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The crossing model for regular A_n-crystals

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Keywords and Phrases: simply-laced algebra; crystal of representation; Gelfand-Tsetlin pattern
Note: An essential part of the research was done while A.V. Karzanov was visiting the group PNA1 in the fall 2006.

The crossing model for regular A_n-crystals[1]

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Abstract. For a positive integer n, regular A_n-crystals are edge-colored directed graphs, with n colors, related to integrable highest weight modules over the quantum algebra $U_q(sl_{n+1})$. Based on Stembridge's local axioms for regular simply-laced crystals and a structural characterization of regular A_2-crystals in [2], we introduce a new combinatorial construction, the so-called *crossing model*, and prove that this model generates precisely the set of regular A_n-crystals. Using it, we obtain a series of results which significantly clarify the structure and demonstrate important ingredients of such crystals K. In particular, we reveal in K a canonical subgraph called the skeleton and a canonical n-dimensional lattice Π of vertices and explain an interrelation of these objects. Also we show that there are exactly $|\Pi|$ maximal (connected) A_{n-1}-subcrystal K' with colors $1, \dots, n-1$ (where neighboring colors do not commute) and that each K' intersects Π at exactly one element, and similarly for the maximal subcrystals with colors $2, \dots, n$.

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1 Introduction

Kashiwara [6, 7] introduced the notion of a (general) *crystal*. It is a certain edge-colored directed graph (digraph), with n colors, in which each connected monochromatic subgraph is a simple finite path and interrelations of the lengths of such paths for pairs of different colors depend on coefficients of an $n \times n$ Cartan matrix M (this matrix is said to describe the type of the crystal). The central role in the theory of Kashiwara is played by the *crystals of representations*; these are associated to integrable highest weight modules (representations) of the quantum enveloping algebra

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related to M. There are several realizations of such crystals for a variety of Cartan matrices, e.g., Littelmann's Path Model [10, 11]. (For important particular cases, see also [1, 9, 12].)

Recently Stembridge [13] pointed out a list of "local" graph-theoretic axioms defining the *regular simply-laced crystals*. These concern the class of simply-laced Cartan matrices, i.e., matrices M having coefficients $m_{ii} = 2$ and $m_{ij} = m_{ji} \in \{0, -1\}$ for $i \neq j$. Relying on the Path Model, Stembridge established a relationship between representations for the corresponding quantum algebras $U_q(g)$ in this case and regular simply-laced crystals K that have a zero-indegree vertex. Such a vertex s, if exists, is unique, and K is shown to be determined by the tuple $\mathbf{c} = (c_1, \ldots, c_n)$ of the lenghts of maximal monochromatic paths $P_1, \ldots, P_n$ beginning at s, where n is the number of colors and P_i concerns color i. So in this case (and when the Cartan matrix is fixed) K may be denoted by $K(\mathbf{c})$. The main theorem in [13] says that $K(\mathbf{c})$ exists for any tuple $\mathbf{c} \in \mathbb{Z}_+^n$ and is just the crystal graph of the irreducible $U_q(g)$-module of highest weight $\sum_i c_i \omega_i$, where ω_i is i-th fundamental weight.

This paper is devoted to a combinatorial study of regular simply-laced crystals of A_n-type, or *regular* A_n*-crystals*; for brevity we throughout call them *RAN-crystals*. In this case the algebra g is sl_{n+1} and the off-diagonal coefficients m_{ij} of the Cartan matrix are equal to -1 if $|i - j| = 1$, and 0 otherwise.

In our previous paper [2] we described the combinatorial structure of regular A_2-crystals K and demonstrated additional combinatorial and polyhedral properties of these crystals and their extensions. It turned out that the structure is rather transparent: K always has a zero-indegree vertex, and therefore, $K = K(c_1, c_2)$ for some $c_1, c_2 \in \mathbb{Z}_+$, and it can be produced by use of a certain operation $\bowtie$ of replicating and gluing together from the crystals $K(c_1, 0)$ and $K(0, c_2)$. The latter crystals are of simple form and are viewed as triangular parts of square grids. We refer to $\bowtie$ as the *diagonal-product* operation and write $K = K(c_1, 0) \bowtie K(0, c_2)$. (It fact, K is the largest component of the tensor product of $K(c_1, 0)$ and $K(0, c_2)$.)

