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First passage process of a Markov additive process, with applications to reflection problems

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ABSTRACT
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2000 Mathematics Subject Classification: 60K25, 60K37
Keywords and Phrases: Levy processes; Fluctuation theory; Markov Additive Processes; Markov Modulated Brownian Motion

First Passage Process of a Markov Additive Process, with Applications to Reflection Problems

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In this paper we consider the first passage process of a spectrally negative Markov additive process (MAP). The law of this process is uniquely characterized by a certain matrix function, which plays a crucial role in fluctuation theory. We show how to identify this matrix using the theory of Jordan chains associated with analytic matrix functions. Importantly, our result also provides us with a technique, which can be used to derive various further identities. We then proceed to show how to compute the stationary distribution associated with a one-sided reflected (at zero) MAP for both the spectrally positive and spectrally negative cases as well as for the two sided reflected Markov modulated Brownian motion; these results can be interpreted in terms of queues with MAP input.

Key words: Lévy processes; Fluctuation theory; Markov Additive Processes; Markov Modulated Brownian Motion

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1. Introduction Continuous-time Markov additive processes (MAPs) with one-sided jumps have proven to be an important modelling tool in various application areas, such as communications networking [21, Ch. 6-7] and finance [3, 12]. Over the past decades a vast body of literature has been developed; see for instance [1, Ch. XI] for a collection of results. A MAP can be thought of as a Lévy process whose Laplace exponent depends on the state of a (finite-state) Markovian background process (with additional jumps at transition epochs of this background process). It is a non-trivial generalization of the standard Lévy process, with many analogous properties and characteristics, as well as new mathematical objects associated to it, posing new challenges. Any Lévy process is characterized by a Laplace exponent $\psi(\alpha)$; its counterpart for MAPs is the matrix exponent $F(\alpha)$, which is essentially a multi-dimensional analogue of $\psi(\alpha)$.

In this paper we consider the *first passage process*. More concretely, with τ_x defined as the first time the process exceeds level x, we study properties of the process τ_x as a function of x. We concentrate on the case of a *spectrally negative* MAP (that is, all jumps are negative), so that the first passage process is a MAP itself. Knowledge of the matrix exponent of this process, which we in the sequel denote by the matrix function $\Lambda(q)$, is of crucial interest when addressing related fluctuation theory issues. Indeed it can be considered as the multi-dimensional generalization of $-\Phi(q)$, where $\Phi(q)$ is the (one-dimensional) right-inverse of $\psi(\alpha)$, as given in [16, Eqn. (3.15)]. Our main result concerns the identification of the matrix function $\Lambda(q)$ in terms of the matrix exponent $F(\alpha)$ of the original MAP. We provide the Jordan normal form of $\Lambda(q)$ relying on the theory of Jordan chains associated with analytic matrix functions.

Importantly, our main result is not only about identification of the matrix exponent of the first passage process. We prefer to see our contribution rather as the development of a new *technique*: the theory of analytic matrix functions, combined with the special structure of the Jordan pairs of $F(\alpha) - q\mathbb{I}$ (with $\mathbb{I}$ being the identity matrix), and their relation to the matrix $\Lambda(q)$, enables the derivation of a set of further identities. These identities, such as (32) and (56), then play an important role in the solution of a number of problems related to the *reflection* of the MAP; here 'reflection' amounts to an adaptation of the MAP in order to ensure that the process attains values in a certain subset of $\mathbb{R}$ only. In this sense, we could reflect at 0 to obtain a process that assumes nonnegative values only (which can be interpreted as a *queue* with infinite buffer capacity), or at both 0 and $b > 0$ to restrict the process to $[0, b]$ (this *double reflection* is essentially a queue with finite buffer b). In the following we discuss in more detail these reflection problems, which we solved using this technique.

In [4] a martingale associated with the MAP was established, being essentially the multidimensional counterpart of the martingale found in [14] for the standard Lévy process. In the same paper, various situations were considered in which this martingale could be applied. Most notably, attention was paid to reflection of a *spectrally positive* MAP at 0, i.e., a queue fed by a spectrally positive MAP; here 'spectrally positive' means that all jumps are positive. For this model the multidimensional martingale allowed to reduce the problem of determining the stationary workload to finding various constants that are solutions to a system of linear equations. The authors of [4] did not succeed, however, in proving that such a system of equations has a unique solution. Also for the situation of doubly reflected Markov-modulated Brownian motion (that is, a finite-buffer queue fed by Markov-modulated Brownian motion), a similar complication arose. In the literature, problems of this type were only partially addressed for special cases (e.g., see [2, 5, 13, 15]). In our paper we tackle these problems using the above outlined technique.

This paper is organized as follows. Section 2 reviews some main results from analytic matrix function theory, while in Section 3 we identify the matrix exponent $\Lambda(q)$ by relating the Jordan pairs of the matrix functions $F(\alpha) - q\mathbb{I}$ and $\alpha\mathbb{I} + \Lambda(q)$ for a fixed $q \geq 0$. The result, which is Theorem 3.1 and which can be considered as the main contribution of our work, is explicit in the sense that it is given in terms of computable quantities associated with $F(\alpha)$. In the second part (Sections 4-6) of the paper we solve a number of open problems related to reflected processes. In Section 4 we study spectrally one-sided MAPs reflected at 0. We succeed in solving the (above mentioned) issues that remained open in [4]. In particular, both for spectrally negative and spectrally positive MAP input, we express the steady-state workload in terms of quantities related to the matrix exponent $\Lambda(q)$ of the first passage process of a spectrally negative MAP. In Section 5 we apply the methodology to identify the stationary distribution of Markov-modulated Brownian motion with two reflecting barriers. We provide a full treatment of this model, also for the case of 0 asymptotic drift. Yet another demonstration of applicability of our technique is given in Section 6, where we present a simple proof of the fact that $\Lambda(q)$ is a unique solution of a certain matrix integral equation. This result (in different degrees of generality) appears in [2, 5, 18, 19, 20, 22], and is commonly considered as the main tool used to numerically identify $\Lambda(q)$. Some spectral considerations (under the assumption that $\Lambda(q)$ has distinct eigenvalues) can be found in [2, 5, 19].

The remainder of this section is devoted to the definition of some of the quantities of interest, with a focus on spectrally negative MAPs and their first passage process.

1.1 Spectrally negative MAP A MAP is a bivariate Markov process $(X(t), J(t))$ defined as follows. Let $J(\cdot)$ be an irreducible continuous-time Markov chain with finite state space $E = \{1, \ldots, N\}$, transition rate matrix $Q = (q_{ij})$ and a (unique) stationary distribution $\boldsymbol{\pi}$. For each state i of $J(\cdot)$ let $X_i(\cdot)$ be a Lévy process with Laplace exponent $\psi_i(\alpha) = \log(\mathbb{E}e^{\alpha X_i(1)})$. Letting T_n and T_{n+1} be two successive transition epochs of $J(\cdot)$, and given that $J(\cdot)$ jumps from state i to state j at T_n, we define the additive process $X(\cdot)$ in the time interval $[T_n, T_{n+1})$ through

$$X(t) = X(T_n-) + U_{ij}^n + [X_j(t) - X_j(T_n)], \tag{1}$$

where (U_{ij}^n) is a sequence of independent and identically distributed random variables with moment generating function

$$\tilde{G}_{ij}(\alpha) = \mathbb{E}e^{\alpha U_{ij}^1}, \quad \text{where} \quad U_{ii}^1 \equiv 0, \tag{2}$$

describing the jumps at transition epochs. To make the MAP spectrally negative, it is required that $U_{ij}^1 \leq 0$ (for all $i, j \in \{1, \ldots, N\}$) and that $X_i(\cdot)$ is allowed to have only negative jumps (for all $i \in \{1, \ldots, N\}$). As a consequence, the moment generation functions $\tilde{G}_{ij}(\alpha)$ are well defined for $\alpha \geq 0$.

