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Computational Geometry 4 (1994) 309–325

Computational Geometry
Theory and Applications

Counting problems relating to a theorem of Dirichlet ⋆

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Abstract

We study weighted versions of Dirichlet's theorem on the probability that two integers, taken at random, are relatively prime. This leads to a uniform approach in the study of several counting problems in discrete and computational geometry relating to incidences between points and lines.

1. Introduction

An old theorem of G. Lejeune Dirichlet, dating back to the year 1849, states that the probability that two integers taken at random are relatively prime is $6/\pi^2$ [12, p. 324], [10, p. 269]. This theorem is also known in an equivalent geometric formulation: the probability that a vertex of the integer lattice (of pairs of integers) is visible from the origin (i.e., the open line segment joining the origin and the vertex meets no lattice points) is $6/\pi^2$. This equivalence is merely due to the simple observation that a vertex of the integer lattice is visible from the origin if and only if its coordinates are relatively prime.

⋆ A preliminary version of this work appears in [13,17].

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[1] Research supported in part by the National Science and Engineering Research Council of Canada.

0925-7721/94/$07.00
SSDI 0925-7721(94)00013-L

Dirichlet's theorem admits several generalizations. One of them concerns the probability that two integer square matrices chosen at random have relatively prime determinants [9]. Another one concerns the probability that a lattice point is visible from each of the points of a fixed subset of the integer lattice. This problem was considered by Rearick [18] who solved the case where the points of the subset are pairwise visible, and Rumsey [19] who solved the case of arbitrary finite subsets. A nontrivial application of Rumsey's theorem is in the study of the camera placement problem, which is concerned with the placement of a fixed number of (point) cameras on an integer lattice in order to maximize their visibility [17]. For more related problems the reader should consult [1,6].

1.1. Results of the paper

This paper is concerned with a geometric formulation of Dirichlet's theorem: let Δ be a bounded region in the plane and let Δ_0 be the set of integer points of Δ visible from the origin; then

$$\sum_{\Delta_0} 1 \sim \frac{6}{\pi^2}\,\mathrm{area}(\Delta)$$

as Δ grows by dilatation to the entire plane [10, p. 409]. Our subsequent analysis of several geometric problems requires the asymptotic evaluation of multidimensional versions of sums of the form

$$\sum_{\Delta_0} f(x) \qquad (1)$$

where f is a real function which is either monotone or satisfies a Lipschitz condition. Intuitively one can think of $f(x)$ as a measure (for an observer sitting at the origin) of the visibility of the point x, while the sum (1) quantifies the total visibility from the origin.

We give a "weighted version" of Dirichlet's theorem and use it to give a unified approach to the study of the asymptotic behavior of several counting problems in discrete and computational geometry. These problems are:

(1) the number of lines passing through at least k integer points of the d-dimensional standard cube of size n, the d-dimensional standard simplex of size n, and more generally of the Cartesian product of such simplexes,

(2) the maximal number of incidences between m points and n lines in the plane [22], [5, Chapter 6],

(3) the maximal complexity of the plane region illuminated by a segment in the presence of other segments [16, pp. 219–223].

We show how to calculate asymptotically optimal bounds for the first problem and constants of known lower bounds for the other two. From a slightly modified version of the first problem we deduce the asymptotic average length (as well as moments of higher order) of maximal integer segments in the cube (cf. [20,8] and problem E 3217 [1987, 549; 1989, 64] in the problem section of *American Mathematical Monthly*).

2. General results

Let Δ be a convex domain (compact and full dimensional) of $\mathbb{R}^d$ ($d \geqslant 2$) of diameter ω; we denote by $\mathrm{vol}(\Delta)$ its volume. It is well known (and easy to prove) that

$$\mathrm{vol}(\Delta) = \mathrm{O}(\omega^d), \tag{2}$$

where the constant implied by the O-notation depends only on the dimension d (according to the 'isodiametric inequality' [7, p. 13] the best constant is the volume of the d-dimensional ball of diameter 1, i.e., $\pi^{d/2}(\frac{1}{2})^d/(\frac{1}{2}d)!$).

We denote by Δ_1 the set of integer points in Δ, and by Δ_0 the set of integer points $x = (x_1, \ldots, x_d)$ in Δ that are visible from the origin, i.e., $x_1, \ldots, x_d$ are relatively prime.

Let f be a real function defined on Δ. The following two theorems give an estimate of the difference between $\sum_{\Delta_0} f(x)$ and the integral $1/\zeta(d) \int_\Delta f(x)\,\mathrm{d}x$, when f is either monotone (i.e., either non-decreasing or non-increasing with respect to all the variables of f simultaneously) or satisfies a Lipschitz condition, respectively. Recall that $\zeta(d)$ denotes the Riemann zeta function $\sum_{n \geqslant 1} n^{-d}$ (see [10, Chapter XVII]).

Theorem 2.1. *Let Δ be a convex domain of $\mathbb{R}^d$ of diameter $\omega \geqslant 1$ and let f be a monotone real positive function defined on Δ. Then*

$$\left| \sum_{\Delta_0} f(x) - \frac{1}{\zeta(d)} \int_\Delta f(x)\,\mathrm{d}x \right| = \mathrm{O}\left(\sup_\Delta f \begin{cases} \omega \log \omega, & \text{if } d = 2 \\ \omega^{d-1}, & \text{otherwise} \end{cases} \right)$$

where the constant implied by the O-notation depends only on the dimension d.

