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AN ELEMENTARY RENEWAL THEOREM FOR RANDOM COMPACT CONVEX SETS

ILYA S. MOLCHANOV,* *CWI, Amsterdam*
EDWARD OMEY,** *EHSAL, Brussels*
EUGENE KOZAROVITZKY,*** *Niiravtoprom, Kiev*

Abstract

A set-valued analog of the elementary renewal theorem for Minkowski sums of random closed sets is considered. The corresponding renewal function is defined as

$$H(K)=\sum_{n=0}^{\infty} P\{S_n \subset K\},$$

where $S_n = A_1 \oplus \cdots \oplus A_n$ are Minkowski (element-wise) sums of i.i.d. random compact convex sets. In this paper we determine the limit of $H(tK)/t$ as t tends to infinity. For K containing the origin as an interior point,

$$\lim_{t\to\infty}\frac{H(tK)}{t}=\inf_{u\in S_A^+}\frac{h_K(u)}{Eh_A(u)},$$

where $h_K(u)$ is the support function of K and S_A^+ is the set of all unit vectors u with $Eh_A(u)>0$. Other set-valued generalizations of the renewal function are also suggested.

AUMANN EXPECTATION; MINKOWSKI ADDITION; RENEWAL FUNCTION

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1. Introduction

This paper continues earlier papers by different authors; see Artstein and Vitale (1975), Cressie (1979), Lyashenko (1982), Weil (1982), Gine et al. (1983), Puri and Ralescu (1985), etc. Their aims were to prove analogs of classical probability results (law of large numbers, central limit theorems, law of the iterated logarithm) for Minkowski sums of random closed sets.

Let A be a random closed set in the Euclidean space $\mathbb{R}^d$, i.e. A is a random element with values in the family $\mathscr{F}$ of all closed subsets of $\mathbb{R}^d$ and measurable with respect to the σ-algebra generated by the classes $\{F\in\mathscr{F}: F\cap K\neq\emptyset\}$ for K running through the family $\mathscr{K}$ of all compacts, see Matheron (1975). The random set A is said to be compact (convex) if almost all its realizations are compact (convex) sets.

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* Present address: Department of Statistics, University of Glasgow, Glasgow, G12 8QQ, UK.
Supported in part by the Alexander von Humboldt-Stiftung, Bonn, Germany.
** Postal address: EHSAL, Stormstraat 2, 1000 Brussels, Belgium.
*** Postal address: Niiravtoprom, Dept. PMO, Predslavinskaja 28, 252145 Kiev, Ukraine.

The Minkowski sum of two sets is defined elementwise:

$$F_1 \oplus F_2 = \{x_1 + x_2 : x_1 \in F_1, x_2 \in F_2\}.$$

Equipped with the operation $\oplus$, the family $\mathcal{F}$ is a semigroup. Furthermore, $\|F\| = \sup\{\|x\| : x \in F\}$ denotes the norm of F.

The law of large numbers for random sets in its simplest form (see Artstein and Vitale (1975)) states that, for any random compact set A with $\boldsymbol{E}\|A\| < \infty$ and $A, A_1, \cdots, A_n$ i.i.d.,

$$\rho_{\mathrm{H}}(n^{-1}(A_1 \oplus \cdots \oplus A_n), \boldsymbol{E}A) \to 0 \quad \text{a.s.} \qquad \text{as} \quad n \to \infty.$$

Here ρ_{H} is the Hausdorff distance, namely

$$\rho_{\mathrm{H}}(K, K_1) = \inf\{r > 0 : K \subset K_1^r, K_1 \subset K^r\},$$

where $K^r = K \oplus B_r(0)$, $B_r(x)$ is the ball of radius r centered at x. The set $\boldsymbol{E}A$ is the Aumann expectation of A, i.e. the convex set having support function

$$h_{\boldsymbol{E}A}(u) = \boldsymbol{E}h_A(u), \qquad u \in \mathbb{S}^{d-1}, \tag{1.1}$$

see Vitale (1988). Here $\mathbb{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$ and $h_A(.)$ is the support function defined as

$$h_A(u) = \sup\{\langle u, x\rangle : x \in A\},$$

where $\langle u, x\rangle$ is the scalar product of u and x.

