notation and terminology introduced in Section 2.1 and in Section 2.2.

5.1 Introduction

System identification is a research topic which deals with the problem of determining systems as realistic models of observed phenomena. The identifiability problem considered in this chapter is one of several problems appearing in system identification. Solution to this problem provides information whether the parameters of a parametrized system can be determined uniquely.

Let us introduce the system identification procedure, for an overview see for example [113, 67]. To model a certain (biological) phenomenon one first deals with modeling issues such as choosing the model structure, experimental design, and data collection. Then the phenomenon is characterized by the collected data (usually in the form of time series) and a system modeling these measurements is proposed. This system is usually not fully specified. It contains parameters the values of which have to be estimated to get a fully specified model. The parameter values can be determined uniquely only if the parametrization is identifiable. Since identifiability is a structural property, it is more efficient to check identifiability of a selected model structure prior to designing an experiment and collecting the measurements rather than to check it afterwards (even though this is possible, too). Only then is it meaningful to continue to estimate the numerical values of the parameters. Consequently one validates the final system modeling the data. Until a desired system is obtained, one repeats this procedure for different experimental designs, different model structures, and different methods of parameter estimation.

The problems of system identification have been studied for the classes of stochastic and deterministic, linear and nonlinear, discrete-time and continuous-time systems, and for time-invariant and time-dependent (time-varying) parameters.

5.1.1 Identifiability

In this chapter, we restrict our attention to the problem of identifiability for deterministic continuous-time parametrized systems whose dynamics is given by polynomial or rational vector fields and whose output function is componentwise given by a polynomial or a rational function of state variables and parameters. There are many approaches to study identifiability of parametrized systems, for example the approach based on power series expansions of outputs [87], differential algebra [73], generating series approach [120], and similarity transformation method [109]. Our approach, which is related to similarity transformation or state isomorphism approach [21, 22, 84, 109, 111, 112, 121], strongly relies on the results of realization theory for polynomial and rational systems presented in Chapter 4 (for rational systems) and in [11] (for polynomial systems).

Many concepts of identifiability are present in the literature. We consider the problem of structural and global identifiability of parametrizations of parametrized polynomial and parametrized rational systems. The first paper introducing the concept of structural identifiability in system theoretic framework is [14]. Structural and algebraic identifiability for a class of nonlinear systems is studied also in [129]. These authors work in a linear algebraic setting which is related to the differential-algebraic approach. The nonlinear systems they consider also include the classes of polynomial and rational systems. Other papers dealing with identifiability of polynomial and rational systems, but without inputs, are [28, 33]. Structural identifiability of the models described by input-output relations (differential-algebraic expression) rather than by state-space forms is studied in differential-algebraic setting in [73]. For an application of this approach to a real-life problem see [43].

Let us point out the main differences between the concepts of identifiability in this chapter and in [129]. First of all, the concept of structural identifiability in [129] is stronger than the concept we consider in this chapter. They define a parametrized system to be structurally identifiable if its outputs corresponding to two different parameter values differ for all inputs of an open dense subset of the set of all admissible inputs. For the concept of structural identifiability considered in this chapter it is sufficient if there exists at least one such input. Because algebraic identifiability implies structural identifiability of [129], see [129, Corollary 1], it implies also structural identifiability considered in this chapter. In general, the converse implication does not hold, see [129, Theorem 3]. Note that the notion of algebraic identifiability is suited for computing the parameter values from inputs and outputs by solving algebraic equations rather than for checking uniqueness of parameter values. Another difference between the concept of structural identifiability in this chapter and the concept used in [129] is the considered class of inputs. In [129] the inputs are considered to be continuously differentiable up to a certain order while we deal with piecewise-constant inputs. Using the terminology of [104] we can formulate the following statement. If, for the systems considered in [129], piecewise-constant inputs are universal inputs which are generic in the class of smooth inputs then our concept of structural identifiability and the concept of structural identifiability considered in [129] are identical. In [104] it is proven that, for analytic systems, analytic inputs are universal and generic in the class of smooth inputs. The proof also implies that piecewise-constant inputs are universal in the class of smooth inputs. But it is not at all clear why piecewise-constant inputs would have to be generic in the class of smooth inputs.

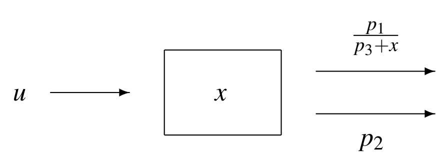
5.1.2 Outline of the chapter

In Section 5.2 we define the concepts of parametrized and structured systems. The problem of global and structural identifiability of the parametrizations of parametrized polynomial and parametrized rational systems is solved in Section 5.3. Section 5.4 contains three examples illustrating the application of the obtained results to check structural identifiability. In the last section we summarize the results presented in this chapter and discuss directions for further research.

5.2 Parametrized and structured systems

By choosing a model structure in the modeling step of the system identification procedure we specify a system which usually contains unknown parameters. Depending on the modeling techniques, the parameters may have a physical or a biological meaning relevant for further investigation of the studied phenomenon. In this section we introduce the concept of parametrized and structured systems within the classes of polynomial and rational systems.

Example 5.1. Let us consider a one-compartment model studied in [74, Example 5] and in [23]. The model is specified by the diagram below.



Here x denotes the concentration of a metabolite observed in a reaction system. We assume that x takes values in $X = \mathbb{R}$ even if it is only meaningful for $[0,\infty)$. The concentration decreases correspondingly to the rates $\frac{p_1}{p_3+x}$ and p_2 of the reactions which are modeled by Michaelis-Menten and mass-action kinetics, respectively. The concentration increase is influenced by the system inflow u. Therefore, the dynamics of this model can be described in state-space form as

$$\dot{x} = -\frac{p_1 x}{p_3 + x} - p_2 x + u, \tag{5.1}$$
$$x(0) = a,\ a \in \mathbb{R} \text{ is a known initial value of the concentration.}$$

The observed (measured) concentration x is taken as the output of the system. Hence, the output function h is considered to be

$$h(x) = x. \tag{5.2}$$

Because the inflow u to the system can be modeled as a piecewise-constant function, we can represent the considered one-compartment model in the framework introduced in Chapter 3. Then the dynamics of the model is given by the family of vector fields $f_\alpha = (-\frac{p_1 x}{p_3+x} - p_2 x + \alpha)\frac{\partial}{\partial x}, \alpha \in \mathbb{R}$ on X, where α corresponds to all possible values of the input u.

The variables p_1, p_2, p_3, which correspond to unknown kinetic constants, are the parameters with their values varying within $\mathbb{R}$. The parameter values have to be determined to get a fully specified model. We call the system given by the dynamics (5.1) and the output function (5.2) with the unknown parameters p_1, p_2, p_3 a parametrized system.

We assume that parameters take values in a set $P \subseteq \mathbb{R}^l$, $l \in \mathbb{N}$ which is an irreducible real affine variety. We refer to such a set P as to a *parameter set*. Then a *parametrized system* is a family of systems such that there is a one-to-one correspondence between the systems of the family and the parameter values of the parameter set. Let us define parametrized systems formally.

Let $P \subseteq \mathbb{R}^l$ be a parameter set determined by an ideal $I^P \subseteq \mathbb{R}[P_1,\dots,P_l]$ such that $I^P = (f_1^P,\dots,f_{n_P}^P)$ for $f_i^P \in \mathbb{R}[P_1,\dots P_l]$, $i = 1,\dots,n_P < +\infty$. Consider an irreducible real affine variety $X \subseteq \mathbb{R}^n$ determined by an ideal $I \subseteq \mathbb{R}[X_1,\dots,X_n]$ where $I = (f_1^X,\dots,f_{n_X}^X)$ for $f_i^X \in \mathbb{R}[X_1,\dots X_n]$, $i = 1,\dots,n_X < +\infty$. Because both varieties P and X can be considered varieties in $\mathbb{R}^{n+l}$ and because the union of two affine varieties is an affine variety, it follows that $X \cup P \subseteq \mathbb{R}^{n+l}$ is the variety determined by the ideal $I^{X\cup P} = (\{f_i^X f_j^P \mid 1 \le i \le n_x; 1 \le j \le n_P\})$. We call the elements of $A^{X\cup P} \cong \mathbb{R}[X_1,\dots,X_n,P_1,\dots,P_l]/I^{X\cup P}$ the *parametrized polynomials on X with the parameter values in P*. Hence, a parametrized polynomial on X with the parameter values in P is a map $w : X \cup P \to \mathbb{R}$ for which there exists a polynomial $w' \in \mathbb{R}[X_1,\dots,X_n,P_1,\dots,P_l]$ such that $w = w'$ on $X \cup P$.

The parametrized polynomials corresponding to a parameter value $p = (p_1,\dots,p_l) \in P$ are the elements of $A^p \cong \mathbb{R}[X_1,\dots,X_n,p_1,\dots,p_l]/I$, i.e. the elements of $A^{X\cup P}$ evaluated for $P_1 = p_1,\dots,P_l = p_l$. We will consider the situations when for different values of parameters we have even different varieties X. We express the dependence of the varieties X on the parameters $p \in P$ by using the notation X^p. Note that X^p does not depend on p explicitly and that it is still allowed that for different $p, p' \in P$ the varieties X^p and $X^{p'}$ are the same. Every irreducible real affine variety $X^p \subseteq \mathbb{R}^{n_p}, p \in P$ is determined by the ideal $I^p \subseteq \mathbb{R}[X_1, \ldots, X_{n_p}]$. Then, the *parametrized polynomials on* X^p, where $p = (p_1, \ldots, p_l) \in P$, are the elements of $A^p \cong \mathbb{R}[X_1, \ldots, X_{n_p}, p_1, \ldots, p_l]/I^p$. Further, the *parametrized rational functions* on X^p, denoted by Q^p, are the elements of the quotient field of A^p. Since A^p is an integral domain, Q^p is well-defined.

Definition 5.2 (Parametrized systems). Let $P \subseteq \mathbb{R}^l$ be a parameter set. By a *parametrized polynomial (rational) system* $\Sigma(P)$ we mean a family $\{\Sigma(p) = (X^p, f^p, h^p, x_0^p) \mid p \in P\}$ of polynomial (rational) systems where, according to Definition 3.4,

(i) $X^p \subseteq \mathbb{R}^{n_p}$ is an irreducible real affine variety,
(ii) $f^p = \{f_\alpha^p \mid \alpha \in U\}$ is a family of polynomial (rational) vector fields f_α^p, i.e. $f_\alpha^p = \sum_{i=1}^{n_p} f_{\alpha,i}^p \frac{\partial}{\partial X_i}$ with $f_{\alpha,i}^p \in A^p$ ($f_{\alpha,i}^p \in Q^p$) for all $\alpha \in U$ and for all $i = 1, \ldots, n_p$,
(iii) $h^p : X^p \to \mathbb{R}^r$ is an output map with the components $h_j^p \in A^p$ ($h_j^p \in Q^p$), $j = 1, \ldots, r$,
(iv) $x_0^p = (x_{0,1}^p, \ldots, x_{0,n_p}^p) \in X^p$ (for a parametrized rational system we assume that all components of h^p and at least one of $f_\alpha^p, \alpha \in U$ are defined at x_0^p).

We assume that the systems $\Sigma(p), p \in P$ have the same input-spaces U and the same output-spaces $\mathbb{R}^r$. The map $\mathscr{P} : P \to \Sigma(P)$ defined as $\mathscr{P}(p) = \Sigma(p)$ for $p \in P$ is called the *parametrization* of $\Sigma(P)$.

Example 5.3. Consider the model presented in Example 5.1. In that case the parameter set P is considered to be $\mathbb{R}^3$ and the parametrized system given by (5.1), (5.2) is a parametrized system $\Sigma(P)$ which is the set of rational systems $\Sigma(p) = (X^p, f^p, h^p, x_0^p)$, $p \in P$ such that

$$\begin{aligned}
X^p &= \mathbb{R}, \\
f_\alpha^p &= (-\frac{p_1 x}{p_3 + x} - p_2 x + \alpha)\frac{\partial}{\partial x}, \alpha \in \mathbb{R}, \\
h^p &= x, \\
x_0^p &= a,\ a \in \mathbb{R} \text{ is known.}
\end{aligned}$$

Let $P \subseteq \mathbb{R}^l$ be a parameter set determined by the ideal $I^P \subseteq \mathbb{R}[P_1, \ldots, P_l]$. The polynomials on P are denoted by $A^P \cong \mathbb{R}[P_1, \ldots, P_l]/I^P$. Let $\Sigma(P)$ be a parametrized polynomial system and let $\Sigma(p) = (X^p \subseteq \mathbb{R}^{n_p}, f^p = \{f_\alpha^p = \sum_{i=1}^{n_p} f_{\alpha,i}^p \frac{\partial}{\partial X_i} \mid \alpha \in U\}, h^p, x_0^p) \in \Sigma(P)$ be a polynomial system determined by a parameter value $p = (p_1, \ldots, p_l) \in P$. Let $\varphi_1, \ldots, \varphi_{n_p} \in A^p$ be the polynomials on X^p corresponding to the polynomials $X_1, \ldots, X_{n_p} \in \mathbb{R}[X_1, \ldots, X_{n_p}]$. Then $\mathbb{R}[\varphi_1, \ldots, \varphi_{n_p}] = A^p$, and the elements $f_{\alpha,i}^p, h_j^p \in A^p$ for $i = 1, \ldots, n_p, j = 1, \ldots, r$, $\alpha \in U$ and the components $x_{0,i}^p$ of $x_0^p \in X^p$ can be written in the form:

$$f^p_{\alpha,i} = \sum_{a_1,\dots,a_{n_p}\in\mathbb{N}\cup\{0\}} q^{f_\alpha}_{i;a_1,\dots,a_{n_p}}(p_1,\dots,p_l)\varphi_1^{a_1}\cdots\varphi_{n_p}^{a_{n_p}}, \tag{5.3}$$

$$h^p_j = \sum_{b_1,\dots,b_{n_p}\in\mathbb{N}\cup\{0\}} q^{h}_{j;b_1,\dots,b_{n_p}}(p_1,\dots,p_l)\varphi_1^{b_1}\cdots\varphi_{n_p}^{b_{n_p}}, \tag{5.4}$$

$$x^p_{0,i} = q^{x_0}_i(p_1,\dots,p_l), \tag{5.5}$$

where $q^{f_\alpha}_{i;a_1,\dots,a_{n_p}}, q^h_{j;b_1,\dots,b_{n_p}}, q^{x_0}_i \in \mathbb{R}[P_1,\dots,P_l]$ are such that for every $i=1,\dots,n_p$ only finitely many $(a_1,\dots,a_{n_p})\in(\mathbb{N}\cup\{0\})^{n_p}$ are such that $q^{f_\alpha}_{i;a_1,\dots,a_{n_p}}(p_1,\dots,p_l)\neq 0$ and for every $j=1,\dots,r$ only finitely many $(b_1,\dots,b_{n_p})\in(\mathbb{N}\cup\{0\})^{n_p}$ are such that $q^h_{j;b_1,\dots,b_{n_p}}(p_1,\dots,p_l)\neq 0$.

Definition 5.4. Let $P\subseteq\mathbb{R}^l$ be a parameter set and let $\Sigma(P)$ be a parametrized polynomial system. We say that $\Sigma(p) = (X^p\subseteq\mathbb{R}^{n_p}, f^p = \{f^p_\alpha = \sum_{i=1}^{n_p} f^p_{\alpha,i}\frac{\partial}{\partial X_i} \mid \alpha\in U\}, h^p, x^p_0)\in\Sigma(P)$ *distinguishes parameters* if the polynomials $q^{f_\alpha}_{i;a_1,\dots,a_{n_p}}$, $q^h_{j;b_1,\dots,b_{n_p}}$, $q^{x_0}_i\in\mathbb{R}[P_1,\dots,P_l]$, $i=1,\dots,n_p$, $j=1,\dots,r$, $\alpha\in U$ determined by the equalities (5.3), (5.4), and (5.5), respectively, distinguish the points of the variety P, i.e. if $\mathbb{R}[\{q^{f_\alpha}_{i;a_1,\dots,a_{n_p}}, q^h_{j;b_1,\dots,b_{n_p}}, q^{x_0}_i \mid i=1,\dots,n_p; j=1,\dots,r; \alpha\in U\}]\cong A^P$. If $\Sigma(p)$ distinguishes parameters for all $p\in P$ then we say that $\Sigma(P)$ *distinguishes parameters*.

Example 5.5. Let $\Sigma(P)=\{\Sigma(p)=(\mathbb{R}, f^p=\{f^p_\alpha=(p_1^2-\alpha p_2^2x)\frac{\partial}{\partial x}\mid\alpha\in\mathbb{R}\}, h^p = 1_{\mathbb{R}}, x^p_0=1)\mid p=(p_1,p_2)\in P=\mathbb{R}^2\}$ be a parametrized polynomial system. Consider a system $\Sigma(p)\in\Sigma(P)$ determined by a parameter value $p=(p_1,p_2)\in P=\mathbb{R}^2$. The only non-zero polynomials of $\{q^{f_\alpha}_{1;a_1}, q^h_{1;b_1}, q^{x_0}_1 \mid a_1,b_1\in\mathbb{N}\cup\{0\}\}$ derived by (5.3), (5.4), (5.5) for $\Sigma(p)$ are

$$\begin{aligned} q^{f_\alpha}_{1;0}(p_1,p_2) &= p_1^2,\ (\text{if } p_1\neq 0), & q^h_{1;1}(p_1,p_2) &= 1,\\ q^{f_\alpha}_{1;1}(p_1,p_2) &= -\alpha p_2^2,\ (\text{if } p_2\neq 0), & q^{x_0}_1(p_1,p_2) &= 1. \end{aligned}$$

Then, if $p_1,p_2\neq 0$, $\mathbb{R}[q^{f_\alpha}_{1;0}, q^{f_\alpha}_{1;1}, q^h_{1;1}, q^{x_0}_1] = \mathbb{R}[P_1^2,P_2^2]\subsetneq\mathbb{R}[P_1,P_2]$. If $p_1=0$ or $p_2=0$ we derive that $\mathbb{R}[q^{f_\alpha}_{1;0}, q^{f_\alpha}_{1;1}, q^h_{1;1}, q^{x_0}_1] = \mathbb{R}[P_2^2]\subsetneq\mathbb{R}[P_1,P_2]$ or $\mathbb{R}[q^{f_\alpha}_{1;0}, q^{f_\alpha}_{1;1}, q^h_{1;1}, q^{x_0}_1] = \mathbb{R}[P_1^2]\subsetneq\mathbb{R}[P_1,P_2]$, respectively. Because $P=\mathbb{R}^2$ and thus $A^P=\mathbb{R}[P_1,P_2]$, the system $\Sigma(p)$ does not distinguish parameters for any $p\in P$. Finally, the parametrized system $\Sigma(P)$ does not distinguish parameters.

The set of rational functions on P, defined as the field of fractions of A^P, is denoted by Q^P. Let $\Sigma(P)$ be a parametrized rational system and let $\Sigma(p) = (X^p\subseteq\mathbb{R}^{n_p}, f^p=\{f^p_\alpha=\sum_{i=1}^{n_p} f^p_{\alpha,i}\frac{\partial}{\partial X_i}\mid\alpha\in U\}, h^p, x^p_0)\in\Sigma(P)$ be a rational system determined by a parameter value $p=(p_1,\dots,p_l)\in P$. For $f^p_{\alpha,i}, h^p_j\in Q^p$, $i=1,\dots,n_p$, $j=1,\dots,r$, $\alpha\in U$ there exist $f^p_{\alpha,i,num}, f^p_{\alpha,i,den}, h^p_{j,num}, h^p_{j,den}\in A^p$ such that $f^p_{\alpha,i} = \frac{f^p_{\alpha,i,num}}{f^p_{\alpha,i,den}}$

and $h^p_j = \frac{h^p_{j,num}}{h^p_{j,den}}$. Let $\varphi_1, \dots, \varphi_{n_p} \in A^p$ be the polynomials on X^p corresponding to the polynomials $X_1, \dots, X_{n_p} \in \mathbb{R}[X_1, \dots, X_{n_p}]$. Then the elements $f^p_{\alpha,i}, h^p_j \in Q^p$ for $i = 1, \dots, n_p$, $j = 1, \dots, r$, $\alpha \in U$, and the components $x^p_{0,i}$ of $x^p_0 \in X^p$ can be written in the form:

$$f^p_{\alpha,i} = \frac{f^p_{\alpha,i,num}}{f^p_{\alpha,i,den}} = \frac{\sum_{a^1_1,\dots,a^1_{n_p} \in \mathbb{N}\cup\{0\}} q^{f\alpha}_{1,i;a^1_1,\dots,a^1_{n_p}}(p_1,\dots,p_l)\varphi_1^{a^1_1}\cdots\varphi_{n_p}^{a^1_{n_p}}}{\sum_{a^2_1,\dots,a^2_{n_p} \in \mathbb{N}\cup\{0\}} q^{f\alpha}_{2,i;a^2_1,\dots,a^2_{n_p}}(p_1,\dots,p_l)\varphi_1^{a^2_1}\cdots\varphi_{n_p}^{a^2_{n_p}}}, \quad (5.6)$$

$$h^p_j = \frac{h^p_{j,num}}{h^p_{j,den}} = \frac{\sum_{b^1_1,\dots,b^1_{n_p} \in \mathbb{N}\cup\{0\}} q^{h}_{1,j;b^1_1,\dots,b^1_{n_p}}(p_1,\dots,p_l)\varphi_1^{b^1_1}\cdots\varphi_{n_p}^{b^1_{n_p}}}{\sum_{b^2_1,\dots,b^2_{n_p} \in \mathbb{N}\cup\{0\}} q^{h}_{2,j;b^2_1,\dots,b^2_{n_p}}(p_1,\dots,p_l)\varphi_1^{b^2_1}\cdots\varphi_{n_p}^{b^2_{n_p}}}, \quad (5.7)$$

$$x^p_{0,i} = q^{x_0}_i(p_1,\dots,p_l), \quad (5.8)$$

where $q^{f\alpha}_{1,i;a^1_1,\dots,a^1_{n_p}}, q^{f\alpha}_{2,i;a^2_1,\dots,a^2_{n_p}}, q^{h}_{1,j;b^1_1,\dots,b^1_{n_p}}, q^{h}_{2,j;b^2_1,\dots,b^2_{n_p}}, q^{x_0}_i \in \mathbb{R}(P_1,\dots,P_l)$ and all sums in (5.6) and (5.7) have only finitely many non-zero summands.

Definition 5.6. Let $P \subseteq \mathbb{R}^l$ be a parameter set and let $\Sigma(P)$ be a parametrized rational system. We say that a system $\Sigma(p) = (X^p \subseteq \mathbb{R}^{n_p}, f^p = \{f^p_\alpha = \sum_{i=1}^{n_p} f^p_{\alpha,i}\frac{\partial}{\partial X_i} \mid \alpha \in U\}, h^p, x^p_0) \in \Sigma(P)$ *distinguishes parameters* if the rational functions $q^{f\alpha}_{1,i;a^1_1,\dots,a^1_{n_p}}$, $q^{f\alpha}_{2,i;a^2_1,\dots,a^2_{n_p}}, q^{h}_{1,j;b^1_1,\dots,b^1_{n_p}}, q^{h}_{2,j;b^2_1,\dots,b^2_{n_p}}, q^{x_0}_i \in \mathbb{R}(P_1,\dots,P_l)$, $i = 1,\dots,n_p$, $j = 1,\dots,r$, $\alpha \in U$ determined by the equalities (5.6), (5.7), (5.8) distinguish the points of the variety P, i.e. if $\mathbb{R}(\{q^{f\alpha}_{1,i;a^1_1,\dots,a^1_{n_p}}, q^{f\alpha}_{2,i;a^2_1,\dots,a^2_{n_p}}, q^{h}_{1,j;b^1_1,\dots,b^1_{n_p}}, q^{h}_{2,j;b^2_1,\dots,b^2_{n_p}}, q^{x_0}_i \mid i = 1,\dots,n_p; j = 1,\dots,r; \alpha \in U\}) \cong Q^P$. If $\Sigma(p)$ distinguishes parameters for all $p \in P$ for which $\Sigma(p)$ is well-defined then we say that $\Sigma(P)$ *distinguishes parameters*.

Example 5.7. In this example we show that the parametrized rational system considered in Examples 5.1 and 5.3 does not distinguish parameters.

Recall that the parameter set P equals $\mathbb{R}^3$ and that the parametrized rational system $\Sigma(P)$ is the set of the systems $\Sigma(p) = (X^p, f^p, h^p, x^p_0)$, $p \in P$ given as:

$$\begin{aligned}
X^p &= \mathbb{R},\\
f^p_\alpha &= \left(-\frac{p_1 x}{p_3 + x} - p_2 x + \alpha\right)\frac{\partial}{\partial x}, \alpha \in \mathbb{R},\\
h^p &= x,\\
x^p_0 &= a,\ a \in \mathbb{R} \text{ is known.}
\end{aligned}$$

Because $f^p_\alpha = (-\frac{p_1 x}{p_3+x} - p_2 x + \alpha)\frac{\partial}{\partial x} = \frac{\alpha p_3 + (\alpha - p_1 - p_2 p_3)x - p_2 x^2}{p_3 + x}\frac{\partial}{\partial x}$ and $h^p = \frac{x}{1}$, it follows that if we write $f^p_{\alpha,1} = f^p_\alpha$ and $h^p_1 = h^p$ in the form of (5.6) and (5.7), respectively, then for $p = (p_1, p_2, p_3) \in P$ the polynomials $q^{f\alpha}_{1,1;0}(p) = \alpha p_3$, $q^{f\alpha}_{1,1;1}(p) =$

$\alpha - p_1 - p_2p_3$, $q^{f_\alpha}_{1,1;2}(p) = -p_2$, $q^{f_\alpha}_{2,1;0}(p) = p_3$, $q^{f_\alpha}_{2,1;1}(p) = 1$, $q^{h}_{1,1;1}(p) = 1$, and $q^{h}_{2,1;0}(p) = 1$ are the only non-zero elements of $\mathbb{R}(P_1,P_2,P_3)$ appearing in the formulas. Since $\mathbb{R}(q^{f_\alpha}_{1,1;1}, q^{f_\alpha}_{1,1;2}, q^{f_\alpha}_{2,1;0}) = \mathbb{R}(P_1,P_2,P_3) = Q^P$ only if $p_2, p_3, p_1 + p_2p_3 \neq 0$, it follows that the system $\Sigma(p)$ distinguishes parameters only for $p \in P \setminus \{p = (p_1, p_2, p_3) \in P \mid p_2p_3(p_1 + p_2p_3) = 0\}$. Note that the set $P \setminus \{p = (p_1, p_2, p_3) \in P \mid p_3 = -a\}$ contains all parameters of P for which $\Sigma(p)$ is well-defined. Therefore, according to Definition 5.6, $\Sigma(P)$ does not distinguish parameters.

To introduce the concept of structured systems we use the notions of polynomial and rational mappings between irreducible varieties and the notions of isomorphic and birationally equivalent irreducible varieties, see Chapter 2. According to the definitions of isomorphic rational systems (Definition 4.38) and of isomorphic polynomial systems ([11]) we introduce the concept of isomorphic systems for parametrized polynomial and parametrized rational systems.