When $n > 2$, the structure of a RAN-crystal becomes much more sophisticated, even for $n = 3$. To explore this structure, in this paper we introduce a certain combinatorial construction, called the *crossing model*. (Another sort of crossing model is constructed in [3] to describe the structure of regular B_2-crystals, i.e., those related to the algebra $U_q(so_5 \simeq sp_4)$.) The crossing model consists of three parts: (i) a finite digraph G, called the *support-graph*, depending only on the number n of colors; (ii) a set $\mathcal{F}$ of *feasible functions* on the vertices of G, depending on parameters $c_1, \ldots, c_n \in \mathbb{Z}_+$; and (iii) n families $\mathcal{E}_1, \ldots, \mathcal{E}_n$ of *transformations* $f \mapsto f'$ of feasible functions. (In fact, the crossing model can be obtained by a certain decomposition of the Gelfand-

Tsetlin pattern model [5].)

Our main working theorem asserts that the n-colored digraph $(\mathcal{F}, \mathcal{E}_1 \cup \ldots \cup \mathcal{E}_n)$ is isomorphic to the RAN-crystal $K(c_1, \ldots, c_n)$. We also show that any RAN-crystal has a zero-indegree vertex. Therefore, the crossing model produces precisely the set of RAN-crystals, or crystals of representations for $U_q(sl_{n+1})$. Our construction and proofs rely on Stembridge's local axioms and combinatorial arguments and do not appeal explicitly to powerful tools, such as the Path Model.

Then we take advantages from the description of RAN-crystals via the crossing model. The support-graph G consists of n pairwise disjoint subgraphs $G^1, \ldots, G^n$. They have the important property that the restrictions of $\mathcal{F}$ and $\mathcal{E}_1 \cup \ldots \cup \mathcal{E}_n$ to each G^i produces the particular n-colored crystal with parameters $c'_i = c_i$ and $c'_j = 0$ for $j \neq i$; we denote it by $K^i(c_i)$. (This is the crystal graph of the representation of $U_q(sl_{n+1})$ with highest weight $c_i\omega_i$.) Using this property, we show that the crystal $K = K(c_1, \ldots, c_n)$ contains a canonical subgraph K' isomorphic to the graph obtained from crystals $K^1(c_1), \ldots, K^n(c_n)$ by use of an n-dimensional analog of the diagonal-product operation $\bowtie$. We call K' the *skeleton* of K; it coincides with the whole crystal K when $n = 2$ (and is typically smaller when $n \geq 3$).

The feasible functions that are constant within each subgraph G^i $(i = 1, \ldots, n)$ are of most interest to us. We refer to the subset Π of vertices v of the crystal K that are determined by these functions as the *principal lattice* in it (each v one-to-one corresponds to an integer tuple $(a_1, \ldots, a_n)$ with $0 \leq a_i \leq c_i$). It turns out that there are exactly $|\Pi|$ maximal (connected) subcrystal of K with colors $1, \ldots, n-1$ and each of them intersects Π at exactly one vertex. A similar property takes place for Π and the maximal subcrystals with colors $2, \ldots, n$. (In [4] we precisely describe the structure of regular A_3-crystals. In this case we are able to characterize, for any two vertices $u, v \in \Pi$, the intersection of the maximal subcrystal with colors 1,2 containing u and the maximal subcrystal with colors 2,3 containing v.)

The crossing model enables us to demonstrate some other properties of RAN-crystals, as well. Also using it, one can derive infinite analogs of RAN-crystals, in which some or all maximal monochromatic paths are infinite (this generalizes the construction of infinite A_2-crystals in [2]).

This paper is organized as follows. Section 2 states Stembridge's axioms (in a slightly different form) for RAN-crystals and exhibits some known properties of crystals. Section 3 gives a brief review of results on A_2-crystals from [2]. Also, relying on a structural characterization of regular A_2-crystals, we explain in this section that any RAN-crystal has a zero-indegree vertex (Corollary 3.4). The crossing model is described in Section 4 (concerning the support-graph and feasible functions) and Section 5 (concerning transformations of feasible functions). The equivalence between the objects generated by the crossing model and the RAN-crystals is proved in Section 6 (Theorem 6.2). Section 7 introduces the principal lattice and the skeleton of a RAN-crystal and explains a relation between these objects. Infinite analogs of RAN-crystals and their properties are discussed in Section 8. The concluding Section 9 proves the above-mentioned relation between the principal lattice and $(n-1)$-colored subcrystals and calculates the parameters and multiplicities for these subcrystals (in Proposition 9.2 and Remark 5).