Theorem 2.2. *Let Δ be a convex domain of $\mathbb{R}^d$ of diameter $\omega \geqslant 1$ and let f be a real, positive function defined on Δ and satisfying the Lipschitz condition $|f(x) - f(y)| \leqslant A|x - y|$, for some constant $A > 0$. Then*

$$\left| \sum_{\Delta_0} f(x) - \frac{1}{\zeta(d)} \int_\Delta f(x)\,\mathrm{d}x \right|$$
$$= \mathrm{O}\left((\omega A + \sup_\Delta f) \begin{cases} \omega \log \omega, & \text{if } d = 2 \\ \omega^{d-1}, & \text{otherwise} \end{cases} \right)$$

where the constant implied by the O-notation depends only on the dimension d.

We begin with a lemma.

Lemma 2.3. *Under the hypothesis of Theorem* 2.1 *we have*

$$\left|\sum_{\Delta_1} f(x) - \int_{\Delta} f(x)\,\mathrm{d}x\right| = \mathrm{O}\left(\omega^{d-1}\sup_{\Delta} f\right).$$

Proof. We first extend f on $\mathbb{R}^d$ by preserving its monotonicity. Without loss of generality we may assume that the function f is non-decreasing. Then we extend f on $\mathbb{R}^d$ by setting $f(x) = \inf\{f(y) \mid y \in \Delta, x \leqslant y\}$ with the convention that $\inf \emptyset = \sup_\Delta f$ (by abuse of notation we use the same symbol for the function f and its extension). It is then easy to verify that the extension is still positive, non-decreasing, upper semi-continuous, and that its supremum does not change, i.e., $\sup_{\mathbb{R}^d} f = \sup_\Delta f$.

The proof that follows generalizes the principle result of Nosarzewska [15]. Let $T(\Delta)$ be the set of points whose distance from the boundary of Δ is less than $\sqrt{d}$ (the thickened boundary), let $\Delta^+ = \Delta \cup T(\Delta)$, and $\Delta^- = \Delta \setminus T(\Delta)$. Let C be the unit cube with vertices the 2^d points $(x_1, \ldots, x_d)$ where $x_i \in \{0, 1\}$.

Using $f(x) = \inf_{x+C} f$ and $\Delta_1 + C \subseteq \Delta^+$ we obtain

$$\sum_{\Delta_1} f(x) \leqslant \int_{\Delta_1+C} f(x)\,\mathrm{d}x \leqslant \int_{\Delta^+} f(x)\,\mathrm{d}x.$$

Similarly, using $f(x) = \sup_{x-C} f$ and $\Delta_1 - C \supseteq \Delta^-$, we get

$$\sum_{\Delta_1} f(x) \geqslant \int_{\Delta_1-C} f(x)\,\mathrm{d}x \geqslant \int_{\Delta^-} f(x)\,\mathrm{d}x.$$

Using $\Delta^- \subseteq \Delta \subseteq \Delta^+$ and $\Delta^+ \setminus \Delta^- \subseteq T(\Delta)$ and the above inequalities we obtain

$$\left|\sum_{\Delta_1} f(x) - \int_{\Delta} f(x)\,\mathrm{d}x\right| \leqslant \int_{\Delta^+} f(x)\,\mathrm{d}x - \int_{\Delta^-} f(x)\,\mathrm{d}x \leqslant \int_{T(\Delta)} f(x)\,\mathrm{d}x.$$

The right-hand side of the above inequality is of course bounded above by $\mathrm{vol}(T(\Delta)) \sup_\Delta f$; so it remains to prove that $\mathrm{vol}(T(\Delta)) = \mathrm{O}(\omega^{d-1})$. For $x \in \mathbb{R}^d$ let $\tau(x)$ be its symmetric point with respect to its orthogonal projection on Δ. Due to the convexity of Δ the restriction to $T(\Delta) \setminus \Delta$ of the mapping τ is non-increasing in distance. Consequently $\mathrm{vol}(T(\Delta)) \leqslant 2\,\mathrm{vol}(T(\Delta) \setminus \Delta)$. According to the formula of Steiner–Minkowski [2, p. 141],

$$\mathrm{vol}(T(\Delta) \setminus \Delta) = \sum_{i=1}^{d} \ell_i(\Delta)\, d^{i/2},$$

where the functions $\ell_1, \ell_2, \ldots, \ell_d$ are bounded on the set of convex subsets of the unit ball and satisfy the identities $\ell_i(k\Delta) = k^{d-i}\ell_i(\Delta)$ for $i = 1, \ldots, d$.

It follows (under the non-restrictive hypothesis $0 \in \Delta$) that $\mathrm{vol}(T(\Delta) \setminus \Delta) = \mathrm{O}(\omega^{d-1})$ where the constant implied by the O-notation depends only on the dimension d. This completes the proof of our lemma. □

Proof of Theorem 2.1. Let Δ_k be the set of points of Δ with integer coordinates, all divisible by k. We observe that

$$\Delta_0 = \Delta_1 \setminus \bigcup_{p \text{ prime}} \Delta_p.$$

Using a standard sieve argument (see for example [14]) we can write

$$\sum_{\Delta_0} f(x) = \sum_{k \geqslant 1} \mu(k) \sum_{\Delta_k} f(x)$$

where μ is the Möbius function. We now use the previous lemma to estimate the sum $\sum_{\Delta_k} f(x)$. It follows from the equality $\Delta_k = k(\Delta/k)_1$ that

$$\sum_{\Delta_k} f(x) = \sum_{(\Delta/k)_1} f(kx).$$

Therefore by applying the lemma we obtain

$$\left| \sum_{(\Delta/k)_1} f(kx) - \int_{\Delta/k} f(kx)\,\mathrm{d}x \right| = \mathrm{O}\left((\omega/k)^{d-1} \sup_{\Delta/k} f(kx) \right)$$