Definition 5.8. Let P be a parameter set and let $\Sigma(P)$ be a parametrized polynomial (rational) system. Recall that all systems $\Sigma(p) \in \Sigma(P)$ have the same input-spaces U and the same output-spaces $\mathbb{R}^r$. Let $\Sigma(p) = (X^p, f^p = \{f^p_\alpha \mid \alpha \in U\}, h^p, x^p_0)$ and $\Sigma(p') = (X^{p'}, f^{p'} = \{f^{p'}_\alpha \mid \alpha \in U\}, h^{p'}, x^{p'}_0)$ be any two systems of $\Sigma(P)$. We say that $\Sigma(p)$ and $\Sigma(p')$ are *isomorphic* if

(i) the state-spaces X^p and $X^{p'}$ are isomorphic (birationally equivalent), i.e. there exist polynomial (rational) mappings $\phi : X^p \to X^{p'}$, $\psi : X^{p'} \to X^p$ such that $\phi \circ \psi = 1_{X^{p'}}$ and $\psi \circ \phi = 1_{X^p}$,
(ii) $\forall \varphi \in A^{p'}(Q^{p'}) \forall \alpha \in U : f^p_\alpha(\varphi \circ \phi) = (f^{p'}_\alpha \varphi) \circ \phi$,
(iii) $h^{p'} \circ \phi = h^p$,
(iv) ϕ is defined at x^p_0, and $\phi(x^p_0) = x^{p'}_0$.

We call ϕ an *isomorphism*.

Definition 5.9 (Structured systems). Let P be a parameter set. We say that a parametrized polynomial (rational) system $\Sigma(P)$ is a *structured system* if for all $\Sigma(p), \Sigma(p') \in \Sigma(P)$ the state-spaces $X^p, X^{p'}$ are isomorphic (birationally equivalent) and thus there exist polynomial (rational) mappings $\phi : X^p \to X^{p'}$, $\psi : X^{p'} \to X^p$ such that $\phi \circ \psi = 1_{X^{p'}}$, $\psi \circ \phi = 1_{X^p}$, and if we symbolically identify $p_1 = p'_1, \ldots, p_l = p'_l$ then the conditions (ii)-(iv) of Definition 5.8 are satisfied. Namely,

(i) $f^p_\alpha(\varphi \circ \phi) = (f^{p'}_\alpha \varphi) \circ \phi$ for all $\alpha \in U$ and for all $\varphi \in A^{p'}(Q^{p'})$,
(ii) $h^p_j = h^{p'}_j \circ \phi$ for all $j = 1, \ldots, r$,
(iii) ϕ is defined at x^p_0, and $\phi(x^p_0) = x^{p'}_0$.

A structured system is a parametrized system such that after symbolic identification of parameters ($p_1 = p'_1, \ldots, p_l = p'_l$) the systems of this structured system, i.e. the systems derived as the evaluations of the structured system for all parameter values, are isomorphic.

Note that allowing different state-spaces for the systems of a structured system is different from the linear case [114] where it is natural to assume that all state-spaces of a structured linear system are the same.

Consider a parametrized polynomial (rational) system $\Sigma(P)$ and let us assume that the state-spaces of all systems of $\Sigma(P)$ are the same. Then all state-spaces are automatically isomorphic (birationally equivalent) and the corresponding isomorphisms and their inverses are the identity maps on particular state-spaces. Therefore, the parametrized system $\Sigma(P)$ is a structured system if after symbolic identification $p_1 = p_1', \ldots, p_l = p_l'$ of the parameters it holds that

(i) $f_\alpha^p = f_\alpha^{p'}$ for all $\alpha \in U$,
(ii) $h_j^p = h_j^{p'}$ for all $j = 1, \ldots, r$,
(iii) $x_{0,i}^p = x_{0,i}^{p'}$ for $i = 1, \ldots, n_p = n_{p'}$.

Any structured system is a parametrized system. The following example proves that not every parametrized system is structured.

Example 5.10. Consider a parameter set $P = \mathbb{R}^3$ and consider a parametrized polynomial system $\Sigma(P)$ such that the systems of $\Sigma(P)$ are given for different parameter values in the following way:

1. for $p = (p_1, p_2, p_3) \in P_1 = \{(p_1, p_2, p_3) \in \mathbb{R}^3 \mid p_1^2 + p_2^2 + p_3^2 = 1; p_3 \geq 0\}$ the system $\Sigma(p) = (X^p, f^p, h^p, x_0^p)$ is given as
$$\begin{aligned} X^p &= \mathbb{R}, \\ f_\alpha^p &= (p_1 - p_2 x^2)\frac{\partial}{\partial x}, \\ h^p &= p_3 x, \\ x_0^p &= p_1, \end{aligned}$$
2. for $p = (p_1, p_2, p_3) \in P_2 = \{(p_1, p_2, p_3) \in \mathbb{R}^3 \mid p_1^2 + p_2^2 + p_3^2 = 1; p_3 < 0\}$ the system $\Sigma(p) = (X^p, f^p, h^p, x_0^p)$ is given as
$$\begin{aligned} X^p &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 - x_2 = 0\}, \\ f_\alpha^p &= (p_1 - p_2 x_2^2)\frac{\partial}{\partial x_1} + (p_1 - p_2 x_1^2)\frac{\partial}{\partial x_2}, \\ h^p &= \frac{p_3}{2}x_1 + \frac{p_3}{2}x_2, \\ x_0^p &= (x_{0,1}^p, x_{0,2}^p) = (p_1, p_1), \end{aligned}$$
3. for $p = (p_1, p_2, p_3) \in P_3 = P \setminus (P_1 \cup P_2)$ the system $\Sigma(p) = (X^p, f^p, h^p, x_0^p)$ is given as
$$\begin{aligned} X^p &= \mathbb{R}, \\ f_\alpha^p &= (p_1 - p_2 x^2)\frac{\partial}{\partial x}, \end{aligned}$$

$$h^p = p_3 x,$$
$$x_0^p = p_2.$$

Because the systems corresponding to the parameter values of P_1 and P_3 differ only in the definition of their initial states so that these do not coincide by identifying the parameter values of P_1 and P_3, we derive that the parametrized system $\Sigma(P) = \{\Sigma(p) \mid p \in \mathbb{R}^3\}$ is not a structured system.

Because the polynomial mappings $\phi : \mathbb{R} \to \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 - x_2 = 0\}$ defined as $\phi(x) = (x,x)$ for $x \in \mathbb{R}$, and $\psi : \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 - x_2 = 0\} \to \mathbb{R}$ defined as $\psi(x_1,x_2) = x_1$ are such that $\phi \circ \psi = 1_{\{(x_1,x_2) \in \mathbb{R}^2 | x_1 - x_2 = 0\}}$ and $\psi \circ \phi = 1_{\mathbb{R}}$, the varieties $\mathbb{R}$ and $\{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 - x_2 = 0\}$ are isomorphic. Moreover, ϕ is such that for every polynomial φ on $\{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 - x_2 = 0\}$ it holds that

$$(p_1 - p_2 x^2)\frac{\partial}{\partial x}(\varphi \circ \phi) = ((p_1 - p_2 x_2^2)\frac{\partial}{\partial x_1}\varphi + (p_1 - p_2 x_1^2)\frac{\partial}{\partial x_2}\varphi) \circ \phi,$$
$$p_3 x = \frac{p_3}{2}x + \frac{p_3}{2}x,$$
$$\phi(p_1) = (p_1, p_1).$$

Therefore, the parametrized system $\Sigma(P_1 \cup P_2) = \{\Sigma(p) \mid p \in P_1 \cup P_2 = \{(p_1, p_2, p_3) \in \mathbb{R}^3 \mid p_1^2 + p_2^2 + p_3^2 = 1\}\}$ is a structured system.

To derive the results of the subsequent sections we introduce the structural concepts of algebraic and rational observability (Definition 3.21), algebraic reachability (Definition 4.21), canonicity (Definition 4.23), and distinguishability of parameters (Definition 5.4 and Definition 5.6).

Definition 5.11. Let P be a parameter set. We say that a parametrized polynomial (rational) system $\Sigma(P)$

(i) is *structurally reachable* if there exists a variety $R \subsetneq P$ such that $\Sigma(p)$ is algebraically reachable (from x_0^p) for all $p \in P \setminus R$,
(ii) is *structurally observable* if there exists a variety $O \subsetneq P$ such that $\Sigma(p)$ is algebraically observable (rationally observable for rational systems) for all $p \in P \setminus O$,
(iii) is *structurally canonical* if there exists a variety $RO \subsetneq P$ such that $\Sigma(p)$ is canonical for all $p \in P \setminus RO$,
(iv) *structurally distinguishes parameters* if there exists a variety $D \subsetneq P$ such that $\Sigma(p)$ distinguishes parameters for all $p \in P \setminus D$.

Proposition 5.12. *Let P be a parameter set. A parametrized polynomial (rational) system $\Sigma(P)$ is structurally canonical if and only if it is structurally reachable and structurally observable.*

Proof. ($\Rightarrow$) A parametrized polynomial (rational) system which is structurally canonical is also structurally reachable and structurally observable. We can consider varieties R and O to be equal to a variety RO which is given by structural canonicity.

($\Leftarrow$) Assume that $\Sigma(P)$ is structurally reachable and structurally observable. There exist varieties $R, O \subsetneq P$ such that $\Sigma(p)$ is algebraically reachable (from x_0^p) for all $p \in P \setminus R$, and $\Sigma(p)$ is algebraically (rationally if $\Sigma(P)$ is a parametrized rational system) observable for all $p \in P \setminus O$. We define a variety RO to be the union of the varieties R and O. Because the union of two varieties is a variety, $RO = R \cup O$ is a variety. Since $P \setminus RO \subseteq P \setminus R$ and $P \setminus RO \subseteq P \setminus O$, the system $\Sigma(p)$ for $p \in P \setminus RO$ is both algebraically (rationally) observable and algebraically reachable (from x_0^p). If $RO = P$ then the variety P would be the union of two non-empty strict subvarieties of P, i.e. P would be reducible. But this contradicts the irreducibility of P. Therefore $RO \subsetneq P$ and thus $\Sigma(P)$ is structurally canonical. $\square$

Corollary 5.13. *Let $\Sigma(P)$ be a parametrized polynomial (rational) system. Let R and O be the smallest strict subvarieties of P such that $\Sigma(p)$ is algebraically reachable for all $p \in P \setminus R$ and algebraically observable (rationally observable for $\Sigma(P)$ being parametrized rational system) for all $p \in P \setminus O$. Then a variety RO for which $\Sigma(p)$ is canonical for all $p \in P \setminus RO$ is such that $R \cup O \subseteq RO \subsetneq P$.*

Example 5.14. Consider the parametrized system $\Sigma(P)$ from Examples 5.1, 5.3, and 5.7. For this parametrized system it holds that it is structurally canonical and that it structurally distinguishes parameters.

For every rational system $\Sigma(p)$, $p \in P = \mathbb{R}^3$ of the parametrized system $\Sigma(P)$ it holds that $h^p(x) = x$ and $X^p = \mathbb{R}$. Then, from Definition 3.21, it follows that $Q_{obs}(\Sigma(p)) = \mathbb{R}(X) = Q^p$ and therefore $\Sigma(p)$ is rationally observable for all $p \in P$. Thereby $\Sigma(P)$ is structurally observable.

For the vector fields f_α^p to be defined at the initial state $x_0^p = a \in \mathbb{R}$, we need to assume that $p_3 \neq -a$. According to Cauchy-Peano theorem on the existence of solutions of ordinary differential equations, for every $p \in P \setminus \{p = (p_1, p_2, p_3) \in P \mid p_3 = -a\}$ and for every $\alpha \in \mathbb{R}$ there is a solution $x(\cdot)$ of $\dot{x}(t) = -\frac{p_1 x(t)}{p_3 + x(t)} - p_2 x(t) + \alpha$, $x(0) = a$ defined on a non-empty open interval $(-T_\alpha, T_\alpha)$. Because for every $p \in P \setminus \{p = (p_1, p_2, p_3) \in P \mid p_3 = -a\}$ there exists $\alpha \in \mathbb{R}$ such that $\dot{x} \neq 0$, there is an open interval $(a-b, a+b) \subseteq \mathbb{R}$, $b \in (0, \infty)$ which is a subset of the set $\{x(t) \mid t \in (-T_\alpha, T_\alpha)\}$. Therefore, for every $p \in P \setminus \{p = (p_1, p_2, p_3) \in P \mid p_3 = -a\}$, the reachable set of the rational system $\Sigma(p)$ contains a non-empty open interval in $\mathbb{R}$. Then, because a non-empty open subset in $\mathbb{R}$ is Z-dense in $\mathbb{R}$, $\Sigma(p)$ is algebraically reachable for every $p \in P \setminus \{p = (p_1, p_2, p_3) \in P \mid p_3 = -a\}$. Since $\{p = (p_1, p_2, p_3) \in P \mid p_3 = -a\}$ is a strict subvariety of P, $\Sigma(P)$ is structurally reachable.

The last two paragraphs above and Proposition 5.12 imply that $\Sigma(P)$ is structurally canonical.

Further, because the set $\{p = (p_1, p_2, p_3) \in \mathbb{R}^3 \mid p_2 p_3 (p_1 + p_2 p_3) = 0\}$ is a strict subvariety of P, it follows from Example 5.7 and Definition 5.11(iv) that $\Sigma(P)$ structurally distinguishes parameters.

Example 5.15. An example of a parametrized system which does not structurally distinguish parameters is the parametrized polynomial system $\Sigma(P) = \{\Sigma(p) =$

$(\mathbb{R}, f^p = \{f^p_\alpha = (p_1^2 - \alpha p_2^2 x)\frac{\partial}{\partial x} \mid \alpha \in \mathbb{R}\}, h^p = 1_{\mathbb{R}}, x_0^p = 1) \mid p = (p_1, p_2) \in P = \mathbb{R}^2\}$. In Example 5.5 we showed that for all $p \in P$ the polynomial systems $\Sigma(p) \in \Sigma(P)$ do not distinguish parameters. Therefore, there does not exist a variety $D \subsetneq P$ such that $\Sigma(p)$ distinguishes parameters for all $p \in P \setminus D$.

5.3 Structural and global identifiability

To understand a (biological) phenomenon one observes its behavior and applies prior knowledge of related fields. The observation of the phenomenon consists in measuring the responses (outputs) of the studied object to stimulating signals (inputs). For example, in a metabolic network one can measure or even change the concentration of glucose input to a reaction system and observe its influence on the change of the concentration of pyruvate. These measurements are usually of the form of a set of tuples (u, y) where u and y are the functions of time with the same time-domain which record the inputs (in the case of u) and outputs (in the case of y) measured for the considered phenomenon. Hence, u can stand for the glucose concentration and y for the pyruvate concentration.

Consider an input $u : [0, T_u] \to \mathbb{R}^m$ and its corresponding output $y : [0, T_u] \to \mathbb{R}^r$. The tuple (u, y) provides the same information about the phenomenon as the set of tuples $\{(u_{[0,t]}, y(t)) \mid t \in [0, T_u]\}$ where $u_{[0,t]}$ is the restriction of the input u to the time-domain $[0,t]$ and $y(t)$ is the value of the output y at the time t which is the end point of the time-domain of the input $u_{[0,t]}$.

We assume that the inputs which can be applied to study the considered phenomenon are given as a set of admissible inputs $\widetilde{\mathscr{U}_{pc}}$ (for polynomial or rational systems depending on the model of the phenomenon). A set $\widetilde{\mathscr{U}_{pc}}$ of *admissible inputs for rational systems* is defined in Definition 4.1 as a subset of $\mathscr{U}_{pc}$ such that:

(i) $\forall u \in \widetilde{\mathscr{U}_{pc}}\ \forall t \in [0, T_u] : u_{[0,t]} \in \widetilde{\mathscr{U}_{pc}}$,
(ii) $\forall u \in \widetilde{\mathscr{U}_{pc}}\ \forall \alpha \in U\ \exists t > 0 : (u)(\alpha, t) \in \widetilde{\mathscr{U}_{pc}}$,
(iii) $\forall u = (\alpha_1, t_1)\cdots(\alpha_k, t_k) \in \widetilde{\mathscr{U}_{pc}}\ \exists \delta > 0\ \forall \overline{t_i} \in [0, t_i + \delta], i = 1, \ldots, k :$

$$\overline{u} = (\alpha_1, \overline{t_1})\cdots(\alpha_k, \overline{t_k}) \in \widetilde{\mathscr{U}_{pc}}.$$

A set of *admissible inputs for polynomial systems* is according to [11] a set of inputs of $\mathscr{U}_{pc}$ which satisfies the conditions (i) and (ii) above. Abusing the notation, we denote it also by $\widetilde{\mathscr{U}_{pc}}$.

Because all restrictions of admissible inputs to shorter time-domains are also admissible inputs, we can further assume that the measurements are provided in the form $(u, y(T_u))$ where $u \in \widetilde{\mathscr{U}_{pc}}$. Then a parametrized system $\Sigma(P)$ which models a phenomenon characterized by the measurements $\{(u, y(T_u)) \mid u \in \widetilde{\mathscr{U}_{pc}}\}$ is such that $\widetilde{\mathscr{U}_{pc}} \subseteq \mathscr{U}_{pc}(\Sigma(p))$ for all $p \in P$, otherwise $\Sigma(P)$ would not be a model of the phenomenon. This property only says that all possible inputs which are meaningful for a real-life phenomenon we investigate should be admissible for the models of this phenomenon.

The assumption on inputs being of a class of admissible inputs allows us to study structural and global identifiability of parametrizations of parametrized polynomial and parametrized rational systems by means of realization theory developed in [11] for polynomial systems and in Chapter 4 for rational systems. Note that parametrized systems are families of realizations of the measurements. Then, because algebraic reachability of polynomial/rational realizations is defined by using the inputs of a set $\widetilde{\mathscr{U}_{pc}}$, see for example Definition 4.21, in the rest of this chapter we mean by an algebraically reachable system a system $\Sigma = (X, f, h, x_0)$ realizing the measurements $\{(u, y(T_u)) \mid u \in \widetilde{\mathscr{U}_{pc}}\}$ such that Z-$cl(\mathscr{R}(x_0)) = Z$-$cl(\{x(T_u; x_0, u) \in X \mid u \in \widetilde{\mathscr{U}_{pc}}\}) = X$. The main results of realization theory which are applied to obtain the characterization of structural and global identifiability are the results formulated in [11, Theorem 4] and in Theorem 4.41. The following theorem states these results in the framework of parametrized systems.

Theorem 5.16. *Let $\Sigma(P)$ be a parametrized polynomial or a parametrized rational system. Any two canonical systems $\Sigma(p), \Sigma(p') \in \Sigma(P)$ are isomorphic, see Definition 5.8.*

5.3.1 Problem description

The problem we treat in this chapter concerns the characterization of structural and global identifiability for the classes of parametrized polynomial and parametrized rational systems. The notion of structural and global identifiability is formally defined as follows:

Definition 5.17. Let $P \subseteq \mathbb{R}^l$ be a parameter set and let $\widetilde{\mathscr{U}_{pc}}$ be a set of admissible inputs for polynomial (rational) systems. Let $\Sigma(P)$ be a parametrized polynomial (rational) system such that $\widetilde{\mathscr{U}_{pc}} \subseteq \mathscr{U}_{pc}(\Sigma(p))$ for all $p \in P$. We say that the parametrization $\mathscr{P} : P \to \Sigma(P)$ is

(i) *globally identifiable* if the map $p \mapsto \{(u, h^p(x^p(T_u; x_0^p, u))) \mid u \in \widetilde{\mathscr{U}_{pc}}\}$ is injective on P,
(ii) *structurally identifiable* if the map $p \mapsto \{(u, h^p(x^p(T_u; x_0^p, u))) \mid u \in \widetilde{\mathscr{U}_{pc}}\}$ is injective on $P \setminus S$ where S is a variety strictly contained in P.

Global identifiability of a parametrization of a parametrized system means that unknown parameters of the parametrized system can be determined uniquely from the measurements. Structural identifiability of a parametrization provides this uniqueness only on a Z-dense subset of a parameter set. Obviously, a globally identifiable parametrization of a parametrized system is structurally identifiable.

5.3.2 Necessary conditions

In this section we specify necessary conditions for a parametrization of a parametrized polynomial or a parametrized rational system to be structurally or globally identifiable.

Theorem 5.18 (Necessary condition for structural identifiability). *Let $P \subseteq \mathbb{R}^l$ be a parameter set and let $\Sigma(P)$ be a parametrized polynomial (rational) system with the parametrization $\mathscr{P} : P \to \Sigma(P)$. We assume that $\Sigma(P)$ is structurally canonical and we denote by RO the smallest strict subvariety of P such that $\Sigma(p) \in \Sigma(P)$ is canonical for all $p \in P \setminus RO$. Then the following statement holds.*

If the parametrization $\mathscr{P}$ is structurally identifiable then there exists a variety S such that $RO \subseteq S \subsetneq P$ and such that for any $p, p' \in P \setminus S$ an isomorphism linking the systems $\Sigma(p), \Sigma(p') \in \Sigma(P)$ is the identity.

Proof. Assume that the parametrization $\mathscr{P}$ is structurally identifiable. Let G be a strict subvariety of P such that the map $p \mapsto \{(u, h^p(x^p(T_u; x_0^p, u))) \mid u \in \widetilde{\mathscr{U}_{pc}}\}$ is injective on $P \setminus G$. Because P is an irreducible variety and because an union of finitely many varieties is a variety, $RO \cup G \subsetneq P$. Let us define $S = RO \cup G$. Then $RO \subseteq S \subsetneq P$.

Consider arbitrary two polynomial (rational) systems $\Sigma(p), \Sigma(p') \in \Sigma(P)$ with $p, p' \in P \setminus S$. They are both canonical and realizing the same data. The realization property means that $h^p(x^p(T_u; x_0^p, u)) = h^{p'}(x^{p'}(T_u; x_0^{p'}, u))$ for all $u \in \widetilde{\mathscr{U}_{pc}}$. Since the map $p \mapsto \{(u, h^p(x^p(T_u; x_0^p, u))) \mid u \in \widetilde{\mathscr{U}_{pc}}\}$ is injective on $P \setminus G$ and hence also on $P \setminus S$, the equality $h^p(x^p(T_u; x_0^p, u)) = h^{p'}(x^{p'}(T_u; x_0^{p'}, u))$ for all $u \in \widetilde{\mathscr{U}_{pc}}$ implies that $p = p'$. From this equality, from Theorem 5.16, and from Definition 5.8 we derive that:

(i) there exist polynomial mappings $\phi, \psi : X^p \to X^p$ such that $\phi \circ \psi = \psi \circ \phi = 1_{X^p}$ (in the case of rational systems the mappings ϕ, ψ are rational; note that they do not have to be defined everywhere, only on Z-dense subsets of X^p),
(ii) $\forall \varphi \in A^p$ (for rational systems: $\forall \varphi \in Q^p$) $\forall \alpha \in U : (f_\alpha^p \varphi) \circ \phi = f_\alpha^p(\varphi \circ \phi)$,
(iii) $h_j^p = h_j^p \circ \phi$ for all $j = 1, \ldots, r$,
(iv) ϕ is defined at x_0^p, and $\phi(x_0^p) = x_0^p$.

Consider arbitrary isomorphism $\phi : X^p \to X^p$ satisfying the conditions above, i.e. ϕ is an isomorphism of the polynomial (rational) system $\Sigma(p)$ to itself.

We finish the proof for polynomial and rational systems separately to illustrate two different arguments.

Polynomial case

Let $\phi^* : A^p \to A^p$ be an $\mathbb{R}$-algebra isomorphism defined as $\phi^*(\varphi) = \varphi \circ \phi$ for all $\varphi \in A^p$. Because a canonical polynomial system is algebraically observable, the observation algebra $A_{obs}(\Sigma(p))$ of the polynomial system $\Sigma(p)$ equals the algebra A^p of all polynomials on X^p. Then the polynomials $h_j^p, f_\alpha^p h_j^p$ with $\alpha = (\alpha_1, \ldots, \alpha_n) \in U^n, n \in \mathbb{N}, j = 1, \ldots, r$ generate the algebra A^p. From (ii) and (iii) above, we get that

$$\phi^* h_j^p = h_j^p \text{ and } \phi^*(f_\alpha^p h_j^p) = f_\alpha^p h_j^p \text{ for } \alpha = (\alpha_1, \ldots, \alpha_n) \in U^n, n \in \mathbb{N}, j = 1, \ldots, r.$$

Since the isomorphism ϕ^* maps the generators of A^p to themselves identically, ϕ^* is the identity on A^p and therefore the isomorphism ϕ is the identity on X^p.

Rational case

From the canonicity of the rational system $\Sigma(p)$ it follows that it is rationally observable and thus $Q_{obs}(\Sigma(p)) = Q^p$. Because the field $Q_{obs}(\Sigma(p))$ is generated by the rational functions $h_j^p, f_\alpha^p h_j^p$ with $\alpha = (\alpha_1, \ldots, \alpha_n) \in U^n, n \in \mathbb{N}, j = 1, \ldots, r$, and because from (ii) and (iii) above $h_j^p \circ \phi = h_j^p, f_\alpha^p h_j^p = f_\alpha^p(h_j^p \circ \phi) = (f_\alpha^p h_j^p) \circ \phi$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in U^n, n \in \mathbb{N}, j = 1, \ldots, r$, we get that

$$\forall \varphi \in Q_{obs}(\Sigma(p)) = Q^p : \varphi = \varphi \circ \phi \text{ on } X^p.$$

Specifically for $\varphi \in Q^p$ defined as $\varphi(x) = x$ it means that $x = \varphi(x) = \varphi(\phi(x)) = \phi(x)$. Therefore, the isomorphism $\phi : X^p \to X^p$ is the identity. $\square$

For Theorem 5.18 the assumption on *RO* being the smallest variety having the desired property can be relaxed. The smallest variety *RO* satisfying the assumptions of Theorem 5.18 provides the minimal lower bound on a variety *S*, i.e. it specifies the smallest variety which can be considered a variety *S*.

Remark 5.19 (Necessary condition for global identifiability). From Theorem 5.18 we derive necessary conditions for a parametrization of a parametrized polynomial or a parametrized rational system to be globally identifiable.

Assume that the systems $\Sigma(p)$ of a parametrized polynomial system $\Sigma(P)$ are canonical for all $p \in P$. It holds that if the parametrization $\mathscr{P}$ of $\Sigma(P)$ is globally identifiable then for every $p, p' \in P$ an isomorphism linking $\Sigma(p)$ and $\Sigma(p')$ is the identity. For parametrized rational systems the same statement holds except for the fact that only the parameters of $P \setminus W$ are considered. Here W denotes a set of such parameters $p^W \in P$ for which the rational system $\Sigma(p^W)$ is not well-defined.

5.3.3 Sufficient conditions

In this section we determine sufficient conditions for a parametrization of a parametrized polynomial or a parametrized rational system to be structurally or globally identifiable.

Theorem 5.20 (Sufficient condition for structural identifiability). *Let $P \subseteq \mathbb{R}^l$ be a parameter set and let $\Sigma(P)$ be a structured polynomial (rational) system with the parametrization $\mathscr{P} : P \to \Sigma(P)$. We assume that $\Sigma(P)$ is structurally canonical and we denote by RO the smallest strict subvariety of P such that $\Sigma(p) \in \Sigma(P)$ is canonical for all $p \in P \setminus RO$. We also assume that $\Sigma(P)$ structurally distinguishes parameters and we denote by D the smallest strict subvariety of P such that $\Sigma(p) \in \Sigma(P)$ distinguishes parameters for all $p \in P \setminus D$. Then the following statement holds.*

If there exists a variety S such that $RO \cup D \subseteq S \subsetneq P$ and such that for any $p, p' \in P \setminus S$ an isomorphism relating the systems $\Sigma(p), \Sigma(p') \in \Sigma(P)$ is the identity then the parametrization $\mathscr{P}$ is structurally identifiable.

Proof. Let a structured polynomial (rational) system $\Sigma(P)$ and varieties $RO, D \subsetneq P$ be as in the theorem. Let S be a variety such that $RO \cup D \subseteq S \subsetneq P$ and such that for any $p, p' \in P \setminus S$ any isomorphism relating the systems $\Sigma(p), \Sigma(p') \in \Sigma(P)$ is the identity. Consider arbitrary $p, p' \in P \setminus S$ and the corresponding systems $\Sigma(p), \Sigma(p')$ of $\Sigma(P)$. It follows from Theorem 5.16 that $\Sigma(p)$ and $\Sigma(p')$ are isomorphic. Let $\phi : X^p \to X^{p'}$ be an isomorphism relating $\Sigma(p)$ and $\Sigma(p')$. Since $p, p' \in P \setminus S$, the assumption on S implies that ϕ is the identity.

