Asymptotic expansions for Riesz fractional derivatives of Airy functions and applications

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Abstract

Riesz fractional derivatives of a function, $D_x^\alpha f(x)$ (also called Riesz potentials), are defined as fractional powers of the Laplacian. Asymptotic expansions for large x are computed for the Riesz fractional derivatives of the Airy function of the first kind, $Ai(x)$, and the Scorer function, $Gi(x)$. Reduction formulas are provided that allow one to express Riesz potentials of products of Airy functions, $D_x^\alpha \{Ai(x)Bi(x)\}$ and $D_x^\alpha \left\{Ai^2(x)\right\}$, via $D_x^\alpha Ai(x)$ and $D_x^\alpha Gi(x)$. Here $Bi(x)$ is the Airy function of the second type. Integral representations are presented for the function $A_2\left(a,b;x\right) = Ai\left(x-a\right)Ai\left(x-b\right)$ with $a, b \in \mathbb{R}$ and its Hilbert transform. Combined with the above asymptotic expansions they can be used for computing asymptotics of the Hankel transform of $D_x^\alpha \left\{A_2\left(a,b;x\right)\right\}$. These results are used for obtaining the weak rotation approximation for the Ostrovsky equation (asymptotics of the fundamental solution of the linearized Cauchy problem as the rotation parameter tends to zero).

1 Introduction

It is well known that fundamental solutions of the linearized Cauchy problems for equations of the Korteweg-de Vries (KdV henceforth) type can be expressed in terms of the Airy function of the first type, $Ai(x)$. Indeed, for the KdV

$$u_t + u_{xxx} + \left(u^2\right)_x = 0,$$

the above fundamental solution has the representation

$$\mathcal{E}_0(x,t) = \frac{1}{\sqrt[3]{3t}} Ai\left(\frac{x}{\sqrt[3]{3t}}\right). \tag{1.1}$$

A close relative of KdV, the Ostrovsky equation takes into account the effect of the weak rotation (Earth's rotation) and, after the appropriate rescaling, it can be written in the form (see [12])

$$u_t + u_{xxx} + \left(u^2\right)_x = \gamma \int_{-\infty}^{x} u(y,t)\, dy,$$

where $\gamma = const > 0$ is a small rotation parameter. It was shown in [19] that the fundamental solution of the Cauchy problem for the linearized Ostrovsky equation can be represented in the form

$$\mathcal{E}(x,t) = \mathcal{E}_0(x,t) + \mathcal{E}_\gamma(x,t), \tag{1.2}$$

where $\mathcal{E}_0(x,t)$ is given by (1.1) and

$$\mathcal{E}_\gamma(x,t) = -\frac{\sqrt{\gamma t}}{\sqrt[3]{3t}} \int_0^\infty Ai\left(\frac{x+y}{\sqrt[3]{3t}}\right) \frac{J_1\left(2\sqrt{\gamma t y}\right)}{\sqrt{y}}\, dy, \tag{1.3}$$

where $J_\nu(x)$ is the Bessel function of order ν.

Riesz fractional derivatives (also called Riesz potentials) are defined as fractional powers of the Laplacian, $D_x^\alpha = (-\Delta)^{\alpha/2}$. Riesz potentials of fundamental solutions of linearized Cauchy problems are of great importance in the study of global solvability, properties and long-time behavior of solutions to initial-value problems (see [14, 8, 9, 10, 7, 20] and the references therein). In the current paper we are concerned with obtaining asymptotic expansions as $x \to \pm\infty$ of the Riesz fractional derivatives of $Ai(x)$ and its conjugate, the Scorer function $Gi(x) = -HAi(x)$. Here H is the Hilbert transform (see

(2.1) below). Riesz potentials of these functions of order $\alpha = 1/2$ stand out as the highest fractional derivatives that are still uniformly bounded on the whole real axis (see [9, 10]). Moreover, all semi-integer derivatives of $Ai(x)$ and $Gi(x)$ can be expressed in terms of the products of the Airy functions (see [20]).

In the next section, we give definitions of Riesz potentials and integral transforms used in the current work. In Section 3, we provide asymptotic expansions of the Riesz potentials of the Airy function of the first kind, $Ai(x)$, and the Scorer function, $Gi(x)$, from which asymptotic estimates of the Riesz fractional derivatives $D_x^\alpha \{Ai(x)Bi(x)\}$, $D_x^\alpha \{Ai^2(x)\}$ and $D_x^\alpha \{Ai\,(x-a)\,Ai\,(x-b)\}$ with $a, b \in \mathbb{R}$ can be obtained. Here $Bi(x)$ is the Airy function of the second type. Section 4 is devoted to integral representations of the Riesz potentials of the products of Airy functions and their Hankel transforms. It can be used for obtaining their asymptotic expansions. In Section 5, we show applications of the above results for obtaining the weak rotation approximation for the Ostrovsky equation (asymptotics of the fundamental solution of the linearized Cauchy problem as $\gamma \to 0$). In the Appendix, we collect integral representations, properties and asymptotic expansions of the Airy functions $Ai(x)$ and $Bi(x)$ and the Scorer function $Gi(x)$ used in the current paper. We also derive asymptotic expansions of the antiderivative $\int_0^x Gi(t)\,dt$ as $x \to \pm\infty$.

2 Definitions and preliminaries

Let $f:\ \mathbb{R} \to \mathbb{R}$. Define the Fourier transform of this function by the formula

$$\hat{f}(\xi) = \mathcal{F}\left\{f(x)\right\}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} f(x)\,dx$$

and the inverse Fourier transform by

$$f(x) = \mathcal{F}^{-1}\left\{\hat{f}(\xi)\right\}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\hat{f}(\xi)\,d\xi.$$

Introduce the Hankel transform of the function f by (see [4, p. 316])

$$\tilde{f}(k) = \mathcal{H}_{x\to k}\left\{f(x)\right\}(k) = \int_0^{\infty} f(x)J_0(kx)x\,dx$$

and the corresponding inverse transform by

$$\mathcal{H}^{-1}_{k\to x}\left\{\tilde{f}(k)\right\}(x)=\int_0^\infty \tilde{f}(k)J_0(kx)k\,dk.$$

Introduce the Hilbert transform of f by the formula (see [16, p. 120])

$$H\left\{f(x)\right\}=\frac{1}{\pi}P.V.\int_{-\infty}^{\infty}\frac{f(y)}{y-x}\,dy, \tag{2.1}$$

where $P.V.$ denotes the Cauchy principal value of an integral. According to our choice of the Fourier transform, $\widehat{(Hf)}(\xi)=i\mathrm{sgn}(\xi)\widehat{f}(\xi)$. Also, $H^2=-I$ on $L_p(\mathbb{R})$, $p\geq 1$, where I is the identity operator (see [5], p 51).

For $x\in\mathbb{R}^n$ Riesz potentials are defined via the Fourier transform (see [15, p. 117] and [5, p. 88])

$$\left((-\Delta)^{\alpha/2}f\right)^{\wedge}(\xi)=|\xi|^{\alpha}\hat{f}(\xi). \tag{2.2}$$

For $x\in\mathbb{R}$ and real $\alpha>-1$

$$D^{\alpha}_x f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}|\xi|^{\alpha}\hat{f}(\xi)e^{i\xi x}\,d\xi, \tag{2.3}$$

provided that the integral in the right-hand side exists.

We shall also use the notation

$$\left(D^{\alpha}f\right)(g(x))=D^{\alpha}_z f\left(z\right)\bigg|_{z=g(x)} \tag{2.4}$$

whenever a fractional derivative is computed first and then its argument is set to equal $g(x)$. Notice that for any $a=const>0$

$$D^{\alpha}_x\left(f(ax)\right)=a^{\alpha}\left(D^{\alpha}f\right)(ax), \tag{2.5}$$

where in the left-hand side D^{α}_x acts on $f(ax)$ and the right-hand side is understood in the sense of (2.4). The proof of (2.5) is based on using (2.3) and the well-known property of the Fourier transform $\mathcal{F}\left\{f(ax)\right\}(\xi)=(1/a)\hat{f}(\xi/a)$ for $a>0$.