$$= \mathrm{O}\left((\omega/k)^{d-1} \sup_{\Delta} f \right).$$

By summing on k and using

$$\int_{\Delta/k} f(kx)\,\mathrm{d}x = \frac{1}{k^d} \int_{\Delta} f(x)\,\mathrm{d}x$$

we get

$$\left| \sum_{\Delta_0} f(x) - \sum_{k \leqslant \omega} \frac{\mu(k)}{k^d} \int_{\Delta} f(x)\,\mathrm{d}x \right| = \mathrm{O}\left(\omega^{d-1} \sup_{\Delta} f \sum_{k \leqslant \omega} \frac{1}{k^{d-1}} \right)$$

which we simplify to

$$\left| \sum_{\Delta_0} f(x) - \sum_{k \leqslant \omega} \frac{\mu(k)}{k^d} \int_{\Delta} f(x)\,\mathrm{d}x \right|$$

$$= \mathrm{O}\left(\sup_{\Delta} f \begin{cases} \omega \log \omega, & \text{if } d = 2 \\ \omega^{d-1} & \text{otherwise} \end{cases} \right).$$

Using the well-known identity $\sum_{k\geqslant 1}\mu(k)/k^d = 1/\zeta(d)$ (see [10, p. 250]), and the inequality $\sum_{k>\omega} 1/k^d \leqslant 1/((d-1)\omega^{d-1})$, the proof of the theorem can be completed without difficulty. □

To prove Theorem 2.2, it is necessary to prove a second lemma, similar to Lemma 2.3.

Lemma 2.4. *Under the hypothesis of Theorem* 2.2 *we have that*

$$\left|\int_{\Delta} f(x)\,\mathrm{d}x - \sum_{\Delta_1} f(x)\right| = \mathrm{O}\left(\omega^d A + \omega^{d-1}\sup_{\Delta} f\right).$$

Proof. We extend f on $\mathbb{R}^d$ by preserving the Lipschitz condition. By the compactness of Δ, for each $x \in \mathbb{R}^d$ there exists a point $x^* \in \Delta$ such that

$$|x - x^*| = \inf_{y\in\Delta} |x - y|. \tag{3}$$

Using the convexity of Δ, it can be shown that for each x the point $x^* \in \Delta$, defined as above, is unique (indeed, if both $x^*, x_1^* \in \Delta$ satisfied (3) then so would $tx^* + (1-t)x_1^*$, for all $0 \leqslant t \leqslant 1$) and the mapping $x \in \mathbb{R}^d \longrightarrow x^* \in \Delta$ is non-increasing on distances (i.e., $|x^* - y^*| \leqslant |x - y|$, for all x, y). This guarantees that the function $x \mapsto f(x) := f(x^*)$ (by abuse of notation we use the same symbol for the function f and its extension) is well defined for $x \in \mathbb{R}^d$ and satisfies the same Lipschitz condition.

We use the thickened boundary $T(\Delta)$ introduced in the proof of the Lemma 2.3. Let $\Delta' = \Delta_1 + D$ where D is the unit cube with center the origin. Since the symmetric difference $(\Delta \setminus \Delta') \cup (\Delta' \setminus \Delta) \subseteq T(\Delta)$, it follows that

$$\left|\int_{\Delta} f(x)\,\mathrm{d}x - \int_{\Delta'} f(x)\,\mathrm{d}x\right| \leqslant \int_{T(\Delta)} f(x)\,\mathrm{d}x,$$

which is in $\mathrm{O}(\omega^{d-1}\sup_{\Delta} f)$ (see the proof of the previous lemma). Due to the Lipschitz condition on f, we can write

$$\left|\sum_{\Delta_1} f(x) - \int_{\Delta'} f(x)\,\mathrm{d}x\right| = \left|\sum_{\Delta_1}\int_{x+D} (f(x) - f(u))\,\mathrm{d}u\right| \leqslant A\,|\Delta_1|.$$

Combining these last two inequalities, together with $|\Delta_1| = \mathrm{O}(\omega^d)$, we obtain the desired result. □

Proof of Theorem 2.2. The proof is similar to that of Theorem 2.1. We need only mention that if the function $x \mapsto f(x)$ satisfies the Lipschitz condition with Lipschitz constant A then the function $x \mapsto f(kx)$ satisfies the Lipschitz condition with Lipschitz constant kA. □

Remark. In the two-dimensional case ($d = 2$) the convexity condition on the domain Δ is not necessary. It is possible to obtain an upper bound on the area of $T(\Delta)$ under the hypothesis that the boundary of Δ is rectifiable; this done as follows. Let $\ell(\Delta)$ be its length. Partition the boundary of Δ in r arcs $A_1, \ldots, A_r$, the first $r-1$ having length $\sqrt{2}$ and the last A_r having length less than $\sqrt{2}$; obviously $r \leqslant \lfloor \ell(\Delta)/\sqrt{2} \rfloor + 1$. Since the boundary of Δ is contained in the union of r discs each of radius less than $\sqrt{2}$, the domain $T(\Delta)$ is contained in the union of r discs each of radius less than $2\sqrt{2}$; consequently, $\mathrm{vol}(T(\Delta)) \leqslant 8\pi r$ which less than $4\sqrt{2}\pi\ell(\Delta) + 8\pi$. Hence the proof of the first lemma is valid in this case as well.

In the next section we use our theorems on relatively complicated functions. Here we only mention the case where the function is a monomial and the summation is over the standard cube.