A Lévy process is called a *downward subordinator* if it has non-increasing paths a.s. We denote the subset of indices of E corresponding to such processes by $E_\downarrow$. Let also $E_+ = E \backslash E_\downarrow$, $N_\downarrow = |E_\downarrow|$ and $N_+ = |E_+|$. We use $\boldsymbol{v}_+$ and $\boldsymbol{v}_\downarrow$ to denote the restrictions of a vector $\boldsymbol{v}$ to the indices from E_+ and $E_\downarrow$ respectively. Finally, in order to exclude trivialities it is assumed that $N_+ > 0$.

Define the matrix $F(\alpha)$ through

$$F(\alpha) = Q \circ \tilde{G}(\alpha) + \text{diag}[\psi_1(\alpha), \ldots, \psi_N(\alpha)], \tag{3}$$

where $\tilde{G}(\alpha) = (\tilde{G}_{ij}(\alpha))$; for matrices A and B of the same dimensions we define $A \circ B = (a_{ij} b_{ij})$. One can see that in the absence of positive jumps $F(\alpha)$ is analytic on $\mathbb{C}^{\text{Re}>0} = \{\alpha \in \mathbb{C} : \text{Re}(\alpha) > 0\}$. Moreover, it is known that

$$\mathbb{E}_i[e^{\alpha X(t)} 1_{\{J(t)=j\}}] = (e^{F(\alpha)t})_{ij}, \tag{4}$$

cf. [1, Prop. XI.2.2], where $\mathbb{E}_i(\cdot)$ denotes expectation given that $J(0) = i$. Hence $F(\alpha)$ can be seen as the multi-dimensional analog of a Laplace exponent, defining the law of the MAP. In the following we call $F(\alpha)$the *matrix exponent* of the MAP $(X(t), J(t))$.

An important quantity associated to a MAP is the *asymptotic drift*:

$$\kappa = \lim_{t \to \infty} \frac{1}{t} \mathbb{E}_i X(t) = \sum_i \pi_i \left(\psi_i'(0) + \sum_{j \neq i} q_{ij} \tilde{G}_{ij}'(0) \right), \tag{5}$$

which does not depend on the initial state i of $J(t)$ [1, Cor. XI.2.7]. Finally for $q \geq 0$ we define $F^q(\alpha) = F(\alpha) - q\mathbb{I}$, with $\mathbb{I}$ being the identity matrix, which can be seen as the matrix exponent of the MAP 'killed' at random time e_q:

$$\mathbb{E}_i[e^{\alpha X(t)} 1_{\{J(t)=j, t<e_q\}}] = (e^{(F(\alpha)-q\mathbb{I})t})_{ij}, \tag{6}$$

where e_q is an exponential random variable of rate q independent of everything else and $e_0 \equiv \infty$ by convention.

1.2 First Passage Process Define the first passage time over level $x > 0$ for the (possibly killed) process $X(t)$ as

$$\tau_x = \inf\{t \geq 0 : X(t) > x\}. \tag{7}$$

It is known that on $\{J(\tau_x) = i\}$ the process $(X(t + \tau_x) - X(\tau_x), J(t + \tau_x)), t \geq 0$ is independent from $(X(t), J(t)), t \in [0, \tau_x]$ and has the same law as the original process under $\mathbb{P}_i$. Therefore, in the absence of positive jumps the time-changed process $J(\tau_x)$ is a time-homogeneous Markov process and hence is a Markov chain. Letting $\{\partial\}$ be an absorbing state corresponding to $J(\infty)$, we note that $J(\tau_x)$ lives on $E_+ \cup \{\partial\}$, because $X(t)$ can not hit new maximum when $J(t)$ is in a state corresponding to a downward subordinator; see also [17]. Let $\Lambda(q)$ be the $N_+ \times N_+$ dimensional transition rate matrix of $J(\tau_x)$ restricted to E_+, that is

$$\mathbb{P}(J(\tau_x) = j, \tau_x < e_q \mid J(\tau_0) = i) = (e^{\Lambda(q)x})_{ij}, \text{ where } i, j \in E_+. \tag{8}$$

It is easy to see that in the absence of positive jumps the first passage process $(\tau_x, J(\tau_x)), x \geq 0$ is a MAP itself. Moreover,

$$\mathbb{E}[e^{-q\tau_x} 1_{\{J(\tau_x)=j\}} \mid J(\tau_0) = i] = \mathbb{P}(J(\tau_x) = j, \tau_x < e_q \mid J(\tau_0) = i) = (e^{\Lambda(q)x})_{ij},$$

so that $\Lambda(q)$ is the matrix exponent of (the negative of) the first passage process. This interpretation, however, is not used in the rest of this paper.

Another matrix of interest is $N \times N_+$ matrix $\Pi(q)$ defined by

$$\Pi(q)_{ij} = \mathbb{P}_i(J(\tau_0) = j, \tau_0 < e_q), \text{ where } i \in E \text{ and } j \in E_+. \tag{9}$$

This matrix specifies initial distributions of the time-changed Markov chain $J(\tau_x)$. Note also that $\Pi(q)$ restricted to the rows in E_+ is the identity matrix, because $\tau_0 = 0$ a.s. when $J(0) \in E_+$ [16, Thm. 6.5]. We note that the case of $q = 0$ is a special case corresponding to no killing. In order to simplify notation we often write Λ and Π instead of $\Lambda(0)$ and $\Pi(0)$.

It is noted that if $q > 0$ or $q = 0, \kappa < 0$ then $\Lambda(q)$ is a defective transition rate matrix: $\Lambda(q)\mathbf{1}_+ \leq \mathbf{0}_+$, with at least one strict inequality. If, however, $\kappa \geq 0$, then Λ is a non-defective transition rate matrix: $\Lambda\mathbf{1}_+ = \mathbf{0}_+$; also $\Pi\mathbf{1}_+ = \mathbf{1}$. These statements follow trivially from [1, Prop. XI.2.10]. Finally, note that Λ is an irreducible matrix, because so is Q. Hence if Λ is non-defective then by Perron-Frobenius theory [1, Thm. I.6.5] the eigenvalue 0 is simple, because it is the eigenvalue with maximal real part.

It is instructive to consider the 'degenerate' MAP, i.e., the one with dimension $N = 1$. Such a MAP is just a Lévy process, and $\Lambda(q) = -\Phi(q)$, where $\Phi(q)$ is the right-inverse of $\psi(\alpha), \alpha \geq 0$. Note also that Λ being non-defective (and hence singular) corresponds to $\Phi(0) = 0$.

2. Preliminaries In this section we review some basic facts from analytic matrix function theory. Let $A(z)$ be an analytic matrix function ($n \times n$ dimensional), defined on some domain $D \subset \mathbb{C}$, where it is assumed that $\det(A(z))$ is not identically zero on this domain. For any $\lambda \in D$ we can write

$$A(z) = \sum_{i=0}^{\infty} \frac{1}{i!} A^{(i)}(\lambda)(z - \lambda)^i, \tag{10}$$

where $A^{(i)}(\lambda)$ denotes the i-th derivative of $A(z)$ at λ. We say that λ is an *eigenvalue* of $A(z)$ if $\det(A(\lambda)) = 0$.

Definition 2.1 *We say that vectors* $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{r-1} \in \mathbb{C}^n$ *with* $\boldsymbol{v}_0 \neq \mathbf{0}$ *form a* Jordan chain *of* $A(z)$ *corresponding to the eigenvalue* λ *if*

$$\sum_{i=0}^{j} \frac{1}{i!} A^{(i)}(\lambda)\boldsymbol{v}_{j-i} = \mathbf{0}, \qquad j = 0, \ldots, r-1. \tag{11}$$

Note that this definition is a generalization of the well-known notion of a Jordan chain for a square matrix A. In this classical case $A(z) = z\mathbb{I} - A$, and (11) reduces to

$$A\boldsymbol{v}_0 = \lambda\boldsymbol{v}_0, \quad A\boldsymbol{v}_1 = \lambda\boldsymbol{v}_1 + \boldsymbol{v}_0, \quad \ldots, \quad A\boldsymbol{v}_{r-1} = \lambda\boldsymbol{v}_{r-1} + \boldsymbol{v}_{r-2}. \tag{12}$$

The following result is well known [9].