Introduce the function

$$A_2(a,b;x)=Ai\left(x-a\right)Ai\left(x-b\right). \tag{2.6}$$

This function appears in the studies of the Gelfand-Levitan-Marchenko equation (see [2, p. 408]), the second Painleve equation (see [18, p. 134]) and the limit at the "edge of the spectrum" of the level spacing distribution functions obtained from scaling random models of Hermitian matrices in the Gaussian Unitary Ensemble [3, 17]. Recently, a new integral representation has been found for $A_2(a, b; x)$ (see (4.1) below). It allows us to compute Riesz fractional derivatives of this function.

The next statement was proved in [21]. It provides projection formulas for the Riesz potentials of the products of Airy functions.

Theorem 1. *Riesz fractional derivatives of the products of Airy functions have the following representations for $\alpha > -1/2$ and $x \in \mathbb{R}$:*

$$D_x^\alpha \left\{Ai^2(x)\right\} = k_\alpha \left[\left(D^{\alpha-1/2}Ai\right)\left(2^{2/3}x\right) - \left(D^{\alpha-1/2}Gi\right)\left(2^{2/3}x\right)\right] \tag{2.7}$$

and

$$D_x^\alpha \left\{Ai(x)Bi(x)\right\} = k_\alpha \left[\left(D^{\alpha-1/2}Ai\right)\left(2^{2/3}x\right) + \left(D^{\alpha-1/2}Gi\right)\left(2^{2/3}x\right)\right], \tag{2.8}$$

where

$$k_\alpha = \frac{2^{2(\alpha-1)/3}}{\sqrt{2\pi}}. \tag{2.9}$$

The fractional derivatives in the right-hand sides are defined by (2.4) and the Scorer function $Gi(x)$ by (6.7).

3 Riesz potentials of $Ai(x)$ and $Gi(x)$

The Riesz potentials of $Ai(x)$ and $Gi(x)$ can be written in the form

$$D_x^\alpha Ai(x) = \Re\left\{F^\alpha(x)\right\}, \quad D_x^\alpha Gi(x) = \Im\left\{F^\alpha(x)\right\}, \tag{3.1}$$

where

$$F^\alpha(x) = \frac{1}{\pi}\int_0^\infty \xi^\alpha \exp\left[i\left(x\xi + \xi^3/3\right)\right] d\xi. \tag{3.2}$$

This integral is defined for real x and $-1 < \Re(\alpha) < 2$. However, we can modify the integral over the positive semi-axis by turning the half line slightly upwards into the complex plane, say, in such a way that $\arg\xi = \pi/6$, with the path running into the valley of $\exp\left(i\xi^3\right)$. In the analysis to follow we shall make this type of modification, and in the new representations we shall take x to be any complex number. This will also remove the upper bound restriction on $\Re(\alpha)$. Hence, in the analysis to follow we shall only assume that $\Re(\alpha) > -1$.

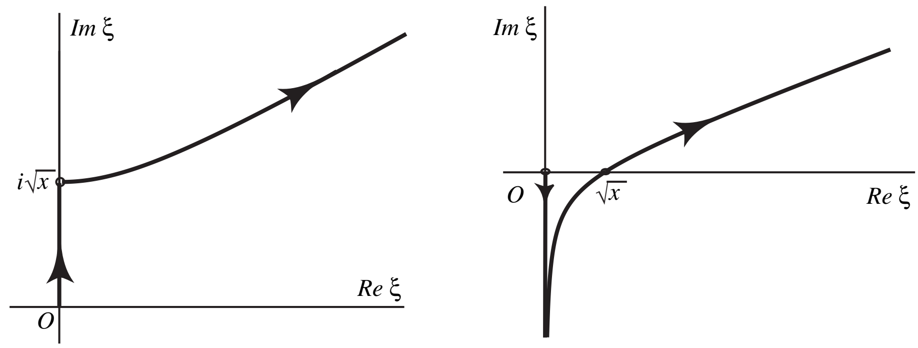

Figure 1: Modification of the paths of integration for the integrals in (3.2) (left) and (3.29) (right), giving the two integrals $F_1^\alpha(x)$ and $F_2^\alpha(x)$ in (3.4) and (3.5), and the two integrals $G_1^\alpha(x)$ and $G_2^\alpha(x)$ in (3.30) and (3.31).

3.1 Asymptotic expansion for $x \to +\infty$

We use a representation of the integral in (3.2) similar to that for $Gi(x)$ in (3.18) of [6]. The exponential function in the integrand of (3.2) has a saddle point at $\xi = i\sqrt{x}$. We integrate from the origin to this saddle point, and from there to infinity inside the valley at $\infty \exp(\pi i/6)$ (see Fig. 1). As a result, we can write

$$F^\alpha(x) = F_1^\alpha(x) + F_2^\alpha(x), \tag{3.3}$$

where

$$F_1^\alpha(x) = \frac{\exp\left[i\pi(\alpha+1)/2\right]}{\pi} \int_0^{\sqrt{x}} v^\alpha \exp\left(-xv + v^3/3\right)\, dv \tag{3.4}$$

and

$$F_2^\alpha(x) = \frac{1}{\pi} \int_{i\sqrt{x}}^{\infty \exp(i\pi/6)} \xi^\alpha \exp\left[i(x\xi + \xi^3/3)\right]\, d\xi. \tag{3.5}$$

3.1.1 Asymptotic expansion of $F_1^\alpha(x)$ for large positive x

Lemma 1. *The following asymptotic expansion holds for $x \to +\infty$:*

$$F_1^\alpha(x) \sim \frac{\exp\left[i\pi(\alpha+1)/2\right]}{\pi\, x^{\alpha+1}} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+3k+1)}{3^k\, k!} \frac{1}{x^{3k}}. \tag{3.6}$$

Proof. The derivation of (3.6) is based on the application of Watson's lemma (see [11]). Expanding the exponential in the integrand of (3.4) into the power series,

$$\exp\left(\frac{v^3}{3}\right) = \sum_{k=0}^{\infty} \frac{v^{3k}}{3^k\, k!},$$

and replacing the upper limit of integration by ∞ we integrate termwise. As a result, we get

$$F_1^{\alpha}(x) \sim \frac{\exp\left[i\pi(\alpha+1)/2\right]}{\pi} \sum_{k=0}^{\infty} \frac{1}{3^k\, k!} \int_0^{\infty} v^{\alpha+3k} e^{-xv}\, dv. \tag{3.7}$$

Evaluating the integrals we deduce (3.6). □

Remark. For $\alpha = 0$ the imaginary part of (3.6) equals the expansion of $Gi(x)$ as given in (6.11). Also, for $\alpha = 0$ the real part of (3.6) vanishes, and the expansion of $Ai(x)$ cannot be recovered from it. Therefore, for $\alpha = 0$ we need the asymptotic expansion of $F_2^{\alpha}(x)$ (see (3.5)) in order to recover the expansion of $Ai(x)$ as given in (6.3). The expansion of $F_2^{\alpha}(x)$ is also important for other non-negative integers α, and we continue to deal with this function for general values of this parameter.

3.1.2 Asymptotic expansion of $F_2^{\alpha}(x)$ for large positive x

Lemma 2. *The integral $F_2^{\alpha}(x)$ has the following asymptotic expansion for $x \to \infty$:*

$$F_2^{\alpha}(x) \sim x^{\alpha/2-1/4} e^{-\zeta(x)} \frac{\exp\left(i\pi\alpha/2\right)}{2\pi} \sum_{k=0}^{\infty} f_k \frac{\Gamma\left((k+1)/2\right)}{x^{3k/4}}, \tag{3.8}$$

where $\zeta(x) = \frac{2}{3}x^{3/2}$.