Theorem 2.5. *Let $a_1, \ldots, a_d$ be d real numbers greater than -1. Then, for fixed d, we have*

$$\sum_{\substack{0 \leqslant x_1, \ldots, x_d \leqslant n \\ \gcd(x_1, \ldots, x_d) = 1}} x_1^{a_1} \cdots x_d^{a_d} \sim \frac{1}{\zeta(d)} \frac{n^{a_1+1}}{a_1+1} \cdots \frac{n^{a_d+1}}{a_d+1}$$

asymptotically in n.

In particular taking the a_i's equal to zero we get the d-dimensional version of the previously mentioned theorem of Dirichlet.

3. Applications

In this section we apply our main theorems to the analysis of several counting problems in discrete and computational geometry.

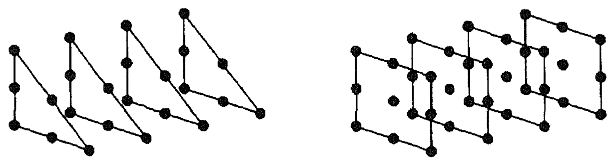
3.1. Counting lines

As a first application of Theorem 2.1 we give an asymptotic evaluation of the number of lines traversing at least $k+1$ ($k \geqslant 1$) integer points of the standard cube

$$C(d,m) = \{ (x_1, \ldots, x_d) \in \mathbb{R}^d \mid 0 \leqslant x_1, \ldots, x_d \leqslant m-1 \},$$

the standard simplex of dimension d and size m

$$S(d,m) = \{ (x_1, \ldots, x_d) \in \mathbb{R}^d \mid 0 \leqslant x_1 + \cdots + x_d \leqslant m-1 \},$$



$S(2,3) \times S(1,4)$ $S(1,3) \times S(1,3) \times S(1,4)$

Fig. 1. Examples of products of simplexes.

and, more generally, a Cartesian product of such simplexes (see Fig. 1). It is convenient to use the following notations. Let $\mathcal{J}$ be a partition of $\{1,\ldots,d\}$, fixed once for all, and let $n = (n_I)_{I\in\mathcal{J}}$ be a function of $\mathcal{J}$ into $\mathbb{Z}$; we assume that $\min_I n_I = \Theta(\max_I n_I)$. Put

$$D(n) = \left\{ (x_1,\ldots,x_d) \in \mathbb{R}^d \mid 0 \leqslant \sum_{i\in I} x_i \leqslant n_I - 1 \quad \forall I \in \mathcal{J} \right\}. \tag{4}$$

For example the cube $C(d,m)$ is obtained for $\mathcal{J} = \{\{1\},\{2\},\ldots,\{d\}\}$ and $n_{\{1\}} = \cdots = n_{\{d\}} = m$, whereas the simplex $S(d,m)$ is obtained for $\mathcal{J} = \{\{1,\ldots,d\}\}$ and $n_{\{1,\ldots,d\}} = m$.

In general the set $D(n)$ is the Cartesian product of $|\mathcal{J}|$ simplexes of dimension $|I|$ and size n_I where $I \in \mathcal{J}$,

$$D(n) = \prod_{I\in\mathcal{J}} S(|I|, n_I).$$

Let $\mathrm{w}(n,k)$ be the number of lines with positive slope passing through at least $k+1$ integer points of the domain $D(n)$. The following theorem gives an asymptotic evaluation of the function $\mathrm{w}(n,k)$.

Theorem 3.1. *Let $\mathcal{J}$ be a partition of $\{1,\ldots,d\}$ and let $n = (n_I)_{I\in\mathcal{J}}$ be a function of $\mathcal{J}$ into $\mathbb{Z}$. The number $\mathrm{w}(n,k)$ of lines with positive slope traversing at least $k+1$ integer points of the domain $D(n)$, defined by (4), is given by the formula*

$$\mathrm{w}(n,k) = \frac{1}{\zeta(d)} \prod_{I\in\mathcal{J}} \frac{n_I^{2|I|}}{(2|I|)!} \left\{ \frac{1}{k^d} - \frac{1}{(k+1)^d} \right\} + \mathrm{O}\left(\begin{cases} \dfrac{|n|^3}{k^2} \log \dfrac{|n|}{k}, & \text{if } d = 2 \\ \dfrac{|n|^{2d-1}}{k^d}, & \text{otherwise} \end{cases} \right)$$

where $|n| = \sup_I n_I$ and where the constant implied by the O-notation depends only on the dimension d.

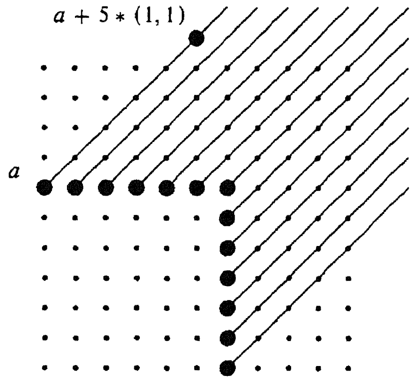


Fig. 2. Lines with slope (1, 1) passing through at least 5 points of the cube $C(11,2)$.

Proof. Let $p = (p_1, \ldots, p_d) \in \mathbb{N}^d$ be a positive slope (i.e., a d-tuple of positive integers such that $\gcd(p_1, \ldots, p_d) = 1$). Let $G_k(p,n)$ be the set of lines with positive slope p each traversing at least $k+1$ integer points of the domain $D(n)$ (see Fig. 2); we denote by $g_k(p,n)$ the cardinal of the set $G_k(p,n)$. By partitioning the set of lines according to the value of their slope, the following equality is obtained

$$\mathrm{w}(n,k) = \sum_{\gcd(p_1,\ldots,p_d)=1} g_k(p,n), \tag{5}$$

which immediately indicates a possible application of the general theorems in the previous paragraph. This is done as follows.