Proposition 2.1 *Let* $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{r-1}$ *be a Jordan chain of* $A(z)$ *corresponding to the eigenvalue* λ*, and let* $C(z)$ *be* $m \times n$ *dimensional matrix. If* $B(z) = C(z)A(z)$ *is* $r - 1$ *times differentiable at* λ*, then*

$$\sum_{i=0}^{j} \frac{1}{i!} B^{(i)}(\lambda)\boldsymbol{v}_{j-i} = \mathbf{0}, \qquad j = 0, \ldots, r-1. \tag{13}$$

Note that if $B(z)$ is a square matrix then $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{r-1}$ is a Jordan chain of $B(z)$ corresponding to the eigenvalue λ. It is, however, not required that $C(z)$ and $B(z)$ be square matrices.

Let m be the multiplicity of λ as a zero of $\det(A(z))$ and p be the dimension of the null space of $A(\lambda) = A_0$. It is known, see e.g. [9], that there exists a *canonical system of Jordan chains* corresponding to λ

$$\boldsymbol{v}_0^{(k)}, \boldsymbol{v}_1^{(k)}, \ldots, \boldsymbol{v}_{r_k-1}^{(k)}, \qquad k = 1, \ldots, p, \tag{14}$$

such that the vectors $\boldsymbol{v}_0^{(1)}, \ldots, \boldsymbol{v}_0^{(p)}$ form the basis of the null space of A_0 and $\sum_{i=1}^{p} r_i = m$. We write such a canonical system of Jordan chains in matrix form:

$$V = [\boldsymbol{v}_0^{(1)}, \boldsymbol{v}_1^{(1)}, \ldots, \boldsymbol{v}_{r_1-1}^{(1)}, \ldots, \boldsymbol{v}_0^{(p)}, \boldsymbol{v}_1^{(p)}, \ldots, \boldsymbol{v}_{r_p-1}^{(p)}], \qquad \Gamma = \text{diag}[\Gamma^{(1)}, \ldots, \Gamma^{(p)}], \tag{15}$$

where $\Gamma^{(i)}$ is the Jordan block of size $r_i \times r_i$ with eigenvalue λ.

Definition 2.2 *A pair of matrices* (V,Γ) *given by* (15) *is called a* Jordan pair *of* $A(z)$ *corresponding to the eigenvalue* λ.

We note that, unlike in the classical case, the vectors forming a Jordan chain are *not* necessarily linearly independent; furthermore a Jordan chain may contain a null vector.

We conclude this section with a result on entire functions of matrices defined through

$$f(M)=\sum_{i=0}^{\infty}\frac{1}{i!}f^{(i)}(0)M^i, \tag{16}$$

for an entire function $f:\mathbb{C}\to\mathbb{C}$ and a square matrix M. The next lemma will be important for applications.

Lemma 2.1 *Let* $f:\mathbb{C}\to\mathbb{C}$ *be an entire function and let* Γ *be a Jordan block of size* k *with* λ *on the diagonal, then for an arbitrary set of vectors* $\boldsymbol{v}_0,\dots,\boldsymbol{v}_{k-1}$ *the* $(j+1)$*-st column of the matrix* $[\boldsymbol{v}_0,\dots,\boldsymbol{v}_{k-1}]f(\Gamma)$ *equals*

$$\sum_{i=0}^{j}\frac{1}{i!}f^{(i)}(\lambda)\boldsymbol{v}_{j-i}, \tag{17}$$

where $j=0,\dots,k-1$.

Proof. Immediate from [6, Thm. 6.6]. □

3. Jordan Normal Form of $\boldsymbol{\Lambda(q)}$ In this section we consider a spectrally negative MAP $(X(t),J(t))$ with matrix exponent $F(\alpha)$ and asymptotic drift κ. Let $\lambda_1,\dots,\lambda_k$ be the eigenvalues of $F^q(\alpha)$, to be understood as the zeros of $\det(F^q(\alpha))$, for a given $q\geq 0$, in its region of analyticity $\mathbb{C}^{\mathrm{Re}>0}$. Let (V_i,Γ_i) be a Jordan pair corresponding to the eigenvalue λ_i. Define the matrices V and Γ in the following way:

$$\begin{aligned} V=&\quad[V_1,\dots,V_k] \\ \Gamma=&\quad\mathrm{diag}[\Gamma_1,\dots,\Gamma_k] \qquad \text{if } q>0 \text{ or } q=0,\kappa<0; \\ V=&\quad[\mathbf{1},V_1,\dots,V_k] \\ \Gamma=&\quad\mathrm{diag}[0,\Gamma_1,\dots,\Gamma_k] \qquad \text{if } q=0,\kappa\geq 0. \end{aligned} \tag{18}$$

and let the matrices V_+ and $V_\downarrow$ be the restrictions of the matrix V to the rows corresponding to E_+ and $E_\downarrow$ respectively.

Theorem 3.1 *It holds that* Γ *and* V_+ *are* $N_+\times N_+$*-dimensional matrices,* V_+ *is invertible, and*

$$\Lambda(q)=-V_+\Gamma V_+^{-1}, \qquad \Pi(q)=VV_+^{-1}. \tag{19}$$

We start by establishing a lemma, which can be considered as a weak analog of Thm. 3.1.

Lemma 3.1 *If* $\boldsymbol{v}^0,\dots,\boldsymbol{v}^{r-1}$ *is a Jordan chain of* $F^q(\alpha)$ *corresponding to the eigenvalue* $\lambda\in\mathbb{C}^{\mathrm{Re}>0}$ *then* $\boldsymbol{v}^0_+,\dots,\boldsymbol{v}^{r-1}_+$ *is a Jordan chain of* $\alpha\mathbb{I}+\Lambda(q)$ *corresponding to the eigenvalue* $\alpha=\lambda$ *and* $\Pi(q)\boldsymbol{v}^i_+=\boldsymbol{v}^i$ *for* $i=0,\dots,r-1$.

Proof. Apply the optional sampling theorem to the martingale $M_\alpha(\cdot)$, $\alpha\in\mathbb{C}^{\mathrm{Re}>0}$ as defined in Lemma A.1 in the Appendix, with the finite stopping time $t\wedge\tau_x$, to obtain

$$C(\alpha)F^q(\alpha)=B(\alpha), \tag{20}$$

with

$$C(\alpha)=\mathbb{E}_k\left[\int_0^{t\wedge\tau_x\wedge e_q}e^{\alpha X(s)}\boldsymbol{e}'_{J(s)}\mathrm{d}s\right],\ B(\alpha)=\mathbb{E}_k\left[e^{\alpha X(t\wedge\tau_x)}1_{\{t\wedge\tau_x<e_q\}}\boldsymbol{e}'_{J(t\wedge\tau_x)}\right]-\boldsymbol{e}'_k, \tag{21}$$

where '$'$' denotes the transposition operation.