Proof. The main contribution to the asymptotics of the integral in (3.5) comes from a neighborhood of the lower limit $i\sqrt{x}$. The first transformation, $\xi = \sqrt{x}(i+\eta)$, gives

$$F_2^{\alpha}(x) = x^{(\alpha+1)/2} e^{-\zeta(x)} \frac{\exp\left(i\pi\alpha/2\right)}{\pi} \int_0^{\infty} (1-i\eta)^{\alpha} e^{-x^{3/2}\left(\eta^2 - i\eta^3/3\right)}\, d\eta. \tag{3.9}$$

The substitution $w = \eta\sqrt{1 - i\eta/3}$ transforms this integral into the standard form

$$F_2^\alpha(x) = x^{(\alpha+1)/2} e^{-\zeta(x)} \frac{\exp(i\pi\alpha/2)}{\pi} \int_0^\infty e^{-x^{3/2}w^2} f(w)\,dw, \tag{3.10}$$

where

$$f(w) = (1 - i\eta)^\alpha \frac{d\eta}{dw}.$$

The asymptotic expansion in question can be obtained from (3.10) by developing $f(w)$ as a power series and term by term integration. First we use an expansion

$$\frac{d\eta}{dw} = \frac{w}{\eta\,(1 - i\eta/3)} = \sum_{k=0}^\infty a_k w^k \tag{3.11}$$

and write the coefficients a_k in the form of a Cauchy integral,

$$a_k = \frac{1}{2\pi i} \int_{C_w} \frac{d\eta}{dw} \frac{dw}{w^{k+1}}, \tag{3.12}$$

where C_w is a small circle around the origin in the $w-$plane. This can be written as an integral in the $\eta-$plane:

$$a_k = \frac{1}{2\pi i} \int_{C_\eta} g(\eta) \frac{d\eta}{\eta^{k+1}}, \tag{3.13}$$

where $g(\eta) = (1 - i\eta/3)^{-(k+1)/2}$ and C_η is a small circle around the origin. We see that the coefficient a_k is the coefficient of η^k in the Taylor expansion of $g(\eta)$. Since

$$g(\eta) = \sum_{j=0}^\infty C^j_{-(k+1)/2} (-\tfrac{1}{3} i\eta)^j, \tag{3.14}$$

where C_n^m are binomial coefficients, we deduce

$$a_k = C^j_{-(k+1)/2} \left(-\tfrac{1}{3} i\right)^k = \frac{i^k}{3^k k!} \frac{\Gamma\left(\frac{3k+1}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)}, \quad k = 0, 1, 2, \ldots. \tag{3.15}$$

Next, we have

$$\eta = \sum_{k=0}^\infty \frac{a_k}{k+1} w^{k+1} \qquad \text{and} \qquad (1 - i\eta)^\alpha = \sum_{k=0}^\infty b_k w^k, \tag{3.16}$$

where a few first coefficients are

$$b_0 = 1, \quad b_1 = -i\alpha, \quad b_2 = \tfrac{1}{2}\alpha(1 - \alpha - ia_1). \tag{3.17}$$

Finally, we can write

$$f(w) = \sum_{k=0}^{\infty} f_k w^k, \tag{3.18}$$

where a few first coefficients are

$$f_0 = 1, \quad f_1 = a_1 - i\alpha, \quad f_2 = (2a_2 - 3i\alpha a_1 - \alpha^2 + \alpha)/2. \tag{3.19}$$

Taking into account (3.15), we can rewrite this in the form

$$f_0 = 1, \quad f_1 = i(1 - 3\alpha)/3, \quad f_2 = (-5 + 24\alpha - 12\alpha^2)/24. \tag{3.20}$$

Using expansion (3.18) for the calculation of the integral in (3.10) we obtain (3.8). □

3.1.3 Asymptotic expansions of $D_x^\alpha Ai(x)$ and $D_x^\alpha Gi(x)$

We summarize here the results for the Riesz fractional derivatives defined by (3.1).

Theorem 2. *The following asymptotic expansions hold for $x \to +\infty$:*

$$\begin{aligned} D_x^\alpha Ai(x) \sim & -\frac{\sin(\pi\alpha/2)}{\pi\, x^{\alpha+1}} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+3k+1)}{3^k\, k!} \frac{1}{x^{3k}} \\ & + \frac{x^{\alpha/2-1/4} e^{-\zeta(x)}}{2\pi} \left[\cos(\pi\alpha/2) S_1(\alpha,x) - \sin(\pi\alpha/2) S_2(\alpha,x)\right] \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} D_x^\alpha Gi(x) \sim & \frac{\cos(\pi\alpha/2)}{\pi\, x^{\alpha+1}} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+3k+1)}{3^k\, k!} \frac{1}{x^{3k}} \\ & + \frac{x^{\alpha/2-1/4} e^{-\zeta(x)}}{2\pi} \left[\sin(\pi\alpha/2) S_1(\alpha,x) + \cos(\pi\alpha/2) S_2(\alpha,x)\right], \end{aligned} \tag{3.22}$$

where

$$S_1(\alpha,x) \sim \sum_{k=0}^{\infty} \frac{f_{2k}\Gamma(k+\frac{1}{2})}{x^{3k/2}}, \quad S_2(\alpha,x) \sim \sum_{k=0}^{\infty} \frac{f_{2k+1}\Gamma(k+1)}{x^{3k/2+3/4}} \tag{3.23}$$

and a few first coefficients f_k are given by (3.20).

Remark. It follows from the construction that the coefficients f_{2k} are real and f_{2k+1} are imaginary. Also, setting $\alpha = 0$ yields $f_k = a_k$, where a_k is given in (3.15). Moreover, for $\alpha = 0$ the real part of the expansion in (3.8) becomes

$$\Re\left\{F_2^0(x)\right\} \sim \frac{\exp\left[-\zeta(x)\right]}{2\sqrt{\pi}x^{1/4}} \sum_{k=0}^{\infty} \frac{a_{2k}\,(1/2)_{\,k}}{x^{3k/2}} \quad \text{for} \quad x \to \infty, \tag{3.24}$$

where $(b)_n = b(b+1)...(b+n-1)$ is the Pochhammer symbol. According to (3.15), we have

$$\frac{a_{2k}\,(1/2)_k}{x^{3k/2}} = \frac{c_k}{\zeta^k}, \quad k = 0, 1, 2, \ldots. \tag{3.25}$$

Thus, we see that (3.24) turns into the expansion for $Ai(x)$ as given by (6.3).

Proof. We have

$$D_x^\alpha \left\{Ai(x)\right\} = \Re\left\{F_1^\alpha(x) + F_2^\alpha(x)\right\}, \quad D_x^\alpha Gi(x) = \Im\left\{F_1^\alpha(x) + F_2^\alpha(x)\right\}. \tag{3.26}$$

Taking the real and imaginary parts of the expansions in (3.6) and (3.8) we get for large positive x

$$\begin{aligned} \Re\left\{F_1^\alpha(x)\right\} &\sim -\frac{\sin(\pi\alpha/2)}{\pi\, x^{\alpha+1}} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+3k+1)}{3^k\, k!}\, \frac{1}{x^{3k}}, \\ \Im\left\{F_1^\alpha(x)\right\} &\sim \frac{\cos(\pi\alpha/2)}{\pi\, x^{\alpha+1}} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+3k+1)}{3^k\, k!}\, \frac{1}{x^{3k}}, \end{aligned} \tag{3.27}$$

$$\begin{aligned} \Re\left\{F_2^\alpha(x)\right\} &= \frac{1}{2\pi} x^{\alpha/2-1/4} e^{-\zeta(x)} \left[\cos(\pi\alpha/2) S_1(\alpha, x) - \sin(\pi\alpha/2) S_2(\alpha, x)\right], \\ \Im\left\{F_2^\alpha(x)\right\} &= \frac{1}{2\pi} x^{\alpha/2-1/4} e^{-\zeta(x)} \left[\sin(\pi\alpha/2) S_1(\alpha, x) + \cos(\pi\alpha/2) S_2(\alpha, x)\right], \end{aligned} \tag{3.28}$$

where $S_1(\alpha, x)$ and $S_2(\alpha, x)$ are defined by (3.23). □

Remark. The expansions for $\Re\left\{F_2^\alpha(x)\right\}$ and $\Im\left\{F_2^\alpha(x)\right\}$ are relevant only for integer values of α. For other values of α they can be neglected.