First, we show that $g_k(p,n)$ admits a simple expression in the coordinates $p_1, \ldots, p_d$ of p, namely

$$g_k(p,n) = \prod_{I\in\mathcal{J}} \frac{1}{|I|!} \prod_{i=0}^{|I|-1} \left(n_I - k\sum_{j\in I} p_j + i \right)^+ - \prod_{I\in\mathcal{J}} \frac{1}{|I|!} \prod_{i=0}^{|I|-1} \left(n_I - (k+1)\sum_{j\in I} p_j + i \right)^+, \tag{6}$$

where t^+ is equal to t if $t \geqslant 0$ and 0 otherwise. This is proved as follows. The mapping which associates to each line $\ell \in G_k(p,n)$ the unique integer point $a \in \ell$ such that $a, a+p, a+2p, \ldots, a+kp \in D(n)$ and $a+(k+1)p \notin D(n)$ is 1–1 and onto the set of integer points a of the domain $D(n)$ such that $a+kp \in D(n)$ and $a+(k+1)p \notin D(n)$. Let E_k be the set of integer points a of the domain $D(n)$ such that $a+kp \in D(n)$. Since $E_{k+1} \subseteq E_k$ we obtain

$$g_k(p,n) = |E_k| - |E_{k+1}|.$$

Now we observe that E_k is the set of integer points of the domain $D(n-kp)$ where $(n-kp)$ is defined by $(n-kp)_I = n_I - k\sum_{i\in I} p_i$ for all $I \in \mathcal{J}$; consequently E_k is the set of integer points of the Cartesian product of simplexes

$$S\left(|I|, n_I - k\sum_{i\in I} p_i\right), \quad I \in \mathcal{J}.$$

It remains to observe that the number of integer points of the simplex $S(d,m)$ is given by the formula $m(m+1)(m+2)\cdots(m+d-1)/d!$ (This result follows from the recurrence relations $a(1,m)=m$, $a(d+1,m)=a(d,0)+a(d,1)+\cdots+a(d,m)$ where $a(d,m)$ is the number of integer points of $S(d,m)$) to obtain the expression of $g_k(p,n)$ given in (6).

Second, we extend the function $p \mapsto g_k(p,n)$ to $\mathbb{R}^d_+$; this is simply done by using the right-hand side of Eq. (6) (by abuse of notation we use the same symbol for $g_k(p,n)$ and its extension). Observe that $g_k(x,n)=0$ for $x \notin D(n/k)$; hence Eq. (5) can be rewritten

$$\mathrm{w}(n,k) = \sum_{x\in D(n/k)_0} g_k(x,n). \tag{7}$$

Third, we estimate the maxima of the function $x \mapsto g_k(x,n)$ and of its partial derivatives. We claim that:

(1) $\sup_{x\in D(n/k)} g_k(x,n) = \mathrm{O}\left(|n|^d/(k+1)\right)$,

(2) $\sup_{x\in D(n/(k+1))} \partial g_k(x,n)/\partial x_i = \mathrm{O}(|n|^{d-1})$,

which are proved as follows. Introduce the functions $h(t)$ and $f(t)$ defined on $\mathbb{R}^d$ by

$$h(t) = \prod_{i=1}^{d}(1-\alpha t_i+\varepsilon_i) \tag{8}$$

and

$$f(t) = \prod_{i=1}^{d}(1-\alpha t_i+\varepsilon_i) - \prod_{i=1}^{d}(1-t_i+\varepsilon_i), \tag{9}$$

where ε_i and α are parameters which we compute in the sequel. By factoring the n_I's in the expression of $g_k(p,n)$ as given by Eq. (6) it follows that

$$g_k(x,n) = Bf(t), \quad \text{for } x \in D(n/(k+1)),$$

and that

$$g_k(x,n) = Bh(t), \quad \text{for } x \in D(n/k) \setminus D(n/(k+1)),$$

where $B = (\prod_{I\in\mathcal{J}} n_I^{|I|}/|I|!)$, which is in $\mathrm{O}(|n|^d)$, $\alpha = k/(k+1)$, $\varepsilon_i = \mathrm{O}(d/n_I)$ (I is the equivalence class of i), and where

$$t_i = \frac{k+1}{n_I} \sum_{j \in I} x_j.$$

Observe that t ranges over $[0,1]^d$ when x ranges over $D(n/(k+1))$ and that t ranges over $[0,1/\alpha]^d \setminus [0,1]^d$ when x ranges over $D(n/k) \setminus D(n/(k+1))$.

It follows that our claims above are a consequence of the affiliation of the sizes of the maxima of h, f and $\partial f/\partial t_i$ to $O(1/(k+1))$ (on the suitable domains) whose proof is given in the next lemma.

Lemma 3.2. *The maxima on the domain $[0,1]^d$ of the function $f(t)$, defined by Eq. (9), and of its partial derivatives are in $1/(k+1) + O(\max_i \varepsilon_i)$.*

The maxima on the domain $[0,1/\alpha]^d \setminus [0,1]^d$ of the function $h(t)$, defined by Eq. (8), is in $1/(k+1) + O(\max_i \varepsilon_i)$.