Polynomial case

Since ϕ is the identity, the polynomial systems $\Sigma(p) = (X^p \subseteq \mathbb{R}^{n_p}, f^p = \{f^p_\alpha \mid \alpha \in U\}, h^p, x^p_0)$ and $\Sigma(p') = (X^{p'} \subseteq \mathbb{R}^{n_{p'}}, f^{p'} = \{f^{p'}_\alpha \mid \alpha \in U\}, h^{p'}, x^{p'}_0)$ are related, according to Definition 5.8, in the following way:

(i) $X^p = X^{p'}$, and then also $A^p = A^{p'}$,
(ii) $\forall \varphi \in A^p\ \forall \alpha \in U : f^{p'}_\alpha \varphi = f^p_\alpha \varphi$,
(iii) $h^p = h^{p'}$,
(iv) $x^p_0 = x^{p'}_0$.

Let $f^p_{\alpha,i}$, h^p_j, x^p_0 and $f^{p'}_{\alpha,i}$, $h^{p'}_j$, $x^{p'}_0$ for $i = 1, \ldots, n_p$, $j = 1, \ldots, r$, $\alpha \in U$ be written in the form of (5.3), (5.4), (5.5) where $\varphi_1, \ldots, \varphi_{n_p}$ are the common generators of A^p and $A^{p'}$ corresponding to the polynomials $X_1, \ldots, X_{n_p} \in \mathbb{R}[X_1, \ldots, X_{n_p}]$. Note that because $X^p = X^{p'}$ we can assume that $n_p = n_{p'}$. Then there exist polynomials $q^{f_\alpha}_{i;a_1,\ldots,a_{n_p}}, q^h_{j;b_1,\ldots,b_{n_p}}, q^{x_0}_i, q^{f'_\alpha}_{i;a_1,\ldots,a_{n_p}}, q^{h'}_{j;b_1,\ldots,b_{n_p}}, q^{x'_0}_i \in \mathbb{R}[P_1, \ldots, P_l], i = 1, \ldots, n_p, j = 1, \ldots, r, \alpha \in U$ such that for every $\alpha \in U$ and for $i = 1, \ldots, n_p, j = 1, \ldots, r$ it holds that

$$\begin{aligned} f^p_{\alpha,i} &= \sum_{a_1,\ldots,a_{n_p} \in \mathbb{N}\cup\{0\}} q^{f_\alpha}_{i;a_1,\ldots,a_{n_p}}(p_1, \ldots, p_l)\varphi_1^{a_1} \cdots \varphi_{n_p}^{a_{n_p}}, \\ h^p_j &= \sum_{b_1,\ldots,b_{n_p} \in \mathbb{N}\cup\{0\}} q^h_{j;b_1,\ldots,b_{n_p}}(p_1, \ldots, p_l)\varphi_1^{b_1} \cdots \varphi_{n_p}^{b_{n_p}}, \\ x^p_{0,i} &= q^{x_0}_i(p_1, \ldots, p_l), \end{aligned} \tag{5.9}$$

and

$$\begin{aligned} f^{p'}_{\alpha,i} &= \sum_{a_1,\ldots,a_{n_p} \in \mathbb{N}\cup\{0\}} q^{f'_\alpha}_{i;a_1,\ldots,a_{n_p}}(p'_1, \ldots, p'_l)\varphi_1^{a_1} \cdots \varphi_{n_p}^{a_{n_p}}, \\ h^{p'}_j &= \sum_{b_1,\ldots,b_{n_p} \in \mathbb{N}\cup\{0\}} q^{h'}_{j;b_1,\ldots,b_{n_p}}(p'_1, \ldots, p'_l)\varphi_1^{b_1} \cdots \varphi_{n_p}^{b_{n_p}}, \\ x^{p'}_{0,i} &= q^{x'_0}_i(p'_1, \ldots, p'_l). \end{aligned} \tag{5.10}$$

The sums in (5.9) and (5.10) have only finitely many non-zero summands. From (ii)-(iv) above and from (5.9), (5.10) it follows that

$$\begin{aligned} q^{f_\alpha}_{i;a_1,\ldots,a_{n_p}}(p_1,\ldots,p_l) &= q^{f'_\alpha}_{i;a_1,\ldots,a_{n_p}}(p'_1,\ldots,p'_l), \\ q^{h}_{j;b_1,\ldots,b_{n_p}}(p_1,\ldots,p_l) &= q^{h'}_{j;b_1,\ldots,b_{n_p}}(p'_1,\ldots,p'_l), \\ q^{x_0}_{i}(p_1,\ldots,p_l) &= q^{x'_0}_{i}(p'_1,\ldots,p'_l), \end{aligned} \tag{5.11}$$

for $i=1,\ldots,n_p,\ j=1,\ldots,r,\ \alpha\in U$. Because $\Sigma(P)$ is a structured system, Definition 5.9 implies that the relations (ii)-(iv) and consequently the equations (5.11) are valid even if we formally identify $p_1=p'_1,\ldots,p_l=p'_l$. Therefore $q^{f_\alpha}_{i;a_1,\ldots,a_{n_p}} = q^{f'_\alpha}_{i;a_1,\ldots,a_{n_p}}, q^{h}_{j;b_1,\ldots,b_{n_p}} = q^{h'}_{j;b_1,\ldots,b_{n_p}}, q^{x_0}_i = q^{x'_0}_i$ for $i=1,\ldots,n_p, j=1,\ldots,r$, and $\alpha\in U$. After substituting these relations into (5.11), we derive that

$$\begin{aligned} q^{f_\alpha}_{i;a_1,\ldots,a_{n_p}}(p_1,\ldots,p_l) &= q^{f_\alpha}_{i;a_1,\ldots,a_{n_p}}(p'_1,\ldots,p'_l), \\ q^{h}_{j;b_1,\ldots,b_{n_p}}(p_1,\ldots,p_l) &= q^{h}_{j;b_1,\ldots,b_{n_p}}(p'_1,\ldots,p'_l), \\ q^{x_0}_{i}(p_1,\ldots,p_l) &= q^{x_0}_{i}(p'_1,\ldots,p'_l), \end{aligned} \tag{5.12}$$

for $i=1,\ldots,n_p,\ j=1,\ldots,r,\ \alpha\in U$. Because $p,p'\in P\setminus S$, the polynomial systems $\Sigma(p)$ and $\Sigma(p')$ distinguish parameters. Thus, $\mathscr{A}=\mathbb{R}[\{q^{f_\alpha}_{i;a_1,\ldots,a_{n_p}}, q^{h}_{j;b_1,\ldots,b_{n_p}}, q^{x_0}_i \mid i=1,\ldots,n_p; j=1,\ldots,r;\alpha\in U\}]\cong A^P$ and it follows that

$$\big(\forall a\in\mathscr{A}: a(p_1,\ldots,p_l)=a(p'_1,\ldots,p'_l)\big)\Rightarrow p_1=p'_1,\ldots,p_l=p'_l.$$

Since the polynomials $\{q^{f_\alpha}_{i;a_1,\ldots,a_{n_p}}, q^{h}_{j;b_1,\ldots,b_{n_p}}, q^{x_0}_i \mid i=1,\ldots,n_p; j=1,\ldots,r;\alpha\in U\}$ generate $\mathscr{A}$, the equalities (5.12) imply that $p_1=p'_1,\ldots,p_l=p'_l$. Therefore $p=p'$ which proves that for the parameters of the set $P\setminus S$ the map $p\mapsto\{(u,h^p(x^p(T_u;x^p_0,u)))\mid u\in\widetilde{\mathscr{U}_{pc}}\}$ is injective. Thus, the parametrization $\mathscr{P}$ is structurally identifiable.

Rational case

Since ϕ is the identity, the rational systems $\Sigma(p)=(X^p\subseteq\mathbb{R}^{n_p}, f^p=\{f^p_\alpha\mid\alpha\in U\}, h^p, x^p_0)$ and $\Sigma(p')=(X^{p'}\subseteq\mathbb{R}^{n_{p'}}, f^{p'}=\{f^{p'}_\alpha\mid\alpha\in U\}, h^{p'}, x^{p'}_0)$ are related, according to Definition 5.8, in the following way:

(i) $X^p\cap X^{p'}$ is a Z-dense subset of both X^p and $X^{p'}$,
(ii) $\forall\varphi\in Q^{p'}\ \forall\alpha\in U: f^p_\alpha\varphi=f^{p'}_\alpha\varphi$,
(iii) $h^{p'}=h^p$,
(iv) $x^p_0=x^{p'}_0$.

As $\Sigma(p)$ and $\Sigma(p')$ are algebraically reachable, $Z\text{-}cl(\mathscr{R}(x^p_0))=X^p$ and $Z\text{-}cl(\mathscr{R}(x^{p'}_0)) = X^{p'}$. Moreover, by (ii) and (iv), the trajectories determining the reachable sets

$\mathscr{R}(x_0^p)$ and $\mathscr{R}(x_0^{p'})$ coincide. Therefore $Z\text{-}cl(\mathscr{R}(x_0^p)) = Z\text{-}cl(\mathscr{R}(x_0^{p'}))$, thus $X^p = X^{p'}$, and consequently $Q^p = Q^{p'}$.

By considering $f_{\alpha,i}^p$, h_j^p, x_0^p and $f_{\alpha,i}^{p'}$, $h_j^{p'}$, $x_0^{p'}$ for $i = 1,\dots,n_p = n_{p'}$, $j = 1,\dots,r$, $\alpha \in U$ written in the form of (5.6), (5.7), (5.8) and by following the steps of the part of this proof concerning polynomial systems, we derive that $p = p'$. Hence, the parametrization $\mathscr{P} : P \to \Sigma(P)$ is structurally identifiable. $\square$

Again, in the same way as for Theorem 5.18, the assumptions on varieties RO, D in Theorem 5.20 being the smallest varieties having the desired properties can be relaxed. The union of the smallest varieties RO and D satisfying the assumptions of Theorem 5.20 provides the minimal lower bound on a variety S, i.e. the union $RO \cup D$ specifies the smallest variety which can be considered a variety S.

Remark 5.21 (Sufficient condition for global identifiability). By a slight modification of Theorem 5.18 we derived a necessary condition for a parametrization of a parametrized polynomial/rational system to be globally identifiable, see Remark 5.19. In the same way we derive a sufficient condition for a parametrization of a structured polynomial/rational system to be globally identifiable from Theorem 5.20. Hence, the following statement holds.

Consider a structured polynomial (rational) system $\Sigma(P)$ such that $\Sigma(p)$ is canonical and distinguishes parameters for all $p \in P$ ($p \in P \setminus W$ where W is a set of such parameters p^W of P for which $\Sigma(p^W)$ is not well-defined). If for every $p, p' \in P$ ($p, p' \in P \setminus W$) an isomorphism linking $\Sigma(p)$ and $\Sigma(p')$ is the identity then the parametrization $\mathscr{P} : P \to \Sigma(P)$ is globally identifiable.

The following example provides an example of a structured system $\Sigma(P)$ which is structurally canonical and for which the parametrization $\mathscr{P} : P \to \Sigma(P)$ is not structurally identifiable.

Example 5.22. Consider a parameter set $P = \mathbb{R}^3$ and the parametrization $\mathscr{P} : P = \mathbb{R}^3 \to \Sigma(P)$ where $\Sigma(P)$ is a parametrized polynomial system such that the systems $\Sigma(p) = (X^p, f^p = \{f_\alpha^p, \alpha \in \mathbb{R}\}, h^p, x_0^p)$ of $\Sigma(P)$ are given as:

$$\begin{aligned} X^p &= \mathbb{R}, \\ f_\alpha^p &= p_1 \alpha \frac{\partial}{\partial x},\ \alpha \in \mathbb{R}, \\ h^p &= p_2 x^2, \\ x_0^p &= p_3. \end{aligned} \tag{5.13}$$

Note that the system $\Sigma(P)$ is a structured system.

The polynomials of the set $\{h^p, f_\alpha^p h^p \mid \alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{R}^n; n \in \mathbb{N}\}$ generate the observation algebra $A_{obs}(\Sigma(p))$. Thus $A_{obs}(\Sigma(p)) = \mathbb{R}[p_1 p_2 X, p_2 X^2]$ and the system $\Sigma(p)$ is algebraically observable for every $p \in \mathbb{R}^3 \setminus \{p = (p_1, p_2, p_3) \in \mathbb{R}^3 \mid p_1 p_2 = 0\}$. Therefore, $\Sigma(P)$ is structurally observable.

The reachable set of the differential equation $\dot{x} = p_1 \alpha$ with the initial state $x(0) = x_0^p = p_3 \in \mathbb{R}$ is the set $\mathscr{R}(x_0^p) = \{p_3 + p_1 \alpha t \mid t \in \mathbb{R}\}$. Because $Z\text{-}cl(\mathscr{R}(x_0^p)) =$

$\mathscr{R}(x_0^p) = \mathbb{R}$ for $p_1 \neq 0$ and because $Z\text{-}cl(\mathscr{R}(x_0^p)) = \{p_3\}$ for $p_1 = 0$, the polynomial system $\Sigma(p)$ is algebraically reachable for $p \in \mathbb{R}^3 \setminus \{p = (p_1, p_2, p_3) \in \mathbb{R}^3 \mid p_1 = 0\}$. Hence, $\Sigma(P)$ is structurally reachable.

Because $\Sigma(P)$ is both structurally observable and structurally reachable, it is according to Proposition 5.12 structurally canonical. A variety $RO \subsetneq P$ such that the systems $\Sigma(p) \in \Sigma(P)$ are canonical for all $p \in \mathbb{R}^3 \setminus RO$ can be chosen as $RO = \{p = (p_1, p_2, p_3) \in \mathbb{R}^3 \mid p_1 p_2 = 0\}$.

Because $P = \mathbb{R}^3$, the polynomials on P are the elements of the algebra $A^P = \mathbb{R}[P_1, P_2, P_3]$. The only polynomials $q^{f_\alpha}_{i;a_1}$, $q^h_{j;b_1}$, $q^{x_0}_i \in \mathbb{R}[P_1, P_2, P_3] = A^P$ given by (5.3), (5.4), (5.5) for the system (5.13) such that $q^{f_\alpha}_{i;a_1}$, $q^h_{j;b_1}$, $q^{x_0}_i \neq 0$ (meaning that they are not zero polynomials) are, if $p_1, p_2, p_3 \neq 0$,

$$q^{f_\alpha}_{1;0}(p_1, p_2, p_3) = \alpha p_1, \alpha \in \mathbb{R},\ q^h_{1;2}(p_1, p_2, p_3) = p_2,\ q^{x_0}_1(p_1, p_2, p_3) = p_3.$$

The algebra generated by $q^{f_\alpha}_{1;0}, \alpha \in \mathbb{R}$, $q^h_{1;2}$, and $q^{x_0}_1$ over $\mathbb{R}$ equals A^P. Therefore, by Definition 5.4, $\Sigma(p)$ distinguishes parameters for every $p \in \mathbb{R}^3 \setminus D$ where $D = \{p = (p_1, p_2, p_3) \in \mathbb{R}^3 \mid p_1 p_2 p_3 = 0\}$. Finally, $\Sigma(P)$ structurally distinguishes parameters.

Since the parametrized polynomial system $\Sigma(P)$ satisfies the assumptions of Theorem 5.20, we check the structural identifiability of the parametrization $\mathscr{P}$ by applying this theorem. Consider $p, p' \in \mathbb{R}^3 \setminus (RO \cup D)$. Both $\Sigma(p)$ and $\Sigma(p')$ are canonical and thus also isomorphic. Let $\phi, \psi : \mathbb{R} \to \mathbb{R}$ be polynomial mappings determining this isomorphic relation. From Definition 5.8, $\phi \circ \psi = \psi \circ \phi = 1_{\mathbb{R}}$ and ϕ has the properties:

(i) $\forall \varphi \in \mathbb{R}[X]\ \forall \alpha \in \mathbb{R} : (p'_1 \alpha \frac{\partial}{\partial x} \varphi) \circ \phi = p_1 \alpha \frac{\partial}{\partial x} (\varphi \circ \phi)$,
(ii) $\forall x \in \mathbb{R} : p_2 x^2 = p'_2 (\phi(x))^2$,
(iii) $\phi(p_3) = p'_3$.

Let us consider a polynomial φ in the condition (i) to be the identity, i.e. $\varphi(x) = x$. Since $p_1, p'_1 \neq 0$, the equality $p'_1 \alpha = p_1 \alpha \frac{\partial}{\partial x} \phi$ for all $\alpha \in \mathbb{R}$ implies that $\frac{p'_1}{p_1} = \frac{\partial}{\partial x} \phi$. Then the isomorphism ϕ has to be of the form

$$\phi(x) = \frac{p'_1}{p_1} x + c \text{ for some } c \in \mathbb{R}.$$

By substituting this form of ϕ into the condition (ii) we derive the equality $p_2 x^2 = p'_2 (\frac{p'^2_1}{p^2_1} x^2 + 2 \frac{p'_1}{p_1} c x + c^2)$ which implies the equations: $p_2 = p'_2 \frac{p'^2_1}{p^2_1}$, $0 = p'_2 \frac{p'_1}{p_1} c$, and $0 = p'_2 c^2$. Because $p_1, p'_1, p_2, p'_2 \neq 0$, we derive that $c = 0$. Therefore,

$$\phi(x) = \frac{p'_1}{p_1} x.$$

From this form of ϕ and from the condition (iii) it follows that $\frac{p'_1}{p_1} p_3 = p'_3$. Since $p, p' \in \mathbb{R}^3 \setminus (RO \cup D)$ it follows that $\frac{p'_1}{p_1} = \frac{p'_3}{p_3}$. To get that ϕ is the identity, i.e. $\phi(x) =$

$\frac{p_1'}{p_1}x = x$, we need to restrict the parameters to be from a subset of $\mathbb{R}^3 \setminus (RO \cup D)$ such that either $p_3 = p_3'$ or $p_1 = p_1'$. Hence, the only sets of parameters for which we can show that ϕ is the identity are

$$S_\zeta = \{(p_1, p_2, p_3) \in \mathbb{R}^3 \setminus (RO \cup D) \mid p_3 = \zeta\} \text{ for } \zeta \in \mathbb{R} \setminus \{0\},$$

$$S_\xi = \{(p_1, p_2, p_3) \in \mathbb{R}^3 \setminus (RO \cup D) \mid p_1 = \xi\} \text{ for } \xi \in \mathbb{R} \setminus \{0\}.$$

Since there does not exist a variety $V \subsetneq \mathbb{R}^3$ such that $\mathbb{R}^3 \setminus V \subseteq S_\zeta$ or $\mathbb{R}^3 \setminus V \subseteq S_\xi$ for some $\zeta, \xi \in \mathbb{R} \setminus \{0\}$, the parametrization $\mathscr{P} : \mathbb{R}^3 \to \Sigma(P)$ is not structurally identifiable.

Note that if we do not restrict the parameters to the parameter sets S_ζ or S_ξ then the parameters $p = (p_1, p_2, p_3)$ and $p' = (p_1', p_2', p_3')$ such that $p_1 = -p_1'$, $p_2 = p_2'$, and $p_3 = -p_3'$ determine two different systems $\Sigma(p), \Sigma(p') \in \Sigma(P)$ modeling the same data, i.e. $h^p(x^p(T_u; x_0^p, u)) = h^{p'}(x^{p'}(T_u; x_0^{p'}, u))$ for all $u \in \widetilde{\mathscr{U}_{pc}}$.

Example 5.23. Let us recall the model of a reaction system considered in Examples 5.1, 5.3, 5.7, 5.14. The parametrized system $\Sigma(P)$ modeling the referenced reaction system is given as a set of rational systems

$$\Sigma(p) : \begin{cases} X^p = \mathbb{R}, \\ f^p = \{f_\alpha^p = (-\frac{p_1 x}{p_3 + x} - p_2 x + \alpha)\frac{\partial}{\partial x} \mid \alpha \in \mathbb{R}\}, \\ h^p = x, \\ x_0^p = a \in \mathbb{R} \text{ known}, \end{cases}$$

for all $p \in P = \mathbb{R}^3$. Because the state-spaces X^p are the same for all $p \in P$ and because f^p, h^p, and x_0^p differ only by the values of parameters p, the parametrized system $\Sigma(P)$ is, according to Definition 5.9, structured. Further, from Example 5.14, $\Sigma(P)$ is structurally canonical and structurally distinguishes parameters. Therefore, $\Sigma(P)$ satisfies the assumptions of Theorem 5.20 which can be then applied to check structural identifiability of the parametrization $\mathscr{P} : P \to \Sigma(P)$.

From Example 5.14, $\Sigma(p)$ is canonical for all $p \in P \setminus RO$ where $RO = \{p = (p_1, p_2, p_3) \in P \mid p_3 + a = 0\}$ and, from Example 5.7, $\Sigma(p)$ distinguishes parameters for $p \in P \setminus D$ where $D = \{p = (p_1, p_2, p_3) \in P \mid p_2 p_3 (p_1 + p_2 p_3) = 0\}$. Even if we do not know whether the varieties RO and D are the smallest varieties having the corresponding properties, we can use them to define a variety S of Theorem 5.20. A variety S has to satisfy the relation $RO \cup D \subseteq S \subsetneq P$. Hence, we can choose S to be the variety $S = \{p = (p_1, p_2, p_3) \in P \mid (p_3 + a) p_2 p_3 (p_1 + p_2 p_3) = 0\}$. Let us consider $p, p' \in P \setminus S$. From Definition 5.8 and Theorem 5.16 we derive that there exists an isomorphism $\phi : X^p \to X^{p'}$ linking the systems $\Sigma(p)$ and $\Sigma(p')$. In particular, ϕ satisfies the equality $h^{p'}\phi = h^p$. Because $h^p(x) = h^{p'}(x) = x$, it follows that $\phi(x) = h^{p'}(\phi(x)) = h^p(x) = x$ and thus ϕ is the identity. From Theorem 5.20 we conclude that the parametrization $\mathscr{P} : P \to \Sigma(P)$ is structurally identifiable.

Note that the rational systems $\Sigma(p)$ are not well-defined for $p \in W = \{p = (p_1, p_2, p_3) \in P \mid p_3 + a = 0\}$. Let us define the set $N = \{p = (p_1, p_2, p_3) \in P \mid$

$p_2p_3 = 0\} \neq \emptyset$ which is a subset of the smallest set D_{min} such that for all $p \in P \setminus D_{min}$ the system $\Sigma(p)$ distinguishes parameters. Since $(P \setminus W) \cap N \neq \emptyset$, there exists a parameter $p = (p_1, p_2, p_3) \in (P \setminus W) \cap N$ such that $\Sigma(p)$ does not distinguish parameters and is well-defined. Therefore, it is not true that $\Sigma(p)$ distinguishes parameters for all $p \in P \setminus W$. This implies that we cannot decide whether the parametrization $\mathscr{P} : P \to \Sigma(P)$ is globally identifiable by using Remark 5.21.

5.3.4 Summary

The following two theorems summarize the results of the preceding sections.

Theorem 5.24 (Structural identifiability). *Let $P \subseteq \mathbb{R}^l$ be a parameter set and let $\Sigma(P)$ be a structured polynomial (rational) system with the parametrization $\mathscr{P} : P \to \Sigma(P)$. We assume that $\Sigma(P)$ is structurally canonical and we denote by RO the smallest strict subvariety of P such that $\Sigma(p) \in \Sigma(P)$ is canonical for all $p \in P \setminus RO$. We also assume that $\Sigma(P)$ structurally distinguishes parameters and we denote by D the smallest strict subvariety of P such that $\Sigma(p)$ distinguishes parameters for all $p \in P \setminus D$. Then the following statements are equivalent:*

(i) the parametrization $\mathscr{P}$ is structurally identifiable,
(ii) there exists a variety S such that $RO \cup D \subseteq S \subsetneq P$, and such that for any $p, p' \in P \setminus S$ an isomorphism linking the systems $\Sigma(p), \Sigma(p')$ is the identity.

Proof. It follows directly from Theorem 5.18 and Theorem 5.20. □

Recall that the properties of a parametrized system $\Sigma(P)$ to be structured and to structurally distinguish parameters are not needed in the proof of the implication $(i) \Rightarrow (ii)$.

Theorem 5.25 (Global identifiability). *Let $\Sigma(P)$ be a structured polynomial (rational) system and let the systems $\Sigma(p) \in \Sigma(P)$ be canonical and distinguish parameters for all $p \in P$ ($p \in P \setminus W$ where W is the set of such parameters p^W of P for which $\Sigma(p^W)$ is not well-defined). Then the following statements are equivalent:*

(i) the parametrization $\mathscr{P}$ of $\Sigma(P)$ is globally identifiable,
(ii) for every $p, p' \in P$ ($p, p' \in P \setminus W$) an isomorphism linking $\Sigma(p)$ and $\Sigma(p')$ is the identity.

Proof. It follows directly from Remark 5.19 and Remark 5.21. □

Remark 5.26. Note the similarity between the Theorems 5.24, 5.25 and [114, Theorem 2.9] which treats the structural identifiability of parametrization from the Markov and initial parameters for linear systems.

5.4 Examples

In this section we apply the results of Section 5.3, namely Theorem 5.20, to study the identifiability properties of the systems modeling different biological phenomena. Theorem 5.20 provides a procedure for checking structural identifiability of a parametrization of a parametrized polynomial/rational system $\Sigma(P)$. Let us recall the main steps which have to be performed to check structural identifiability of a parametrization $\mathscr{P} : P \to \Sigma(P)$ by applying this theorem.

1. $\Sigma(P)$ **is a structured system.** To apply Theorem 5.20 for checking structural identifiability of $\mathscr{P}$ the considered parametrized system has to be structured. In most biological examples the parametrized systems consist only of systems which all have the same state-spaces and which differ only by the values of parameters. Because the parametrized systems having these properties are automatically structured, see Definition 5.9, in most realistic examples the chosen model given as a parametrized system is already structured.

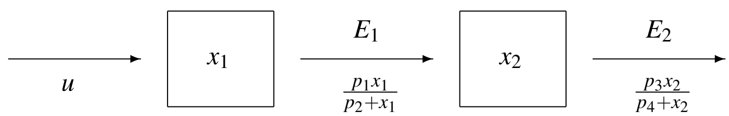
   Remark 5.27. From the above-mentioned reason the parametrized systems of all three examples presented in this section are structured systems.

2. $\Sigma(P)$ **is structurally canonical.** We need to verify whether $\Sigma(p)$ is algebraically reachable and algebraically/rationally observable for almost all $p \in P$. We proceed to check these properties by various methods illustrated in Examples 5.14, 5.28, 5.29, 5.30. Note that these methods are the same as the methods for checking algebraic reachability and algebraic/rational observability of polynomial and rational systems. The presence of parameters only leads to constraints in the form of polynomial equations which then define the varieties *RO*, see Definition 5.11.
3. $\Sigma(P)$ **structurally distinguishes parameters.** To check whether the system $\Sigma(p) = (X^p, f^p, h^p, x_0^p)$, $p \in P$ distinguishes parameters we rewrite the formulas in f^p, h^p, and x_0^p in the form of polynomials (rational functions) so that the same monomials M in the state variables are coupled together by deriving a new coefficient for the monomial M. The new coefficients are given as polynomials (rational functions) in the variables corresponding to parameters. If these polynomials (rational functions) generate all polynomials (rational functions) on P then the system $\Sigma(p)$ distinguishes parameters, see Definition 5.4 and Definition 5.6. A variety $D \subseteq P$ which determines the parameters p such that $\Sigma(p)$ does not distinguish parameters, see Definition 5.11, is derived as a by-product.
4. **Existence of a variety $S \subsetneq P$.** The last step deals with the construction of a variety $S \subsetneq P$ such that any isomorphism linking the systems $\Sigma(p)$ and $\Sigma(p')$ with $p, p' \in P \setminus S$ is the identity.
   Consider an arbitrary variety $S \subsetneq P$ containing the variety *RO* determined in Step 2. From Theorem 5.16 and from Definition 5.8 we obtain the characterization of all isomorphisms ϕ linking the systems $\Sigma(p)$ and $\Sigma(p')$ for $p, p' \in P \setminus S \subseteq P \setminus RO$. Further, to be able to prove that ϕ is the identity, we exclude also the parameters of the variety D determined in Step 3 from P. Hence, $S \subsetneq P$ is now considered to be an arbitrary variety containing both *RO* and *D*.