3.2 Asymptotic expansions for large negative x

In this case we write

$$G^\alpha(x) = F^\alpha(-x) = \frac{1}{\pi}\int_0^\infty \xi^\alpha e^{i(-x\xi+\xi^3/3)}\,d\xi, \quad x>0. \tag{3.29}$$

There is a positive stationary point (saddle point) at $\xi_0 = \sqrt{x}$, which gives a contribution to the asymptotic expansion, but one should take into account a contribution from $\xi = 0$ as well. In order to handle both of them, we replace the original path of integration by the two new contours (see Fig. 1). This leads to

$$G^\alpha(x) = G_1^\alpha(x) + G_2^\alpha(x),$$

where

$$G_1^\alpha(x) = \frac{1}{\pi}\int_0^{-i\infty} \xi^\alpha e^{i(-x\xi+\xi^3/3)}\,d\xi \tag{3.30}$$

and

$$G_2^\alpha(x) = \frac{1}{\pi}\int_{-i\infty}^{\infty e^{\pi i/6}} \xi^\alpha e^{i(-x\xi+\xi^3/3)}\,d\xi. \tag{3.31}$$

Notice that the contour for $G_2^\alpha(x)$ runs from the valley at $-i\infty$ to the valley at $\infty\exp(i\pi/6)$, and we can take this contour through the saddle point $\xi_0 = \sqrt{x}$.

3.2.1 Asymptotic expansion of $G_1^\alpha(x)$

Lemma 3. *The integral $G_1^\alpha(x)$ has the following asymptotic expansion for $x \to +\infty$:*

$$G_1^\alpha(x) \sim \frac{\exp\left(-i\pi(\alpha+1)/2\right)}{\pi\, x^{\alpha+1}} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+3k+1)}{3^k\, k!}\,\frac{(-1)^k}{x^{3k}}, \quad x\to+\infty. \tag{3.32}$$

Proof. We set $\xi = -iv$ with $v>0$ in the integral representation of $G_1^\alpha(x)$ and get

$$G_1^\alpha(x) = \frac{\exp\left(-i\pi(\alpha+1)/2\right)}{\pi}\int_0^\infty v^\alpha e^{-\left(xv+v^3/3\right)}\,dv. \tag{3.33}$$

Conducting the same arguments as in the proof of Lemma 1 we deduce (3.32). □

3.2.2 Asymptotic expansion of $G_2^\alpha(x)$

Lemma 4. *$G_2^\alpha(x)$ has the following asymptotic expansion for $x \to +\infty$:*

$$G_2^\alpha(x) \sim x^{\alpha/2-1/4} \frac{\exp\left[i(\pi/4 - \zeta(x))\right]}{\pi} \sum_{k=0}^{\infty} g_{2k} \frac{i^k\,\Gamma\left(k+1/2\right)}{x^{3k/2}}. \tag{3.34}$$

Proof. First, we set $\xi = \eta\sqrt{x}$ in (3.31) which gives

$$G_2^\alpha(x) = \frac{x^{(\alpha+1)/2}}{\pi} \int_{-i\infty}^{\infty e^{\pi i/6}} \eta^\alpha \exp\left[-x^{3/2}\phi(\eta)\right]\, d\eta, \tag{3.35}$$

where

$$\phi(\eta) = i\left(\eta - \tfrac{1}{3}\eta^3\right).$$

Notice that $\phi(1) = 2i/3$ and $\phi''(1) = -2i$.

Performing the transformation $\phi(\eta) = \phi(1) + \frac{1}{2}\phi''(1)w^2$ we get

$$w^2 = \tfrac{2}{3} - \eta + \tfrac{1}{3}\eta^3 \quad \text{and} \quad w = (\eta - 1)\sqrt{(\eta+2)/3}. \tag{3.36}$$

Next, we integrate over the neighborhood of the saddle point at $w = 0$ along the straight line passing through the origin, having an angle of $\pi/4$ with the positive $w-$axis. This yields

$$G_2^\alpha(x) = x^{(\alpha+1)/2} \frac{\exp\left[-i\zeta(x)\right]}{\pi} \int_{\infty\exp(-i3\pi/4)}^{\infty\exp(i\pi/4)} g(w)\exp\left(ix^{3/2}w^2\right)\, dw, \tag{3.37}$$

where $g(w) = \eta^\alpha d\eta/dw$ and $\zeta(x)$ is given by (6.4).

After that we expand $g(w)$ into the power series, $g(w) = \sum_{k=0}^{\infty} g_k w^k$, and get

$$G_2^\alpha(x) \sim x^{(\alpha+1)/2} \frac{\exp\left[-i\zeta(x)\right]}{\pi} \sum_{k=0}^{\infty} g_{2k} \int_{\infty\exp(-i3\pi/4)}^{\infty\exp(i\pi/4)} w^{2k}\exp\left(ix^{3/2}w^2\right)\, dw. \tag{3.38}$$

In order to evaluate these integrals we set $w = te^{i\pi/4}$, which gives

$$\begin{aligned} &\exp\left[\tfrac{1}{4}\pi i(2k+1)\right] \int_{-\infty}^{\infty} t^{2k}\exp\left(-x^{3/2}t^2\right)\, dt \\ &\qquad = \exp\left[\tfrac{1}{4}\pi i(2k+1)\right]\Gamma\left(k+\tfrac{1}{2}\right)\, x^{-3(k+1/2)/2}. \end{aligned} \tag{3.39}$$

Finally, we obtain (3.34), where a few first coefficients are

$$\begin{aligned} &g_0 = 1, \quad g_2 = \tfrac{1}{24}\left(12\alpha^2 - 24\alpha + 5\right), \\ &g_4 = \tfrac{1}{3456}(144\alpha^4 - 1344\alpha^3 + 3864\alpha^2 - 3504\alpha + 385). \end{aligned} \tag{3.40}$$

$\square$

3.2.3 Asymptotic expansions of $D_x^\alpha Ai(x)$ and $D_x^\alpha Gi(x)$ for large negative arguments

Theorem 3. *The following asymptotic expansions hold for $x \to -\infty$:*

$$\begin{aligned} D_x^\alpha Ai(x) \sim &-\frac{\sin(\pi\alpha/2)}{\pi\,|x|^{\alpha+1}} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+3k+1)}{3^k\,k!}\,\frac{(-1)^k}{|x|^{3k}} \\ &+\frac{|x|^{\alpha/2-1/4}}{\pi}\left[\sin\psi(|x|)T_1(\alpha,|x|) - \cos\psi(|x|)T_2(\alpha,|x|)\right] \end{aligned} \tag{3.41}$$

and

$$\begin{aligned} D_x^\alpha Gi(x) \sim &-\frac{\cos(\pi\alpha/2)}{\pi\,|x|^{\alpha+1}} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+3k+1)}{3^k\,k!}\,\frac{(-1)^k}{|x|^{3k}} \\ &+\frac{|x|^{\alpha/2-1/4}}{\pi}\left[\cos\psi(|x|)T_1(\alpha,|x|) + \sin\psi(x)T_2(\alpha,|x|)\right], \end{aligned} \tag{3.42}$$

where $\psi(x) = \zeta(x) + \pi/4$, $\zeta(x)$ is given by (6.4),

$$\begin{aligned} T_1(\alpha,y) &\sim \sum_{k=0}^{\infty}(-1)^k\,\frac{g_{4k}\Gamma(2k+1/2)}{y^{3k}}, \\ T_2(\alpha,y) &\sim \sum_{k=0}^{\infty}(-1)^k\,\frac{g_{4k+2}\Gamma(2k+3/2)}{y^{3k+3/2}} \end{aligned} \tag{3.43}$$

and a few first coefficients g_k are given by (3.40).