Later on Cressie (1979), Lyashenko (1982) and Weil (1982) established the central limit theorem for Minkowski sums of random closed sets. The main difficulty in its formulation is caused by the fact that Minkowski sums of sets cannot be centered, since $(\mathcal{F}, \oplus)$ is a semigroup only. On the other hand, a random closed set with zero expectation coincides with the origin o almost surely. The limit theorem of Weil (1982) states that if $\boldsymbol{E}\|A\|^2 < \infty$, then

$$n^{1/2}\rho_{\mathrm{H}}(n^{-1}(A_1 \oplus \cdots \oplus A_n), \boldsymbol{E}A) \xrightarrow{\mathrm{d}} \sup_{u \in \mathbb{S}^{d-1}} |\zeta(u)| \qquad \text{as } n \to \infty,$$

where ζ is a continuous centered Gaussian process on the unit sphere, such that

$$\boldsymbol{E}\zeta(u)\zeta(v) = \boldsymbol{E}h_A(u)h_A(v) - h_{\boldsymbol{E}A}(u)h_{\boldsymbol{E}A}(v).$$

If the random function ζ is the support function of a certain random closed set Z, then Z is said to be Gaussian. It admits the representation $Z = \xi + M$ for a Gaussian vector ξ and non-random convex M, see Lyashenko (1983), Vitale (1984). Similar results for random closed sets in Banach spaces were proved by Gine et al. (1983) and Puri and Ralescu (1985).

Note that the Minkowski addition of sets corresponds to the addition of their support functions. Therefore, the strong law of large numbers and the central limit theorem for Minkowski sums of random sets can be proved by invoking the corresponding results for Banach spaces. On the other hand, Banach space variants of renewal theorems are not known up to now.

2. Multidimensional renewal theorem

The aim of this paper is to prove a random set analog of the *elementary renewal theorem.* This theorem states that

$$H(t)/t \to 1/E\xi_1 \qquad \text{as } t \to \infty,$$

where

$$H(t) = \sum_{n=0}^{\infty} P\{S_n \leqq t\}$$

is the renewal function, $S_n = \xi_1 + \cdots + \xi_n$, $n \geqq 1$, are partial sums of i.i.d. non-negative random variables, $S_0 = 0$.

A two-dimensional vector analog of this theorem was established by Omey (1989), see also Hunter (1974) and Bickel and Yahav (1965). We give below its d-dimensional variant.

Let $\xi_i = (\xi_{i1}, \cdots, \xi_{id})$, $i \geqq 1$, be a sequence of i.i.d. random vectors with the distribution function $F(x_1, \cdots, x_d) = P\{\xi_{11} \leqq x_1, \cdots, \xi_{1d} \leqq x_d\}$, and let $S_n = \sum_{i=1}^{n} \xi_i$ denote the corresponding partial sums. Define the counting process

$$N(x) = N(x_1, \cdots, x_d) := \text{card}\,\{n : S_{n1} \leqq x_1, \cdots, S_{nd} \leqq x_d\}$$

and the renewal function

$$\begin{aligned} H(x) = H(x_1, \cdots, x_d) &:= E(N(x_1, \cdots, x_d) + 1) \\ &= \sum_{n=0}^{\infty} P\{S_{n1} \leqq x_1, \cdots, S_{nd} \leqq x_d\}. \end{aligned} \tag{2.1}$$

Theorem 2.1. Assume that $E\xi_{1i} = \mu_i$ are positive and finite, $1 \leqq i \leqq d$. Furthermore, let $E(\xi_{1i}^-)^2 < \infty$, $1 \leqq i \leqq d$, where ξ_{1i}^- is the negative part of ξ_{1i}. Then, for all finite positive $x_1, \cdots, x_d$,

$$\lim_{t \to \infty} \frac{1}{t} H(tx_1, \cdots, tx_d) = \min\left(\frac{x_1}{\mu_1}, \cdots, \frac{x_d}{\mu_d}\right). \tag{2.2}$$

If $\max(\mu_i) = \infty$, then the limit is equal to zero.