If the considered parametrization is indeed structurally identifiable then we get this result by substituting the formulas defining the dynamics, output functions, and initial states of $\Sigma(p)$ and $\Sigma(p')$ into the relations of Definition 5.8(ii)-(iv), by considering the polynomials $\varphi(X_1,\ldots,X_{n^{p'}}) = X_i$, $i = 1,\ldots n^{p'}$ in Definition 5.8(ii), and by solving the equations derived in this way. In Examples 5.14, 5.28, 5.29, and 5.30 we illustrate how to deal with these computations.

The same steps are performed also for checking global identifiability of a parametrization by applying Remark 5.21. Then the varieties RO, D, and S determined in Step 2, 3, and 4, respectively, have to be the empty set (in the case of polynomial systems) or to be equal to a variety $W \subseteq P$ of all parameter values p^W such that $\Sigma(p^W)$ is not well-defined (in the case of rational systems).

Example 5.28. Consider a chain of two enzyme-catalyzed irreversible reactions represented by the following diagram:



Here x_1 and x_2 denote the concentrations of the respective reactants. We assume that the inflow u to the system is modeled by piecewise-constant functions. The corresponding rates $\frac{p_1x_1}{p_2+x_1}$ and $\frac{p_3x_2}{p_4+x_2}$ of the reactions catalyzed by the enzymes E_1 and E_2 are modeled by Michaelis-Menten kinetics.

Let U be the set of all real values of admissible inputs u and let $P = \mathbb{R}^4$ be the parameter set. The parametrized system $\Sigma(P)$ modeling this chain of reactions consists of rational systems $\Sigma(p) = (X^p, f^p = \{f^p_\alpha \mid \alpha \in U\}, h^p, x^p_0)$, $p \in P$ given as:

$$\begin{aligned}
X^p &= \mathbb{R}^2,\\
f^p_\alpha &= \left(-\frac{p_1x_1}{p_2+x_1}+\alpha\right)\frac{\partial}{\partial x_1} + \left(\frac{p_1x_1}{p_2+x_1} - \frac{p_3x_2}{p_4+x_2}\right)\frac{\partial}{\partial x_2},\ \alpha \in U,\\
h^p(x_1,x_2) &= (h^p_1(x_1,x_2), h^p_2(x_1,x_2)) = \left(\frac{p_1x_1}{p_2+x_1}, \frac{p_3x_2}{p_4+x_2}\right),\\
x_1(0) &= x^p_{0,1} = 1,\ \ x_2(0) = x^p_{0,2} = 1.
\end{aligned}$$

Note that, from Remark 5.27, the parametrized system $\Sigma(P)$ is structured.

Rational observability of the rational system $\Sigma(p)$, $p \in P$ is determined by the equality $Q_{obs}(\Sigma(p)) = Q^p$, see Definition 3.21. We show that there exists a variety $O \subsetneq P$ such that $\Sigma(p)$ is rationally observable for all $p \in P \setminus O$. Because the state-space X^p of the system $\Sigma(p)$ equals $\mathbb{R}^2$ for all $p \in P$, to prove that $Q_{obs}(\Sigma(p)) = Q^p$ it is sufficient to prove that the polynomials $q_1(x_1,x_2) = x_1$ and $q_2(x_1,x_2) = x_2$ belong to $Q_{obs}(\Sigma(p))$. Consider an arbitrary $p \in P$. From the definition of the observation field $Q_{obs}(\Sigma(p))$ we know that $h^p_1, h^p_2 \in Q_{obs}(\Sigma(p))$. Hence, $\frac{p_1x_1}{p_2+x_1}, \frac{p_3x_2}{p_4+x_2} \in Q_{obs}(\Sigma(p))$. If $p_1, p_3 \neq 0$ then also $\frac{x_1}{p_2+x_1}, \frac{x_2}{p_4+x_2} \in Q_{obs}(\Sigma)$. As $Q_{obs}(\Sigma(p))$ is closed with respect to the derivations given by the vector fields f^p_α, $\alpha \in U$, we get that $f^p_\alpha(\frac{x_2}{p_4+x_2}) = \frac{p_4}{(p_4+x_2)^2}(h^p_1 - h^p_2) \in Q_{obs}(\Sigma(p))$. Because $Q_{obs}(\Sigma(p))$ is a field, every element of $Q_{obs}(\Sigma(p))$ has its inverse in $Q_{obs}(\Sigma(p))$, the product and the difference of two elements of $Q_{obs}(\Sigma(p))$ are also elements of $Q_{obs}(\Sigma(p))$. Specifically, $h^p_1 - h^p_2$, $\frac{1}{h^p_1 - h^p_2}$, and then also $\frac{p_4}{(p_4+x_2)^2}$ are the elements of $Q_{obs}(\Sigma(p))$. Further, if $p_4 \neq 0$, $(p_4+x_2)^2 \in Q_{obs}(\Sigma(p))$. By multiplying the elements $(p_4+x_2)^2$ and $\frac{x_2^2}{(p_4+x_2)^2}$ of $Q_{obs}(\Sigma)$, we derive that $x_2^2 \in Q_{obs}(\Sigma)$. Therefore, if $p_4 \neq 0$, the element $\frac{(p_4+x_2)^2 - x_2^2 - p_4^2}{2p_4} = x_2$ belongs to $Q_{obs}(\Sigma)$. By taking $f^p_\alpha(\frac{x_1}{p_2+x_1}) \in Q_{obs}(\Sigma)$ we derive in an analogous way that if $p_2 \neq 0$ then $x_1 \in Q_{obs}(\Sigma)$. Thus, for $p \in P \setminus O$ where $O = \{p = (p_1, p_2, p_3, p_4) \in \mathbb{R}^4 \mid p_1 p_2 p_3 p_4 = 0\}$ the system $\Sigma(p)$ is rationally observable. Because O is a strict subvariety of P, we showed that the parametrized system $\Sigma(P)$ is structurally observable.

To check algebraic reachability of a system $\Sigma(p) \in \Sigma(P)$ it is sufficient to prove that its reachable set from the initial state contains a non-empty open set, see Section 3.3. Then the Zariski closure of the reachable set equals $\mathbb{R}^2$ implying that $\Sigma(p)$ is algebraically reachable. Let us assume that $p_2, p_4 \neq -1$. Then there exists an open set $S \subseteq \mathbb{R}^2$ which contains the point $x^p_0 = (x_1(0), x_2(0)) = (1,1)$. Because the system $\Sigma(p)$ is a smooth affine nonlinear control system (on S), we can apply for example [79, Theorem 3.9] or [55, Theorem 2.2.4] to prove that the reachable set of $\Sigma(p)$ contains a non-empty open set in $\mathbb{R}^2$. By using the terminology of [79], it is sufficient to prove that the accessibility distribution C at x^p_0 has dimension 2. Since the accessibility algebra is spanned by the vector fields $f = -\frac{p_1 x_1}{p_2+x_1}\frac{\partial}{\partial x_1} + (\frac{p_1 x_1}{p_2+x_1} - \frac{p_3 x_2}{p_4+x_2})\frac{\partial}{\partial x_2}$, $g = \frac{\partial}{\partial x_1}$, and their Lie brackets, it follows for $C(x^p_0) = C((x_1(0), x_2(0))) = C((1,1))$ that

$$\dim C(x^p_0) = \dim \begin{pmatrix} \frac{-p_1}{p_2+1} & 1 \\ \frac{p_1}{p_2+1} - \frac{p_3}{p_4+1} & 0 \end{pmatrix}.$$

Recall that we assumed $p_2, p_4 \neq -1$. Let us also assume that $p_1(p_4+1) - p_3(p_2+1) \neq 0$. From these assumptions and from the formula characterizing the dimension of C at x^p_0 we derive that $\dim C(x^p_0) = 2$. Hence, for all $p \in P \setminus R$ where $R = \{p = (p_1, p_2, p_3, p_4) \in \mathbb{R}^4 \mid (p_2+1)(p_4+1)(p_1(p_4+1) - p_3(p_2+1)) = 0\}$ it holds that $\Sigma(p)$ is algebraically reachable from the initial state x^p_0. Because the set R is a strict subvariety of P, we showed that $\Sigma(P)$ is structurally reachable.

As $\Sigma(P)$ is structurally observable and structurally reachable, Proposition 5.12 implies that the parametrized system $\Sigma(P)$ is structurally canonical. Let us define the variety $RO = \{p = (p_1, \ldots, p_4) \in \mathbb{R}^4 \mid p_1 p_2 p_3 p_4 (p_2+1)(p_4+1)(p_1(p_4+1) - p_3(p_2+1)) = 0\}$. It holds that $R \cup O \subseteq RO \subsetneq P$ and that $\Sigma(p)$ is canonical for all $p \in P \setminus RO$.

The systems $\Sigma(p)$ with $p \in P \setminus D$, where $D = \{(p_1, \ldots, p_4) \in P \mid p_1 p_2 p_3 p_4 = 0\}$, distinguish parameters. This can be checked in the following way. Consider the components h^p_1, h^p_2 of h^p. From (5.7) we derive that $q^{h^p}_{1,1;1,0}(p_1, \ldots, p_4) = p_1$, $q^{h^p}_{2,1;0,0}(p_1, \ldots, p_4) = p_2$, $q^{h^p}_{1,2;0,1}(p_1, \ldots, p_4) = p_3$, and $q^{h^p}_{2,2;0,0}(p_1, \ldots, p_4) = p_4$. Therefore $\mathbb{R}(q^{h^p}_{1,1;1,0}, q^{h^p}_{2,1;0,0}, q^{h^p}_{1,2;0,1}, q^{h^p}_{2,2;0,0}) = \mathbb{R}(P_1,\ldots,P_4)$ if $p \in P \setminus D$. Because $P = \mathbb{R}^4$ and thus $Q^P = \mathbb{R}(P_1,\ldots,P_4)$ and because D is a strict subvariety of P, the parametrized system $\Sigma(P)$ structurally distinguishes parameters.

Let us denote $S = \{p = (p_1,\ldots,p_4) \in \mathbb{R}^4 \mid p_1p_2p_3p_4(p_2+1)(p_4+1)(p_1(p_4+1) - p_3(p_2+1)) = 0\}$. The variety S is such that $RO \cup D \subseteq S \subsetneq P$. For arbitrary $p, p' \in P \setminus S$ it holds that the systems $\Sigma(p)$ and $\Sigma(p')$ are canonical and such that $h^p(x^p(T_u; x_0^p, u)) = h^{p'}(x^{p'}(T_u; x_0^{p'}, u))$ for all $u \in \widetilde{\mathscr{U}_{pc}}$. Hence, according to Theorem 5.16, the systems $\Sigma(p)$ and $\Sigma(p')$ are isomorphic. By Definition 5.8 there exists an isomorphism $\phi = (\phi_1, \phi_2) : \mathbb{R}^2 \to \mathbb{R}^2$ such that:

(i) $\forall \varphi \in Q^{p'} \forall \alpha \in U : f^p_\alpha(\varphi \circ \phi) = (f^{p'}_\alpha \varphi) \circ \phi$,
(ii) $h^{p'}\phi = h^p$,
(iii) $\phi(x_0^p) = x_0^{p'}$.

By substituting the explicit forms of h^p and $h^{p'}$ into (ii) we derive the equations

$$\frac{p'_1\phi_1(x_1,x_2)}{p'_2 + \phi_1(x_1,x_2)} = \frac{p_1x_1}{p_2+x_1}, \tag{5.14}$$

$$\frac{p'_3\phi_2(x_1,x_2)}{p'_4 + \phi_2(x_1,x_2)} = \frac{p_3x_2}{p_4+x_2}. \tag{5.15}$$

Let us consider the polynomial $\varphi(x_1,x_2) = x_1$ in (i). Then for every $\alpha \in U$ it holds that $\frac{-p'_1\phi_1(x_1,x_2)}{p'_2+\phi_1(x_1,x_2)} + \alpha = \left(\frac{-p_1x_1}{p_2+x_1} + \alpha\right)\frac{\partial\phi_1(x_1,x_2)}{\partial x_1} + \left(\frac{p_1x_1}{p_2+x_1} - \frac{p_3x_2}{p_4+x_2}\right)\frac{\partial\phi_1(x_1,x_2)}{\partial x_2}$. By substituting (5.14) into this equation we obtain the equation

$$\left(\frac{p_1x_1}{p_2+x_1} - \alpha\right)\frac{\partial\phi_1}{\partial x_1} + \left(\frac{p_3x_2}{p_4+x_2} - \frac{p_1x_1}{p_2+x_1}\right)\frac{\partial\phi_1}{\partial x_2} = \frac{p_1x_1}{p_2+x_1} - \alpha. \tag{5.16}$$

We solve this linear partial differential equation by the method of characteristics, see for example [32]. The equations for the characteristic curves are:

$$\frac{d\xi}{ds} = \frac{p_1x_1}{p_2+x_1} - \alpha = \frac{p_1\xi(s)}{p_2+\xi(s)} - \alpha, \tag{5.17}$$

$$\frac{d\eta}{ds} = \frac{p_3x_2}{p_4+x_2} - \frac{p_1x_1}{p_2+x_1} = \frac{p_3\eta(s)}{p_4+\eta(s)} - \frac{p_1\xi(s)}{p_2+\xi(s)}. \tag{5.18}$$

The restriction $\phi_{1,\gamma}(s) = \phi_1(\xi(s),\eta(s))$ of $\phi_1(x_1,x_2)$ to $\gamma = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 = \xi(s); x_2 = \eta(s); \xi,\eta \in \mathscr{C}^1(I,\mathbb{R})\}$ is given by the equation $\frac{d\phi_{1,\gamma}}{ds} = \frac{d\phi_1}{dx_1}\frac{d\xi}{ds} + \frac{d\phi_1}{dx_2}\frac{d\eta}{ds}$. From (5.16) and (5.17) it follows that

$$\frac{d\phi_{1,\gamma}}{ds} = \frac{p_1\xi(s)}{p_2+\xi(s)} - \alpha = \frac{d\xi}{ds}$$

and consequently $\phi_1(\xi(s),\eta(s)) = \xi(s) + c$ where $c \in \mathbb{R}$ is a constant. Therefore, $\phi_1(x_1,x_2) = x_1 + c$. Since (iii) implies that $\phi_1(x_0^p) = \phi_1(x^p_{0,1}, x^p_{0,2}) = x^p_{0,1}$, we deduce

$$\phi_1(x_1,x_2) = x_1. \tag{5.19}$$

By considering the polynomial $\varphi(x_1,x_2) = x_2$ in (i) and by substituting (5.15) into the derived relation, we obtain the following equation valid for all $\alpha \in U$:

$$\left(\frac{p_1x_1}{p_2+x_1} - \alpha\right)\frac{\partial\phi_2}{\partial x_1} + \left(\frac{p_3x_2}{p_4+x_2} - \frac{p_1x_1}{p_2+x_1}\right)\frac{\partial\phi_2}{\partial x_2} = \frac{p_3x_2}{p_4+x_2} - \frac{p_1x_1}{p_2+x_1}.$$

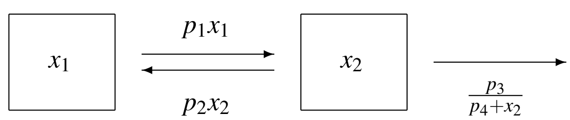
Again by the method of characteristics we derive the solution of this linear partial differential equation in the form $\phi_2(x_1,x_2) = x_2 + c$ for $c \in \mathbb{R}$. Because $\phi_2(x_0^p) = \phi_2(x_{0,1}^p, x_{0,2}^p) = x_{0,2}^p$ from (iii), we conclude

$$\phi_2(x_1,x_2) = x_2. \tag{5.20}$$

Finally, from (5.19) and (5.20), the isomorphism ϕ is the identity. Because ϕ was an arbitrary isomorphism linking $\Sigma(p)$ and $\Sigma(p')$ with $p, p' \in P \setminus S$, the parametrization $\mathscr{P}$ is, according to Theorem 5.20 or Theorem 5.24, structurally identifiable.

For a comparison of the method derived in this chapter and the differential-algebraic methods used in [74] for determining structural and global identifiability of the parametrizations of structured rational systems, we consider the following example of a two-compartment model with Michaelis-Menten elimination kinetics studied in [74, Section 7].

Example 5.29. Consider a reaction system described by the diagram below.



We assume that there are no external inputs to the system. The concentrations of the respective reactants in the reaction system are denoted by x_1 and x_2. The reversible reaction between the two reactants is simulated by two irreversible reactions which are modeled by mass-action kinetics. Further dissociation of the second reactant is modeled by Michaelis-Menten kinetics.

Since the parameters p_1, p_2, p_3, p_4 could take the values in $\mathbb{R}$, the parameter set P equals $\mathbb{R}^4$ and the reaction system is modeled by the parametrized rational system $\Sigma(P) = \{\Sigma(p) = (X^p, f^p, h^p, x_0^p = (x_{0,1}^p, x_{0,2}^p)) \mid p \in P = \mathbb{R}^4\}$ where

$$\begin{aligned}
X^p &= \mathbb{R}^2,\\
f^p &= (-p_1x_1 + p_2x_2)\frac{\partial}{\partial x_1} + \left(p_1x_1 - p_2x_2 - \frac{p_3x_2}{p_4+x_2}\right)\frac{\partial}{\partial x_2},\\
h^p &= x_1,\\
x_{0,1}^p &= a \in \mathbb{R},\ x_{0,2}^p = 0.
\end{aligned}$$

Note that we assume that the concentration x_1 can be observed. Further, according to Remark 5.27, the parametrized system $\Sigma(P)$ is a structured system.

From the definition of the observation field $Q_{obs}(\Sigma(p))$ it follows that $h^p(x_1,x_2) = x_1$ and $f^p h^p(x_1,x_2) = -p_1x_1 + p_2x_2$ are the elements of $Q_{obs}(\Sigma(p))$. Therefore, if we assume that $p_2 \neq 0$, the polynomials $q_1(x_1,x_2) = x_1$ and $q_2(x_1,x_2) = x_2$, which generate all polynomials on $X^p = \mathbb{R}^2$, belong to $Q_{obs}(\Sigma(p))$. Thus $Q_{obs}(\Sigma(p)) = Q^p$ for all $p \in P \setminus O$ where $O = \{p = (p_1,p_2,p_3,p_4) \in P \mid p_2 = 0\}$. Because O is a strict subvariety of P, we conclude that the parametrized system $\Sigma(P)$ is structurally observable.

The dynamics of the system $\Sigma(p)$ rewritten in the state-space form is given as

$$\begin{aligned} \dot{x}_1 &= -p_1x_1 + p_2x_2, \\ \dot{x}_2 &= p_1x_1 - p_2x_2 - \frac{p_3x_2}{p_4+x_2}, \\ x_1(0) &= a \in \mathbb{R},\ x_2(0) = 0. \end{aligned} \tag{5.21}$$

If $p_3 \neq 0$, it follows that $(p_4+x_2)\dot{x}_1 + (p_4+x_2)\dot{x}_2 + p_3x_2 = 0$. Since there does not exist a non-zero polynomial $M(x_1,x_2) \in \mathbb{R}[x_1,x_2]$ such that $\frac{d}{dt}M(x_1,x_2) = (p_4+x_2)\dot{x}_1 + (p_4+x_2)\dot{x}_2 + p_3x_2$ (note that x_1,x_2 are time-dependent variables), the reachable set of $\Sigma(p)$ is not a variety and it is not a subset of a strict subvariety of $\mathbb{R}^2$. According to [3, Theorem 2.3], there exists a solution of (5.21) defined on a non-empty time interval. Because this solution cannot be described in the state-space as an algebraic curve or as a finite set of points (otherwise the reachable set would be a variety), it implies that the smallest variety containing the reachable set is the whole state-space $X^p = \mathbb{R}^2$. Therefore, the system $\Sigma(p)$ is algebraically reachable from the initial state $(x^p_{0,1} = a \in \mathbb{R}, x^p_{0,2} = 0)$ for all $p \in P \setminus R$ where $R = \{p = (p_1,p_2,p_3,p_4) \in P \mid p_3 = 0\}$. Because $R \subsetneq P$ is a variety, the parametrized system $\Sigma(P)$ is structurally reachable.

Let us define a variety $RO = \{p = (p_1,p_2,p_3,p_4) \in P \mid p_2p_3 = 0\}$. Then $O \cup R \subseteq RO \subsetneq P$ and $\Sigma(p)$ is canonical for all $p \in P \setminus RO$. This implies that the parametrized system $\Sigma(P)$ is structurally canonical.

Consider the vector field f^p of a system $\Sigma(p) \in \Sigma(P)$. If we rewrite f^p_1 and f^p_2 in the form of (5.6) we derive that $q^f_{1,1;1,0}(p_1,\dots,p_4) = -p_1$, $q^f_{1,1;0,1}(p_1,\dots,p_4) = p_2$, $q^f_{1,2;0,1}(p_1,\dots,p_4) = -p_2p_4 - p_3$, $q^f_{2,2;0,0}(p_1,\dots,p_4) = p_4$, and consequently that $\mathbb{R}(q^f_{1,1;1,0}, q^f_{1,1;0,1}, q^f_{1,2;0,1}, q^f_{2,2;0,0}) = \mathbb{R}(P_1,P_2,P_3,P_4) = Q^P$ for all $p \in P \setminus D$ where $D = \{p = (p_1,p_2,p_3,p_4) \in P \mid p_1p_2p_3p_4 = 0\}$. Therefore, $\Sigma(P)$ structurally distinguishes parameters.

Consider a variety $S = D$. It holds that $RO \cup D \subseteq S \subsetneq P$. Let $p, p' \in P \setminus S$ be arbitrary. We show that any isomorphism linking $\Sigma(p)$ and $\Sigma(p')$ is the identity. Note that because $p, p' \in P \setminus S$, the systems $\Sigma(p)$ and $\Sigma(p')$ are canonical and distinguish parameters. Further, $h^p(x^p(T_u;x^p_0,u)) = h^{p'}(x^{p'}(T_u;x^{p'}_0,u))$ for all $u \in \widetilde{\mathcal{U}_{pc}}$. Therefore, from Theorem 5.16, the systems $\Sigma(p), \Sigma(p')$ are isomorphic which means (according to Definition 5.8) that

(i) there exist rational mappings ϕ, ψ such that $\phi : X^p \to X^{p'}$, $\psi : X^{p'} \to X^p$, and $\phi \circ \psi = 1_{X^p}$, $\psi \circ \phi = 1_{X^{p'}}$; moreover, $\phi(x_1,x_2) = (\phi_1(x_1,x_2), \phi_2(x_1,x_2))$ for $x_1,x_2 \in X^p$ and $\psi(x_1,x_2) = (\psi_1(x_1,x_2), \psi_2(x_1,x_2))$ for $x_1,x_2 \in X^{p'}$,
(ii) $\forall \varphi \in Q^{p'} : f^p(\varphi \circ \phi) = (f^{p'}\varphi) \circ \phi$,
(iii) $\phi_1(x_1,x_2) = h^{p'}(\phi_1(x_1,x_2), \phi_2(x_1,x_2)) = h^{p'}(\phi(x_1,x_2)) = h^p(x_1,x_2) = x_1$,
(iv) because the initial state is independent of the parameters, we have that $x^p_{0,1} = x^{p'}_{0,1} = a \in \mathbb{R}$, $x^p_{0,2} = x^{p'}_{0,2} = 0$ and therefore $\phi(x^p_0) = x^{p'}_0 = x^p_0$.

Consider the relation in (ii) formulated only for the polynomial $\varphi(x_1,x_2) = x_1$. By substituting $\phi_1(x_1,x_2) = x_1$ derived in (iii) to this relation we obtain the equality $-p_1x_1 + p_2x_2 = -p'_1x_1 + p'_2\phi_2(x_1,x_2)$. Because $p'_2 \neq 0$, it follows that

$$\phi_2(x_1,x_2) = \frac{(p'_1 - p_1)x_1 + p_2x_2}{p'_2}. \tag{5.22}$$

According to (iv), $\phi(x^p_0) = x^p_0$ and hence $\phi_2(x^p_{0,1}, x^p_{0,2}) = x^p_{0,2} = 0$. Therefore, the equality (5.22), for $x_1 = x^p_{0,1}$ and $x_2 = x^p_{0,2}$, implies

$$0 = \frac{(p'_1 - p_1)x^p_{0,1}}{p'_2}.$$

Let us assume that $x^p_{0,1} = a \neq 0$. Then, from the last equation, $p'_1 = p_1$. Consequently, by (5.22),

$$\phi_2(x_1,x_2) = \frac{p_2x_2}{p'_2}. \tag{5.23}$$

By considering (ii) stated only for the polynomial $\varphi(x_1,x_2) = x_2$, by substituting (5.23) and $\phi_1(x_1,x_2) = x_1$ from (iii) into the derived relation, and by evaluating the received equality at the point $(x^p_{0,1}, x^p_{0,2})$, we obtain the equality

$$p_1\frac{p_2}{p'_2}x^p_{0,1} = p'_1x^p_{0,1}.$$

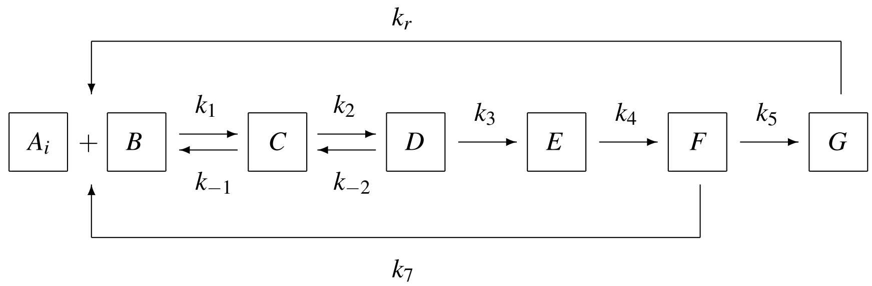
Since $p_1 = p'_1$ and $x^p_{0,1} = a \neq 0$, it follows that $p_2 = p'_2$. Hence, from (5.23), $\phi_2(x_1,x_2) = x_2$. This together with (iii) proves that ϕ is the identity. To sum up, for all $p, p' \in P \setminus S$ and for $a \neq 0$ an isomorphism relating $\Sigma(p)$ and $\Sigma(p')$ is the identity. Therefore, by Theorem 5.20 or Theorem 5.24, in the case $a \neq 0$ the parametrization $\mathscr{P} : P \to \Sigma(P)$ is structurally identifiable.

The last example treats the model of a peptide chain elongation described in [51] and consequently in [62, Chapter 8.3.3]. It is a bilinear system with parameters which is a special case of a parametrized polynomial and thus also a parametrized rational system.

Example 5.30. The model of a peptide chain elongation from [51, 62] is given by the equations:

$$
\begin{aligned}
\dot{B} &= -k_1 A_i B + k_{-1} C + k_r G + k_7 F, \quad (i = 1,2)\\
\dot{C} &= k_1 A_i B - k_{-1} C - k_2 C + k_{-2} D,\\
\dot{D} &= k_2 C - k_{-2} D - k_3 D,\\
\dot{E} &= k_3 D - k_4 E,\\
\dot{F} &= k_4 E - k_5 F - k_7 F,\\
\dot{G} &= k_5 F - k_r G.
\end{aligned}
$$

The state variables B, C, D, E, F, G correspond to ribosome, initial binding, codon recognition, GTPase activation and GTP hydrolysis, EF-Tu released, and accommodation and peptide transfer, respectively. A_1 and A_2 stand for correct and wrong form of tRNA, respectively, the correct one provides an amino acid to be the next in the peptide sequence. The process of the elongation of a peptide chain can be described also by the following diagram.