Proof. Taking the real and imaginary parts of (3.32) and (3.34) we get for $x < 0$

$$D_x^\alpha Ai(x) = \Re\left\{G_1^\alpha(|x|) + G_2^\alpha(|x|)\right\}, \quad D_x^\alpha Gi(x) = \Im\left\{G_1^\alpha(|x|) + G_2^\alpha(|x|)\right\}. \tag{3.44}$$

Summing the results we obtain (3.41) and (3.42). $\square$

Remark. For $\alpha = 0$ these expansions simplify to give those for $Ai(x)$ and $Gi(x)$ as $x \to -\infty$. First, we observe that the coefficients g_{2k} can be written as

$$g_{2k} = \frac{1}{3^{2k}(2k)!}\,\frac{\Gamma(3k+1/2)}{\Gamma(k+1/2)}, \quad k = 0,\, 1,\, 2, \ldots. \tag{3.45}$$

This can be shown in the same way as for proving (3.15). Next, for $Ai(x)$ the first item in (3.41) vanishes and the first term in (3.42) gives the first item in (6.6). For $Gi(x)$ we can use (6.10), (6.11) and the expansion for $Bi(x)$ in (6.6) to verify that the expansions in the second lines of (3.41) and (3.42) reduce to the known expansions.

4 Riesz potentials of products of Airy functions

In this section we provide integral representations for the products of Airy functions, their Riesz potentials and the Hankel transforms of the latter. They can be used for obtaining asymptotic expansions for large values of the arguments.

The next statement was proved in [23].

Theorem 4. *The following representation holds for $x \in \mathbb{R}$; a, b, ω_1, $\omega_2 \in \mathbb{R}$ and ω_1, $\omega_2 \neq 0$:*

$$Ai\left(\frac{x-a}{\omega_1}\right) Ai\left(\frac{x-b}{\omega_2}\right)$$

$$= -\frac{2}{\Omega_1}\int_0^\infty \frac{d}{dx}\left[Ai^2\left(\Omega_1 x - A + \eta^2\right)\right] J_0\left(2\left(\Omega_2 x + B\right)\eta\right)\eta\, d\eta, \tag{4.1}$$

where

$$\begin{aligned} \Omega_1 &= \frac{\omega_1+\omega_2}{2\omega_1\omega_2}, \qquad & \Omega_2 &= \frac{\omega_2-\omega_1}{2\omega_1\omega_2}, \\ A &= \frac{a\omega_1+b\omega_2}{2\omega_1\omega_2}, \qquad & B &= \frac{b\omega_1-a\omega_2}{2\omega_1\omega_2}. \end{aligned} \tag{4.2}$$

We list here several important corollaries that allow us to get the Hankel transforms of the function $A_2(a, b; x)$ and its fractional derivatives. Notice that

$$Ai\left(x-a\right) Ai\left(x-b\right) = Ai\left(x-Y-Z\right) Ai\left(x-Y+Z\right),$$

where

$$Y = \frac{a+b}{2} \qquad \text{and} \qquad Z = \frac{b-a}{2}. \tag{4.3}$$

Corollary 1. *The following formulas hold for $x \in \mathbb{R}$ and a, $b \in \mathbb{R}$:*

$$A_2(a,b;x) = -2\frac{d}{dx}\int_0^\infty Ai^2\left(x - Y + \eta^2\right) J_0\left(2Z\eta\right)\eta\, d\eta \tag{4.4}$$

and

$$\begin{aligned} -H\left\{A_2(a,b;x)\right\} = -2\frac{d}{dx}\int_0^\infty & Ai\left(x - Y + \eta^2\right) \\ & \times Bi\left(x - Y + \eta^2\right) J_0\left(2Z\eta\right)\eta\, d\eta. \end{aligned} \tag{4.5}$$

Proof. Evidently, (4.4) is a particular case of (4.1) when $\omega_1 = \omega_2 = 1$. Taking into account that [18]

$$-H\left\{Ai^2(x)\right\} = Ai(x)Bi(x)$$

and computing the Hilbert transform of (4.4) with respect to x yield (4.5). □

Corollary 2. *For $\alpha, a, b \in \mathbb{R}$ fractional derivatives of the function $A_2(a,b;x)$ are given by the formula*

$$\begin{aligned} D_x^\alpha\left\{A_2(a,b;x)\right\} = -k_\alpha\frac{d}{dx}\int_0^\infty & \left[\left(D^{\alpha-1/2}Ai\right)\left(2^{2/3}\left(x - Y + \eta^2\right)\right)\right. \\ & \left. - \left(D^{\alpha-1/2}Gi\right)\left(2^{2/3}\left(x - Y + \eta^2\right)\right)\right] J_0\left(2Z\eta\right)\eta\, d\eta, \end{aligned} \tag{4.6}$$

$$\begin{aligned} H\left\{D_x^\alpha\left\{A_2(a,b;x)\right\}\right\} = k_\alpha\frac{d}{dx}\int_0^\infty & \left[\left(D^{\alpha-1/2}Ai\right)\left(2^{2/3}\left(x - Y + \eta^2\right)\right)\right. \\ & \left. + \left(D^{\alpha-1/2}Gi\right)\left(2^{2/3}\left(x - Y + \eta^2\right)\right)\right] J_0\left(2Z\eta\right)\eta\, d\eta, \end{aligned} \tag{4.7}$$

where k_α is given by (2.9) and the integrals in the right-hand sides exist at least in the sense of distributions.

Proof. The proof follows from (4.4), (4.5). □

Corollary 3. *The following relations hold for $\alpha > -1/2$:*

$$\begin{aligned} & 2\mathcal{H}^{-1}_{Z\to\zeta}\left\{D_x^{\alpha-1}\left(Ai(x-Z)Ai(x+Z)\right)\right\} \\ & \qquad = k_\alpha\left[\left(D^{\alpha-1/2}Ai\right)(X) + \left(D^{\alpha-1/2}\right)Gi\left(X\right)\right], \end{aligned} \tag{4.8}$$

$$2\mathcal{H}_{Z\to\zeta}^{-1}\left\{D_x^{\alpha-1}H_x\left(Ai(x-Z)Ai(x+Z)\right)\right\}$$
$$= k_\alpha\left[\left(D^{\alpha-1/2}Ai\right)(X)-\left(D^{\alpha-1/2}\right)Gi\,(X)\right], \tag{4.9}$$

where k_α *is defined by (2.9) and* $X = X(x,\zeta) = 2^{2/3}\,(x+\zeta^2/4)$.

Remark. Combining the asymptotic expansions (3.21), (3.22), (3.41) and (3.42) and Corollary 3 we can obtain asymptotic expansions of the Hankel transforms (4.8) and (4.9) for $x \to \pm\infty$ or $\zeta \to \infty$.

5 Weak rotation approximation for the Ostrovsky equation

In this section we shall establish a pointwise estimate as $\gamma \to 0$ for the fundamental solution $\mathcal{E}(x,t)$ of the Cauchy problem for the linearized Ostrovsky equation. This asymptotic estimate is referred to as the weak rotation approximation.