Proof. The result is trivial for the function h. For the function f we proceed as follows. Let $f^*(t)$ be the function obtained by substituting in the expression of $f(t)$ the ε_i by 0. Since the function $f(t)$ is bounded on the unit cube we have

$$f(t) = f^*(t) + O\Big(\sup_i \varepsilon_i\Big).$$

Similarly,

$$\frac{\partial f(t)}{\partial t_i} = \frac{\partial f^*(t)}{\partial t_i} + O\Big(\sup_i \varepsilon_i\Big).$$

The expressions of f^* and $\partial f^*/\partial t_i$ are sufficiently simple to allow the calculation of the respective maxima. Since the calculations are elementary we give directly the results. The search for an extremum on the open unit cube, by setting to zero the partial derivatives, shows the existence of a unique extremum obtained for $t_i = t_j$. We denote by τ_1 the extremum of f^* and by τ_2 the extremum of $\partial f^*/\partial t_i$, respectively; their values are

$$\tau_1 = \frac{(1-\alpha)^d}{(1-\alpha^{d/(d-1)})^{d-1}}, \qquad \tau_2 = \alpha(1-\alpha)\left\{\frac{1-\alpha}{1-\alpha^{(d+1)/(d-1)}}\right\}^{d-1}.$$

Using the fact that the function $a \to \alpha^a$ is non-decreasing it can be verified that τ_1 and τ_2 are less than $1-\alpha$. By examining the values of the functions on the boundary of the domain it follows that their maximum is exactly $1-\alpha = 1/(k+1)$. □

Fourth, we mention without proof (the calculation is tedious but elementary) that

$$\int_{\mathbb{R}^d} g_k(x,n)\,\mathrm{d}x = \prod_{I \in \mathcal{J}} \frac{n_I^{2|I|}}{(2|I|)!} \left\{\frac{1}{k^d} - \frac{1}{(k+1)^d}\right\} + O\left(\frac{|n|^{2d-1}}{k^d}\right).$$

Fifth, we apply Theorem 2.1 on the domain $D(n/k)\backslash D(n/(k+1))$ ($g_k(x,n)$ is decreasing on this domain) and Theorem 2.2 on the domain $D(n/(k+1))$. This completes the proof of our theorem. □

The following result generalizes the preceding theorem to the case where the lines are assigned weights by a function which is homogeneous with respect to their slopes. The particular form of this generalization concerns the study of the length of segments of the cube $C(d,n)$. The proof of the following theorem resembles the proof of the theorem just derived. Since the technical difficulties were resolved during the proof of the previous theorem, we only state the result leaving it to the interested reader to verify that the function $p \to h(p)\,|G_k(p)|$ satisfies on the ad hoc domain a Lipschitz condition with a Lipschitz constant in $O(n^{a+d-1}/k^a)$.

Theorem 3.3. *Let h be a real function, which is homogeneous of degree $a \geqslant 1$ and of class C^1 on $(\mathbb{R}^d_+)^*$. Let $G_k(p)$ be the set of lines of positive slope $p = (p_1,\ldots,p_d) \in \mathbb{N}^d$ traversing at least $k+1$ integer points of the cube $C(d,n)$. The number*

$$\mathrm{w}(h,n,k) = \sum_{\gcd(p_1,\ldots,p_d)=1} h(p)\,|G_k(p)|$$

is given by the formula

$$\mathrm{w}(h,n,k) = \frac{n^{a+2d}}{\zeta(d)}\left(\frac{1}{k^{a+d}} - \frac{1}{(k+1)^{a+d}}\right)\omega(h) + \mathrm{O}\left(\begin{cases}\dfrac{n^{a+3}}{k^{a+2}}\log\dfrac{n}{k}, & \text{if } d=2\\[2mm] \dfrac{n^{a+2d-1}}{k^{a+d}}, & \text{otherwise}\end{cases}\right)$$

where $\omega(h) = \int_{[0,1]^d}\prod_i(1-x_i)h(x_1,\ldots,x_d)\,\mathrm{d}x_1\cdots\mathrm{d}x_d$. □

From Theorem 3.3 we can derive the average length l_n and standard deviation σ_n of the maximal segments of the cube $C(d,n)$. A segment of the cube with slope $p \in \mathbb{N}^d$ and endpoints A_1 and A_2 is called maximal, if for each i one of the points $A_i \pm p$ is not a point on the cube. Thus using Theorem 3.3 we can show that the previous quantities are given by

$$l_n \sim \frac{2^{2d}}{2^d-1}\,\omega(\|\ \|)\,n,$$

$$\sigma_n{}^2 \sim \frac{2^{2d}}{2^d-1}\left((2\zeta(d+2)-1)\,\omega(\|\ \|^2) - \frac{2^{2d}}{2^d-1}\omega(\|\ \|)^2\right)n^2$$

where $\|\ \|$ is a norm on $\mathbb{R}^d$. For example, in two dimensions and using the Euclidean norm $\sqrt{x^2+y^2}$ we get $l_n \sim (0.695\ldots)\,n$ and $\sigma_n \sim (0.185\ldots)\,n$.

3.2. The counting incidence problem

In [22] it is shown that the maximal number of incidences $I(m,n)$ between m points and n lines in the plane is less than $10^{60}(m^{2/3}n^{2/3}+m+n)$. With a different approach [3,4] reduces the constant and shows that

$$I(m,n) \leqslant 3\sqrt[3]{6}\, m^{2/3}n^{2/3} + 25n + 2m.$$

The above bound is tight in the sense that $I(m,n) = \Omega(m^{2/3}n^{2/3}+m+n)$. The value of $I(m,n)$ is difficult to estimate only when the term $m^{2/3}n^{2/3}$ is the largest, i.e., larger than m or n. And this term is the largest precisely when $m = o(n^2)$ and $n = o(m^2)$. The example of [5, Theorem 6.18], used to prove the lower bound, is based on sets of points and lines of a square grid. Using Theorem 2.1 we can make the exact calculations. This leads to the following result.