We will study the model only for one of A_1, A_2. Hence, let us consider either A_1 or A_2 as the inflow to the system. We denote it by u. We assume that u can be modeled by piecewise-constant functions with the values in $\mathbb{R}$. To study structural identifiability of the model we need to specify the initial state and the outputs of the system. Let us assume that the initial state is given as $(B(0), C(0), D(0), E(0), F(0), G(0)) = (1,1,1,1,1,1)$ and that the outputs are given as the outflows k_2C, k_7F, k_rG of the system. Further, we assume that the parameters $k_1, k_{-1}, k_2, k_{-2}, k_3, k_4, k_5, k_7, k_r$ of the model take values in $\mathbb{R}$ and thus the considered parameter set is given as $P = \mathbb{R}^9$. Finally, the parametrized system $\Sigma(P) = \{\Sigma(k) = (X^k, f^k = \{f^k_\alpha \mid \alpha \in \mathbb{R}\}, h^k, x^k_0) \mid k = (k_1, k_{-1}, k_2, k_{-2}, k_3, k_4, k_5, k_7, k_r) \in P = \mathbb{R}^9\}$, where

$$
X^k = \mathbb{R}^6,
$$

$$
f^k_\alpha = \left[\begin{pmatrix} -k_1\alpha & k_{-1} & 0 & 0 & k_7 & k_r \\ k_1\alpha & -k_{-1}-k_2 & k_{-2} & 0 & 0 & 0 \\ 0 & k_2 & -k_{-2}-k_3 & 0 & 0 & 0 \\ 0 & 0 & k_3 & -k_4 & 0 & 0 \\ 0 & 0 & 0 & k_4 & -k_5-k_7 & 0 \\ 0 & 0 & 0 & 0 & k_5 & -k_r \end{pmatrix} \begin{pmatrix} B \\ C \\ D \\ E \\ F \\ G \end{pmatrix} \right]^T \begin{pmatrix} \frac{\partial}{\partial B} \\ \frac{\partial}{\partial C} \\ \frac{\partial}{\partial D} \\ \frac{\partial}{\partial E} \\ \frac{\partial}{\partial F} \\ \frac{\partial}{\partial G} \end{pmatrix},
$$

$$
h^k = (h^k_1, h^k_2, h^k_3)^T = (k_2 C, k_7 F, k_r G)^T,
$$

$$
(B(0), C(0), D(0), E(0), F(0), G(0))^T = (x^k_{0,1}, \ldots, x^k_{0,6})^T = (1,1,1,1,1,1)^T,
$$

models the elongation of a peptide chain. From Remark 5.27, the parametrized system $\Sigma(P)$ is a structured system.

Because $\mathbb{R}[h_1^k, h_2^k, h_3^k, f_\alpha^k(h_2^k), f_\alpha^k(f_\alpha^k(h_2^k)), f_\alpha^k(h_1^k)] = \mathbb{R}[B,C,D,E,F,G]$ for all $\alpha \in \mathbb{R}\setminus\{0\}$ and for all $k \in P \setminus O$ where $O = \{k = (k_1, k_{-1}, k_2, k_{-2}, k_3, k_4, k_5, k_7, k_r) \in P = \mathbb{R}^9 \mid k_1 k_2 k_3 k_4 k_7 k_r = 0\}$ is a strict subvariety of P, the parametrized system $\Sigma(P)$ is structurally observable. Note that $\Sigma(P)$ would be structurally observable even if we considered only one output, namely $h^k = h_3^k$. Then $\mathbb{R}[\{(f_\alpha^k)^i h_3^k \mid i = 0, \ldots, 5\}] = \mathbb{R}[B,C,D,E,F,G]$ for all $k \in P$ up to a variety.

By defining new inputs as $v = Bu$ we derive a parametrized linear system $\Sigma_{lin}(P)$ from the parametrized system $\Sigma(P)$. To show that $\Sigma(P)$ is structurally reachable it is sufficient to prove that $\Sigma_{lin}(P)$ is structurally reachable by checking controllability rank condition for the systems $\Sigma_{lin}(k)$ where $k \in P \setminus R$ for a variety $R \subsetneq P$, see Section 3.3. The linear system $\Sigma_{lin}(k)$, $k \in P$ is given as

$$
\begin{aligned}
X^k &= \mathbb{R}^6, \\
\begin{pmatrix} \dot{B} \\ \dot{C} \\ \dot{D} \\ \dot{E} \\ \dot{F} \\ \dot{G} \end{pmatrix} &= \underbrace{\begin{pmatrix} 0 & k_{-1} & 0 & 0 & k_7 & k_r \\ 0 & -k_{-1}-k_2 & k_{-2} & 0 & 0 & 0 \\ 0 & k_2 & -k_{-2}-k_3 & 0 & 0 & 0 \\ 0 & 0 & k_3 & -k_4 & 0 & 0 \\ 0 & 0 & 0 & k_4 & -k_5-k_7 & 0 \\ 0 & 0 & 0 & 0 & k_5 & -k_r \end{pmatrix}}_{\mathbf{M}} \begin{pmatrix} B \\ C \\ D \\ E \\ F \\ G \end{pmatrix} + \underbrace{\begin{pmatrix} -k_1 \\ k_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\mathbf{N}} v, \\
h^k &= (h_1^k, h_2^k, h_3^k)^T = (k_2 C, k_7 F, k_r G)^T, \\
(B(0), C(0), &D(0), E(0), F(0), G(0))^T = (1,1,1,1,1,1)^T.
\end{aligned}
$$

It is controllable if $\mathrm{rank}(N\ MN\ \ldots\ M^5 N) = 6$. We show that this rank condition is satisfied for all $k \in P \setminus R$ where R is a strict subvariety of P. First, note that

$$
(N\ MN\ \ldots\ M^5 N) = \begin{pmatrix} * & * & * & * & * & * \\ k_1 & * & * & * & * & * \\ 0 & k_2 & * & * & * & * \\ 0 & 0 & k_3 & * & * & * \\ 0 & 0 & 0 & k_4 & * & * \\ 0 & 0 & 0 & 0 & k_5 & * \end{pmatrix}
$$

where $*$ stands for polynomials in 9 variables $k_1, k_{-1}, k_2, k_{-2}, k_3, k_4, k_5, k_7, k_r$ with real coefficients. Thus, if $k_1, k_2, k_3, k_4, k_5 \neq 0$ then $\mathrm{rank}(N\ MN\ \ldots\ M^5 N) \geq 5$. By Gaussian elimination we transform the matrix $(N\ MN\ \ldots\ M^5 N)$ into a matrix $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \bullet \\ k_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_5 & 0 \end{pmatrix}$ of the same rank. If the polynomial $\bullet$ is a non-zero polynomial then $\mathrm{rank}(N\ MN\ \ldots\ M^5N) = 6$. Therefore, $\Sigma_{lin}(k)$ is controllable for all $k \in P \setminus R$ where $R = \{k = (k_1, k_{-1}, k_2, k_{-2}, k_3, k_4, k_5, k_7, k_r) \in P = \mathbb{R}^9 \mid k_1k_2k_3k_4k_5\bullet = 0\}$. Because $R \subsetneq P$ is a variety, $\Sigma(P)$ is structurally reachable.

By Proposition 5.12, the parametrized system $\Sigma(P)$ is structurally canonical. We can define a variety $RO \subsetneq P$ such that $\Sigma(k)$ is canonical for all $k \in P \setminus RO$ by the union $R \cup O$.

From the definition of the vector fields f^k_α, $\alpha \in \mathbb{R}$ it is easy to see how the nonzero polynomials $q^{f_\alpha}_{i;a_1,\ldots,a_6}$ defined by (5.3) look like. Specifically,

$$
\begin{aligned}
&q^{f_\alpha}_{1;0,1,0,0,0,0}(k_1, k_{-1}, k_2, k_{-2}, k_3, k_4, k_5, k_7, k_r) = k_{-1},\\
&q^{f_\alpha}_{1;0,0,0,0,1,0}(k_1, \ldots, k_r) = k_7, \qquad q^{f_\alpha}_{3;0,1,0,0,0,0}(k_1, \ldots, k_r) = k_2,\\
&q^{f_\alpha}_{1;0,0,0,0,0,1}(k_1, \ldots, k_r) = k_r, \qquad q^{f_\alpha}_{4;0,0,1,0,0,0}(k_1, \ldots, k_r) = k_3,\\
&q^{f_\alpha}_{2;1,0,0,0,0,0}(k_1, \ldots, k_r) = k_1\alpha, \qquad q^{f_\alpha}_{5;0,0,0,1,0,0}(k_1, \ldots, k_r) = k_4,\\
&q^{f_\alpha}_{2;0,0,1,0,0,0}(k_1, \ldots, k_r) = k_{-2}, \qquad q^{f_\alpha}_{6;0,0,0,0,1,0}(k_1, \ldots, k_r) = k_5.
\end{aligned}
$$

These polynomials generate the algebra of all polynomials on $P = \mathbb{R}^9$ for all $\alpha \in \mathbb{R} \setminus \{0\}$ and $k \in P \setminus D$ where $D = \{k = (k_1, k_{-1}, k_2, k_{-2}, k_3, k_4, k_5, k_7, k_r) \in P = \mathbb{R}^9 \mid k_1k_{-1}k_2k_{-2}k_3k_4k_5k_7k_r = 0\}$. Therefore, the system $\Sigma(k)$ distinguishes parameters for all $k \in P \setminus D$. Since $D \subsetneq P$ is a variety, we conclude that $\Sigma(P)$ structurally distinguishes parameters.

From the irreducibility of P it follows that $S = O \cup R \cup D \subsetneq P$. Let us consider arbitrary $\Sigma(k), \Sigma(k') \in \Sigma(P)$ such that $k, k' \in P \setminus S$. Both systems are canonical and distinguish parameters. Further, because they are realizing the same measurements, they are according to Theorem 5.16 isomorphic. From Definition 5.8, there exist polynomial mappings $\phi, \psi : \mathbb{R}^6 \to \mathbb{R}^6$ such that $\phi \circ \psi = \psi \circ \phi = 1_{\mathbb{R}^6}$ and

(i) $f^k_\alpha(\varphi \circ \phi) = (f^{k'}_\alpha \varphi) \circ \phi$ for all $\alpha \in \mathbb{R}$ and for all $\varphi \in \mathbb{R}[X_1, \ldots, X_6]$,
(ii) $h^k_i = h^{k'}_i \circ \phi$ for $i = 1, 2, 3$,
(iii) $\phi(B(0), C(0), D(0), E(0), F(0), G(0)) = \phi(1, 1, 1, 1, 1, 1) = (1, 1, 1, 1, 1, 1)$.

We prove that $\phi = (\phi_1, \ldots, \phi_6)^T : \mathbb{R}^6 \to \mathbb{R}^6$ is the identity. Since $k_2, k_7, k_r \neq 0$ (because $k \in P \setminus S = P \setminus (O \cup R \cup D)$ and the parameters $k \in P$ such that $k_2, k_7, k_r = 0$ are all contained in D), it follows from (ii) that $k_2 = k'_2$, $k_7 = k'_7$, $k_r = k'_r$ and furthermore that

$$
\begin{aligned}
\phi_2(B, C, D, E, F, G) &= C,\\
\phi_5(B, C, D, E, F, G) &= F,\\
\phi_6(B, C, D, E, F, G) &= G.
\end{aligned} \tag{5.24}
$$

If we consider the polynomial $\varphi(B, C, D, E, F, G) = F$ in (i), we derive that $k_4 = k'_4$ (because $k_4 \neq 0$ since $k \in P \setminus S$) and

$$\phi_4(B, C, D, E, F, G) = E. \tag{5.25}$$

As $k_3 \neq 0$ (again because $k \in P \setminus S$), we obtain by considering the polynomial $\varphi(B,C,D,E,F,G) = E$ in (i) that $k_3 = k_3'$ and

$$\phi_3(B,C,D,E,F,G) = D. \tag{5.26}$$

For the polynomial $\varphi(B,C,D,E,F,G) = D$ the relation (i) and the equality $k_2 = k_2'$ imply that $k_{-2} = k_{-2}'$. Then, since $k_1 \neq 0$ (from $k \in P \setminus S$), by considering the polynomial $\varphi(B,C,D,E,F,G) = C$ in (i) we derive that $(-k_{-1} + k_{-1}')C + k_1 B\alpha = k_1' \alpha \varphi_1(B,C,D,E,F,G)$ for all $\alpha \in \mathbb{R}$. From (iii) follows the equality $-k_{-1} + k_{-1}' = \alpha(k_1' - k_1)$ for all $\alpha \in \mathbb{R}$. Therefore $k_{-1} = k_{-1}'$, $k_1 = k_1'$, and consequently

$$\phi_1(B,C,D,E,F,G) = B. \tag{5.27}$$

Finally, from (5.24), (5.25), (5.26), (5.27), an isomorpism ϕ is the identity. Because $\Sigma(k)$, $\Sigma(k')$ were arbitrary systems of $\Sigma(P)$ with $k, k' \in P \setminus S$, the parametrization $\mathscr{P} : P \to \Sigma(P)$ is according to Theorem 5.20 or Theorem 5.24 structurally identifiable.

Note that if we consider $h_1^k = k_1 B$ instead of $h_1^k = k_2 C$ then we can prove in the same way that the parametrization $\mathscr{P}$ of the modified parametrized system is structurally identifiable.

5.5 Concluding remarks

We have provided the characterization of structural and global identifiability of parametrizations of parametrized polynomial and parametrized rational systems. The basic objects used are polynomial and rational maps on or between irreducible varieties. Therefore, the main results of this chapter make it possible to apply the results of computational algebra to obtain procedures and algorithms for checking structural and global identifiability for the classes of parametrized polynomial and parametrized rational systems.

We assumed that the parameter sets are irreducible real affine varieties. We could also work with arbitrary subsets of $\mathbb{R}^l$ and consider Euclidean topology on $\mathbb{R}^l$. Then the structural properties defined in Definition 5.11 have to be considered as properties valid for all parameter values except for a set of parameter values of measure zero.

As demonstrated in [88, p. 248] and in [74, p. 14], the parametrized systems parametrizations of which are such that the rational combinations of parameters are present as coefficients in the vector fields, output functions, or initial conditions are realistic and very often necessary to faithfully describe the biological character of the studied process. Our approach allows for such parametrizations once the condition on distinguishability of parameters is satisfied.

In Example 5.22 we provided an example of a parametrized system which is structurally canonical but the parametrization of which is not structurally identifiable. Therefore, structural canonicity is not a sufficient condition for a parametrization of a parametrized polynomial/rational system to be structurally identifiable. Finding a parametrized system which is not structurally canonical but the parametrization of which is structurally identifiable would imply that structural canonicity is not even a necessary condition. The corresponding result holds for linear systems, see [53].

There are still many open problems concerning system identification for polynomial and rational systems. One of them is the problem of determining the classes of inputs which are exciting polynomial and rational systems sufficiently to be able to determine their identifiability properties and consequently estimate the values of the parameters. For bilinear systems, the problem of characterizing sufficiently exciting inputs is considered in [100]. The problem of determining the numerical values of parameters from measurements is itself a major open problem. Further, structural indistinguishability which deals with the uniqueness of a model structure is of interest. It is treated for example in [34] for uncontrolled nonlinear analytic systems by generalizing the results of structural identifiability from [33]. In the case of polynomial and rational systems it should be easily solvable by means of realization theory developed for these classes of systems in [8, 11] and in Chapter 4.

Chapter 6
Nash Realizations

In this chapter we deal with Nash systems which are dynamical systems with Nash submanifolds of $\mathbb{R}^n$ as state-spaces and with the dynamics and output functions defined by Nash functions. A Nash submanifold of $\mathbb{R}^n$ is a manifold which is defined by polynomial equalities and/or inequalities. By a Nash function we mean an analytic semi-algebraic function, i.e. an analytic function satisfying an algebraic equation.

The basic notions, such as Nash function/submanifold, and the basic facts of real algebraic geometry are introduced in Section 2.3. The main topic of this chapter concerns realization theory for Nash systems. The presented approach extends the approach of [11] for polynomial systems and the approach presented in Chapter 4 for rational systems.

6.1 Motivation

The class of Nash systems lies between polynomial/rational systems and analytic systems. While more general than the former, it still allows for a constructive description by means of finitely many polynomial equalities and inequalities. Hence, it might still be possible to derive computational methods for control and analysis of Nash systems.

It is well-known that (piecewise-)polynomial systems can be used to approximate the behavior of nonlinear systems, see [66]. Since polynomial systems are Nash, it follows that Nash systems might be used to approximate general nonlinear systems.

Polynomial and rational systems, and thus Nash systems, are used in systems biology to model metabolic, signaling, and genetic networks. In [91, 92] Savageau proposes the power-law framework, also known as Biochemical Systems Theory, for modeling metabolic and gene-regulatory networks by dynamical systems which still belong to the class of Nash systems, but are more general than polynomial and rational systems. In this framework, see also [93, 119], all processes are represented as products of power-law functions, i.e. sums of products $x_1^{q_1} x_2^{q_2} \cdots x_n^{q_n}$ of state variables $x_1,\dots,x_n$ taken to a rational power (q_i, $i=1,\dots,n$ are rational numbers). Such functions are a special case of Nash functions. Note that the values of rational exponents (kinetic orders) in power-law systems can be related to parameters of different rate laws such as for example Michaelis-Menten kinetics mentioned in Section 3.1, see [119]. The role of power-law systems in modeling of signal transduction pathways is discussed in [116]. For further examples and details on application of power-law systems in biology see [118, 62].

There are several frameworks for modeling metabolic networks. Some of them (including power-law framework) are compared in [48]. The tendency modeling framework which extends the power-law framework combines mass-action and power-law kinetics into tendency kinetics, [117]. It also leads to Nash systems.

Further, considering Nash submanifolds as the state-spaces allows a natural implementation of possible conservation laws and restrictions on the state variables. For example, one can capture the assumption on positivity of state variables which represent the concentrations of reactants of a chemical system by defining the state-space as the positive orthant of $\mathbb{R}^n$ which is a Nash submanifold.

6.2 Nash systems

Nash system is a dynamical system with inputs and outputs such that its dynamics is specified by Nash functions. We consider Nash systems with the following fixed input- and output-spaces. The *input-space* U is a subset of $\mathbb{R}^m$. The *output-space* is $\mathbb{R}^r$. As the *space of input functions* for a Nash system with a given input-space U we consider the set $\mathscr{U}_{pc}$ of piecewise-constant functions $u:[0,T_u]\to U$, see Section 3.2.

Definition 6.1. A *Nash system* Σ with an input-space U and an output-space $\mathbb{R}^r$ is a quadruple (X,f,h,x_0) where

(i) the state-space X is a Nash submanifold of $\mathbb{R}^n$ which is semi-algebraically connected,
(ii) the dynamics of the system is given by $\dot{x}(t)=f(x(t),u(t))$ for an input $u\in\mathscr{U}_{pc}$, where $f:X\times U\to\mathbb{R}^n$ is such that for every input value $\alpha\in U$ the components $f_{\alpha,i}:X\to\mathbb{R}$, $i=1,\dots,n$ of the map $f(x,\alpha)=(f_{\alpha,1}(x),\dots,f_{\alpha,n}(x))$ are Nash functions on X ($f_{\alpha,i}$ is the ith coordinate of the vector field $f_\alpha : X\ni x\mapsto f(x,\alpha)\in\mathbb{R}^n$),
(iii) the output of the system is specified by the map $h:X\to\mathbb{R}^r$, the components $h_1,\dots,h_r$ of h are Nash functions on X,
(iv) $x_0=x(0)\in X$ is the initial state of Σ.

As an example of a Nash system we provide a simplified model of glycolysis in Lactococcus lactis introduced in [118].

Example 6.2. The following model of glycolysis and lactate production in Lactococcus lactis is adopted from the supplements of [118]:

$$
\begin{aligned}
\dot{x}_1 &= 0.3592 x_1^{-1.2906} x_4^{0.2168} Glc^{1.1287} - 0.3115 x_1^{2.17} ATP^{0.8152}, \\
\dot{x}_2 &= 0.3115 x_1^{2.17} ATP^{0.8152} - 0.4698 x_2^{1.0297} P_i^{0.2377}, \\
\dot{x}_3 &= 0.9396 x_2^{1.0297} P_i^{0.2377} + 1.1452 x_4^{3.5453} - 2.167 x_3^{2.1649}, \\
\dot{x}_4 &= 2.167 x_3^{2.1649} - 0.3592 x_1^{-1.2906} x_4^{0.2168} Glc^{1.1287} - 1.1452 x_4^{3.5453} - 0.2087 x_4^{0.0002} \\
&\quad - 0.9375 x_2^{0.8744} x_4^{0.0991} Pi^{-0.0005}, \\
\dot{x}_5 &= 0.3592 x_1^{-1.2906} x_4^{0.2168} Glc^{1.1287} - 0.0417 x_5^{0.6202} x_2^{0.9264} - 1.3258 x_5^{1.5255} \\
&\quad + 0.9375 x_2^{0.8744} x_4^{0.0991} Pi^{-0.0005}, \\
\dot{x}_6 &= 0.0417 x_5^{0.6202} x_2^{0.9264}.
\end{aligned}
$$

Here $x_1, \ldots, x_6$ are the concentrations of the respective metabolites (G6P, FBP, PGA3, PEP, pyruvate, lactate). The state-space $X = (0, +\infty)^6$ is a Nash submanifold of $\mathbb{R}^6$ which is semi-algebraically connected. The initial state is $x_1(0) = \cdots = x_6(0) = 1$. The output function is the outflow of the pathway, i.e. $h(x_1, \ldots, x_6) = x_6$. The inputs are the concentrations of Glc, ATP and Pi of external glucose, ATP, and inorganic phosphate, respectively.

Note that this system is neither polynomial nor rational, but still Nash. Indeed, h is linear and hence Nash function of the state. The right-hand sides of the differential equations are linear combinations of terms of the form $x_1^{q_1} \cdots x_6^{q_6}$ where q_i, $i = 1, \ldots, 6$ is a rational number. Since Nash functions are closed under linear combination, multiplication and division, it remains to be shown that the map $g(x) = x^{\frac{n}{d}}$, where $x \in (0, +\infty)$ and $n, d \in \mathbb{N}$, is Nash. Since $g(x)$ is analytic, it is sufficient to show that g is semi-algebraic. For all $x \in (0, +\infty)$ and for all $n, d \in \mathbb{N}$ it holds that $y = g(x)$ if and only if $P(y, x) = 0$, where P is the polynomial defined as $P(Y, X) = Y^d - X^n$. Therefore, g is a semi-algebraic function.

Definition 6.3. The *state trajectory* of a Nash system $\Sigma = (X, f, h, x_0)$ corresponding to an input $u = (\alpha_1, t_1) \cdots (\alpha_k, t_k) \in \mathscr{U}_{pc}$, $u : [0, T_u] \to U$ is a continuous piecewise-differentiable function $x_\Sigma(.; x_0, u) : [0, T_u] \to X$ such that $x_\Sigma(0; x_0, u) = x_0$ and

$$
\frac{d}{dt} x_\Sigma(t; x_0, u) = f(x_\Sigma(t; x_0, u), u(t)) \tag{6.1}
$$

for $t \in [\sum_{j=0}^{i} t_j, \sum_{j=0}^{i+1} t_j]$, $i = 0, \ldots, k-1$, $t_0 = 0$.

The trajectory of a Nash system Σ does not need to exist for every input $u \in \mathscr{U}_{pc}$. In order to deal with this phenomenon we introduce the notion of *admissible inputs for Nash systems* in the same way as in the case of rational systems.

Definition 6.4. A set $\mathscr{U}_{pc}(\Sigma)$ of *admissible inputs for a Nash system* $\Sigma = (X, f, h, x_0)$ is a subset of $\mathscr{U}_{pc}$ such that for all $u \in \mathscr{U}_{pc}(\Sigma)$ there exists a trajectory $x_\Sigma(\cdot; x_0, u) : [0, T_u] \to X$ of Σ corresponding to the input u.

Let $\Sigma = (X, f, h, x_0)$ be a Nash system with an input-space U. For any $\alpha \in U$ such that the Nash functions $f_{\alpha,i}, i = 1, \ldots, n$ are defined at x_0 there exists a unique trajectory of Σ corresponding to the constant input $u = (\alpha, T)$ defined on the maximal interval $[0, T)$ (T may be infinite). This follows from the fact $f(x, \alpha)$ is smooth with respect to x for every $\alpha \in U$.

6.3 Realization theory for Nash systems

In this section we formulate and partially solve the realization problem for the class of Nash systems. First, further properties of response maps and dual input-to-state maps, which are introduced in Chapter 4, are derived. Then we deal with the existence, canonicity and minimality of Nash realizations.

6.3.1 Problem formulation

Response maps characterize external behavior of a system by evaluating the outputs of the system after applying the inputs to the system only for finite time. For our purposes, the response maps introduced in Chapter 4 represent the right formalization of the external behavior of a Nash system. Thus, the response maps considered in this chapter are the maps with the components of $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ where $\widetilde{\mathscr{U}_{pc}}$ is a set of admissible inputs.

Definition 6.5. Let $\widetilde{\mathscr{U}_{pc}}$ be a set of admissible inputs. Consider a response map $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$. A Nash system $\Sigma = (X, f, h, x_0)$ such that

$$\widetilde{\mathscr{U}_{pc}} \subseteq \mathscr{U}_{pc}(\Sigma) \text{ and } p(u) = h(x_\Sigma(T_u; x_0, u)) \text{ for all } u \in \widetilde{\mathscr{U}_{pc}}$$

is called a *Nash realization* of p.

The realization problem for Nash systems consists of the same subproblems as the realization problem for rational systems stated in Section 4.3. Hence, it can be split into the subproblems which concern the existence and minimality of Nash realizations and the algorithms for computing them.

Problem 6.6. Let $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$ be a response map. Determine necessary and sufficient conditions for the existence of a (canonical, minimal) Nash realization of p. Provide the characterization of canonical and minimal Nash realizations of p. Formulate the algorithms for computing (canonical, minimal) Nash realizations of p (from finite data directly obtainable from p).

In this chapter we deal with the existence and minimality problems for Nash realizations. We do not address the algorithmic part of the realization problem.

6.3.2 Further properties of response maps

Proposition 6.7. *The zero ideal of $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ is a real ideal. Consequently, if $\mathscr{A} \subseteq \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ is a subalgebra of $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ then the field $\mathscr{Q}(\mathscr{A})$ of fractions of $\mathscr{A}$ is a real field.*

Proof. Since $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ is a commutative ring, we prove that the zero ideal $(0) \subseteq \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ is a real ideal by showing that for every sequence $\varphi_1, \dots, \varphi_k \in \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ it holds that $\sum_{i=1}^{k} \varphi_i^2 \in (0)$ implies that $\varphi_i \in (0)$ for all $i = 1, \dots, k$.

Let $k \in \mathbb{N}$. Consider arbitrary $\varphi_1, \dots, \varphi_k \in \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ such that $\sum_{i=1}^{k} \varphi_i^2 \in (0)$. Then $\sum_{i=1}^{k} (\varphi_i(u))^2 = 0$ for every $u \in \widetilde{\mathscr{U}_{pc}}$. Because $\varphi_i(u) \in \mathbb{R}$ for all $i = 1, \dots, k$, $u \in \widetilde{\mathscr{U}_{pc}}$, it follows that $\varphi_i(u) = 0$ for all $i = 1, \dots, k$, $u \in \widetilde{\mathscr{U}_{pc}}$. Thus, $\varphi_i \in (0)$ for all $i = 1, \dots, k$.