Recall representation (1.2) for the above fundamental solution. Computing the Riesz potential for $\mathcal{E}(x,t)$ we can write

$$D_x^\alpha\mathcal{E}(x,t) = D_x^\alpha\mathcal{E}_0(x,t) + D_x^\alpha\mathcal{E}_\gamma(x,t),$$

where

$$D_x^\alpha\mathcal{E}_0(x,t) = \frac{1}{(3t)^{(1+\alpha)/3}}\left(D^\alpha Ai\right)\left(\frac{x}{\sqrt[3]{3t}}\right),$$

$$D_x^\alpha\mathcal{E}_\gamma(x,t) = -\frac{a}{(3t)^{(1+\alpha)/3}}\int_0^\infty\left(D^\alpha Ai\right)\left(\frac{x+\eta^2}{\sqrt[3]{3t}}\right)J_1\left(a\eta\right)d\eta,$$

$a = 2\sqrt{\gamma t}$ and $\gamma > 0$ is a small rotation parameter.

The next statement is a modification of Lemma 1 of [22].

Lemma 5. For $\alpha > 0$ and $x \in \mathbb{R}$

$$\int_0^x D_t^\alpha Ai(t)dt = D_x^{\alpha-1}Gi(x) - D_x^{\alpha-1}Gi(0) \tag{5.1}$$

and

$$\int_0^x D_t^\alpha Gi(t)dt = -D_x^{\alpha-1}Ai(x) + D_x^{\alpha-1}Ai(0). \tag{5.2}$$

Proof. Using the relations $d/dx = H \circ D$, $Ai(x) = HGi(x)$ and $Gi(x) = -HAi(x)$ (see [18, p. 71]) and integrating the identities

$$D_x^\alpha Ai(x) = \frac{d}{dx}\left(D_x^{\alpha-1} Gi(x)\right) \qquad \text{and} \qquad D_x^\alpha Gi(x) = -\frac{d}{dx}\left(D_x^{\alpha-1} Ai(x)\right)$$

we establish (5.1) and (5.2). □

Remark. It follows from (5.1), (5.2), (3.21), (3.22), (3.41) and (3.42) that for $0 \le \alpha \le 3/2$, $x \in \mathbb{R}$

$$\left| \int_0^x D_t^\alpha Ai(t)\, dt \right| \le C_1, \tag{5.3}$$

for $0 < \alpha \le 3/2$, $x \in \mathbb{R}$

$$\left| \int_0^x D_t^\alpha Gi(t)\, dt \right| \le C_2, \tag{5.4}$$

and for $x \in \mathbb{R}$

$$\left| \int_0^x Gi(t)\, dt \right| \le C_3 \ln(1 + |x|) \tag{5.5}$$

where the constants C_i, $i = 1, 2, 3$, are independent of x. Estimate (5.3) for $\alpha = 0$ follows from the properties of $\int_0^x Ai(t)dt$ (see [1, p. 449]) and the inequality (5.5) follows from the arguments presented in the Appendix.

Now consider a Cauchy problem for the linearized Ostrovsky equation

$$\begin{aligned} u_t + u_{xxx} = \gamma \textstyle\int_{-\infty}^x u(y,t)dy, \qquad & x \in \mathbf{R},\ \ t > 0, \\ u(x,0) = \phi(x), \qquad & x \in \mathbf{R}, \end{aligned} \tag{5.6}$$

and the corresponding Cauchy problem for the linearized KdV with the same initial data

$$\begin{aligned} v_t + v_{xxx} = 0, \qquad & x \in \mathbf{R},\ \ t > 0, \\ v(x,0) = \phi(x), \qquad & x \in \mathbf{R}. \end{aligned} \tag{5.7}$$

We are interested in obtaining pointwise estimates of the difference $D_x^\alpha \mathcal{E}(x,t) - D_x^\alpha \mathcal{E}_0(x,t)$ as $\gamma t \to 0$, where $\mathcal{E}(x,t)$ and $\mathcal{E}_0(x,t)$ are the fundamental solutions for the linearized Cauchy problems (5.6) and (5.7), respectively (see (1.1) and (1.2)).

Theorem 5. *The following estimate holds for* $0 \le \alpha \le 3/2$, $\gamma > 0$, $x \in \mathbb{R}$ *and* $t > 0$*:*

$$|D_x^\alpha \mathcal{E}(x,t) - D_x^\alpha \mathcal{E}_0(x,t)| \le C\gamma t^{1-\alpha/3}. \tag{5.8}$$

where $C = const > 0$ *is independent of* x, t *and* γ.

Proof. Notice that for $\alpha = 0$, the estimate (5.8) follows from the results of [19]. Let $\alpha > 0$. Using (2.5) and setting $\sqrt{y} = \eta$ we get

$$D_x^\alpha \mathcal{E}_0(x,t) = \frac{1}{(3t)^{(1+\alpha)/3}} \left(D^\alpha Ai\right)\left(\frac{x}{\sqrt[3]{3t}}\right)$$

and

$$D_x^\alpha \mathcal{E}_\gamma(x,t) = -\frac{a}{(3t)^{(1+\alpha)/3}} \int_0^\infty \left(D^\alpha Ai\right)\left(\frac{x+\eta^2}{\sqrt[3]{3t}}\right) J_1\left(a\eta\right)\, d\eta.$$

First consider $x > 0$. Introducing the notation $\chi = x/\sqrt[3]{3t}$ and making the change of variable $\zeta = \chi + \eta^2/\sqrt[3]{3t}$ we get

$$D_x^\alpha \mathcal{E}_\gamma(x,t) = -\frac{a}{(3t)^{1/6+\alpha/3}} \int_\chi^\infty D_\zeta^\alpha Ai\left(\zeta\right) \frac{J_1\left(a\sqrt[6]{3t}\sqrt{\zeta-\chi}\right)}{2\sqrt{\zeta-\chi}} d\zeta.$$

Using the inequality (see [24, p. 49])

$$\left|\frac{J_1\left(x\right)}{x}\right| \le \frac{1}{2} \qquad \text{for} \qquad x \in \mathbb{R}$$

and the asymptotics (3.21) with $\alpha > 0$ we obtain

$$\begin{aligned} |D_x^\alpha \mathcal{E}_\gamma(x,t)| &\le \frac{a^2}{(3t)^{\alpha/3}} \int_\chi^\infty \left|D_\zeta^\alpha Ai\left(\zeta\right)\right| d\zeta \\ &\le \frac{a^2}{(3t)^{\alpha/3}} \int_0^\infty \left|D_\zeta^\alpha Ai\left(\zeta\right)\right| d\zeta \le C\frac{a^2}{(3t)^{\alpha/3}}. \end{aligned} \tag{5.9}$$

Consider now $x < 0$. In this case we can write

$$D_x^\alpha \mathcal{E}_\gamma(x,t) = -\frac{a}{(3t)^{(1+\alpha)/3}} \left(I_1 + I_2\right), \tag{5.10}$$

where

$$I_1 = \int_0^{\sqrt{|x|}} \left(D^\alpha Ai\right)\left(-\left(|\chi| - \frac{\eta^2}{\sqrt[3]{3t}}\right)\right) J_1\left(a\eta\right) d\eta \tag{5.11}$$

and

$$I_2 = \int_{\sqrt{|x|}}^{\infty} (D^\alpha Ai)\left(\frac{\eta^2}{\sqrt[3]{3t}} - |\chi|\right) J_1(a\eta)\, d\eta. \tag{5.12}$$

First we deal with the integral I_1. Making the change of variable $\zeta = |\chi| - \eta^2/\sqrt[3]{3t}$ and setting $b = a\sqrt[6]{3t}$ we can rewrite it in the form

$$I_1 = \sqrt[6]{3t}\int_0^{|\chi|} (D^\alpha Ai)(-\zeta)\, \frac{J_1\left(b\sqrt{|\chi| - \zeta}\right)}{2\sqrt{|\chi| - \zeta}} d\zeta$$