Noting that $X(\cdot) \le x$ on $[0, \tau_x]$ and using usual dominated convergence argument we conclude that $B(\alpha)$ is infinitely differentiable in $\alpha \in \mathbb{C}^{\mathrm{Re}>0}$. Apply Prop. 2.1 to (20) to see that for all $j = 0, \ldots, r-1$ the following holds true:

$$\sum_{i=0}^{j} \frac{1}{i!} \mathbb{E}_k \left[X^i(t \wedge \tau_x) e^{\lambda X(t \wedge \tau_x)} 1_{\{t \wedge \tau_x < e_q\}} \boldsymbol{e}'_{J(t \wedge \tau_x)} \right] \boldsymbol{v}^{j-i} - \boldsymbol{e}'_k \boldsymbol{v}^j = 0. \tag{22}$$

Letting $t \to \infty$ we obtain

$$\sum_{i=0}^{j} \frac{1}{i!} x^i e^{\lambda x} \mathbb{P}_k(J(\tau_x), \tau_x < e_q) \boldsymbol{v}^{j-i} - \boldsymbol{e}'_k \boldsymbol{v}^j = 0, \tag{23}$$

where $\mathbb{P}_k(J(\tau_x), \tau_x < e_q)$ denotes a row vector with ℓ-th element given by $\mathbb{P}_k(J(\tau_x) = \ell, \tau_x < e_q)$. Note that the case when $q = 0$ and $\mathbb{P}_k(\tau_x = \infty) > 0$ should be treated with care. In this case $\kappa < 0$ and thus $\lim_{t\to\infty} X(t) = -\infty$ a.s. [1, Prop. XI.2.10], so the above limit is still valid.

Considering (23) for all $k \in E$ and choosing $x = 0$ we indeed obtain $\Pi(q)\boldsymbol{v}^j_+ = \boldsymbol{v}^j$. If, however, we pick $k \in E_+$, then

$$\sum_{i=0}^{j} \frac{1}{i!} x^i e^{(\lambda \mathbb{I} + \Lambda(q))x} \boldsymbol{v}^{j-i}_+ - \boldsymbol{v}^j_+ = \boldsymbol{0}_+. \tag{24}$$

Take the right derivative in x at 0 of both sides to see that

$$(\lambda \mathbb{I} + \Lambda(q))\boldsymbol{v}^j_+ + \boldsymbol{v}^{j-1}_+ = \boldsymbol{0}_+, \tag{25}$$

which shows that $\boldsymbol{v}^0_+, \ldots, \boldsymbol{v}^{r-1}_+$ is a Jordan chain of $\alpha\mathbb{I} + \Lambda(q)$ corresponding to the eigenvalue λ. □

We are now ready to give a proof of our main result, Thm. 3.1.

Proof of Theorem 3.1. Lemma 3.1 states that $\boldsymbol{v}^0_+, \ldots, \boldsymbol{v}^{r-1}_+$ is a classical Jordan chain of the matrix $-\Lambda(q)$. Recall that if $q = 0, \kappa \ge 0$ then $\Lambda(q)\boldsymbol{1}_+ = \boldsymbol{0}_+$ and $\Pi(q)\boldsymbol{1}_+ = \boldsymbol{1}$. Therefore the columns of V_+ are linearly independent [8, Prop. 1.3.4] and

$$-\Lambda(q)V_+ = V_+\Gamma, \qquad \Pi(q)V_+ = V. \tag{26}$$

Consider the case when $q > 0$. Now [11, Thm. 1] states that $\det(F^q(\alpha))$ has N_+ zeros (counting multiplicities) in $\mathbb{C}^{\mathrm{Re}>0}$; see also [11, Rem. 1.2], so the matrices V_+ and Γ are of size $N_+ \times N_+$ by construction (18). Note there is one-to-one correspondence between the zeros of $\det(F^q(\alpha))$ in $\mathbb{C}^{\mathrm{Re}>0}$ and the eigenvalues of $-\Lambda(q)$ when $q > 0$.

Assume now that $q = 0$. Pick a sequence of q_n converging to 0 and consider a sequence of matrix exponents $F^{q_n}(\alpha) = F(\alpha) - q_n\mathbb{I}$. In view of [11, Thm. 10] it only remains to show that exactly $1_{\{\kappa \ge 0\}}$ eigenvalues of $\Lambda(q_n)$ converge to 0. From (8) it follows that $e^{\Lambda(q_n)} \to e^{\Lambda}$, hence the eigenvalues of $\Lambda(q_n)$ converge to the eigenvalues of Λ (preserving multiplicities) as $n \to \infty$. Finally recall that 0 is an eigenvalue of Λ if and only if $\kappa \ge 0$, in which case it is a simple eigenvalue. □

The above proof strengthens [11, Thm. 2]; we remove the assumption that κ is non-zero and finite.

Corollary 3.1 *It holds that* $\det(F(\alpha))$ *has* $N_+ - 1_{\{\kappa \ge 0\}}$ *zeros (counting multiplicities) in* $\mathbb{C}^{\mathrm{Re}>0}$.

4. One-sided Reflection Problems For a given MAP $(X(t), J(t))$ the *reflected process* $(W(t), J(t))$ is defined through

$$W(t) = X(t) - \inf_{0 \le s \le t} X(s) \wedge 0; \tag{27}$$

we say that $W(t)$ is the (single-sided) reflection of $X(t)$ at 0. It is well known that this process has a unique stationary distribution if the asymptotic drift κ is negative, which we assume in the sequel. Let a pair of random variables (W, J) have the stationary distribution of $(W(t), J(t))$, and denote the all-time maximum attained by $X(t)$ through $\overline{X} = \sup_{t \ge 0} X(t)$. It is an immediate consequence of [1, Prop. XI.2.11] that

$$(W \mid J = i) \text{ and } \left(\hat{\overline{X}} \mid \hat{J}(0) = i\right) \text{ have the same distribution,} \tag{28}$$

where $(\hat{X}(t), \hat{J}(t))$ is the time-reversed process characterized by the matrix exponent $\hat{F}(\alpha) = \Delta_{\boldsymbol{\pi}}^{-1} F(\alpha)' \Delta_{\boldsymbol{\pi}}$. In the following we identify the distribution of the random variables appearing in (28) for two important classes: spectrally negative MAPs and spectrally positive MAPs.

4.1 Spectrally negative MAP Let $(X(t), J(t))$ be a spectrally negative MAP with negative asymptotic drift: $\kappa < 0$. It is crucial to observe that $\left(\overline{X} \mid J(0) = i\right)$ is the life-time of $J(\tau_x)$, thus it has a phase-type distribution [1, Section III.4] with transition rate matrix Λ, exit vector $-\Lambda \mathbf{1}_+$ and initial distribution given by $\boldsymbol{e}_i' \Pi$. It is noted that if $i \in E_\downarrow$, then $X(t)$ never hits the interval $(0, \infty)$ with probability $\mathbb{P}_i(\tau_0 = \infty)$, hence $\left(\overline{X} \mid J(0) = i\right)$ has a mass at zero.

The time-reversed process is again a spectrally negative MAP with negative asymptotic drift and $E_\downarrow$ being the set of indices of associated downward subordinators. Thus, with self-evident notation, the vector of densities of $(W \mid J)$ at $x > 0$ is given by

$$\boldsymbol{p}(x) = \hat{\Pi} e^{\hat{\Lambda} x} (-\hat{\Lambda} \mathbf{1}_+). \tag{29}$$

We conclude that we can express the distribution of the stationary workload in terms of quantities that uniquely follow from Thm. 3.1.

4.2 Spectrally positive MAP Let $(Y(t), J(t))$ be a spectrally positive MAP with negative asymptotic drift. Define $X(t) = -Y(t)$ and note that $(X(t), J(t))$ is a spectrally negative MAP with positive asymptotic drift: $\kappa > 0$. Let $F(\alpha)$ and $E_+ \cup E_\downarrow$ be the matrix exponent and the partition of the state space of the latter process. The Laplace-Stieltjes transform of (W, J) is identified in [4] up to a vector $\boldsymbol{\ell}$ of unknown constants:

$$\mathbb{E}[e^{-\alpha W} \boldsymbol{e}_J'] = \alpha \boldsymbol{\ell}' F(\alpha)^{-1}, \tag{30}$$

where $\boldsymbol{\ell}_\downarrow = \mathbf{0}$, $\boldsymbol{\ell}' \mathbf{1} = \kappa$.