Theorem 3.4. *Assume that $m = o(n^2)$ and $n = o(m^2)$. Then*

$$\liminf_{n,m\to\infty} \frac{I(m,n)}{m^{2/3}n^{2/3}} \geqslant \sqrt[3]{6/\pi^2}.$$

Proof. Let ℓ be a line in the grid of size $p \times p$. Denote by $I(\ell)$ the number of points of the grid lying on the line ℓ. Let L be the set of lines of the grid with positive slope $(p_1,p_2) \leqslant \alpha(p,p)$, and let n_α be the number of such lines. Put $I(L) = \sum_{\ell\in L} I(\ell)$. It is clear that $I(L)$ is a lower bound for $I(p^2, n_\alpha)$. To calculate precisely the numbers n_α and $I(L)$ we proceed as follows. Let $u_k(p_1,p_2) = (p-kp_1)^+(p-kp_2)^+$ and $g_k(p_1,p_2) = u_k(p_1,p_2) - u_{k+1}(p_1,p_2)$ be the number of lines of slope (p_1,p_2), with $\gcd(p_1,p_2)=1$, each traversing at least $k+1$ points of the grid of size p (see Eq. (5)). Then we have

$$n_\alpha = \sum_{\substack{\gcd(p_1,p_2)=1\\ p_1,p_2\leqslant\alpha p}} g_1(p_1,p_2)$$

and

$$\begin{aligned} I(L) &= \sum_{\substack{\gcd(p_1,p_2)=1\\ p_1,p_2\leqslant\alpha p}} \sum_{k=1}^{\infty} (k+1)\{g_k(p_1,p_2) - g_{k+1}(p_1,p_2)\} \\ &= \sum_{\substack{\gcd(p_1,p_2)=1\\ p_1,p_2\leqslant\alpha p}} \{2u_1(p_1,p_2) - u_2(p_1,p_2)\}. \end{aligned}$$

Trivial calculations show that $\sup g_1(p_1,p_2) = O(\alpha p^2)$ and $\sup(2u_1 - u_2) = O(p^2)$. According to Theorem 2.1 and assuming that $\alpha p \to \infty$, we obtain

$$n_\alpha = \frac{1}{\zeta(2)} p^4 f_1(\alpha) + O(\alpha^2 p^3 \log \alpha p),$$

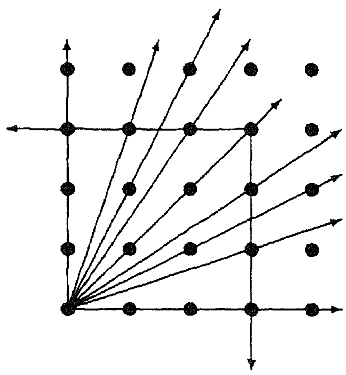


Fig. 3. Grid of size $p = 6$ and the slopes $\leqslant (3,3)$.

and

$$I(L) = \frac{1}{\zeta(2)} p^4 f_2(\alpha) + O(\alpha p^3 \log \alpha p)$$

where $f_1(\alpha) = \alpha^3(1 - 3/4\alpha)$, $f_2(\alpha) = \alpha^2(1 - 1/2\alpha^2)$ for $\alpha \leqslant 1/2$ and $f_1(\alpha) = \alpha^2(1 - 1/2\alpha)^2 - 1/16$, $f_2(\alpha) = 2\alpha^2(1 - 1/2\alpha)^2 - 1/16$ for $\alpha \geqslant 1/2$. Combining these last two equations we obtain

$$I(L) = \zeta(2)^{-1/3} n_\alpha^{2/3} p^{4/3} \frac{f_2(\alpha)}{f_1(\alpha)^{2/3}} \left\{ 1 + O\left(\frac{\log \alpha p}{\alpha p} \right) \right\}. \tag{10}$$

Let now m, n be as in the theorem and let $p = \lfloor \sqrt{m} \rfloor$. Since $n_{1/p} \approx p \approx \sqrt{m}$ and $n_1 \approx p^4 \approx m^2$, and since n_α is an increasing function of α we get an $0 < \alpha < 1$ such that $n_\alpha \leqslant n < n_{\alpha + 1/p}$. Furthermore, we can easily verify that $\alpha p \to \infty$ and that $\alpha \to 0$ from which we deduce that $n_{\alpha+1/p} \sim n_\alpha \sim n$. Then according to (10) we can write $I(L) \sim \zeta(2)^{-1/3} n^{2/3} m^{2/3}$, and we use the fact that $I(L)$ is a lower bound for $I(m,n)$ to conclude. □

3.3. Complexity of the edge visibility region

The problem is to calculate the plane region illuminated by a segment in the presence of other segments. Suri and O'Rourke [21] have established an $\Omega(n^4)$ lower bound on the complexity of any algorithm that calculates explicitly the boundary of the illuminated region. The example used by [16] is illustrated in Fig. 4. It consists of a horizontal luminescent edge ($y = 0$); at the vertices $(0,0), (\pm 1, 0), (\pm 2, 0), \ldots, (\pm(n-1), 0)$, light sources are located emitting light in all directions. Above and parallel to this edge place two rows ($y = 1$, $y = 2$) each consisting of n closely spaced line segments, thus permitting $\Theta(n^2)$ beams of light to emerge above them. Since these beams intersect in $\Theta(n^4)$ points above the second row we obtain a region with $\Theta(n^4)$ vertices and edges. The mathematical analysis that determines the $\Omega(n^4)$ lower