Assume that $\mathscr{A}$ is a subalgebra of $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$. Then the zero ideal of $\mathscr{A}$ is real as well. Moreover, since $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ and hence also $\mathscr{A}$ are integral domains, the zero ideal of $\mathscr{A}$ is prime. By Lemma 2.20, the field $\mathscr{Q}(\mathscr{A})$ of fractions of $\mathscr{A} = \mathscr{A}/(0)$ is a real field. $\square$

We define the notion of Nash extension of a finite subset of maps analytic in switching times.

Definition 6.8. Let X be a Nash submanifold of $\mathbb{R}^n$ which is semi-algebraically connected, let $\widetilde{\mathscr{U}_{pc}}$ be a set of admissible inputs, and let $\mathscr{A} = \{\varphi_1, \dots, \varphi_n\}$ be a subset of $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$. Assume that for all $u \in \widetilde{\mathscr{U}_{pc}}$, $(\varphi_1(u), \dots, \varphi_n(u)) \in X$. The *Nash extension* $\mathscr{A}^{Nash}(X)$ of $\mathscr{A}$ with respect to X is the subalgebra of $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ generated by the maps $\widetilde{\mathscr{U}_{pc}} \ni u \mapsto q(\varphi_1(u), \dots, \varphi_n(u)) \in \mathbb{R}$, where $q \in \mathscr{N}(X)$.

Intuitively, the Nash extension of $\mathscr{A}$ is obtained by substituting the elements of $\mathscr{A}$ into Nash functions defined on X. Note that the set of substitutions into linear forms (polynomials) yields the linear space (algebra) generated by $\mathscr{A}$. Then, the Nash extension of $\mathscr{A}$ can be thought of as the generalization of the notions of linear space and algebra generated by $\mathscr{A}$.

Based on Definition 6.8 we introduce Nash extensions of observation algebras and observation fields of response maps.

Definition 6.9. Let $\widetilde{\mathscr{U}_{pc}}$ be a set of admissible inputs and let $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$ be a response map. We define

$$A_{obs}^{Nash}(p) = \{g : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R} \mid \exists k \in \mathbb{N}\ \exists \varphi_1, \dots, \varphi_k \in A_{obs}(p)\ \exists q \in \mathscr{N}(\mathbb{R}^k)\ \forall u \in \widetilde{\mathscr{U}_{pc}} : \\ g(u) = q(\varphi_1(u), \dots, \varphi_k(u))\},$$
$$Q_{obs}^{Nash}(p) = \mathscr{Q}(A_{obs}^{Nash}(p)).$$

Corollary 6.10. *Let $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$ be a response map. Recall that the observation algebra $A_{obs}(p)$ of p and its Nash extension $A_{obs}^{Nash}(p)$ are subalgebras of $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to$*

$\mathbb{R}$). *Then, from Proposition 6.7 and from the definitions of* $Q_{obs}(p)$ *and* $Q_{obs}^{Nash}(p)$, *it follows that the observation field* $Q_{obs}(p)$ *and the field* $Q_{obs}^{Nash}(p)$ *are real fields.*

Proposition 6.11. *For every response map p it holds that* $Q_{obs}(p) \subseteq Q_{obs}^{Nash}(p)$.

Proof. This follows from Definition 6.9 and the fact that the identity on $\mathbb{R}$ is a Nash function. □

Because $A_{obs}^{Nash}(p) \subseteq (Q_{obs}(p))^{Nash} \subseteq Q_{obs}^{Nash}(p)$ and consequently $Q_{obs}^{Nash}(p) = \mathscr{Q}(A_{obs}^{Nash}(p)) \subseteq \mathscr{Q}((Q_{obs}(p))^{Nash}) \subseteq \mathscr{Q}(Q_{obs}^{Nash}(p)) = Q_{obs}^{Nash}(p)$, there follows the equality $Q_{obs}^{Nash}(p) = \mathscr{Q}((Q_{obs}(p))^{Nash})$. If it holds that the Nash extension of $Q_{obs}(p)$ is closed with respect to taking fractions then $Q_{obs}^{Nash}(p) = (Q_{obs}(p))^{Nash}$.

We will argue that the transcendence degree of $A_{obs}^{Nash}(p)$ and $A_{obs}(p)$ coincide. This follows from a more general lemma which is also used for the derivation of a necessary condition for the existence of Nash realizations, see Proposition 6.25.

Lemma 6.12. *Let* $S \subseteq \mathbb{R}^n$ *be a semi-algebraic set, let* $\psi_1, \dots, \psi_n \in \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ *be arbitrary maps such that* $\forall u \in \widetilde{\mathscr{U}_{pc}} : \xi(u) = (\psi_1(u), \dots, \psi_n(u)) \in S$, *and let* $f : S \to \mathbb{R}$ *be a semi-algebraic map. Consider a map* $\psi \in \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ *defined as*

$$\forall u \in \widetilde{\mathscr{U}_{pc}} : \psi(u) = f(\xi(u)) = f(\psi_1(u), \dots, \psi_n(u)).$$

Then ψ *is algebraic over* $\mathbb{R}[\psi_1, \dots, \psi_n]$, *i.e. there exists* $P \in \mathbb{R}[X_1, \dots, X_n, Y]$ *such that* $P(\psi_1, \dots, \psi_n, Y)$ *is a non-zero polynomial in* Y *and* $P(\psi_1, \dots, \psi_n, \psi) = 0$. *In other words, there exists an input* $u \in \widetilde{\mathscr{U}_{pc}}$ *such that* $P(\psi_1(u), \dots, \psi_n(u), Y)$ *is a non-zero one-variable polynomial with real coefficients, and*

$$\forall u \in \widetilde{\mathscr{U}_{pc}} : P(\psi_1(u), \dots, \psi_n(u), \psi(u)) = 0.$$

Proof. From Lemma 2.23, there exist semi-algebraic subsets $S_1, \dots, S_m$ of S and polynomials $g_i(X_1, \dots, X_n, Y) \in \mathbb{R}[X_1, \dots, X_n, Y]$, $i = 1, \dots, m$ such that $S = \bigcup_{i=1}^m S_i$, $S_i \cap S_j = \emptyset$ for all $i \neq j \in \{1, \dots, m\}$ and such that for all $x \in S_i$, $g_i(x, Y)$ is a non-zero polynomial and $g_i(x, f(x)) = 0$.

For every $u \in \widetilde{\mathscr{U}_{pc}}$ there exists a unique index $i(u) \in \{1, \dots, m\}$ such that $\xi(u) = (\psi_1(u), \dots, \psi_n(u)) \in S_{i(u)}$. Indeed, $\xi(u) \in S$, and hence $\xi(u)$ belongs to one of the disjoint sets $S_{i(u)}$. Then $g_{i(u)}(\xi(u), Y) \neq 0$ and $g_{i(u)}(\xi(u), f(\xi(u)) = g_{i(u)}(\xi(u), \psi(u)) = 0$. Consider the subset $I = \{i(u) \mid u \in \widetilde{\mathscr{U}_{pc}}\} \subseteq \{1, \dots, m\}$. Assume that $I = \{i_1, \dots, i_k\}$. Consider the polynomial $P = g_{i_1} \cdots g_{i_k} \in \mathbb{R}[X_1, \dots, X_n, Y]$. It follows that $P(\xi(u), \psi(u)) = g_{i_1}(\xi(u), \psi(u)) \cdots g_{i_k}(\xi(u), \psi(u)) = 0$ for all $u \in \widetilde{\mathscr{U}_{pc}}$. If we prove that P is a non-zero polynomial in Y, i.e. $P(\xi(u), Y) \neq 0$ for some $u \in \widetilde{\mathscr{U}_{pc}}$, then the proof is complete.

The fact that there exists $u \in \widetilde{\mathscr{U}_{pc}}$ such that $P(\xi(u), Y) \neq 0$ is equivalent to the requirement that $P(\psi_1, \dots, \psi_n, Y) \in \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})[Y]$, the polynomial in one variable Y with coefficients in $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$, is not identically zero. Since $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ is an integral domain, Theorem 2.1 implies that the algebra $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})[Y]$ is also an integral domain. Hence, if $P(\psi_1,\dots,\psi_n,Y) = \Pi_{j=1}^{k} g_{i_j}(\psi_1,\dots,\psi_n,Y)$, where $g_{i_j}(\psi_1,\dots,\psi_n,Y) \in \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})[Y]$, equals zero as an element of $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})[Y]$ then there exists at least one $i \in I$ such that $g_i(\psi_1,\dots,\psi_n,Y) = 0$. The latter means that $g_i(\psi_1(u),\dots,\psi_n(u),Y) = g_i(\xi(u),Y) = 0$ for all inputs $u \in \widetilde{\mathscr{U}_{pc}}$. But there exists at least one input $\hat{u}$ such that $i = i(\hat{u})$, i.e. such that $\xi(\hat{u}) \in S_i$ and $g_i(\xi(\hat{u}),Y) \neq 0$. This is a contradiction. Therefore, we get that $P(\psi_1,\dots,\psi_n,Y) \neq 0$. □

Proposition 6.13. *Let $\widetilde{\mathscr{U}_{pc}}$ be a set of admissible inputs and let $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$ be a response map. Then $A_{obs}^{Nash}(p)$ is algebraic over $A_{obs}(p)$ and consequently* trdeg $A_{obs}^{Nash}(p) =$ trdeg $A_{obs}(p)$.

Proof. Consider an arbitrary $g \in A_{obs}^{Nash}(p)$. According to Definition 6.9 there exist $k \in \mathbb{N}$, $\varphi_1,\dots,\varphi_k \in A_{obs}(p)$, and $q \in \mathscr{N}(\mathbb{R}^k)$ such that

$$g(u) = q(\varphi_1(u),\dots,\varphi_k(u)) \text{ for all } u \in \widetilde{\mathscr{U}_{pc}}.$$

Then, by Lemma 6.12, g is algebraic over $\mathbb{R}[\varphi_1,\dots,\varphi_k] \subseteq A_{obs}(p)$. Since $g \in A_{obs}^{Nash}(p)$ was arbitrary, $A_{obs}^{Nash}(p)$ is algebraic over $A_{obs}(p)$. This and the definition of transcendence degree imply that trdeg $A_{obs}^{Nash}(p) =$ trdeg $A_{obs}(p)$. □

Remark 6.14. Once $Q_{obs}^{Nash}(p) = (Q_{obs}(p))^{Nash}$, one can prove in the same way as in the proof of Proposition 6.13 that $Q_{obs}^{Nash}(p)$ is algebraic over $Q_{obs}(p)$ and derive that trdeg $Q_{obs}^{Nash}(p) =$ trdeg $Q_{obs}(p)$. Then, from Corollary 6.10, it would follow that for every response map p the field $Q_{obs}^{Nash}(p)$ is a subfield of the real closure of $Q_{obs}(p)$. Moreover, it would hold that the real closures of $Q_{obs}(p)$ and $Q_{obs}^{Nash}(p)$ are the same.

6.3.3 Properties of dual input-to-state maps

In this section we introduce the notions of input-to-state map and dual input-to-state map of a Nash system. See Definition 3.7 and Definition 4.9 for the same concepts in the case of polynomial and rational systems.

Definition 6.15. Let $\Sigma = (X,f,h,x_0)$ be a Nash realization of a response map $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$ and let $x_\Sigma(\cdot;x_0,u)$ be the trajectory of Σ corresponding to an input $u \in \widetilde{\mathscr{U}_{pc}}$. The map $\tau_\Sigma : \widetilde{\mathscr{U}_{pc}} \to X$ defined as $\tau_\Sigma(u) = x_\Sigma(T_u;x_0,u)$ for all $u \in \widetilde{\mathscr{U}_{pc}}$ is called *input-to-state map*. By the *dual input-to-state map* of Σ we mean the map $\tau_\Sigma^* : \mathscr{N}(X) \to \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ such that for every Nash function $g \in \mathscr{N}(X)$ it holds that $\tau_\Sigma^*(g) = g \circ \tau_\Sigma$, i.e.

$$\forall u \in \widetilde{\mathscr{U}_{pc}} : \tau_\Sigma^*(g)(u) = g(\tau_\Sigma(u)) = g(x_\Sigma(T_u;x_0,u)).$$

Intuitively, the dual input-to-state map of a Nash system $\Sigma = (X, f, h, x_0)$ maps each Nash function g on X to the response map which is generated by the Nash system $\Sigma_2 = (X, f, g, x_0)$ where g is used instead of h as the output map.

The dual input-to-state map plays a role similar to the observability Gramian of linear systems. In particular, it allows us to relate the properties of a response map p to the properties of the ring of Nash functions on the state-space X of a Nash realization $\Sigma = (X, f, h, x_0)$ of p.

Definition 6.16. The *observation algebra* $A_{obs}(\Sigma)$ of a Nash system $\Sigma = (X, f, h, x_0)$ is the smallest subalgebra of $\mathcal{N}(X)$ which contains h_i, $i = 1, \dots, r$ and which is closed under taking Lie-derivatives with respect to the vector fields $f_\alpha : X \ni x \mapsto f(x, \alpha) \in \mathbb{R}^n$, $\alpha \in U$.

Proposition 6.17. *Let $\Sigma = (X, f, h, x_0)$ be a Nash realization of a response map $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$ and let $\tau_\Sigma^* : \mathcal{N}(X) \to \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ be as in Definition 6.15. Then,*

(i) the ideal Ker τ_Σ^* *is a finitely generated prime ideal of $\mathcal{N}(X)$,*
(ii) the set $\mathscr{Z}_X(\text{Ker } \tau_\Sigma^*) = \{x \in X \mid \forall g \in \text{Ker } \tau_\Sigma^* : g(x) = 0\}$ of zeros of the ideal Ker τ_Σ^* *is a Nash subset of X which is semi-algebraically connected,*
(iii) $\forall g \in \mathcal{N}(X)\ \forall \alpha \in U : D_\alpha \tau_\Sigma^*(g) = \tau_\Sigma^*(L_{f_\alpha} g)$, where $L_{f_\alpha} g$ is the Lie-derivative of the map g with respect to the vector field $f_\alpha : X \ni x \mapsto f(x, \alpha)$, and D_α is the derivation defined in Definition 4.2,
(iv) $A_{obs}(p) \subseteq \tau_\Sigma^*(\mathcal{N}(X))$, especially $A_{obs}(p) = \tau_\Sigma^*(A_{obs}(\Sigma))$,
(v) $\forall f \in \mathcal{N}(\mathbb{R}^k)\ \forall \varphi_1, \dots, \varphi_k \in \mathcal{N}(X)\ \forall u \in \widetilde{\mathscr{U}_{pc}}$:

$$\tau_\Sigma^*(f(\varphi_1, \dots, \varphi_k))(u) = f(\tau_\Sigma^*(\varphi_1)(u), \dots, \tau_\Sigma^*(\varphi_k)(u)).$$

Proof. (i) Because $\tau_\Sigma^* : \mathcal{N}(X) \to \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ is a homomorphism and because $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ is an integral domain, the ideal Ker τ_Σ^* is prime. Further, since X is a Nash submanifold, it follows from Theorem 2.28 that any ideal of $\mathcal{N}(X)$ has a finite system of generators. Therefore, the ideal Ker $\tau_\Sigma^* \subseteq \mathcal{N}(X)$ is finitely generated.

(ii) From (i), Ker τ_Σ^* is a prime ideal of $\mathcal{N}(X)$. By Theorem 2.27, the set $\mathscr{Z}_X(\text{Ker } \tau_\Sigma^*)$ is a Nash subset of X which is semi-algebraically connected.

(iii) We show that $\forall g \in \mathcal{N}(X)\ \forall \alpha \in U\ \forall u \in \widetilde{\mathscr{U}_{pc}} : D_\alpha \tau_\Sigma^*(g)(u) = \tau_\Sigma^*(L_{f_\alpha} g)(u)$. Let $g \in \mathcal{N}(X)$, $\alpha \in U$, $u \in \widetilde{\mathscr{U}_{pc}}$ be arbitrary. Then,

$$\begin{aligned}
&D_\alpha \tau_\Sigma^*(g)(u) \\
&= D_\alpha g(\tau_\Sigma(u)) = D_\alpha g(x_\Sigma(T_u; x_0, u)) = \frac{d}{dt} g(x_\Sigma(T_u + t; x_0, (u)(\alpha, t)))|_{t=0+} \\
&= \sum_{i=1}^{n} \left[\frac{d}{dx_{\Sigma,i}} g(x_\Sigma(T_u + t; x_0, (u)(\alpha, t))) \frac{d}{dt} x_{\Sigma,i}(T_u + t; x_0, (u)(\alpha, t)) \right]_{t=0+} \\
&= \sum_{i=1}^{n} \frac{d}{dx_{\Sigma,i}} g(x_\Sigma(T_u; x_0, u)) f_i(x_\Sigma(T_u; x_0, u), \alpha) = (L_{f_\alpha} g)(x_\Sigma(T_u; x_0, u)) \\
&= (L_{f_\alpha} g)(\tau_\Sigma(u)) = \tau_\Sigma^*(L_{f_\alpha} g)(u).
\end{aligned}$$

(iv) From Definition 6.16, $A_{obs}(\Sigma) \subseteq \mathscr{N}(X)$. To prove (iv) it is sufficient to prove that $A_{obs}(p) = \tau_\Sigma^*(A_{obs}(\Sigma))$.

Because $\Sigma = (X, f, h, x_0)$ is a Nash realization of a response map $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$, it holds that $p_i(u) = h_i(\tau_\Sigma(u))$ for all $u \in \widetilde{\mathscr{U}_{pc}}$ and $i = 1, \ldots, r$. Thus,

$$p_i = \tau_\Sigma^*(h_i) \text{ for all } i = 1, \ldots, r. \tag{6.2}$$

Further, since $h_i \in \mathscr{N}(X)$ for $i = 1, \ldots, r$, from (iii) it follows that $D_\alpha p_i(u) = D_\alpha h_i(\tau_\Sigma(u)) = D_\alpha \tau_\Sigma^*(h_i)(u) = \tau_\Sigma^*(L_{f_\alpha} h_i)(u)$ for all $\alpha \in U$, $u \in \widetilde{\mathscr{U}_{pc}}$, and $i = 1, \ldots, r$. Hence,

$$D_\alpha p_i = \tau_\Sigma^*(L_{f_\alpha} h_i) \text{ for all } i = 1, \ldots, r, \alpha \in U. \tag{6.3}$$

Because $A_{obs}(p)$ is generated by the elements of the set $\{p_i, D_{\alpha_1} \cdots D_{\alpha_k} p_i \mid i = 1, \ldots, r; k \in \mathbb{N}; \alpha_1, \ldots, \alpha_k \in U\}$, because $A_{obs}(\Sigma)$ is generated by the elements of the set $\{h_i, L_{f_{\alpha_1}} \cdots L_{f_{\alpha_k}} h_i \mid i = 1, \ldots, r; k \in \mathbb{N}; \alpha_1, \ldots, \alpha_k \in U\}$, and because τ_Σ^* is a homomorphism, (6.2) and (6.3) imply that $A_{obs}(p) = \tau_\Sigma^*(A_{obs}(\Sigma))$.

(v) Because $\tau_\Sigma^*(f(\varphi_1, \ldots, \varphi_k))(u) = f(\varphi_1(x_\Sigma(T_u; x_0, u)), \ldots, \varphi_k(x_\Sigma(T_u; x_0, u)))$, this statement follows directly from the definition of τ_Σ^*. □

Corollary 6.18. *Proposition 6.17(iv) implies that* trdeg $A_{obs}(p) \leq$ trdeg $\tau_\Sigma^*(\mathscr{N}(X))$ *and that* trdeg $A_{obs}(p) \leq$ trdeg $A_{obs}(\Sigma)$.

In particular, we can state the following relationship between the transcendence degree of the observation algebra of a response map p and the transcendence degree of the observation algebra of a Nash realization of p.

Corollary 6.19. *Let Σ be a Nash realization of a response map p. Then, τ_Σ^* is injective if and only if* trdeg $A_{obs}(p) =$ trdeg $A_{obs}(\Sigma)$.

The result above is important for linking canonicity and minimality of Nash realizations, see Theorem 6.44.

Corollary 6.20. *Let $\Sigma = (X \subseteq \mathbb{R}^n, f, h, x_0)$ be a Nash realization of a response map $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$. Let $\varphi_i \in \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$, $i = 1, \ldots, n$ be the maps defined as $\varphi_i(u) = x_{\Sigma,i}(T_u; x_0, u)$ for all $u \in \widetilde{\mathscr{U}_{pc}}$, i.e. $\varphi_i(u)$ is the ith component of the state of Σ at time T_u under the input $u \in \widetilde{\mathscr{U}_{pc}}$. For $i = 1, \ldots, n$ and for $\alpha_1, \ldots, \alpha_k \in U$, $k \in \mathbb{N}$ it holds that*

$$D_{\alpha_k} \cdots D_{\alpha_1} \varphi_i = (L_{f_{\alpha_k}} \cdots L_{f_{\alpha_1}} pr_i)(\varphi_1, \ldots, \varphi_n),$$

where $pr_i : X \to \mathbb{R} : (x_1, \ldots, x_n) \mapsto x_i$ is the restriction of the projection on the ith coordinate to X.

Proof. Let us prove this corollary by induction in k.

Base case: $k = 1$. Let $\alpha_1 \in U$ be arbitrary. For all $i = 1, \ldots, n$ it holds that $D_{\alpha_1} \tau_\Sigma^*(pr_i) = D_{\alpha_1}(pr_i \circ \tau_\Sigma) = D_{\alpha_1}(pr_i(\varphi_1, \ldots, \varphi_n)) = D_{\alpha_1} \varphi_i$. Because pr_i is a Nash function on X, Proposition 6.17(iii) implies that $D_{\alpha_1} \tau_\Sigma^*(pr_i) = \tau_\Sigma^*(L_{f_{\alpha_1}} pr_i)$. Then, $D_{\alpha_1} \varphi_i = \tau_\Sigma^*(L_{f_{\alpha_1}} pr_i) = L_{f_{\alpha_1}} pr_i(\varphi_1, \ldots, \varphi_n)$ for all $i = 1, \ldots, n$ and $\alpha_1 \in U$.

Inductive hypothesis: Let $D_{\alpha_k} \cdots D_{\alpha_1} \varphi_i = (L_{f_{\alpha_k}} \cdots L_{f_{\alpha_1}} pr_i)(\varphi_1, \dots, \varphi_n)$ for $i = 1, \dots, n$ and for $\alpha_1, \dots, \alpha_k \in U$. Consider arbitrary $\alpha_1, \dots, \alpha_k, \alpha_{k+1} \in U$ and arbitrary $i = 1, \dots, n$. Because $pr_i \in \mathscr{N}(X)$, we obtain that $L_{f_{\alpha_k}} \cdots L_{f_{\alpha_1}} pr_i \in \mathscr{N}(X)$. Then, from Proposition 6.17(iii),

$$\begin{aligned} D_{\alpha_{k+1}} \tau_\Sigma^*(L_{f_{\alpha_k}} \cdots L_{f_{\alpha_1}} pr_i) &= \tau_\Sigma^*(L_{f_{\alpha_{k+1}}} L_{f_{\alpha_k}} \cdots L_{f_{\alpha_1}} pr_i) \\ &= (L_{f_{\alpha_{k+1}}} L_{f_{\alpha_k}} \cdots L_{f_{\alpha_1}} pr_i)(\varphi_1, \dots, \varphi_n). \end{aligned}$$

The induction hypothesis implies that

$$\begin{aligned} D_{\alpha_{k+1}} \tau_\Sigma^*(L_{f_{\alpha_k}} \cdots L_{f_{\alpha_1}} pr_i) &= D_{\alpha_{k+1}} \Big((L_{f_{\alpha_k}} \cdots L_{f_{\alpha_1}} pr_i)(\varphi_1, \dots, \varphi_n) \Big) \\ &= D_{\alpha_{k+1}} (D_{\alpha_k} \cdots D_{\alpha_1} \varphi_i) = D_{\alpha_{k+1}} \cdots D_{\alpha_1} \varphi_i. \end{aligned}$$

Therefore, $D_{\alpha_{k+1}} \cdots D_{\alpha_1} \varphi_i = (L_{f_{\alpha_{k+1}}} \cdots L_{f_{\alpha_1}} pr_i)(\varphi_1, \dots, \varphi_n)$ for all $i = 1, \dots, n$ and for all $\alpha_1, \dots, \alpha_{k+1} \in U$. $\square$

Let us formulate an extension of Proposition 6.17(iii) which generalizes Corollary 6.20. To prove this proposition we proceed in the same way as in the proof of Corollary 6.20.

Proposition 6.21. *Let $\Sigma = (X, f, h, x_0)$ be a Nash system such that $\widetilde{\mathscr{U}_{pc}} \subseteq \mathscr{U}_{pc}(\Sigma)$ for a set $\widetilde{\mathscr{U}_{pc}}$ of admissible inputs. Let $\tau_\Sigma^* : \mathscr{N}(X) \to \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ be as in Definition 6.15. Then,*

$$\forall g \in \mathscr{N}(X)\ \forall k \in \mathbb{N}\ \forall \alpha_1, \dots, \alpha_k \in U :\ D_{\alpha_k} \cdots D_{\alpha_1} \tau_\Sigma^*(g) = \tau_\Sigma^*(L_{f_{\alpha_k}} \cdots L_{f_{\alpha_1}} g).$$

Proof. Let $g \in \mathscr{N}(X)$ be arbitrary. The statement for $k = 1$ follows from Proposition 6.17(iii). Let us assume as an inductive hypothesis that $D_{\alpha_k} \cdots D_{\alpha_1} \tau_\Sigma^*(g) = \tau_\Sigma^*(L_{f_{\alpha_k}} \cdots L_{f_{\alpha_1}} g)$ for some $k \in \mathbb{N}$ and for all $\alpha_1, \dots, \alpha_k \in U$. We show that the same holds for $k+1$ and for all $\alpha_1, \dots, \alpha_{k+1} \in U$.

Consider arbitrary $\alpha_1, \dots, \alpha_{k+1} \in U$. It holds that $L_{f_{\alpha_k}} \cdots L_{f_{\alpha_1}} g \in \mathscr{N}(X)$ because $g \in \mathscr{N}(X)$. Then Proposition 6.17(iii) implies that $\tau_\Sigma^*(L_{f_{\alpha_{k+1}}}(L_{f_{\alpha_k}} \cdots L_{f_{\alpha_1}} g)) = D_{\alpha_{k+1}} \tau_\Sigma^*(L_{f_{\alpha_k}} \cdots L_{f_{\alpha_1}} g)$. Next, by the inductive hypothesis, $D_{\alpha_{k+1}} \tau_\Sigma^*(L_{f_{\alpha_k}} \cdots L_{f_{\alpha_1}} g) = D_{\alpha_{k+1}}(D_{\alpha_k} \cdots D_{\alpha_1} \tau_\Sigma^*(g)) = D_{\alpha_{k+1}} D_{\alpha_k} \cdots D_{\alpha_1} \tau_\Sigma^*(g)$. Therefore, $\tau_\Sigma^*(L_{f_{\alpha_{k+1}}} \cdots L_{f_{\alpha_1}} g) = \tau_\Sigma^*(L_{f_{\alpha_{k+1}}}(L_{f_{\alpha_k}} \cdots L_{f_{\alpha_1}} g)) = D_{\alpha_{k+1}} \cdots D_{\alpha_1} \tau_\Sigma^*(g)$. This completes the proof. $\square$

6.3.4 Existence of Nash realizations

In this section we derive necessary and sufficient conditions for the existence of a Nash realization of a response map.