Proof. Note that the conditions of the theorem yield the finiteness of $H(x)$ for all $x \in \mathbb{R}^d$. Suppose that $\max(\mu_i) < \infty$. For each $a > 0$ and each i define $N_i(a)$ and $M(a)$

as follows:

$$N_i(a) = \text{card}\,\{n : S_{ni} \leqq a\},$$

$$M(a) = \text{card}\,\{n : |S_n| \leqq a\},$$

where $|S_n| = \max_{1 \leqq i \leqq d} |S_{ni}|$ denotes the max-norm. Clearly, we have

$$M(a) \leqq N(a, \cdots, a) \leqq \min_{1 \leqq i \leqq d} N_i(a)$$

for all $a > 0$. Also, replacing ξ_{ki} with ξ_{ki}/x_i yields

$$M(x_1, \cdots, x_d, t) \leqq N(xt) \leqq \min_{1 \leqq i \leqq d} N_i(tx_i), \tag{2.3}$$

where

$$M(x_1, \cdots, x_d, t) = \text{card}\left\{n : \max_{1 \leqq i \leqq d} |S_{ni}|/x_i \leqq t\right\}.$$

It was shown by Lai (1975) that

$$N_i(tx_i)/t \to x_i/\mu_i \quad \text{a.s.} \quad \text{as } t \to \infty, \tag{2.4}$$

and

$$\lim_{t \to \infty} EN_i(tx_i)/t = x_i/\mu_i. \tag{2.5}$$

Expression (2.4) holds if the mean is positive; expression (2.5) holds if the second moment of the negative part of ξ_{1i} is finite.

As to $M(x)$, we use the strong law of large numbers to obtain that

$$|S_n|/n \to |\mu| \quad \text{a.s.} \qquad \text{as } n \to \infty,$$

where $\mu = (\mu_1, \cdots, \mu_d)$. From here it follows that

$$M(t)/t \to |\mu|^{-1} \quad \text{a.s.} \qquad \text{as } t \to \infty,$$

and consequently that

$$M(x_1, \cdots, x_d, t)/t \to |\mu/x|^{-1} = \min_{1 \leqq i \leqq d} x_i/\mu_i \quad \text{a.s.} \qquad \text{as } t \to \infty. \tag{2.6}$$

Combining (2.3), (2.4) and (2.6) yields

$$N(tx)/t \to \min_{1 \leqq i \leqq d} x_i/\mu_i \quad \text{a.s.} \qquad \text{as } t \to \infty. \tag{2.7}$$

Using (2.5), (2.7) and Pratt's extension of Lebesgue's theorem (see for example Johns (1957)), we obtain that

$$\lim_{t\to\infty} EN(tx)/t = \min_{1\leq i\leq d} x_i/\mu_i. \tag{2.8}$$

Hence (2.2) follows. If $\max(\mu_i)=\infty$, then it is necessary to truncate the components of ξ with a truncation bound going to infinity.

Corollary 2.2. Suppose that in Theorem 2.1 all μ_i are finite and that $\max(\mu_i)>0$. If $E(\xi_{1i}^-)^2<\infty$, then, for all finite positive $x_1,\cdots,x_d$,

$$\lim_{t\to\infty}\frac{1}{t}H(tx_1,\cdots,tx_d)=\min\left(\frac{x_i}{\mu_i}: 1\leqq i\leqq d,\ \mu_i>0\right). \tag{2.9}$$

Proof. Without loss of generality suppose that only $\mu_1=E\xi_{11}\leqq 0$ (note that $\mu_1>-\infty$ due to the condition $E(\xi_{1i}^-)^2<\infty$). Consider a new random vector $\tilde{\xi}_1=(\xi_{11}-a,\xi_{12},\cdots,\xi_{1d})$ for $a<\mu_1$. Then $\tilde{\xi}_1$ satisfies the condition of Theorem 2.1, that is

$$\lim_{t\to\infty}\frac{1}{t}\tilde{H}(tx_1,\cdots,tx_d)=\min\left(\frac{x_1}{\mu_1-a},\frac{x_2}{\mu_2},\cdots,\frac{x_d}{\mu_d}\right), \tag{2.10}$$

where $\tilde{H}$ is the renewal function constructed by the vector $\tilde{\xi}_1$ and (2.1). Furthermore,

$$\lim_{t\to\infty}\frac{1}{t}\bar{H}(tx_1,\cdots,tx_d)=\min\left(\frac{x_2}{\mu_2},\cdots,\frac{x_d}{\mu_d}\right), \tag{2.11}$$

where $\bar{H}$ is the renewal function constructed by realizations of the random vector $\bar{\xi}_1=(0,\xi_{12},\cdots,\xi_{1d})$. Evidently,

$$\tilde{H}(x)\leqq H(x)\leqq \bar{H}(x).$$

Now the statement can be easily deduced from (2.10) and (2.11) by letting $a\uparrow\mu_1$.