We determine the vector $\boldsymbol{\ell}_+$ as follows. Let $\boldsymbol{v}^0, \ldots, \boldsymbol{v}^{r-1}$ be a Jordan chain of $F(\alpha)$ associated with an eigenvalue $\lambda \in \mathbb{C}^{\mathrm{Re}>0}$. Right multiply both sides of Eqn. (30) by $F(\alpha)$ and use Prop. 2.1 to see that $\lambda \boldsymbol{\ell}' \boldsymbol{v}^j + \boldsymbol{\ell}' \boldsymbol{v}^{j-1} = 0$ for all $j = 0, \ldots, r-1$; where $\boldsymbol{v}^{-1} = \mathbf{0}$. It trivially follows that $\boldsymbol{\ell}' V = \kappa \boldsymbol{e}_1'$ or equivalently

$$(\boldsymbol{\ell}_+)' = \kappa \boldsymbol{e}_1' (V_+)^{-1}. \tag{31}$$

It is easy to verify now using Thm. 3.1 that

$$\boldsymbol{\ell}_+ = \kappa \boldsymbol{\pi}_\Lambda, \tag{32}$$

where $\boldsymbol{\pi}_\Lambda$ is the stationary distribution of Λ, cf. [5, Lemma 2.2]. Observe that we again succeeded in expressing the distribution of the stationary workload in terms of quantities that can be determined by applying Thm. 3.1.

5. Two-sided Reflection of MMBM In this section we consider a Markov modulated Brownian motion (MMBM) $(X(t), J(t))$ of dimension N. In this case the matrix exponent (3) has the special form of a matrix polynomial of second order, that is,

$$F(\alpha) = \frac{1}{2} \Delta_{\boldsymbol{\sigma}}^2 \alpha^2 + \Delta_{\boldsymbol{a}} \alpha + Q, \tag{33}$$

where $\alpha_i \in \mathbb{R}$, $\sigma_i \geq 0$ for $i = 1, \ldots, N$; for any given vector $\boldsymbol{x}$ we define $\Delta_{\boldsymbol{x}} = \mathrm{diag}[x_1, \ldots, x_N]$.

We are interested in a two-sided reflection [4] of $(X(t), J(t))$ with 0 and $b > 0$ being lower and upper barriers respectively. This reflected process can be interpreted as a workload process of a queue fed by MMBM, where b is the capacity of the buffer. Let the pair of random variables (W, J) have the stationary distribution of this process. It is shown in [4] that

$$\mathbb{E}\left[e^{\alpha W} \boldsymbol{e}_J'\right] \cdot F(\alpha) = \alpha (e^{\alpha b} \boldsymbol{u}' - \boldsymbol{\ell}'), \tag{34}$$

where $\boldsymbol{u}$ and $\boldsymbol{\ell}$ are column vectors of unknown constants. In the following we show how to compute these constants. This result completes the investigation on MMBM contained in [4] and extends the previous works [13, 15, 22].

In order to uniquely characterize the vectors $\boldsymbol{u}$ and $\boldsymbol{\ell}$ we exploit the special structure of MMBM. As MMBM is a continuous process almost surely, both $(X(t), J(t))$ and $(-X(t), J(t))$ are spectrally negative MAPs. Regarding the downward subordinators of $(X(t), J(t))$, let the sets E_+ and $E_\downarrow$, and the cardinalities N_+ and $N_\downarrow$ be defined as before. The downward subordinators of the process $(-X(t), J(t))$ correspond to the upward subordinators of the process $(X(t), J(t))$. Denote the subset of states of E associated to such processes by $E_\uparrow$. Similarly we denote by E_- the set of the states where the process $(-X(t), J(t))$ can reach positive records. We note that it is possible to have that $E_\uparrow \cap E_\downarrow \neq \emptyset$, as the intersection is given by the states where the process stays constant. Finally the cardinalities of E_- and $E_\uparrow$ are denoted through N_- and $N_\uparrow$ respectively.

Let $\lambda_0, \ldots, \lambda_k$ be the zeros of $\det(F(\alpha))$ in $\mathbb{C}$ with $\lambda_0 = 0$. Let also (V_i, Γ_i) be a Jordan pair of $F(\alpha)$ corresponding to λ_i. Define

$$V = (V_0, \ldots, V_k), \qquad \Gamma = \text{diag}(\Gamma_0, \ldots, \Gamma_k). \tag{35}$$

THEOREM 5.1 *The unknown vectors $\boldsymbol{u}$ and $\boldsymbol{\ell}$ in Eqn. (34) can be uniquely identified in the following way: $\boldsymbol{u}_\downarrow = \boldsymbol{0}$ and $\boldsymbol{\ell}_\uparrow = \boldsymbol{0}$ while the vectors $\boldsymbol{u}_+$ and $\boldsymbol{\ell}_-$ are the solutions of the system of equations*

$$(\boldsymbol{u}_+', \boldsymbol{\ell}_-') \begin{pmatrix} V_+ e^{b\Gamma} \\ -V_- \end{pmatrix} = (\boldsymbol{k}', 0, \ldots, 0), \tag{36}$$

where

$$\boldsymbol{k}' = \boldsymbol{\pi}'(\Delta_{\boldsymbol{a}}, \frac{1}{2}\Delta_{\boldsymbol{\sigma}}^2) \begin{pmatrix} V_0 \\ V_0 \Gamma_0 \end{pmatrix} \tag{37}$$

is a vector with dimension equal to the multiplicity of the 0 *root of* $\det(F(\alpha))$.

Before we give a proof of Thm. 5.1 we provide some comments on the structure of the pair (V, Γ). The following simple lemma identifies the number of zeros of $\det(F(\alpha))$ in different parts of the complex plane, see also [13] for the case when $\kappa \neq 0$.

LEMMA 5.1 $\det(F(\alpha))$ *has $N_+ - 1_{\{\kappa \geq 0\}}$ zeros in $\mathbb{C}^{\text{Re}>0}$, $N_- - 1_{\{\kappa \leq 0\}}$ zeros in $\mathbb{C}^{\text{Re}<0}$ and a zero at 0 of multiplicity $1 + 1_{\{\kappa = 0\}}$.*

PROOF. It is easy to see that $\det(F(\alpha))$ is a polynomial of degree $N_+ + N_-$, hence the total number of zeros of $\det(F(\alpha))$ in $\mathbb{C}$ counting their multiplicities is $N_+ + N_-$. On the other hand Corollary 3.1 states that $\det(F(\alpha))$ has $N_+ - 1_{\{\kappa \geq 0\}}$ zeros in $\mathbb{C}^{\text{Re}>0}$ and $N_- - 1_{\{\kappa \leq 0\}}$ zeros in $\mathbb{C}^{\text{Re}<0}$, because $F(-\alpha)$ is the matrix exponent of a spectrally negative process $(-X(t), J(t))$ having asymptotic drift $-\kappa$. The result follows from [11], where it is shown that $\det(F(\alpha))$ does not have zeros on the imaginary axis which are distinct from 0. □

Next we note that the null-space of $F(0) = Q$ is spanned by $\boldsymbol{1}$, because Q is an irreducible transition rate matrix. Moreover, scaling matrix V amounts to scaling both sides of Eqn. (36) by the same constant, hence we can assume that the first vector in V_0, and thus also in V, is $\boldsymbol{1}$. If $\kappa = 0$ then according to Lemma 5.1 the zero λ_0 has multiplicity 2, in which case we have that Γ_0 is a 2×2 matrix, and V_0 an $N \times 2$ matrix, given by

$$\Gamma_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad V_0 = (\boldsymbol{1}, \boldsymbol{h}), \tag{38}$$

where the vector $\boldsymbol{h}$ solves

$$F(0)\boldsymbol{h} + F'(0)\boldsymbol{1} = Q\boldsymbol{h} + \Delta_{\boldsymbol{a}}\boldsymbol{1} = \boldsymbol{0} \tag{39}$$

due to (11). Since $\boldsymbol{\pi}'\Delta_{\boldsymbol{a}}\boldsymbol{1}$ equals the asymptotic drift κ, Eqn. (37) reduces to

$$\boldsymbol{k}' = \begin{cases} \kappa, & \text{if } \kappa \neq 0 \\ (0, \boldsymbol{\pi}'(\frac{1}{2}\Delta_{\boldsymbol{\sigma}}^2\boldsymbol{1} + \Delta_{\boldsymbol{a}}\boldsymbol{h})), & \text{if } \kappa = 0. \end{cases} \tag{40}$$

We now prove the following technical lemma that further specifies $\boldsymbol{k}$ in the case $\kappa = 0$.