Theorem 6.22. *A response map $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$ has a Nash realization if and only if there exist a finite subset $\mathscr{A} = \{\varphi_1, \dots, \varphi_n\}$ of $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ and a Nash submanifold $X \subseteq \mathbb{R}^n$ which is semi-algebraically connected such that $(\varphi_1(u), \dots, \varphi_n(u)) \in X$ for all $u \in \widetilde{\mathscr{U}_{pc}}$ and such that*

(i) $p_i \in \mathscr{A}^{Nash}(X)$ for $i = 1, \dots, r$,
(ii) $\forall \varphi \in \mathscr{A}^{Nash}(X)\ \forall \alpha \in U : D_\alpha \varphi \in \mathscr{A}^{Nash}(X)$.

Proof. ($\Rightarrow$) Let $\Sigma = (X \subseteq \mathbb{R}^n, f, h, x_0)$ be a Nash realization of a response map $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$. Let $\varphi_i \in \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ and $pr_i : X \to \mathbb{R}$ for $i = 1, \dots, n$ be as in Corollary 6.20. Let us define the set $\mathscr{A} = \{\varphi_1, \dots, \varphi_n\}$. We show that $\mathscr{A}$ satisfies the conditions (i) and (ii) of the theorem.

Because Σ is a Nash realization of p, $h_i \in \mathscr{N}(X)$ and $p_i(u) = h_i(\varphi_1(u), \dots, \varphi_n(u))$ for $i = 1, \dots, r$ and for all $u \in \widetilde{\mathscr{U}_{pc}}$. Hence, by Definition 6.8, $p_i \in \mathscr{A}^{Nash}(X)$ for all $i = 1, \dots, r$.

Let $\varphi \in \mathscr{A}^{Nash}(X)$ and $\alpha \in U$ be arbitrary. We prove that $D_\alpha \varphi \in \mathscr{A}^{Nash}(X)$. From Definition 6.8 and from the definition of the maps φ_i, $i = 1, \dots, n$, there exists $q \in \mathscr{N}(X)$ such that $\varphi = q(\varphi_1, \dots, \varphi_n) = \tau_\Sigma^*(q)$. Then, Proposition 6.17$(iii)$ implies that

$$D_\alpha \varphi = D_\alpha \tau_\Sigma^*(q) = \tau_\Sigma^*(L_{f_\alpha} q) = (L_{f_\alpha} q)(\varphi_1, \dots, \varphi_n).$$

As $L_{f_\alpha} q \in \mathscr{N}(X)$, from the last equality above it follows that $D_\alpha \varphi \in \mathscr{A}^{Nash}(X)$.

($\Leftarrow$) Let $\mathscr{A} = \{\varphi_1, \dots, \varphi_n\} \subseteq \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ and $X \subseteq \mathbb{R}^n$ be as in the theorem. We will define a Nash system Σ realizing p. From (i) and (ii) of the theorem it follows that for all $i = 1, \dots, r$, $j = 1, \dots, n$, $\alpha \in U$, and $u \in \widetilde{\mathscr{U}_{pc}}$ there exist $h_i, f_{\alpha,j} \in \mathscr{N}(X)$ such that

$$p_i(u) = h_i(\varphi_1(u), \dots, \varphi_n(u)) \text{ and } D_\alpha \varphi_j(u) = f_{\alpha,j}(\varphi_1(u), \dots, \varphi_n(u)).$$

Consider the Nash system $\Sigma = (X, f, h, x_0)$ such that for all $x \in X$ and for all $\alpha \in U$ it holds that $f(x, \alpha) = (f_{\alpha,1}(x), \dots, f_{\alpha,n}(x))$, $h(x) = (h_1(x), \dots, h_r(x))$ and $x_0 = (\varphi_1(e), \dots, \varphi_n(e))$. Recall that e stands for the empty input. We prove that this system is a Nash realization of p.

For every input $u = (\alpha_1, t_1) \cdots (\alpha_k, t_k) \in \widetilde{\mathscr{U}_{pc}}$ we define the map $\psi_u : [0, T_u] \to X$ as $\psi_u(t) = (\varphi_1, \dots, \varphi_n)(u_{[0,t]})$ for $t \in [0, T_u]$. Then, for $i = 1, \dots, k$ and for $t \in [\sum_{j=0}^{i-1} t_j, \sum_{j=0}^{i} t_j]$, note that $t_0 = 0$, it holds that

$$\begin{aligned}
\dot{\psi}_u(t) &= \frac{d}{dt}\psi_u(t) = \frac{d}{d\tau}\psi_u(t+\tau)|_{\tau=0+} = \frac{d}{d\tau}(\varphi_1(u_{[0,t+\tau]}), \dots, \varphi_n(u_{[0,t+\tau]}))|_{\tau=0+} \\
&= \frac{d}{d\tau}(\varphi_1((\alpha_1, t_1) \cdots (\alpha_{i-1}, t_{i-1})(\alpha_i, t + \tau - \sum_{j=0}^{i-1} t_j)), \dots \\
&\qquad \dots, \varphi_n((\alpha_1, t_1) \cdots (\alpha_{i-1}, t_{i-1})(\alpha_i, t + \tau - \sum_{j=0}^{i-1} t_j)))|_{\tau=0+} \\
&= (D_{\alpha_i}\varphi_1(u_{[0,t]}), \dots, D_{\alpha_i}\varphi_n(u_{[0,t]}))
\end{aligned}$$

$$
\begin{aligned}
&= \Big((L_{f_{\alpha_i}} pr_1)(\varphi_1, \ldots, \varphi_n)(u_{[0,t]}), \ldots, (L_{f_{\alpha_i}} pr_n)(\varphi_1, \ldots, \varphi_n)(u_{[0,t]}) \Big) \\
&= \Big((L_{f_{\alpha_i}} pr_1)(\psi_u)(t), \ldots, (L_{f_{\alpha_i}} pr_n)(\psi_u)(t) \Big) \\
&= \left(\sum_{j=1}^{n} \frac{d}{d\psi_{u,j}} pr_1(\psi_u)(t) \cdot f_j(\psi_u(t), \alpha_i), \ldots \right. \\
&\qquad\qquad\qquad \left. \ldots, \sum_{j=1}^{n} \frac{d}{d\psi_{u,j}} pr_n(\psi_u)(t) \cdot f_j(\psi_u(t), \alpha_i) \right) \\
&= (f_1(\psi_u(t), \alpha_i), \ldots, f_n(\psi_u(t), \alpha_i)) = f(\psi_u(t), \alpha_i) = f(\psi_u(t), u(t))
\end{aligned}
$$

and

$$\psi_u(0) = (\varphi_1(e), \ldots, \varphi_n(e)) = x_0.$$

Therefore, by Definition 6.3, ψ_u is the state trajectory of Σ corresponding to the input u, i.e. $\psi_u(t) = x_\Sigma(t; x_0, u)$ for $t \in [0, T_u]$. Hence, for arbitrary $u \in \widetilde{\mathscr{U}_{pc}}$ and for all $i = 1, \ldots, r$,

$$h_i(x_\Sigma(T_u; x_0, u)) = h_i(\psi_u(T_u)) = h_i((\varphi_1, \ldots, \varphi_n)(u)) = h_i(\varphi_1(u), \ldots, \varphi_n(u)) = p_i(u).$$

Thus, Σ is indeed a Nash realization of p. $\square$

6.3.4.1 Necessary conditions

The conditions of Theorem 6.22 are rather difficult to check. In this section we derive a necessary condition for the existence of a Nash realization which is easier to check.

First, let us state a useful corollary of Lemma 2.23. Note that the proof is very similar to the proof of Lemma 6.12.

Lemma 6.23. *Let X be a Nash submanifold of $\mathbb{R}^n$ which is semi-algebraically connected. The ring $\mathscr{N}(X)$ of Nash functions on X is algebraic over the ring $\mathbb{R}[X]$ of polynomials on X.*

Proof. Consider an arbitrary $f \in \mathscr{N}(X)$. By Lemma 2.23, there exist semi-algebraic sets S_i, $i = 1, \ldots, k$ such that $X = \bigcup_{i=1}^{k} S_i$ and $S_i \cap S_j = \emptyset$ for $i \neq j$, and there exist $g_i \in \mathbb{R}[X_1, \ldots, X_n, Y]$, $i = 1, \ldots, k$ such that for each $x \in S_i$, $g_i(x, Y) \neq 0$ and $g_i(x, f(x)) = 0$. Consider the polynomial $G = g_1 \cdots g_k$. Let $pr_j : X \to \mathbb{R} : x \mapsto x_j$ be the restriction of the projection map on the jth coordinate to X. Then pr_j belongs to $\mathbb{R}[X] \subseteq \mathscr{N}(X)$ and $G(pr_1, \ldots, pr_n, Y) = \Pi_{i=1}^{k} g_i(pr_1, \ldots, pr_n, Y) \in \mathscr{N}(X)[Y]$. Because $\mathscr{N}(X)$ is an integral domain, the ring $\mathscr{N}(X)[Y]$ is an integral domain, too. Assume that $G(pr_1, \ldots, pr_n, Y) = 0$. Since $\mathscr{N}(X)[Y]$ is an integral domain, this implies that there exists $i \in \{1, \ldots, k\}$ such that $g_i(pr_1, \ldots, pr_n, Y) = 0$. Thus, $\forall x \in S_i : g_i(x, Y) = 0$ which contradicts the assumption $g_i(x, Y) \neq 0$ for $x \in S_i$.

Therefore, $G(pr_1,\ldots,pr_n,Y) \neq 0$. On the other hand, $G(pr_1,\ldots,pr_n,f) = 0$ because $\forall x \in X : G(pr_1(x),\ldots,pr_n(x),f(x)) = G(x,f(x)) = 0$. This proves that f is algebraic over pr_i, $i = 1,\ldots,n$, and thus over $\mathbb{R}[X]$. $\square$

Theorem 6.24. *Let $\Sigma = (X,f,h,x_0)$ be a Nash realization of a response map $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$. Then,* trdeg $A_{obs}(p) \leq$ dim X. *Hence, if there exists a Nash realization of a response map p then* trdeg $A_{obs}(p) < +\infty$.

Proof. Let $\Sigma = (X,f,h,x_0)$ be a Nash realization of a response map $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$ such that the state-space X is a Nash submanifold of $\mathbb{R}^n$ which is semi-algebraically connected. Let V be the Zariski closure of X. Since V is an irreducible variety, see Proposition 2.24, dim V = trdeg $\mathbb{R}[V]$. Further, from Proposition 2.22, dim V = dim X. The algebra $\mathbb{R}[X]$ of all polynomials on X coincides with the coordinate ring $\mathbb{R}[V]$ of the variety V. Because every polynomial is a Nash function, $\mathbb{R}[X] \subseteq \mathscr{N}(X)$. From Lemma 6.23 it follows that $\mathscr{N}(X)$ is algebraic over $\mathbb{R}[X]$. Then, trdeg $\mathscr{N}(X)$ = trdeg $\mathbb{R}[X]$ = trdeg $\mathbb{R}[V]$ = dim V = dim X. Finally, by applying Corollary 6.18, we derive that trdeg $A_{obs}(p) \leq$ trdeg $\tau_{\Sigma}^{*}(\mathscr{N}(X)) \leq$ trdeg $\mathscr{N}(X)$ = dim X. $\square$

This necessary condition is analogous to the finite Hankel-rank condition for linear systems. In particular, the transcendence degree of $A_{obs}(p)$ can be seen as a generalization of the rank of the Hankel-matrix for linear systems.

The proof of the following proposition is an alternative proof of the necessary condition for the existence of a Nash realization stated in Theorem 6.24.

Proposition 6.25. *Let $\Sigma = (X,f,h,x_0)$ be a Nash realization of a response map $p : \widetilde{\mathscr{U}}_{pc} \to \mathbb{R}^r$. Then* trdeg $A_{obs}(p) \leq n$ *where n is the number of components of f.*

Proof. Let $\Sigma = (X \subseteq \mathbb{R}^n,f,h,x_0)$ be a Nash realization of p. For each $i = 1,\ldots,n$ we define the map $\varphi_i \in \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ as $\varphi_i(u) = x_{\Sigma,i}(T_u;x_0,u)$ for all $u \in \widetilde{\mathscr{U}_{pc}}$, see Corollary 6.20.

Consider the algebra A_Σ generated by the maps $D_{\alpha_1}\cdots D_{\alpha_k}\varphi_i$, $D_{\beta_1}\cdots D_{\beta_l}p_j$, $i = 1,\ldots,n$, $j = 1,\ldots,r$, $\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_l \in U$, $k,l \in \mathbb{N}$. Thus, A_Σ is the smallest subalgebra of $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ which contains φ_i, p_j, $i = 1,\ldots,n$, $j = 1,\ldots,r$ and which is closed under the derivations D_α, $\alpha \in U$. Because $A_{obs}(p)$ is a subalgebra of A_Σ, it holds that if trdeg $A_\Sigma \leq n$ then trdeg $A_{obs}(p) \leq$ trdeg $A_\Sigma \leq n$.

To prove that trdeg $A_\Sigma \leq n$ it is sufficient to show that every element of A_Σ is algebraic over $\mathbb{R}[\varphi_1,\ldots,\varphi_n]$. Furthermore, as $A_\Sigma = \mathbb{R}[\{D_{\alpha_1}\cdots D_{\alpha_k}\varphi_i, D_{\beta_1}\cdots D_{\beta_l}p_j \mid i = 1,\ldots,n; j = 1,\ldots,r; \alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_l \in U; k,l \in N\}]$, it is sufficient to show that elements of the form $D_{\alpha_1}\cdots D_{\alpha_k}\varphi_i$ and $D_{\beta_1}\cdots D_{\beta_l}p_j$ are algebraic over $\mathbb{R}[\varphi_1,\ldots,\varphi_n]$.

From Corollary 6.20,

$$D_{\alpha_1}\cdots D_{\alpha_k}\varphi_i = (L_{f_{\alpha_1}}\cdots L_{f_{\alpha_k}}pr_i)(\varphi_1,\ldots,\varphi_n) \tag{6.4}$$

for $i = 1,\ldots,n$ and for $\alpha_1,\ldots,\alpha_k \in U$, $k \in \mathbb{N}$.

Because Σ is a realization of p, it holds that $p(u) = h(x_\Sigma(T_u; x_0, u))$ for all $u \in \widetilde{\mathscr{U}_{pc}}$. Thus, $p = \tau_\Sigma^* h$. Then, by Proposition 6.21,

$$D_{\beta_1} \cdots D_{\beta_l} p_j = (L_{f_{\beta_1}} \cdots L_{f_{\beta_l}} h_j)(\varphi_1, \ldots, \varphi_n) \tag{6.5}$$

for $j = 1, \ldots, r$ and for $\beta_1, \ldots, \beta_l \in U$, $l \in \mathbb{N}$. As pr_i and h_j in (6.4) and (6.5) are Nash functions on X, the maps $L_{f_{\alpha_1}} \cdots L_{f_{\alpha_k}} pr_i$ and $L_{f_{\beta_1}} \cdots L_{f_{\beta_l}} h_j$ are Nash and hence semi-algebraic functions on X. Since $(\varphi_1(u), \ldots, \varphi_n(u)) = x_\Sigma(T_u; x_0, u) \in X$ for all inputs $u \in \widetilde{\mathscr{U}_{pc}}$, Lemma 6.12 implies that the left-hand sides of (6.4) and (6.5) are algebraic over $\mathbb{R}[\varphi_1, \ldots, \varphi_n]$. This completes the proof. □

6.3.4.2 Sufficient conditions

In [124] it is shown that the condition trdeg $A_{obs}(p) < +\infty$ on a response map p is a sufficient condition for the existence of a realization of p by an input-affine rational system, if p itself has a representation by Fliess-series expansion. The class of input-affine rational systems considered in [124] is a subclass of Nash systems. Since in [124] response maps are required to admit a Fliess-series expansion and the notion of a realization is slightly different from the one adopted in this chapter, the results of [124] are only a strong indication that the condition of Theorem 6.24 might be a sufficient one too.

The question whether the condition of Theorem 6.24 is also a sufficient condition for the existence of a Nash realization is still an open problem. However, it is perhaps easier to prove for the case of so-called local Nash realizations introduced below.

Definition 6.26. A Nash system $\Sigma = (X, f, h, x_0)$ is *input-analytic* if the input-space U equals $\mathbb{R}^m$ and if all components of the map $(X \times U) \ni (x, \alpha) \mapsto f(x, \alpha) \in \mathbb{R}^n$ are analytic in both variables x and α.

Polynomial and rational systems are examples of input-analytic Nash systems. The property of analyticity in inputs is also inherited by response maps.

Definition 6.27. A response map $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$ is *input-analytic* if all components of the map

$$(U \times \mathbb{R}_+)^k \ni u = (\alpha_1, t_1) \cdots (\alpha_k, t_k) \mapsto p(u) \in \mathbb{R}^r$$

are analytic in both $\alpha_1, \ldots, \alpha_k \in U$ and $t_1, \ldots, t_k \in \mathbb{R}_+ = [0, \infty)$ for all $k \in \mathbb{N}$.

If a response map p has a realization by an input-analytic Nash system then p is also input-analytic. Let us introduce the notion of local Nash realizations for input-analytic response maps.

Definition 6.28. A Nash system $\Sigma = (X, f, h, x_0)$ is said to be a *local Nash realization* of an input-analytic response map $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$ if for all $k \in \mathbb{N}$ there exists an open neighborhood $\mathscr{U}$ of $0 \in (U \times \mathbb{R}_+)^k$ such that $p(u) = h(x_\Sigma(T_u; x_0, u)$ for all $u = (\alpha_1, t_1) \cdots (\alpha_k, t_k) \in \mathscr{U} \subseteq \widetilde{\mathscr{U}_{pc}} \subseteq \mathscr{U}_{pc}(\Sigma)$.

Note that if Σ is a Nash realization of a reponse map $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$ then it is also a local Nash realization of p. On the other hand, if Σ is a local Nash realization of p then the output generated by Σ and by p coincide for the inputs of $\widetilde{\mathscr{U}_{pc}}$ with sufficiently small input values and with sufficiently small switching times. If, in addition, Σ is input-analytic then, by the principle of analytic continuation, one derives that the response of Σ and that of p are equal on any open connected set of inputs containing the origin. For many practical situations arising in analysis and control design, the concept of local Nash realization is likely to be sufficient.

6.3.5 Semi-algebraic reachability

We define semi-algebraic reachability of Nash realizations by a slight modification of the usual concept of reachability. Namely, instead of the requirement that the whole state-space is reachable from the initial state, we only require that the set of reachable states is, in a sense, a sufficiently large subset of the state-space.

Definition 6.29. Let $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$ be a response map and let $\Sigma = (X, f, h, x_0)$ be a Nash realization of p. We denote by $\mathscr{R}(x_0)$ the set of states of Σ reachable from x_0 by the inputs of $\widetilde{\mathscr{U}_{pc}}$, i.e.

$$\mathscr{R}(x_0) = \{x_\Sigma(T_u; x_0, u) \mid u \in \widetilde{\mathscr{U}_{pc}} \subseteq \mathscr{U}_{pc}(\Sigma)\}.$$

Since $\widetilde{\mathscr{U}_{pc}} \subseteq \mathscr{U}_{pc}(\Sigma)$, the reachable set $\mathscr{R}(x_0)$ is potentially smaller than the set of all states of Σ reachable from x_0 by the inputs of $\mathscr{U}_{pc}(\Sigma)$.

Proposition 6.30. *Let $\Sigma = (X, f, h, x_0)$ be a Nash realization of a response map $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$ and let $\tau_\Sigma^* : \mathscr{N}(X) \to \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ be as in Definition 6.15. Then,*

(i) $\mathscr{R}(x_0) \subseteq \mathscr{Z}_X(\text{Ker } \tau_\Sigma^*)$,
(ii) $\text{Ker } \tau_\Sigma^* \subseteq \mathscr{I}_{\mathscr{N}(X)}(\mathscr{Z}_X(\text{Ker } \tau_\Sigma^*))$, *where* $\mathscr{I}_{\mathscr{N}(X)}(S) = \{g \in \mathscr{N}(X) \mid \forall s \in S : g(s) = 0\}$ *for a subset* $S \subseteq X$.

Proof. (i) From the definition of the kernel of τ_Σ^* and of the reachable set $\mathscr{R}(x_0)$ we obtain that $\text{Ker } \tau_\Sigma^* = \{g \in \mathscr{N}(X) \mid \tau_\Sigma^*(g) = 0\} = \{g \in \mathscr{N}(X) \mid g(\tau_\Sigma(u)) = 0 \text{ for all } u \in \widetilde{\mathscr{U}_{pc}}\} = \{g \in \mathscr{N}(X) \mid g = 0 \text{ on } \mathscr{R}(x_0)\}$. Therefore, all functions of $\text{Ker } \tau_\Sigma^*$ are zero on $\mathscr{R}(x_0)$. Since $\mathscr{Z}_X(\text{Ker } \tau_\Sigma^*) = \{x \in X \mid \forall g \in \text{Ker } \tau_\Sigma^* : g(x) = 0\}$, this implies that $\mathscr{R}(x_0) \subseteq \mathscr{Z}_X(\text{Ker } \tau_\Sigma^*)$.

(ii) This statement follows from the facts that $\text{Ker } \tau_\Sigma^* \subseteq \mathscr{N}(X)$ and that all functions of $\text{Ker } \tau_\Sigma^*$ vanish on $\mathscr{Z}_X(\text{Ker } \tau_\Sigma^*)$. □

Remark 6.31. Let $\mathbb{R}[X]$ be the algebra of all real polynomials on X. Because the polynomials of $\mathbb{R}[X]$ which vanish on $\mathscr{R}(x_0)$ belong to $\text{Ker } \tau_\Sigma^*$, it holds that $\mathscr{Z}_X(\text{Ker } \tau_\Sigma^*) \subseteq Z\text{-}cl(\mathscr{R}(x_0))$.

Next we define the notion of semi-algebraic reachability for Nash realizations.

Definition 6.32. We say that a Nash realization $\Sigma = (X, f, h, x_0)$ of a response map $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$ is *semi-algebraically reachable* if no non-zero element of $\mathscr{N}(X)$ vanishes on the set of states of Σ reachable from x_0, i.e.

$$\forall g \in \mathscr{N}(X) : (g = 0 \text{ on } \mathscr{R}(x_0) \Rightarrow g = 0).$$

Proposition 6.33. *Let $\Sigma = (X, f, h, x_0)$ be a Nash realization of a response $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$. The following statements are equivalent:*

(i) Σ is semi-algebraically reachable,
(ii) the dual input-to-state map $\tau_\Sigma^* : \mathscr{N}(X) \to \mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ is injective,
(iii) the ideal Ker τ_Σ^* *is the zero ideal in $\mathscr{N}(X)$,*
(iv) trdeg $A_{obs}(p) =$ trdeg $A_{obs}(\Sigma)$.

Proof. The equivalence $(ii) \Leftrightarrow (iii)$ is obvious. From Corollary 6.19 the relation $(ii) \Leftrightarrow (iv)$ follows. We prove that $(i) \Leftrightarrow (iii)$.

By Definition 6.29 and Definition 6.32, Σ is semi-algebraically reachable if and only if

$$\forall g \in \mathscr{N}(X) : \Big(\forall u \in \widetilde{\mathscr{U}_{pc}} : g(x_\Sigma(T_u; x_0, u)) = 0\Big) \Rightarrow g = 0.$$

Using the notation of τ_Σ^* one can reformulate this characterization as $\forall g \in \mathscr{N}(X) : \tau_\Sigma^*(g) = 0 \Rightarrow g = 0$. Thus, Σ is semi-algebraically reachable if and only if Ker $\tau_\Sigma^* = (0) \subseteq \mathscr{N}(X)$. □

If a Nash system Σ is reachable, i.e. if every state of Σ can be reached by a suitable input, then Σ is also semi-algebraically reachable. Recall from [98, Section 4.3] that a nonlinear smooth system is called accessible if the set of reachable states contains an open set. We link accessibility with semi-algebraic reachability in the following proposition.

Proposition 6.34. *Let $\Sigma = (X, f, h, x_0)$ be a Nash realization of a response $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$. If there exists a non-empty open subset S of X such that $S \subseteq \mathscr{R}(x_0)$ then Σ is semi-algebraically reachable.*

Proof. Let $g \in \mathscr{N}(X)$ be arbitrary. Because $S \subseteq \mathscr{R}(x_0)$, $g = 0$ on $\mathscr{R}(x_0)$ implies that $g = 0$ on S. Since S is a non-empty open subset of X and since the Nash submanifold X is semi-algebraically connected, from Proposition 2.26 it follows that $g = 0$. Thus, Σ is semi-algebraically reachable. □

Accessibility of a nonlinear system allows for a characterization in terms of the rank of the Lie-algebra generated by the vector fields of the system. Since Nash systems are smooth nonlinear systems, Proposition 6.34 implies that Lie-rank condition yields a sufficient condition for semi-algebraic reachability of Nash systems.

6.3.6 Semi-algebraic observability

The definition of the observation algebra of a Nash system, see Definition 6.16, is analogous to the one of rational and polynomial systems presented in Chapter 4. The notion of the Nash extension of the observation algebra of a Nash system is derived from Definition 6.8 below.

Definition 6.35. The *Nash extension* $A_{obs}^{Nash}(\Sigma)$ of the observation algebra $A_{obs}(\Sigma)$ of a Nash system $\Sigma = (X, f, h, x_0)$ is given as

$$A_{obs}^{Nash}(\Sigma) = \{g : X \to \mathbb{R} \mid \\ \exists k \in \mathbb{N}\ \exists \varphi_1, \ldots, \varphi_k \in A_{obs}(\Sigma)\ \exists q \in \mathscr{N}(\mathbb{R}^k) : g = q(\varphi_1, \ldots, \varphi_k)\}.$$

Definition 6.36. We say that a Nash system $\Sigma = (X, f, h, x_0)$ is *semi-algebraically observable* if $A_{obs}^{Nash}(\Sigma) = \mathscr{N}(X)$.

A system-theoretic interpretation of semi-algebraic observability is provided by the following proposition.