The elementary renewal theorem for almost surely positive random variables ($d=1$) follows from the Blackwell theorem. The latter result was generalized for the multidimensional case in Nagaev (1979) and Gafurov (1980), see also Gut (1988).

3. Renewal theorem for random sets

In what follows we shall consider a renewal theorem for Minkowski sums of random compact convex sets in $\mathbb{R}^d$. Recall that the distribution of a general random closed set A is determined uniquely by the so-called capacity (or hitting) functional

$$T(K)=P\{A\cap K\neq\varnothing\},\qquad K\in\mathcal{K},$$

see Matheron (1975), Stoyan et al. (1987). It is also known (see Vitale (1983) and

Molchanov (1983)) that the distribution of a random *compact convex* set is determined by the *containment* functional

$$H(K) = \mathbf{P}\{A \subset K\}$$

defined on the class $\mathcal{C}_0$ of convex compact sets.

Let $A, A_1, A_2, \cdots$ be i.i.d. random compact convex sets, and let

$$S_n = A_1 \oplus \cdots \oplus A_n \tag{3.1}$$

be their partial sums ($S_0 = \{o\}$ is the origin). For a closed set K define the *containment renewal function*

$$H(K) = \sum_{n=0}^{\infty} \mathbf{P}\{S_n \subset K\}. \tag{3.2}$$

If $d = 1$, $K = [0, t]$, and $A = \{\xi\}$ for a non-negative random variable ξ, then $H(K)$ is the classical renewal function. Our aim is to find the limit of $H(tK)/t$ as $t \to \infty$.

The set-valued renewal theorem may be of use when exact locations of the summands (random variables or vectors) are not known. Then the random variable ξ is replaced by the random set A and the sum of random variables turns into the Minkowski sum (3.1).

The following theorem is the most simple of its kind. For any set $M \subset \mathbb{R}^d$ let $\operatorname{int} M$ and $\operatorname{conv}(M)$ denote the interior and the convex hull of M respectively.

Theorem 3.1. Suppose that $\mathbf{E}\,\|A\| < \infty$ and $\mathbf{E}\rho(o, \operatorname{conv}(A))^2 < \infty$, where $\rho(o, A)$ is the minimum distance between points of A and the origin. Consider a convex compact K, such that $o \in \operatorname{int} K$. Then

$$\lim_{t\to\infty} \frac{H(tK)}{t} = \inf_{u \in S_A^+} \frac{h_K(u)}{h_{EA}(u)}, \tag{3.3}$$

where $S_A^+ = \{x \in \mathbb{S}^{d-1} : h_{EA}(u) > 0\}$.

Proof. First, note that $S_n \subset tK$ iff the support function of S_n is not greater than the support function of tK, that is

$$h_{S_n}(u) \leqq t h_K(u), \qquad u \in \mathbb{S}^{d-1}.$$

The left-hand side is equal to $\sum_{i=1}^{n} h_{A_i}(u)$. Note also that

$$\sup_{u \in \mathbb{S}^{d-1}} h_A(u)^- < \rho(o, \operatorname{conv}(A)),$$

where $h_A(u)^-$ is the negative part of the support function $h_A(u)$.

Choose an ε-net $u_1, \cdots, u_m$ on the unit sphere $\mathbb{S}^{d-1}$. Then

$$\Gamma'_\varepsilon = \bigcap_{j=1}^{m} l_{u_j}(x'_j) \subset K \subset \Gamma''_\varepsilon = \bigcap_{j=1}^{m} l_{u_j}(x''_j)$$

for suitable positive reals x_j', x_j'', $1 \leqq j \leqq m$. Here $l_u(x) = \{y \in \mathbb{R}^d : \langle y, u \rangle \leqq x\}$, i.e. K can be approximated by polyhedrons with facets orthogonal to u_j, $1 \leqq j \leqq m$, and

$$\rho_{\mathrm{H}}(\Gamma_\varepsilon', \Gamma_\varepsilon'') \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{3.4}$$