Lemma 5.2 *If $\kappa = 0$, then $V_0 = (\mathbf{1}, \boldsymbol{h})$ and*

$$b\boldsymbol{u}'\mathbf{1} + (\boldsymbol{u}' - \boldsymbol{\ell}')\boldsymbol{h} = \boldsymbol{\pi}'(\frac{1}{2}\Delta^2_{\boldsymbol{\sigma}}\mathbf{1} + \Delta_{\boldsymbol{a}}\boldsymbol{h}) \neq 0. \tag{41}$$

Proof. Differentiating Eqn. (34) at 0 and right multiplying by $\boldsymbol{h}$, we obtain the identity

$$(\boldsymbol{u}' - \boldsymbol{\ell}')\boldsymbol{h} = \mathbb{E}[W\boldsymbol{e}'_J]Q\boldsymbol{h} + \mathbb{E}[\boldsymbol{e}'_J]\Delta_{\boldsymbol{a}}\boldsymbol{h} = -\mathbb{E}[W\boldsymbol{e}'_J]\Delta_{\boldsymbol{a}}\mathbf{1} + \mathbb{E}[\boldsymbol{e}'_J]\Delta_{\boldsymbol{a}}\boldsymbol{h}, \tag{42}$$

where the second equality follows from (39). Differentiating Eqn. (34) twice at 0 and multiplying by $\mathbf{1}$, we find

$$b\,\boldsymbol{u}'\mathbf{1} = \mathbb{E}[W\boldsymbol{e}'_J]\Delta_{\boldsymbol{a}}\mathbf{1} + \mathbb{E}[\boldsymbol{e}'_J]\frac{1}{2}\Delta^2_{\boldsymbol{\sigma}}\mathbf{1}, \tag{43}$$

which summed with the previous equation gives (41). We conclude the proof by showing that the resulting expression cannot equal 0.

It is known, see [7], that the maximum length of a Jordan chain cannot exceed the algebraic multiplicity of the associated eigenvalue. We therefore have that for any vector $\boldsymbol{v}$ it holds that

$$\frac{1}{2}\Delta^2_{\boldsymbol{\sigma}}\,\mathbf{1} + \Delta_{\boldsymbol{a}}\,\boldsymbol{h} + Q\boldsymbol{v} \neq \mathbf{0}, \tag{44}$$

because otherwise $(\mathbf{1}, \boldsymbol{h}, \boldsymbol{v})$ would be a Jordan chain associated with λ_0, which has multiplicity 2 (Lemma 5.1). This implies that $\frac{1}{2}\Delta^2_{\boldsymbol{\sigma}}\,\mathbf{1} + \Delta_{\boldsymbol{a}}\,\boldsymbol{h}$ is not in the column space of Q, which is known to be of dimension $N-1$, because Q is irreducible. Moreover, $\boldsymbol{\pi}'Q = \mathbf{0}'$, thus $\boldsymbol{\pi}'(\frac{1}{2}\Delta^2_{\boldsymbol{\sigma}}\,\mathbf{1} + \Delta_{\boldsymbol{a}}\,\boldsymbol{h}) \neq 0$. □

We now construct pairs (V^+, Γ^+) and (V^-, Γ^-) in the same way as we constructed (V, Γ), but we use only those (V_i, Γ_i) for which $\lambda_i \in \mathbb{C}^{\mathrm{Re}>0}$ and $\lambda_i \in \mathbb{C}^{\mathrm{Re}<0}$, respectively. Moreover, an additional pair $(\mathbf{1}, 0)$ is used (as the first pair) in the construction of (V^+, Γ^+) and (V^-, Γ^-) if $\kappa \geq 0$ and $\kappa \leq 0$, respectively. Note that in view of Lemma 5.1 the matrix $V^\pm$ has exactly $N_\pm$ columns.

In the following we use $\Lambda^\pm$ and $\Pi^\pm$ to denote the matrices associated to the first passage process of $(\pm X(t), J(t))$, and defined according to Eqns. (8) and (9). Next we present a consequence of Thm. 3.1.

Lemma 5.3 *The following holds*

$$\begin{array}{ll} \Lambda^+ = -V^+_+\Gamma^+(V^+_+)^{-1} & \Pi^+ = V^+(V^+_+)^{-1} \\ \Lambda^- = \quad V^-_-\Gamma^-(V^-_-)^{-1} & \Pi^- = V^-(V^-_-)^{-1}. \end{array} \tag{45}$$

Proof. The first line is immediate from Thm. 3.1 and the second line follows by noting that if $\boldsymbol{v}^0, \ldots, \boldsymbol{v}^{r-1}$ is a Jordan chain of $F(\alpha)$ then $\boldsymbol{v}^0, -\boldsymbol{v}^1, \ldots, (-1)^{r-1}\boldsymbol{v}^{r-1}$ is a Jordan chain of $F(-\alpha)$. Lemma 3.1 applied to the process $(-X(t), J(t))$ entails that $(-\lambda\mathbb{I} + \Lambda^-)\boldsymbol{v}^j_- - \boldsymbol{v}^{j-1}_- = \mathbf{0}$, where λ is a zero of $\det(F(\alpha))$ in $\mathbb{C}^{\mathrm{Re}<0}$. Hence $\Lambda^- V^-_- = V^-_-\Gamma^-$. □

We now proceed with the proof of Thm. 5.1.

Proof of Theorem 5.1. The proofs of the facts that $\boldsymbol{u}_\downarrow = \mathbf{0}$ and $\boldsymbol{\ell}_\uparrow = \mathbf{0}$ and, moreover, $(\boldsymbol{u}' - \boldsymbol{\ell}')\mathbf{1} = \boldsymbol{\pi}'\Delta_{\boldsymbol{a}}\mathbf{1} = \kappa$ were already given in [4]. The rest of the proof is split into two steps. First we show that $(\boldsymbol{u}'_+, \boldsymbol{\ell}'_-)$ solves (36), and then we show that the solution is unique.