Integrating by parts we get

$$\begin{aligned} I_1 &= \sqrt[6]{3t}\int_0^{|\chi|} \frac{J_1\left(b\sqrt{|\chi| - \zeta}\right)}{2\sqrt{|\chi| - \zeta}}\, d_\zeta \left(\int_0^\zeta (D^\alpha Ai)(-y)\, dy\right) \\ &= \sqrt[6]{3t}\left[\frac{J_1\left(b\sqrt{|\chi| - \zeta}\right)}{2\sqrt{|\chi| - \zeta}} \int_0^\zeta (D^\alpha Ai)(-y)\, dy \Bigg|_{\zeta=0}^{\zeta=|\chi|} \right. \\ &\left. - \int_0^{b\sqrt{|\chi|}} \left(\int_0^{|\chi| - z^2/b^2} (D^\alpha Ai)(-y)\, dy\right) \frac{d}{dz}\left(\frac{J_1(z)}{z}\right)\, dz\right]. \end{aligned}$$

Using the formula (see [24, p. 46])

$$\frac{d}{dz}\left(\frac{J_\nu(z)}{z^\nu}\right) = -\frac{J_{\nu+1}(z)}{z^\nu}$$

we can get

$$\begin{aligned} I_1 = b\sqrt[6]{3t}&\left[\frac{1}{4}\int_0^{|\chi|} (D^\alpha Ai)(-y)\ dy \right. \\ &\left. + \int_0^{b\sqrt{|\chi|}} \left(\int_0^{|\chi| - z^2/b^2} (D^\alpha Ai)(-y)\, dy\right) \frac{J_2(z)}{z}\, dz\right]. \end{aligned} \tag{5.13}$$

Recalling (5.3) and using the estimate (see [24])

$$\left|\frac{J_2(x)}{x}\right| \leq \frac{c}{|x|^{3/2}} \qquad \text{for} \qquad x \to \infty$$

we can see that for $0 < \alpha \le 3/2$

$$|I_1| \le Ca\sqrt[3]{t}. \tag{5.14}$$

Making the change of variable $\rho = \eta^2/\sqrt[3]{3t} - |\chi|$ we can rewrite the integral (5.12) in the form

$$I_2 = \sqrt[6]{3t}\int_0^{\infty} D_\rho^\alpha Ai\,(\rho)\,\frac{J_1\left(b\sqrt{\rho+|\chi|}\right)}{2\sqrt{\rho+|\chi|}}\,d\rho.$$

In view of the asymptotics (3.21) with $\alpha > 0$ we can estimate the integral I_2 in the following way:

$$\begin{aligned} |I_2| &\le b\,\sqrt[6]{3t}\left|\int_0^{\infty} D_\rho^\alpha Ai\,(\rho)\,\frac{J_1\left(b\sqrt{\rho+|\chi|}\right)}{2b\sqrt{\rho+|\chi|}}\,d\rho\right| \\ &\le Cb\,\sqrt[6]{3t}\int_0^{\infty}\left|D_\rho^\alpha Ai\,(\rho)\right|\,d\rho \le Ca\,\sqrt[3]{3t}. \end{aligned} \tag{5.15}$$

Combining (5.9), (5.10), (5.14) and (5.15) we establish (5.8). □

Remark. We observe that for $1/2 < \alpha \le 3/2$ fractional derivatives of the KdV fundamental solution, $D_x^\alpha \mathcal{E}_0(x,t)$, are unbounded. In order to avoid this difficulty, we can assume that $\phi \in W_1^1(\mathbb{R})$ and integrate by parts. As a result we obtain

$$\begin{aligned} D_x^\alpha u &= \int_{-\infty}^{\infty} D_x^\alpha \mathcal{E}_0(x-y,t)\phi(y)\,dy \\ &= \frac{1}{\sqrt[3]{3t}}\int_{-\infty}^{\infty} HD_x^{\alpha-1}\partial_x\left(Ai\left(\frac{x-y}{\sqrt[3]{3t}}\right)\right)\phi(y)\,dy \\ &= \frac{1}{\sqrt[3]{3t}}\left[-\frac{1}{(3t)^{(\alpha-1)/3}}\left(D^{\alpha-1}Gi\right)\left(\frac{x-y}{\sqrt[3]{3t}}\right)\phi(y)\Big|_{y=-\infty}^{y=\infty}\right. \\ &\left.+\frac{1}{(3t)^{(\alpha-1)/3}}\int_{-\infty}^{\infty}\left(D^{\alpha-1}Gi\right)\left(\frac{x-y}{\sqrt[3]{3t}}\right)\phi'(y)\,dy\right] \end{aligned} \tag{5.16}$$

Here the first term in the brackets vanishes since $D_x^{\alpha-1}Gi(x)$ is bounded for $0 < \alpha \le 3/2$ (see (3.22) and (3.42)) and $\phi(x) \to 0$ for $|x| \to \infty$ due to the imposed smoothness condition. Various linear estimates can be obtained from (5.16) with the help of the asymptotics of $D_x^\alpha Gi\,(x)$.

6 Appendix: Airy and Scorer functions

We summarize below the main properties of the Airy and Scorer functions that are used in this paper.

6.1 Airy functions

Linearly independent solutions of the homogeneous Airy equation $w'' - zw = 0$ are denoted by $Ai(z)$ and $Bi(z)$. They have integral representations

$$\begin{aligned} Ai(z) &= \frac{1}{\pi}\int_0^\infty \cos\left(z\xi + \xi^3/3\right)\, d\xi, \\ Bi(z) &= \frac{1}{\pi}\int_0^\infty \sin\left(z\xi + \xi^3/3\right)\, d\xi + \frac{1}{\pi}\int_0^\infty e^{z\xi - \xi^3/3}\, d\xi, \end{aligned} \tag{6.1}$$

where z is assumed to be real. Initial values are

$$\begin{aligned} Ai(0) &= Bi(0)/\sqrt{3} = 3^{-2/3}/\Gamma(2/3), \\ Ai'(0) &= -Bi'(0)/\sqrt{3} = -3^{-1/3}/\Gamma(1/3). \end{aligned} \tag{6.2}$$

For large positive z we have asymptotic expansions

$$Ai(z) \sim \frac{e^{-\zeta}}{2\pi^{1/2} z^{1/4}} \sum_{k=0}^\infty (-1)^k \frac{c_k}{\zeta^k}, \quad Bi(z) \sim \frac{e^{\zeta}}{\pi^{1/2} z^{1/4}} \sum_{k=0}^\infty \frac{c_k}{\zeta^k}, \tag{6.3}$$

where

$$\zeta = \zeta(z) = \tfrac{2}{3} z^{3/2}, \quad c_k = \frac{\Gamma(3k + 1/2)}{54^k k!\, \Gamma(k + 1/2)}, \quad k = 0, 1, 2, \ldots. \tag{6.4}$$

A few first coefficients are

$$c_0 = 1, \quad c_1 = \frac{5}{72}, \quad c_2 = \frac{385}{10368}, \quad c_3 = \frac{85085}{2239488}. \tag{6.5}$$

For complex values of z the expansion of $Ai(z)$ in (6.3) is valid for $-\pi < \arg z < \pi$, and the expansion for $Bi(z)$ holds for $-\pi/3 < \arg z < \pi/3$.