Then, by (2.9),

$$\begin{aligned} \frac{H(tK)}{t} &\geqq t^{-1} \sum_{n=0}^{\infty} \boldsymbol{P}\left\{\sum_{i=1}^{n} h_{A_i}(u_j) \leqq tx_j'; 1 \leqq j \leqq m\right\} \\ &\to \min_{1 \leqq j \leqq m;\, u_j \in S_A^+} \frac{x_j'}{\boldsymbol{E}h_A(u_j)} \\ &\geqq \min_{1 \leqq j \leqq m;\, u_j \in S_A^+} \frac{h_K(u_j) - \rho_{\mathrm{H}}(\Gamma_\varepsilon', K)}{\boldsymbol{E}h_A(u_j)}. \end{aligned}$$

A similar bound from above yields

$$\min_{1 \leqq j \leqq m;\, u_j \in S_A^+} \frac{h_K(u_j) - \rho_{\mathrm{H}}(\Gamma_\varepsilon', K)}{\boldsymbol{E}h_A(u_j)} \leqq \lim_{t \to \infty} \frac{H(tK)}{t} \leqq \min_{1 \leqq j \leqq m;\, u_j \in S_A^+} \frac{h_K(u_j) + \rho_{\mathrm{H}}(\Gamma_\varepsilon'', K)}{\boldsymbol{E}h_A(u_j)}.$$

The continuity of the support function and (3.4) finish the proof.

Corollary 3.2. If $K = B_1(o)$, then

$$\lim_{t \to \infty} H(tK)/t = (\|\boldsymbol{E}A\|)^{-1}.$$

Corollary 3.3. Suppose that A is isotropic (i.e. its distribution is invariant under rotations). Then

$$\lim_{t \to \infty} H(tK)/t = a^{-1} \sup\{r : B_r(o) \subset K\},$$

where

$$a = \frac{1}{db_d} \boldsymbol{E}\left[\int_{\mathbb{S}^{d-1}} h_A(u)\, du\right]$$

and b_d is the volume of the unit ball in $\mathbb{R}^d$.

Proof. Evidently, $\boldsymbol{E}A$ is a ball centered at the origin. Its radius a can be evaluated using the fact that the mean width of the expectation is equal to the expected mean width of the set in question, see Vitale (1988). Then the result follows from Theorem 3.1. Note that if $d = 2$, then $\boldsymbol{E}\int_{\mathbb{S}^{d-1}} h_A(u)\, du$ is the mean perimeter of A.

In particular, if $A = B_\xi(\eta)$ is the ball with rotation invariant distribution of its center, then $a = \boldsymbol{E}\xi$, and

$$\lim_{t \to \infty} H(tK)/t = (\boldsymbol{E}\xi)^{-1} \sup\{r : B_r(o) \subset K\}.$$

The result does not depend on the distribution of η.

Example 3.4. Let $A = \{\xi\}$ be a random singleton. If $\boldsymbol{E}\,\|\xi\|^2 < \infty$, then Theorem 3.1 yields

$$\lim_{t\to\infty} \frac{H(tK)}{t} = \sup\{r : r\boldsymbol{E}\xi \in K\} = \frac{1}{g(K, \boldsymbol{E}\xi)},$$

where $g(K, x) = \inf\{r \geqq 0 : x \in rK\}$ is the *gauge function* of K; see Schneider (1993), p. 43.

In the following the assumption $o \in \operatorname{int} K$ will be dropped. First, suppose that the convex set K in Theorem 3.1 does not contain the origin.

Theorem 3.5. *Suppose that* $\boldsymbol{E}\,\|A\| < \infty$, *and* $\boldsymbol{E}\rho(o, \operatorname{conv}(A))^2 < \infty$. *If* K *is convex and* $o \notin K$, *then*

$$\lim_{t\to\infty} \frac{H(tK)}{t} = \alpha_K - \min(\alpha_K, \beta_K), \tag{3.5}$$

where

$$\alpha_K = \inf\left\{\frac{h_K(u)}{h_{\boldsymbol{E}A}(u)} : u \in \mathbb{S}^{d-1};\, h_K(u) > 0,\, h_{\boldsymbol{E}A}(u) > 0\right\}, \tag{3.6}$$

$$\beta_K = \inf\left\{\frac{h_K(u)}{h_{\boldsymbol{E}A}(u)} : u \in \mathbb{S}^{d-1};\, h_K(u) < 0,\, h_{\boldsymbol{E}A}(u) < 0\right\}. \tag{3.7}$$

Here $\inf \varnothing = \infty$ *and* $\infty - \infty = 0$.