Step 1: Lemma 2.1 and Prop. 2.1 applied to Eqn. (34) imply

$$\boldsymbol{u}'Ve^{b\Gamma}\Gamma - \boldsymbol{\ell}'V\Gamma = \mathbf{0}'. \tag{46}$$

Let $\hat{\Gamma}$ be the matrix Γ with Jordan block Γ_0 replaced with $\mathbb{I}$. Suppose first that $\kappa \neq 0$. Then $(V_0, \Gamma_0) = (\mathbf{1}, 0)$ and so (46) can be rewritten as

$$\boldsymbol{u}'Ve^{b\Gamma}\hat{\Gamma} - \boldsymbol{\ell}'V\hat{\Gamma} = ([\boldsymbol{u}' - \boldsymbol{\ell}']\mathbf{1}, 0, \ldots, 0). \tag{47}$$

Multiply by $\hat{\Gamma}^{-1}$ from the right to obtain (36). Supposing that $\kappa = 0$, we have that adding

$$\hat{\boldsymbol{k}}' = \boldsymbol{u}'V_0e^{b\Gamma_0}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} - \boldsymbol{\ell}'V_0\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \tag{48}$$

to the first two elements of the vectors appearing on the both sides of (46), leads to

$$\boldsymbol{u}'Ve^{b\Gamma}\hat{\Gamma}-\boldsymbol{\ell}'V\hat{\Gamma}=(\hat{\boldsymbol{k}}',0,\ldots,0). \tag{49}$$

To complete Step 1 it is now enough to show that $\hat{\boldsymbol{k}}=\boldsymbol{k}$. A simple computation reveals that $\hat{\boldsymbol{k}}'=\boldsymbol{u}'(\mathbf{1},(b-1)\mathbf{1}+\boldsymbol{h})-\boldsymbol{\ell}'(\mathbf{1},-\mathbf{1}+\boldsymbol{h})=(0,b\boldsymbol{u}'\mathbf{1}+(\boldsymbol{u}'-\boldsymbol{\ell}')\boldsymbol{h})$, where we used that $(\boldsymbol{u}'-\boldsymbol{\ell}')\mathbf{1}=\kappa=0$. Use Lemma 5.2 and (40) to see that $\hat{\boldsymbol{k}}=\boldsymbol{k}$.

Step 2 (uniqueness): Without loss of generality we assume that $\kappa\geq 0$. It is easy to see that $(\boldsymbol{u}'_+,\boldsymbol{\ell}'_-)$ solves (36) if and only if

$$(\boldsymbol{u}'_+,-\boldsymbol{\ell}'_-)\begin{pmatrix} V_+^+e^{b\Gamma^+} & V_+^-e^{b\Gamma^-}\\ V_-^+ & V_-^-\end{pmatrix}=\kappa\boldsymbol{e}_1' \tag{50}$$

and, in addition when $\kappa=0$ Eqn. (41) holds true because of the construction of the matrices $V^{\pm}$ and $\Gamma^{\pm}$. Note also that we can right multiply both sides of the above display by the same matrix to obtain

$$(\boldsymbol{u}'_+,-\boldsymbol{\ell}'_-)\begin{pmatrix} V_+^+ & V_+^-e^{b\Gamma^-}\\ V_-^+e^{-b\Gamma^+} & V_-^-\end{pmatrix}=\kappa\boldsymbol{e}_1'\begin{pmatrix} e^{-b\Gamma^+} & \mathbb{O}\\ \mathbb{O} & \mathbb{I}\end{pmatrix}. \tag{51}$$

Lemma 5.3 shows that

$$\begin{aligned} V_-^+&=\Pi_-^+V_+^+, & V_+^+e^{-b\Gamma^+}&=e^{b\Lambda^+}V_+^+,\\ V_+^-&=\Pi_+^-V_-^-, & V_-^-e^{+b\Gamma^-}&=e^{b\Lambda^-}V_-^-,\end{aligned} \tag{52}$$

so we obtain

$$\begin{pmatrix} V_+^+ & V_+^-e^{b\Gamma^-}\\ V_-^+e^{-b\Gamma^+} & V_-^-\end{pmatrix}=\begin{pmatrix} \mathbb{I} & \Pi_+^-e^{b\Lambda^-}\\ \Pi_-^+e^{b\Lambda^+} & \mathbb{I}\end{pmatrix}\begin{pmatrix} V_+^+ & \mathbb{O}\\ \mathbb{O} & V_-^-\end{pmatrix}. \tag{53}$$

Theorem 3.1 states that V_+^+ and V_-^- are invertible matrices. Moreover, $\Pi_-^+e^{b\Lambda^+}$ and $\Pi_+^-e^{b\Lambda^-}$ are irreducible transition probability matrices, so the first matrix on the right hand side of (53), call it M, is an irreducible non-negative matrix, which is non-strictly diagonally dominant. If $\kappa>0$, then $\Pi_+^-e^{b\Lambda^-}$ is sub-stochastic, which implies that M is irreducibly diagonally dominant and hence invertible [10]. If $\kappa=0$, then M has a simple eigenvalue at 0 by Perron-Frobenius, so

$$(\boldsymbol{u}'_+,-\boldsymbol{\ell}'_-)\begin{pmatrix} \mathbb{I} & \Pi_+^-e^{b\Lambda^-}\\ \Pi_-^+e^{b\Lambda^+} & \mathbb{I}\end{pmatrix}=\mathbf{0}' \tag{54}$$

determines the vector $(\boldsymbol{u}'_+,-\boldsymbol{\ell}'_-)$ up to a scalar, which is then identified using (41):

$$(\boldsymbol{u}'_+,-\boldsymbol{\ell}'_-)\begin{pmatrix} b\mathbf{1}+\boldsymbol{h}\\ \boldsymbol{h}\end{pmatrix}=\boldsymbol{\pi}'(\Delta_{\boldsymbol{a}}\boldsymbol{h}+\frac{1}{2}\Delta_{\boldsymbol{\sigma}}^2\mathbf{1}), \tag{55}$$

which is non-zero by Lemma 5.2. □

Finally, we state a corollary, which identifies (in the case of non-zero asymptotic drift) vectors $\boldsymbol{u}_+$ and $\boldsymbol{\ell}_-$ in terms of matrices $\Lambda^{\pm}$ and $\Pi^{\pm}$. We believe that there should exist a direct probabilistic argument leading to this identity. Moreover, it is interesting to investigate if such a result holds in the case of countably infinite state space E.

Corollary 5.1 *It holds that*

$$(\boldsymbol{u}'_+,-\boldsymbol{\ell}'_-)\begin{pmatrix} \mathbb{I} & \Pi_+^-e^{b\Lambda^-}\\ \Pi_-^+e^{b\Lambda^+} & \mathbb{I}\end{pmatrix}=\begin{cases}\kappa(\boldsymbol{\pi}'_{\Lambda^+},\mathbf{0}'_-), & \text{if } \kappa>0\\ \mathbf{0}', & \text{if } \kappa=0\\ \kappa(\mathbf{0}'_+,\boldsymbol{\pi}'_{\Lambda^-}), & \text{if } \kappa<0\end{cases}, \tag{56}$$

where $\boldsymbol{\pi}_{\Lambda^{\pm}}$ is the unique stationary distribution of $\Lambda^{\pm}$, which is well-defined if $\kappa\neq 0$.

Proof. Assume that $\kappa>0$. From the above proof we know that

$$(\boldsymbol{u}'_+,-\boldsymbol{\ell}'_-)\begin{pmatrix} \mathbb{I} & \Pi_+^-e^{b\Lambda^-}\\ \Pi_-^+e^{b\Lambda^+} & \mathbb{I}\end{pmatrix}\begin{pmatrix} V_+^+ & \mathbb{O}\\ \mathbb{O} & V_-^-\end{pmatrix}=\kappa(\boldsymbol{e}_1',\mathbf{0}'_-). \tag{57}$$

Hence it is enough to check that $\boldsymbol{\pi}_{\Lambda^+}=\boldsymbol{e}_1'(V_+^+)^{-1}$, which is immediate in view of Thm. 3.1. The case of $\kappa<0$ is symmetric, and the case of $\kappa=0$ is trivial. □

6. Matrix Integral Equation In this section we demonstrate how our technique can be used to show in a simple way that $\Lambda(q)$ is a unique solution of a certain matrix integral equation. This equation appears in [2, 5, 18, 19, 20, 22] and is commonly considered as the main tool in numerically computing $\Lambda(q)$. Throughout this section it is assumed that $N_+ = N$. We use the following notation

$$\begin{aligned} F^q(M) &= \Delta_{\boldsymbol{a}} M + \frac{1}{2}\Delta_{\boldsymbol{\sigma}}^2 M^2 + \int_{-\infty}^0 \Delta_{\boldsymbol{\nu}}(\mathrm{d}x)\left(e^{Mx} - \mathbb{I} - Mx1_{\{x>-1\}}\right) \\ &\quad + \int_{-\infty}^0 Q \circ G(\mathrm{d}x) e^{Mx} - q\mathbb{I} \end{aligned} \tag{58}$$

where $(a_i, \sigma_i, \nu_i(\mathrm{d}x))$ are the Lévy triplets corresponding to the Lévy processes $X_i(\cdot)$ and $G_{ij}(\mathrm{d}x)$ is the distribution of U_{ij}.