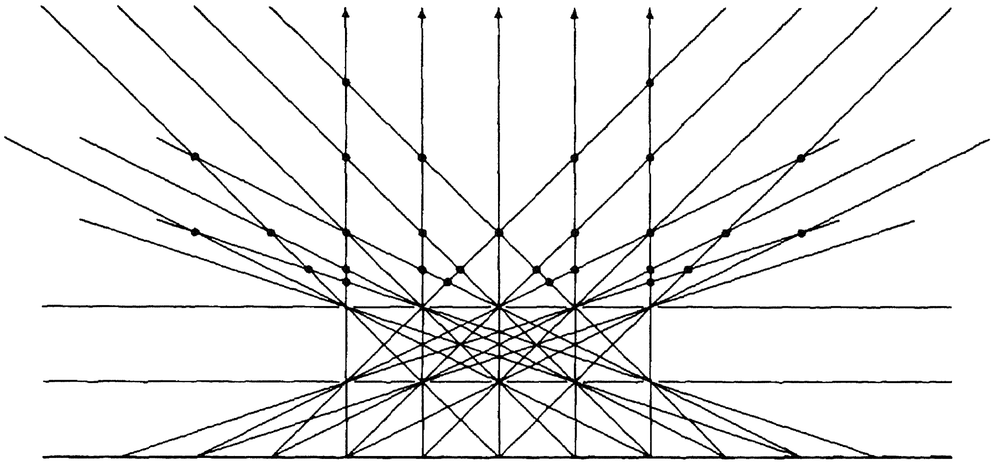


Fig. 4. The visibility region.

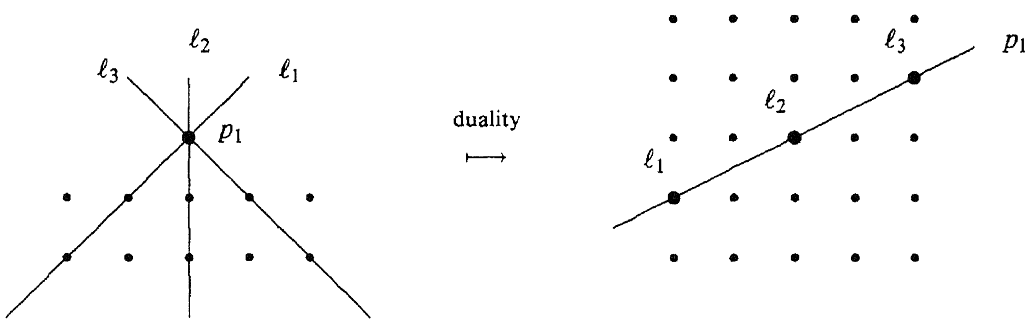


Fig. 5. The duality.

bound is based on the evaluation of the number $N(n)$ of distinct intersections located on the half-plane $y > 2$ between lines passing through the points $(\pm i, 1)$ and $(\pm j, 2)$ for $0 \leqslant i, j < n$. Theorem 3.1 can be used to give an asymptotic evaluation of the number $N(n)$. Indeed, by using the duality which maps the line passing through the points $(x, 1)$ and $(y, 2)$ to the point (x, y) (see Fig. 5) we see that the number $2N(n)$ is also the number of lines with positive slope passing through at least two points of the grid of size n. Hence, we get using Theorem 3.1

$$N(n) \sim \frac{1}{\zeta(2)} \frac{3}{32} n^4.$$

For generalizations and further applications of this duality to computational geometry the reader may consult [11].

4. Conclusion

We have given two general theorems which facilitate calculations of asymptotic evaluations on the number of incidences between points (with integer coordinates) and lines of a cube and more generally of a product of simplexes. Two natural questions arise. Is it possible to generalize our results to more general classes of convex sets, like spheres and polyhedra? What is the number of incidences between points with integer coordinates and d-dimensional subspaces, ($d \geqslant 2$)?

It is clear that our results generalize to convex sets C for which it is possible to express "simply" (as a function of the slope and the integer k) the number of points with integer coordinates included in the domain

$$C \cap (-kp + C) \setminus (-(k+1)p + C).$$

However, the class of convex sets for which this is possible still remains to be determined. The answer to the second question seems to be more delicate since a subspace of dimension $\geqslant 2$ is not uniquely defined by a "slope" as is in the case of lines.

A natural generalization of our theorems concerns the asymptotic evaluation of sums over visibility sets. If $V(S)$ is the set of lattice points which are visible from each of the points of S then define

$$s(f,S,\Delta) := \sum_{\Delta \cap V(S)} f(x).$$

Rumsey [19] treats the case where the "weight measure" f is constant and the set S is fixed: the density of the set $V(S)$ is given by the infinite product

$$\prod_{p \text{ prime}} \left(1 - \frac{|S/p|}{p^d}\right),$$

where S/p denotes the set of equivalence classes of the relation of equality modulo p on S. Using techniques similar to those developed in Section 2 it can be shown (under ad hoc hypothesis on the function f and for a fixed finite set S) that

$$s(f,S,\Delta) \sim \prod_{p \text{ prime}} \left(1 - \frac{|S/p|}{p^d}\right) \int_\Delta f(x)\,\mathrm{d}x$$

as Δ grows by dilatation to the entire plane. Eventual applications of this result as well as its generalization to the case of "variable" sets S (for example when it is assumed that the points of S are located on the boundary of the domain Δ) are still to be explored.

Acknowledgements

We thank the anonymous referee for helpful suggestions.

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