Proof. Similarly to the proof of Theorem 3.1 and using the same notation we get

$$\frac{H(tK)}{t} \geqq t^{-1} \sum_{n=0}^{\infty} \boldsymbol{P}\left\{\sum_{i=1}^{n} h_{A_i}(u_j)/x'_j \leqq t : 1 \leqq j \leqq m,\, x'_j > 0\right\}$$
$$- t^{-1} \sum_{n=0}^{\infty} \boldsymbol{P}\left\{\sum_{i=1}^{n} h_{A_i}(u_j)/x'_j \leqq t : 1 \leqq j \leqq m\right\}.$$

Note that the approximation of K can be chosen in such a way that all x'_i are non-vanishing. Therefore, the proof can be finished similarly to the proof of Theorem 3.1.

If $o \in A$ a.s., then $\beta_K = \infty$, whence $H(tK)/t$ tends to zero. Furthermore,

$$H(t\boldsymbol{E}A)/t \to \begin{cases} 1 & \text{if } o \in \operatorname{Int} \boldsymbol{E}A \\ 0 & \text{if } o \notin \boldsymbol{E}A \end{cases} \quad \text{as } t \to \infty.$$

›m Theorems 3.1 and 3.5 it is easy to obtain a corollary for the renewal function

$$H(K) = \sum_{n=0}^{\infty} \boldsymbol{P}\{S_n \in K\},$$

of a random singleton $A = \{\xi\}$, since in this case $h_{EA}(u) = \langle u, E\xi \rangle$; see also Example 3.4.

Similar results as in Theorems 3.1 and 3.5 can be obtained for K being the finite union of disjoint convex compacts, since

$$H\left(\bigcup_{i=1}^{m} K_i\right) = \sum_{i=1}^{m} H(K_i).$$

Results for the case when the origin is a *boundary* point of K can be deduced from Theorems 3.1 and 3.5. Denote

$$S_K^+ = \{u \in \mathbb{S}^{d-1} : h_K(u) > 0\}.$$

Furthermore, let $\mathbb{C}$ be the minimum cone containing K.

Theorem 3.6. Suppose that $E\,\|A\| < \infty$, $E\rho(o, \operatorname{conv}(A))^2 < \infty$, and the origin is a boundary point of a convex compact K. If $A \subset \mathbb{C}$ a.s., then

$$\lim_{t\to\infty} \frac{H(tK)}{t} = \inf_{u \in S_A^+} \frac{h_K(u)}{h_{EA}(u)}. \tag{3.8}$$

The same is valid if $S_A^+ \subset S_K^+$ and $o \notin EA$. If $S_A^+ \not\subset S_K^+$, then the limit in (3.8) is equal to zero.

Proof. For each $\delta > 0$, the set K^δ satisfies the conditions of Theorem 3.1. Then

$$\frac{H(tK)}{t} \leqq \frac{H(tK^\delta)}{t} \to \inf_{u \in S_A^+} \frac{h_K(u) + \delta}{h_{EA}(u)} \qquad \text{as } s \to \infty. \tag{3.9}$$

If $h_0 = h_{EA}(u_0) > 0$ for some $u_0 \notin S_K^+$, then the limit is less than δ / h_0. This proves the last assertion of the theorem.

It follows from (3.9) that the limit in (3.8) is less than the corresponding right-hand side. Let us suppose that $S_A^+ \subset S_K^+$ and $o \notin EA$. Pick $u_0 \in (\operatorname{int} \mathbb{C}) \cap \mathbb{S}^{d-1}$ and put $K_\delta = \{x \in K : \langle x, u_0 \rangle \geqq \delta\}$. Since $o \notin K_\delta$, Theorem 3.5 is applicable with

$$\alpha_{K_\delta} \to \inf_{u \in S_A^+} h_K(u)/h_{EA}(u) \qquad \text{as } \delta \to 0,$$

and

$$\beta_{K_\delta} \leqq \rho(o, K_\delta)/\rho(o, EA) \to 0 \qquad \text{as } \delta \to 0.$$

Then (3.8) follows from (3.9) and (3.5).