Define $\mathcal{M}$ to be a set of all $N \times N$ matrices Q, such that Q is a transition rate matrix of an irreducible Markov chain. Let also $\{\mathcal{M}_0, \mathcal{M}_1\}$ be a partition of $\mathcal{M}$ into the sets of defective and non-defective matrices respectively.

Theorem 6.1 $\Lambda(q)$ *is the unique solution of* $F^q(-M) = \mathbb{O}$*, where* $M \in \mathcal{M}_i$ *and* $i = 1_{\{q=0,\kappa\geq 0\}}$.

Proof. In the proof we drop the superscript q to simplify notation. Let $-M = V\Gamma V^{-1}$ be a Jordan decomposition of the matrix $-M$. Let also $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{r-1}$ be the columns of V corresponding to some Jordan block of size r and eigenvalue λ. Note that $\lambda \in \mathbb{C}^{\mathrm{Re}>0}$ or $\lambda = 0$ in which case it must be simple, because $M \in \mathcal{M}$. Right multiply (58) by V, note that $g(-M) = Vg(\Gamma)V^{-1}$ for an entire function $g : \mathbb{C} \to \mathbb{C}$, and finally use Lemma 2.1 to see that the column of $F(-M)V$ corresponding to $\boldsymbol{v}_j$ equals

$$\sum_{i=0}^{j} \frac{1}{i!} F^{(i)}(\lambda) \boldsymbol{v}_{j-i}, \tag{59}$$

where we also used the fact that differentiation of $F(\alpha)$ at $\lambda, \mathrm{Re}(\lambda) > 0$ can be done under the integral signs and no differentiation is needed for a simple eigenvalue $\lambda = 0$ if such exists.

If $M = \Lambda$, then the matrices V and Γ can be chosen as in (18). Hence (59) becomes $\mathbf{0}$, because $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{r-1}$ is a Jordan chain of $F(\alpha)$, see (11). But V is an invertible matrix, so that $F(-\Lambda) = \mathbb{O}$.

Suppose now that $F(-M) = \mathbb{O}$ and $M \in \mathcal{M}_i$. Then the vectors $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{r-1}$ form a Jordan chain of $F(\alpha)$ corresponding to an eigenvalue $\lambda \in \mathbb{C}^{\mathrm{Re}>0}$ or $\lambda = 0$. Finally, use Lemma 3.1 to see that $\Lambda V = -V\Gamma$, and hence we have $M = \Lambda$. □

Appendix A.

Lemma A.1 *For any* $i \in E$ *and* $\alpha \in \mathbb{C}^{\mathrm{Re}>0}$ *the right-continuous process*

$$M_\alpha(t) = \left[\int_0^{t\wedge e_q} e^{\alpha X(s)} \boldsymbol{e}'_{J(s)} \mathrm{d}s\right] \cdot F^q(\alpha) + \boldsymbol{e}'_k - e^{\alpha X(t)} 1_{\{t<e_q\}} \boldsymbol{e}'_{J(t)} \tag{60}$$

is a row vector valued zero mean martingale under the probability measure $\mathbb{P}_i$.

Proof. It trivially follows from [4, Lemma 2.1] that

$$M^W(\alpha, t) = e^{\alpha X(t)} 1_{\{t<e_q\}} \boldsymbol{e}'_{J(t)} e^{-F^q(\alpha)t} \tag{61}$$

is a row vector valued martingale under $\mathbb{P}_i$. Repeat the steps of the proof of [4, Thm. 2.1] with $M^W(\alpha, t)$ defined above and $Y(\cdot) \equiv 0$ to conclude. □

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References

[1] S. Asmussen, *Applied Probability and Queues*, Applications of Mathematics. Springer-Verlag New York, Inc., 2nd edition, 2003.

[2] S. Asmussen, *Stationary distributions for fluid flow models with or without Brownian noise*, Stochastic Models **11** (1995), 21–49.

[3] S. Asmussen, F. Avram and M. Pistorius, *Russian and American put options under exponential phase-type Lévy models*, Stochastic Processes and their Applications **109** (2004), 79–111.

[4] S. Asmussen and O. Kella, *A multi-dimensional martingale for Markov additive processes and its applications*, Advances in Applied Probability **32** (2000), 376–393.

[5] A. B. Dieker and M. Mandjes, *Extremes of Markov-additive processes with one-sided jumps, with queueing applications*, Methodology and Computing in Applied Probability (to appear, 2009). `http://www2.isye.gatech.edu/~adieker3/publications/modulatedfluid.pdf`

[6] E. Doolittle, *Analytic Functions of Matrices* (1998). `http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.51.2968`

[7] I. Gohberg, P. Lancaster and L. Rodman, *Matrix Polynomials*, Academic Press, New York, 1982.

[8] I. Gohberg, P. Lancaster and L. Rodman, *Invariant Subspaces of Matrices with Applications*, Classics in Applied Mathematics, **51**, SIAM, Philadelphia, 2nd edition, 2006.

[9] I. Gohberg and L. Rodman, *Analytic matrix functions with prescribed local data*, Journal d'Analyse Mathématique **40** (1981), 90–128.

[10] R. A. Horn, *Matrix Analysis*, Cambridge University Press, 1990.

[11] J. Ivanovs, O. Boxma, and M. Mandjes, *Singularities of the generator of a Markov additive process with one-sided jumps* (submitted, 2008). `http://www.cwi.nl/ftp/CWIreports/PNA/PNA-R0813.pdf`

[12] A. Jobert and L. C. G. Rogers, *Option pricing with Markov-modulated dynamics*, SIAM Journal on Control and Optimization **44** (2006), 2063–2078.

[13] R. L. Karandikar and V. G. Kulkarni, *Second-order fluid flow models: reflected Brownian motion in a random environment*, Operations Research, **43** (1995), 77–88.

[14] O. Kella and W. Whitt, *Useful martingales for stochastic storage processes with Lévy input*, Journal of Applied Probability **29** (1992), 396–403.

[15] O. Kella and W. Stadje, *A Brownian motion with two reflecting barriers and Markov modulated speed*, Journal of Applied Probability **41** (2004), 1237–1243.

[16] A. E. Kyprianou, *Introductory Lectures on Fluctuations of Lévy Processes with Applications*, Springer-Verlag Berlin Heidelberg, 2006.

[17] A. E. Kyprianou and Z. Palmowski, *Fluctuations of spectrally negative Markov additive processes*, Séminaire de Probabilité (to appear, 2008).

[18] M. Miyazawa and H. Takada, *A matrix exponential form for hitting probabilities and its application to a Markov modulated fluid queue with downward jumps*, Journal of Applied Probability **39** (2002), 604–618.

[19] A. Pacheco and N. U. Prabhu, *A Markovian storage model*, Ann. Appl. Prob. **6** (1996), 76–91.

[20] M. Pistorius, *On maxima and ladder processes for a dense class of Lévy processes*, Journal of Applied Probability **43** (2006), 208–220.

[21] N. U. Prabhu, *Stochastic Storage Processes*, 2nd. edition, Springer-Verlag, 1998.

[22] L. C. G. Rogers, *Fluid models in queueing theory and Wiener-Hopf factorization of Markov chains*, Annals of Applied Probability **4** (1994), 390–413.

[23] E. Seneta, *Non-Negative Matrices and Markov Chains*, Springer Series in Statistics, Springer Inc., New York, 2nd edition, 1981.