For large negative arguments the expansions are

$$
\begin{aligned}
Ai(-z) &\sim \pi^{-1/2} z^{-1/4} \left(\sin\psi(z) \sum_{k=0}^{\infty} (-1)^k \frac{c_{2k}}{\zeta^{2k}} - \cos\psi(z) \sum_{k=0}^{\infty} (-1)^k \frac{c_{2k+1}}{\zeta^{2k+1}} \right), \\
Bi(-z) &\sim \pi^{-1/2} z^{-1/4} \left(\cos\psi(z) \sum_{k=0}^{\infty} (-1)^k \frac{c_{2k}}{\zeta^{2k}} + \sin\psi(z) \sum_{k=0}^{\infty} (-1)^k \frac{c_{2k+1}}{\zeta^{2k+1}} \right),
\end{aligned}
\tag{6.6}
$$

where $\psi(z) = \zeta(z) + \pi/4$. For complex values of z these expansions hold in the sector $-2\pi/3 < \arg z < 2\pi/3$.

6.2 Scorer functions

The Scorer function $Gi(z)$ is a particular solution of the non-homogeneous Airy differential equation $w'' - zw = -1/\pi$. For $z \in \mathbb{R}$ we have the representation

$$
Gi(z) = \frac{1}{\pi} \int_0^{\infty} \sin\left(z\xi + \xi^3/3\right)\, d\xi. \tag{6.7}
$$

For the same z a particular solution of the equation $w'' - z\,w = 1/\pi$ is given by

$$
Hi(z) = \frac{1}{\pi} \int_0^{\infty} e^{z\xi - \xi^3/3}\, d\xi. \tag{6.8}
$$

Initial values are

$$
Gi(0) = \tfrac{1}{2} Hi(0) = 3^{-7/6}/\Gamma(\tfrac{2}{3}), \quad Gi'(0) = \tfrac{1}{2} Hi'(0) = 3^{-5/6}/\Gamma(\tfrac{1}{3}). \tag{6.9}
$$

From (6.1), (6.7), and (6.8) it follows that

$$
Gi(z) + Hi(z) = Bi(z). \tag{6.10}
$$

We have the asymptotic expansions (see [11, pp. 431–432])

$$
Gi(z) \sim \frac{1}{\pi z} \left(1 + \frac{1}{z^3} \sum_{s=0}^{\infty} \frac{(3s+2)!}{s!(3z^3)^s} \right), \quad -\frac{\pi}{3} < \arg z < \frac{\pi}{3}, \tag{6.11}
$$

and

$$
Hi(-z) \sim \frac{1}{\pi z} \left(1 - \frac{1}{z^3} \sum_{s=0}^{\infty} (-1)^s \frac{(3s+2)!}{s!(3z^3)^s} \right), \quad -\frac{2\pi}{3} < \arg z < \frac{2\pi}{3}. \tag{6.12}
$$

Other relations are (see [6])

$$Hi(z) = e^{\pm 2\pi i/3} Hi\left(ze^{\pm 2\pi i/3}\right) + 2e^{\mp \pi i/6} Ai\left(ze^{\mp 2\pi i/3}\right). \tag{6.13}$$

and

$$Gi(z) = -e^{\pm 2\pi i/3} Hi\left(ze^{\pm 2\pi i/3}\right) \pm iAi(z). \tag{6.14}$$

The proofs follow easily by verifying that the right-hand sides satisfy the differential equations, and from the initial values given in (6.2) and (6.9).

With the connection formulas (6.13) and (6.14) and with (6.10) asymptotic relations in other sectors of the complex plane can be derived.

6.3 Asymptotics of the antiderivative of $Gi(x)$

In Section 5 we dealt with the estimates of the integrals $\int_0^x Gi(t)\,dt$ for $|x| \to \infty$. It follows from the expansions in (6.10)–(6.12) that this integral has a logarithmic estimate shown in (5.5). In this section we would like to treat this issue in more detail using the asymptotic expansions of the Riesz potentials $D_x^\alpha Ai(x)$ with $\alpha > -1$ obtained above.

Theorem 6. *The following asymptotic expansions hold for the antiderivative of the Scorer function $Gi(x)$:*

$$\begin{aligned}\int_0^x Gi(t)\,dt \sim & -\frac{2\gamma + 3\ln x + \ln 3}{3\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{\Gamma(3k)}{3^k\,k!}\,\frac{1}{x^{3k}} \\ & + \frac{e^{-\zeta(x)}}{2\pi x^{3/4}} S_2(-1,x) \qquad \text{for} \qquad x \to +\infty\end{aligned} \tag{6.15}$$

and

$$\begin{aligned}\int_0^x Gi(t)\,dt \sim & -\frac{2\gamma + 3\ln|x| + \ln 3}{3\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{\Gamma(3k)}{3^k\,k!}\,\frac{(-1)^k}{|x|^{3k}} \\ & + \frac{1}{\pi|x|^{1/4}}\left[\sin\psi(|x|)T_1(-1,|x|) - \cos\psi(|x|)T_2(-1,|x|)\right] \\ & \text{for} \qquad x \to -\infty,\end{aligned} \tag{6.16}$$

where $\gamma = 0.57721...$ is the Euler constant, $S_2(-1,x)$ is given by (3.20) and (3.23) with $\alpha = -1$, $\psi(x) = \zeta(x) + \pi/4$, and $T_1(-1,|x|)$ and $T_2(-1,|x|)$ are defined by (3.43) with $\alpha = -1$.

Remark. Since the notation γ for the Euler constant is used only in the current subsection, it cannot be confused with the rotation parameter in the Ostrovsky equation.

Proof. We have from (6.7)

$$\int_0^x Gi(t)\,dt = \frac{1}{\pi}\int_0^\infty \frac{\cos\left(x\xi+\xi^3/3\right)-\cos\left(\xi^3/3\right)}{\xi}\,d\xi. \tag{6.17}$$

It is not possible to break up this integral into two with a single cosine term in the integrand because of the divergence of the resulting integrals at $\xi = 0$. Instead we split it up in the following way:

$$\int_0^x Gi(t)\,dt = \lim_{\alpha\downarrow -1}\left[\Phi_1(\alpha,x)-\Phi_2(\alpha)\right], \tag{6.18}$$

where for $\alpha > -1$

$$\begin{aligned}\Phi_1(\alpha,x) &= \frac{1}{\pi}\int_0^\infty \xi^\alpha \cos\left(x\xi+\xi^3/3\right)\,d\xi,\\ \Phi_2(\alpha) &= \frac{1}{\pi}\int_0^\infty \xi^\alpha \cos\left(\xi^3/3\right)\,d\xi.\end{aligned} \tag{6.19}$$

Taking into account (3.1) and (3.2) we see that

$$\Phi_1(\alpha,x) = D_x^\alpha Ai(x) \tag{6.20}$$

and, according to (2.5.3.10) of [13],

$$\Phi_2(\alpha) = \frac{3^{(\alpha-2)/3}}{\pi}\cos\left(\frac{\pi(1+\alpha)}{6}\right)\Gamma\left(\frac{1+\alpha}{3}\right). \tag{6.21}$$

Here both Φ_1 and Φ_2 are singular at $\alpha = -1$.

For $\Phi_1(\alpha,x)$ we use the asymptotic expansions given in Theorems 2 and 3 (see (3.21) and (3.41)). We notice that in these expansions only the terms with $k = 0$ in the infinite series become singular in the limit as $\alpha \downarrow -1$. These terms should be combined with $\Phi_2(\alpha)$ in order to get regular expressions when finding limit (6.18).

Thus, in order to obtain the asymptotic representation for $x \to \pm\infty$ we have to compute the limit

$$L(x) = \lim_{\alpha\to -1}\left[-\frac{\sin(\pi\alpha/2)}{\pi|x|^{\alpha+1}}\Gamma(\alpha+1)-\Phi_2(\alpha)\right]. \tag{6.22}$$

After some manipulations with computer algebra it turns out to be

$$L(x) = -\frac{2\gamma + 3\ln|x| + \ln 3}{3\pi}, \tag{6.23}$$

where γ is the Euler constant. Using the other terms in (3.21) and (3.41) with $\alpha = -1$ we obtain asymptotic expansions for large x (6.15) and (6.16). $\square$

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