If $A \subset \mathbb{C}$ a.s., then $S_A^+ \subset S_K^+$ and either $o \notin EA$ or $o \in A$ with probability 1. The first case has been already considered. If $o \in A$ a.s., then $h_A(u) \geqq 0$ for all $u \in \mathbb{S}^{d-1}$, and the proof can be finished exactly like the proof of Theorem 3.1 using approximations Γ'_ε of K from below such that $o \in \Gamma'_\varepsilon$ for all ε.

4. Another variant of the renewal function in the set-valued case

Let us define the *inclusion renewal function*

$$I(K)=\sum_{n=0}^{\infty} \boldsymbol{P}\{K \subset S_n\}$$

for a convex compact set K. If $o \in K$, then $I(tK)$ decreases, whence either $I(tK)/t \to 0$ as $t \to \infty$ or $I(K)=\infty$. Thus we have to consider only the case $o \notin K$. The following result is proved similarly to Theorem 3.5.

Theorem 4.1. Under the conditions and with the notation of Theorem 3.5

$$\lim_{t\to\infty} \frac{I(tK)}{t} = \beta_K - \min(\alpha_K, \beta_K),$$

if $o \notin \boldsymbol{E}A$. If $o \in \operatorname{int} \boldsymbol{E}A$, then $I(K)$ is infinite. If o belongs to the boundary of $\boldsymbol{E}A$, then the limit of $I(tK)/t$ is equal to zero in case $h_K(u) \neq 0$ and $h_{\boldsymbol{E}A}(u)=0$ for some u. Otherwise $I(K)$ is infinite.

5. Concluding remarks

1. It is interesting also to consider the *hitting renewal function*

$$U(K)=\sum_{n=0}^{\infty} \boldsymbol{P}\{S_n \cap K \neq \emptyset\}.$$

In this case the situation is more complicated. Clearly, if A is a singleton, then U coincides with the containment renewal function H. Similarly, for a one-point compact K Theorem 4.1 can be applied.

If $A=[\xi, \eta]$ is a convex subset of the line ($d=1$), then a renewal theorem for the hitting function easy follows from Theorem 2.1. It is easily seen that

$$\boldsymbol{P}\{A \cap [a, b] \neq \emptyset\} = \boldsymbol{P}\{\xi \leqq a, \eta \geqq a\} + \boldsymbol{P}\{\xi \in (a, b]\}.$$

First, suppose that $a, b > 0$, and $\boldsymbol{E}\xi, \boldsymbol{E}\eta > 0$. Then

$$\lim_{t\to\infty} U(t[a, b])/t = b/\boldsymbol{E}\xi - \min(b/\boldsymbol{E}\eta, a/\boldsymbol{E}\xi).$$

Similar results can be obtained for other a, b. Let $a<0$, $b>0$. If $\boldsymbol{E}\xi<0$ and $\boldsymbol{E}\eta>0$, then $U([a, b])$ is infinite. If both $\boldsymbol{E}\xi$, $\boldsymbol{E}\eta$ are positive, then

$$\lim_{t\to\infty} U(t[a, b])/t = b/\boldsymbol{E}\xi.$$

Moreover, further formulae can be obtained for K being a finite union of disjoint segments.

2. It was mentioned in Smith (1964) that the finiteness of the second moment of

the negative part of the random variable in Theorem 2.1 is necessary and sufficient for the finiteness of the corresponding renewal function.

If in Theorem 2.1 we assume that $E(\xi_{1i}^-)^r < \infty$ for $r > 2$, then (2.2) can be replaced by

$$\lim_{t \to \infty} E\left(\frac{N(tx)}{t}\right)^r = \left(\min\left(\frac{x_i}{\mu_i}\right)\right)^r.$$

A similar result can be formulated for the containment renewal counting process $N(K) = \text{card}\,\{n\colon S_n \subset K\}$.

3. The famous Boolean model in stochastic geometry (see Stoyan et al. (1987)) is defined to be the union of 'points' of the Poisson point process on the space $\mathcal{K}$ of compact sets. A general point process on $\mathcal{K}$ yields the so-called grain-germ model.

Similarly to the points of the renewal point process on the line, the sets S_n, $n \geqq 0$, from (3.1) can be viewed as 'points' of a renewal set-valued point process on $\mathcal{K}$. Then their union $\bigcup_{n=0}^{\infty} S_n$ serves as a model of a random set. This model can be applied to describe a kind of growth both of location and size, since with each new summand in (3.1) the result is shifted away (if $o \notin EA$) and grows simultaneously.

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