Proposition 6.37. *If a Nash system Σ is semi-algebraically observable then any two states $x_1 \neq x_2$ of Σ are distinguishable by an element of $A_{obs}(\Sigma)$, i.e. $\exists g \in A_{obs}(\Sigma)$: $g(x_1) \neq g(x_2)$.*

Proof. Let $\Sigma = (X, f, h, x_0)$ be a semi-algebraically observable Nash system. Because Nash functions on X distinguish the points of X, it holds that all states of Σ are distinguishable by the elements of $A_{obs}^{Nash}(\Sigma) = \mathscr{N}(X)$. That is,

$$\forall x_1, x_2 \in X, x_1 \neq x_2\ \exists g \in A_{obs}^{Nash}(\Sigma) : g(x_1) \neq g(x_2).$$

We prove that the same statement holds even if we can choose g only from $A_{obs}(\Sigma)$. Let us assume by contradiction that for some $x_1 \neq x_2 \in X$ there exists only $g \in A_{obs}^{Nash}(\Sigma) \setminus A_{obs}(\Sigma)$ which distinguishes x_1 and x_2. Then, by Definition 6.35, there exist $k \in \mathbb{N}$, $\varphi_1, \ldots, \varphi_k \in A_{obs}(\Sigma)$, and $q \in \mathscr{N}(\mathbb{R}^k)$ such that

$$g(x_1) = q(\varphi_1(x_1), \ldots, \varphi_k(x_1)) \neq q(\varphi_1(x_2), \ldots, \varphi_k(x_2)) = g(x_2).$$

If $\varphi_i(x_1) = \varphi_i(x_2)$ for all $i = 1, \ldots, k$ then it would imply that $q(\varphi_1(x_1), \ldots, \varphi_k(x_1)) = q(\varphi_1(x_2), \ldots, \varphi_k(x_2))$. Therefore, since $g(x_1) \neq g(x_2)$, it follows that there is at least one $i \in \{1, \ldots, k\}$ such that $\varphi_i(x_1) \neq \varphi_i(x_2)$. Thus, $\varphi_i \in A_{obs}(\Sigma)$ distinguishes x_1 and x_2 which is a contradiction. $\square$

Corollary 6.38. *Let $\Sigma = (X, f, h, x_0)$ be a semi-algebraically observable Nash system. Let us assume that for all $x \in X$ and for all $u \in \mathscr{U}_{pc}$ the trajectory $x_\Sigma(\cdot; x, u)$: $[0, T_u] \to X$ is well-defined. Then Σ is observable in the sense that it has no indistinguishable states. Formally, if $x_1 \neq x_2 \in X$ then there exists $u \in \mathscr{U}_{pc}$ such that $h(x_\Sigma(T_u; x_1, u)) \neq h(x_\Sigma(T_u; x_2, u))$.*

Proof. Consider arbitrary $x_1 \neq x_2 \in X$. We denote by τ^*_{Σ,x_1} and by τ^*_{Σ,x_2} the dual input-to-state maps corresponding to the Nash systems $\Sigma_1 = (X,f,h,x_1)$ and $\Sigma_2 = (X,f,h,x_2)$ derived from Σ, respectively. Let us assume by contradiction that $h(x_\Sigma(T_u;x_1,u)) = h(x_\Sigma(T_u;x_2,u))$ for all $u \in \mathscr{U}_{pc}$. Then,

$$D_{\alpha_k}\cdots D_{\alpha_1}\tau^*_{\Sigma,x_1}(h_i) = D_{\alpha_k}\cdots D_{\alpha_1}\tau^*_{\Sigma,x_2}(h_i) \tag{6.6}$$

for $i = 1,\ldots,r$ and for all $k \in \mathbb{N}$, $\alpha_1,\ldots,\alpha_k \in U$. Because $\mathscr{U}_{pc}$ is a set of admissible inputs and because all components h_i, $i = 1,\ldots,r$ of h are Nash functions on X, we derive by Proposition 6.21 that

$$D_{\alpha_k}\cdots D_{\alpha_1}\tau^*_{\Sigma,x_j}(h_i) = \tau^*_{\Sigma,x_j}(L_{f_{\alpha_k}}\cdots L_{f_{\alpha_1}}h_i) \tag{6.7}$$

for $j = 1,2$ and for all $i = 1,\ldots,r$, $k \in \mathbb{N}$, $\alpha_1,\ldots,\alpha_k \in U$. From (6.6) and (6.7) it follows that $\tau^*_{\Sigma,x_1}(L_{f_{\alpha_k}}\cdots L_{f_{\alpha_1}}h_i) = \tau^*_{\Sigma,x_2}(L_{f_{\alpha_k}}\cdots L_{f_{\alpha_1}}h_i)$ for $i = 1,\ldots,r$, $k \in \mathbb{N}$, $\alpha_1,\ldots,\alpha_k \in U$. Thus, $(L_{f_{\alpha_k}}\cdots L_{f_{\alpha_1}}h_i)(x_\Sigma(T_u;x_1,u)) = (L_{f_{\alpha_k}}\cdots L_{f_{\alpha_1}}h_i)(x_\Sigma(T_u;x_2,u))$ for all $u \in \mathscr{U}_{pc}$ and hence also for the empty input $e \in \mathscr{U}_{pc}$. Then,

$$(L_{f_{\alpha_k}}\cdots L_{f_{\alpha_1}}h_i)(x_1) = (L_{f_{\alpha_k}}\cdots L_{f_{\alpha_1}}h_i)(x_2) \tag{6.8}$$

for all $i = 1,\ldots,r$, $k \in \mathbb{N}$, $\alpha_1,\ldots,\alpha_k \in U$. Because $\mathbb{R}[\{L_{f_{\alpha_k}}\cdots L_{f_{\alpha_1}}h_i \mid i = 1,\ldots,r; k \in \mathbb{N}; \alpha_1,\ldots,\alpha_k \in U\}] = A_{obs}(\Sigma)$, (6.8) implies that $\forall g \in A_{obs}(\Sigma) : g(x_1) = g(x_2)$ which contradicts the fact that Σ is semi-algebraically observable. Therefore, there exists $u \in \mathscr{U}_{pc}$ such that $h(x_\Sigma(T_u;x_1,u)) \neq h(x_\Sigma(T_u;x_2,u))$. $\square$

Corollary 6.38 implies that differential-geometric conditions for observability of nonlinear systems, see for example [55], also yield necessary conditions for semi-algebraic observability of Nash systems.

The next proposition corresponds to Proposition 6.13 where the same result is stated for response maps.

Proposition 6.39. *Let $\Sigma = (X,f,h,x_0)$ be a Nash system. $A^{Nash}_{obs}(\Sigma)$ is algebraic over $A_{obs}(\Sigma)$ and consequently* $\operatorname{trdeg} A^{Nash}_{obs}(\Sigma) = \operatorname{trdeg} A_{obs}(\Sigma)$.

Proof. Consider an arbitrary $g \in A^{Nash}_{obs}(\Sigma)$. According to Definition 6.35 there exist $k \in \mathbb{N}$, $\varphi_1,\ldots,\varphi_k \in A_{obs}(\Sigma)$, and $q \in \mathscr{N}(\mathbb{R}^k)$ such that

$$g(x) = q(\varphi_1(x),\ldots,\varphi_k(x)) \text{ for all } x \in X.$$

In the same way as it is done for the maps of $\mathscr{A}(\widetilde{\mathscr{U}_{pc}} \to \mathbb{R})$ in the proof of Lemma 6.12, we derive that g is algebraic over $\mathbb{R}[\varphi_1,\ldots,\varphi_k] \subseteq A_{obs}(\Sigma)$. Let us state the detailed proof for the completeness.

From Lemma 2.23 there exist semi-algebraic subsets $S_1,\ldots,S_m$ of $\mathbb{R}^k$ and polynomials $g_i(X_1,\ldots,X_k,Y) \in \mathbb{R}[X_1,\ldots,X_k,Y]$, $i = 1,\ldots,m$ such that $\mathbb{R}^k = \bigcup_{i=1}^m S_i$, $S_i \cap S_j = \emptyset$ for all $i \neq j \in \{1,\ldots,m\}$ such that for all $z \in S_i$, $g_i(z,Y)$ is a non-zero polynomial and $g_i(z,g(z)) = 0$. For each $x \in X$ there exists unique $i(x) \in \{1,\ldots,m\}$ such that $\xi(x) := (\varphi_1(x),\ldots,\varphi_k(x)) \in S_{i(x)}$. Consider the set $I = \{i(x) \mid$

$x \in X\}$. We assume that $I = \{i_1, \ldots, i_r\} \subseteq \{1, \ldots, m\}$. Let us define the polynomial $G = g_{i_1} \cdots g_{i_r} \in \mathbb{R}[X_1, \ldots, X_k, Y]$. It follows that for all $x \in X$, $G(\xi(x), g(x)) = g_{i_1}(\xi(x), g(x)) \cdots g_{i_r}(\xi(x), g(x)) = 0$. To complete the proof we show that $G(\xi(x), Y) \neq 0$ for some $x \in X$.

There exists $x \in X$ such that $G(\xi(x), Y) \neq 0$ if and only if $G(\varphi_1, \ldots, \varphi_k, Y) \in \mathscr{N}(X)[Y]$ is a non-zero polynomial. Note that since X is semi-algebraically connected, $\mathscr{N}(X)$ is an integral domain, and hence so is the algebra $\mathscr{N}(X)[Y]$, see Theorem 2.1. If $G(\varphi_1, \ldots, \varphi_k, Y) = \Pi_{j=1}^{r} g_{i_j}(\varphi_1, \ldots, \varphi_k, Y)$ equals zero as an element of $\mathscr{N}(X)[Y]$ then there exists at least one $i \in I$ such that $g_i(\varphi_1(x), \ldots, \varphi_k(x), Y) = g_i(\xi(x), Y) = 0$ for all $x \in X$. But there is at least one $x \in X$ such that $i = i(x)$, i.e. $\xi(x) \in S_i$ and $g_i(\xi(x), Y) \neq 0$. This is a contradiction.

Thus, g is algebraic over $\mathbb{R}[\varphi_1, \ldots, \varphi_k] \subseteq A_{obs}(\Sigma)$. Because $g \in A_{obs}^{Nash}(\Sigma)$ was arbitrary, $A_{obs}^{Nash}(\Sigma)$ is algebraic over $A_{obs}(\Sigma)$. This implies that trdeg $A_{obs}^{Nash}(\Sigma) =$ trdeg $A_{obs}(\Sigma)$. □

Proposition 6.40. *If a Nash system $\Sigma = (X, f, h, x_0)$ is semi-algebraically observable then* trdeg $A_{obs}(\Sigma) = \dim X$.

Proof. Because X is a Nash submanifold of $\mathbb{R}^n$ which is semi-algebraically connected, it is a semi-algebraic set. Therefore, by Proposition 2.22, the dimension of X equals the dimension of a variety given as $V = Z\text{-}cl(X)$. Moreover, $\mathbb{R}[X] = \mathbb{R}[V]$ and, from Proposition 2.24, V is irreducible. Then,

$$\dim X = \dim V = \text{trdeg } \mathbb{R}[V] = \text{trdeg } \mathbb{R}[X]. \tag{6.9}$$

As Σ is semi-algebraically observable, we derive from Definition 6.36 that $\mathscr{N}(X) = A_{obs}^{Nash}(\Sigma)$. Consequently, trdeg $\mathscr{N}(X) =$ trdeg $A_{obs}^{Nash}(\Sigma)$. Further, from Lemma 6.23 and Proposition 6.39, trdeg $\mathbb{R}[X] =$ trdeg $\mathscr{N}(X) =$ trdeg $A_{obs}^{Nash}(\Sigma) =$ trdeg $A_{obs}(\Sigma)$. Then (6.9) implies that trdeg $A_{obs}(\Sigma) = \dim X$. □

6.3.7 Canonicity and minimality

In this section we introduce and link the concepts of canonical and minimal Nash realizations. By the dimension of a Nash system $\Sigma = (X, f, h, x_0)$ we mean the dimension of its state-space X which is a semi-algebraically connected Nash submanifold of $\mathbb{R}^n$, i.e. $\dim \Sigma = \dim X$.

Definition 6.41. We say that a Nash realization $\Sigma = (X, f, h, x_0)$ of a response map p is a *minimal* Nash realization of p if for any Nash realization Σ' of p it holds that $\dim \Sigma \leq \dim \Sigma'$.

Note that this concept of minimality does not automatically lead to the minimal number of state variables.

Definition 6.42. We say that a Nash realization Σ of a response map p is *canonical* if it is both semi-algebraically reachable and semi-algebraically observable.

Proposition 6.43. *Let $\Sigma = (X, f, h, x_0)$ be a canonical Nash realization of a response map $p : \widetilde{\mathscr{U}_{pc}} \to \mathbb{R}^r$. Then, $\tau_\Sigma^* : \mathscr{N}(X) \to A_{obs}^{Nash}(p)$ is an isomorphism.*

Proof. Since Σ is semi-algebraically reachable, τ_Σ^* is injective by Proposition 6.33. Hence, it remains to be shown that $\mathscr{N}(X)$ is mapped by τ_Σ^* onto $A_{obs}^{Nash}(p)$. Because Σ is semi-algebraically observable, i.e $\mathscr{N}(X) = A_{obs}^{Nash}(\Sigma)$, it is sufficient to prove that $\tau_\Sigma^*(A_{obs}^{Nash}(\Sigma)) = A_{obs}^{Nash}(p)$.

First, let us prove that $\tau_\Sigma^*(A_{obs}^{Nash}(\Sigma)) \subseteq A_{obs}^{Nash}(p)$. Let $g \in A_{obs}^{Nash}(\Sigma)$ be arbitrary. From Definition 6.35,

$$\exists k \in \mathbb{N}\ \exists \varphi_1, \ldots, \varphi_k \in A_{obs}(\Sigma)\ \exists q \in \mathscr{N}(\mathbb{R}^k) : g = q(\varphi_1, \ldots, \varphi_k).$$

Since $\varphi_1, \ldots, \varphi_k \in A_{obs}(\Sigma) \subseteq \mathscr{N}(X)$ by Definition 6.16, it follows from Proposition 6.17(v) that

$$\forall u \in \widetilde{\mathscr{U}_{pc}} : \tau_\Sigma^*(g)(u) = \tau_\Sigma^*(q(\varphi_1, \ldots, \varphi_k))(u) = q(\tau_\Sigma^*(\varphi_1)(u), \ldots, \tau_\Sigma^*(\varphi_k)(u)).$$

Because $\tau_\Sigma^*(A_{obs}(\Sigma)) = A_{obs}(p)$ by Proposition 6.17(iv), and therefore especially $\tau_\Sigma^*(\varphi_1), \ldots, \tau_\Sigma^*(\varphi_k) \in A_{obs}(p)$, Definition 6.9 implies that $\tau_\Sigma^* g \in A_{obs}^{Nash}(p)$.

Conversely, let $\hat{g} \in A_{obs}^{Nash}(p)$ be arbitrary. By Definition 6.9,

$$\exists k \in \mathbb{N}\ \exists \hat{\varphi}_1, \ldots, \hat{\varphi}_k \in A_{obs}(p)\ \exists q \in \mathscr{N}(\mathbb{R}^k)\ \forall u \in \widetilde{\mathscr{U}_{pc}} : \hat{g}(u) = q(\hat{\varphi}_1(u), \ldots, \hat{\varphi}_k(u)).$$

Because $\tau_\Sigma^*(A_{obs}(\Sigma)) = A_{obs}(p)$, see Proposition 6.17($iv$), there exist $\varphi_1, \ldots, \varphi_k \in A_{obs}(\Sigma)$ such that $\tau_\Sigma^* \varphi_i = \hat{\varphi}_i$ for $i = 1, \ldots, k$. Then, by Proposition 6.17(v),

$$q(\hat{\varphi}_1(u), \ldots, \hat{\varphi}_k(u)) = q(\tau_\Sigma^* \varphi_1(u), \ldots, \tau_\Sigma^* \varphi_k(u)) = \tau_\Sigma^*(q(\varphi_1, \ldots, \varphi_k)(u))$$

for all $u \in \widetilde{\mathscr{U}_{pc}}$ and consequently $\hat{g} = \tau_\Sigma^*(q(\varphi_1, \ldots, \varphi_k))$. Since $q(\varphi_1, \ldots, \varphi_k) \in A_{obs}^{Nash}(\Sigma)$, it follows that $A_{obs}^{Nash}(p) \subseteq \tau_\Sigma^*(A_{obs}^{Nash}(\Sigma))$. □

Theorem 6.44. *A canonical Nash realization Σ of a response map p is minimal.*

Proof. Let Σ be a canonical Nash realization of a response map p. Then Σ is semi-algebraically reachable which means, by Proposition 6.33(iv), that trdeg $A_{obs}(\Sigma) =$ trdeg $A_{obs}(p)$. Further, because Σ is also semi-algebraically observable, Proposition 6.40 implies that dim $\Sigma =$ trdeg $A_{obs}(p)$. From Proposition 6.24 we derive that for any Nash realization Σ' of p it holds that dim $\Sigma =$ trdeg $A_{obs}(p) \leq$ dim Σ'. Hence, Σ is a minimal Nash realization of p. □

Let us point out that if the dimension of a Nash realization Σ of a response map p equals trdeg $A_{obs}(p)$ then Σ is a minimal realization of p, see Theorem 6.24. Then, if the converse implication of Theorem 6.44 holds, i.e. all minimal Nash realizations are canonical, we derive by Propositions 6.43, 6.40, 6.39, and 6.13 that a Nash realization $\Sigma = (X, f, h, x_0)$ of a response map p is minimal if and only if

$$\dim\, \Sigma = \dim\, X = \operatorname{trdeg} A_{obs}^{Nash}(p) = \operatorname{trdeg} A_{obs}(p).$$

This characterization would be useful since a response map alone would be sufficient for the specification of the dimension of its minimal Nash realizations.

6.4 Conclusions

Nash systems are a subclass of the class of semi-algebraic systems, i.e. systems described by polynomial equalities and inequalities. This class contains also, for example, the class of semi-algebraic hybrid systems, [78, 20, 69]. In [85] realization theory of discrete-time semi-algebraic hybrid systems is investigated. Since the class of systems in [85] is different from the class of Nash systems we consider, the results of this chapter neither imply nor are implied by those of [85].

In this chapter we have introduced the class of Nash systems and we have formulated the realization problem for this class. The presented results, which concern necessary and sufficient conditions for the existence of Nash realizations and minimality and canonicity of Nash realizations, are not yet complete.

Further research aims at extending these results to necessary and sufficient conditions for the existence of canonical and minimal Nash realizations, characterizing the relations between Nash realizations of the same response maps, and obtaining a realization algorithm. In addition, one can generalize the results for arbitrary real closed fields (instead of $\mathbb{R}$) and investigate their applications to system identification, model reduction, observer and control design of Nash systems.

Chapter 7
Further research

In the preceding chapters we studied problems concerning system properties, realization theory and system identification for the classes of rational and Nash systems. However, many problems for these classes of systems remain open. Several of these open problems were already discussed in the preceding chapters. Below we list other open problems and add information on the problems mentioned before.

Control and system theoretic properties of rational and Nash systems are still under investigation. Linking the concepts of algebraic, rational, semi-algebraic reachability and observability with the usual concepts of nonlinear control theory is one of the issues. Concerning algebraic reachability, one can study the smoothness properties of the Zariski closures of reachable sets. Finding the conditions under which the Zariski closure of the reachable set of an initialized rational system is smooth would lead to the characterization of minimal rational realizations with the state-spaces being smooth irreducible varieties. Comparing the concepts of algebraic and semi-algebraic reachability with the same concepts but allowing piecewise-constant inputs with infinitely many switches is also of interest, mainly from the computability point of view. Concerning algebraic, rational and semi-algebraic observability properties, one could study the differences between observability given by considering all admissible inputs and observability given by the so-called universal inputs. But specifying the class of universal inputs for rational and Nash systems is an interesting problem itself. Control and observer design, optimal control problems, control with partial observations for rational and Nash systems are further examples of unresolved topics.

Another important system theoretic property is stability. For the special subclass of rational systems relevant results can be found in [99]. For dealing with the stability problem of rational systems [47, 46, 86] could be useful.

Arising especially from biological applications, positivity of rational and Nash systems is of interest. By positivity we refer to the property that the positive orthant is a forward invariant set. Positive linear systems are considered for example in [37]. However, for rational and Nash systems the problem of characterizing positive systems within these classes seems to be more difficult. The positivity of polynomial, rational and Nash systems is related to Hilbert's 17th problem which deals with the representation of the polynomials nonnegative on $\mathbb{R}^n$ as the sum of squares of rational functions. Hilbert's 17th problem can be stated also for other rings of functions than polynomials. The results for polynomial, real analytic and Nash functions are reviewed for example in [15, Chapter 6 and 8].

In Chapter 4 we studied rational realizations with respect to the equivalence relations given by birational maps. It is not yet clear whether one can determine even the number of birationally equivalent rational realizations of the same response map. The answer may be derived by knowing the number of rational maps relating different irreducible varieties, see [6] and others. Birational equivalence of rational realizations is related to the problem of deriving the conditions under which the state-spaces of rational realizations are unirational varieties. Some results in differential-algebraic framework are given in [41, Theorem 5.6]. It also remains to study the properties of rational realizations with respect to isomorphisms which specify stronger equivalence classes than the ones given by birational maps. For the relation between birational equivalence and isomorphism see for example [95, I.1.4.3].

With the solutions to the problems listed above and in the preceding chapters it will be easier to deal with the problems concerning system reduction and invertibility of rational and Nash systems. The former problem deals with finding a system of lower dimension and/or complexity which preserves the input-output behavior of the original system. The latter one deals with revealing an input and/or switching times of an input by knowing only an output of a system, see for example [38, 54]. Within system identification, for example, the way how to estimate the dimension of a system which would model measurements sufficiently well is not yet fully understood.

Since rational and Nash systems we considered are defined only with the output functions independent of inputs, it is natural extension of the presented approaches to consider output functions explicitly dependent on inputs. To conclude the list of open problems for the classes of rational and Nash systems let us mention the still missing comparison between the algebraic-geometric approach presented in the preceding chapters and the differential-algebraic one, extension in the line of behavioral theory, generalization to arbitrary fields (instead of $\mathbb{R}$), development of the corresponding algebraic techniques for nonlinear systems and development of new algorithms based on computational algebra and readily available computer algebra systems.

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Summary

Title of the thesis: Rational Systems in Control and System Theory

In this thesis an algebraic approach to realization theory for the class of rational systems is presented. The results are applied in system identification and generalized for the class of so-called Nash systems.

Rational systems are dynamical systems whose dynamics and output functions are determined by rational functions. They are widely used as models of phenomena in life sciences, economy, physics, and engineering. The framework and motivation to study rational systems are presented in Chapter 3. The notions of algebraic reachability and of algebraic/rational observability are introduced. For algebraic reachability of rational systems we provide a characterization in terms of polynomial ideals satisfying certain conditions. Both concepts, of reachability and of observability, are related to different notions of controllability, accessibility and observability of linear and nonlinear systems.

Realization theory is one of the central topics of control and system theory. Restricted to rational systems, it deals with the characterization of all rational systems which have a specified input-output behavior. Apart from the existence issues, the realization problem concerns properties of rational realizations such as canonicity and minimality, relations between different rational realizations of the same map, algorithms and procedures for constructing rational realization of desired properties. Furthermore, realization theory serves as a theoretical foundation for model reduction, system identification and control/observer design.

We deal with the realization problem for rational systems in Chapter 4. We derive necessary and sufficient conditions for a response map to be realizable by a rational system. The characterization of the existence of rationally observable, canonical, and minimal rational realizations for a given response map is provided as well. We relate minimality of rational realizations to their rational observability, algebraic reachability, and canonicity. The relations between birational equivalence of rational realizations and their canonicity and minimality properties are determined. Namely, we show that all canonical rational realizations of the same response map are bira-

tionally equivalent, and that birational equivalence preserves minimality of rational realizations.

In Chapter 6 we investigate realization theory of Nash systems. In particular, we introduce the class of Nash systems and then formulate and partially solve the realization problem for them. In analogy with results of Chapter 4 we derive necessary and sufficient conditions for the existence of Nash realizations of a response map. Further, the concepts of semi-algebraic observability and semi-algebraic reachability of Nash realizations are defined and their relationship with minimality is explained.

The problems of system identification deal with modeling a phenomenon based on the observed measurements. This involves the selection of a model structure, experimental design, identifiability analysis, parameter estimation and evaluation methods. In this thesis we consider only the identifiability problem for the deterministic classes of polynomial and rational systems and for noise-free data. Namely, in Chapter 5 we provide the characterization of structural and global identifiability of parametrizations of parametrized polynomial and parametrized rational systems. The corresponding method for checking identifiability is employed to investigate identifiability properties of systems modeling certain biological phenomena. Identifiability of a parametrization is a necessary condition for the uniqueness of parameter values determining a model fitting measurements. Without the existence of a unique solution to the parameter estimation problem it could happen that the methods for estimating parameters will never find the true values of the parameters. Therefore, verification of identifiability of a parametrization precedes estimation of numerical values of parameters, and thus formulation of a fully specified model of a phenomenon.

The thesis is concluded by Chapter 7 which provides directions for further research.

Samenvatting

De titel van dit proefschrift: Rationale Systemen in de Systeem- en Regeltheorie

In dit proefschrift wordt een algebraïsche aanpak tot realisatietheorie voor de klasse van rationale systemen gepresenteerd. De resultaten worden in systeemidentificatie toegepast en voor de klasse van zogenaamde Nash-systemen gegeneraliseerd.

Rationale systemen zijn dynamische systemen waar van de dynamica en output functies door rationale functies worden bepaald. Ze worden vaak gebruikt als modellen voor verschillende fenomenen in de levenswetenschappen, de economie, de natuurkunde en de techniek. Het algemene kader en de motivatie voor het studeren van rationale systemen worden in de hoofdstuk 3 gepresenteerd. Daar worden ook de begrippen algebraïsche bereikbaarheid en algebraïsche/rationale waarneembaarheid geïntroduceerd. Wij karakteriseren algebraïsche bereikbaarheid met behulp van polynomische idealen, die aan bepaalde voorwaarden voldoen. Beide concepten, bereikbaarheid en waarneembaarheid, hebben betrekking op de verschillende begrippen regelbaarheid, toegankelijkheid en waarneembaarheid van lineaire en niet-lineaire systemen.

Realisatietheorie is een van de hoofdthema's van de systeem- en regeltheorie. Wanneer het beperkt is tot rationale systemen, behandelt het de karakterisering van alle rationale systemen, die het gegeven ingangs-uitgangsgedrag hebben. Behalve problemen met existentie, behandelt het onderwerp van realisatie eigenschappen van rationale realisaties zoals canoniciteit en minimaliteit, betrekkingen tussen verschillende rationale realisaties van een bepaalde afbeelding, algoritmen en procedures, die de rationale realisatie van de gevraagde eigenschappen construeert. Bovendien dient de realisatietheorie als een theoretische basis voor modelreductie, systeemidentificatie en ontwerp van regelwetten en van waarnemers.

Hoofdstuk 4 behandelt het ontwerp van realisatie voor rationale systemen. Wij leiden de noodzakelijke en voldoende voorwaarden af voor de realiseerbaarheid van een responsieafbeelding door een rationaal systeem. De karakterisering van de existentie van rationaal waarneembare, kanonieke en minimaal rationale realisaties voor een gegeven responsieafbeelding is ook aanwezig. Wij leiden de verbanden af tussen de minimaliteit van rationale realisaties en hun rationale waarneembaarheid, algebraïsche bereikbaarheid en kanoniciteit. Bovendien bewijzen wij de relaties tussen birationale equivalentie van rationale realisaties en hun eigenschappen van kanoniciteit en minimaliteit. Wij bewijzen namelijk, dat alle kanonische rationale realisaties van een bepaalde responsieafbeelding birationaal equivalent zijn en dat de birationale equivalentie de minimaliteit van rationale realisaties behoudt.

In de hoofdstuk 6 bestuderen wij de realisatietheorie van Nash-systemen. Wij definiëren de klasse van Nash-systemen, formuleren het realisatie probleem hiervoor en presenteren een gedeeltelijke oplossing. In analogie met de resultaten van de hoofdstuk 4 leiden wij noodzakelijke en voldoende voorwaarden voor de existentie van Nash-realisaties van een gegeven responsieafbeelding af. Bovendien definiëren wij de begrippen van semi-algebraïsche waarneembaarheid en semi-algebraïsche bereikbaarheid van Nash realisaties en hun betrekking op minimaliteit wordt uitgelegd.

De problemen van systeemidentificatie behandelen het modelleren van verschijnselen gebaseerd op geobserveerde metingen. Dit modelleren bestaat uit de selectie van een modelstructuur, ontwerp van het experiment, analyse van identificeerbaarheid, methode van de parameter schatting en evaluatie van de resultaten. In dit proefschrift behandelen wij alleen het probleem van identificeerbaarheid voor deterministische klassen van polynomische en rationale systemen en voor ruis-vrije data. Hoofdstuk 5 karakteriseert de structurele en globale identificeerbaarheid van de parametrisaties van geparametriseerde polynomische en geparametriseerde rationale systemen. De desbetreffende methode voor het controleren van identificeerbaarheid wordt gebruikt voor het bestuderen van de identificeerbaarheidseigenschappen van systemen, die bepaalde biologische verschijnselen modelleren. Identificeerbaarheid van een parametrisatie is een noodzakelijke voorwaarde voor de eenduidigheid van de parameter waarden, die het model bepalen, welk bij de metingen past. Zonder een eenduidige oplossing, hoeven de parameter schatting methoden geen correcte parameter waarden te geven. Het controleren van identificeerbaarheid van een parametrizatie komt dus voor de schatting van numerieke waarden van parameters en daarom dus ook voor de volledige gespecificeerd model van een fenomeen.

In hoofdstuk 7 worden enkele suggesties gedaan voor toekomstige onderzoeksthema's.