2 The definition and some properties of RAN-crystals

Throughout, by an n-colored digraph we mean a (finite or infinite) directed graph $K = (V(K), E(K))$ with vertex set $V(K)$ and with edge set $E(K)$ partitioned into n subsets $E_1, \ldots, E_n$. We say that an edge in E_i has *color* i and for brevity call it an *i-edge*. Stembridge [13] pointed out local graph-theoretic axioms that precisely characterize the set of regular simply-laced crystals among such K. The RAN-crystals K (which form a subclass of regular simply-laced crystals) are defined by axioms (A1)–(A5) below; we give the list of axioms in a slightly different, but equivalent, form compared with [13]. W.l.o.g., we assume that K is (weakly) *connected*, i.e., it is not representable as the disjoint union of two nonempty digraphs.

The first axiom concerns the structure of monochromatic subgraphs (V, E_i).

(A1) For $i = 1, \ldots, n$, each maximal connected subgraph (component) of (V, E_i) is a simple *finite* path, i.e., a sequence of the form $(v_0, e_1, v_1, \ldots, e_k, v_k)$, where $v_0, v_1, \ldots, v_k$ are distinct vertices and each e_i is an edge from v_{i-1} to v_i.

In particular, for each i, each vertex has at most one incoming i-edge and at most one outgoing i-edge, and therefore, one can associate to the set E_i partial inversible operator F_i acting on vertices: (u, v) is an i-edge if and only if F_i is applicable to u and $F_i(u) = v$. Since K is connected, one can use the operator notation to express any vertex via another one. For example, the expression $F_1^{-1}F_3^2F_2(v)$ (where F_p^{-1} stands for the partial operator inverse to F_p) determines the vertex w obtained from a vertex v by traversing 2-edge (v, v'), followed by traversing 3-edges (v', u) and (u, u'), followed by traversing 1-edge (w, u') in backward direction. Emphasize that every time we use such an operator expression in what follows, this automatically indicates that all involved edges do exist in K.

For convenience, we refer to a *maximal* monochromatic path with color i on the edges as an *i-line*. The i-line passing through a given vertex v (possibly consisting of the only vertex v) is denoted by $P_i(v)$, its part from the first vertex to v by $P_i^{\rm in}(v)$, and its part from v to the last vertex by $P_i^{\rm out}(v)$. The lengths of $P_i^{\rm in}(v)$ and of $P_i^{\rm out}(v)$ (i.e., the numbers of edges in these paths) are denoted by $t_i(v)$ and $h_i(v)$, respectively.

Axioms (A2)–(A5) tell us about interrelations of different colors i,j. Taken together, they are equivalent to saying that each component of the digraph $(V(K), E_i \cup E_j)$ forms a regular A_2-crystal when colors i,j are *neighboring*, i.e., $|i-j|=1$, and forms a regular $A_1 \times A_1$-crystal (the Cartesian product of two paths) otherwise.

The second axiom (which is of standard form for regular crystals) indicates possible changes of the head and tail part lengths of j-lines when one traverses an edge of another color i; these changes depend on the Cartan matrix.

(A2) For any two colors $i \ne j$ and for any edge (u,v) with color i, one holds $t_j(v) \le t_j(u)$ and $h_j(v) \ge h_j(u)$. Furthermore, the value $(t_j(v)-t_j(u))+(h_j(u)-h_j(v))$ is equal to the coefficient m_{ij} in the Cartan matrix M.

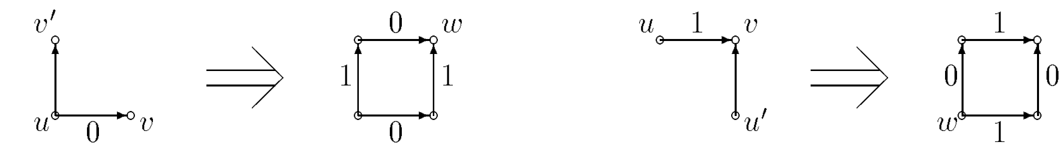
This can be rewritten in a more convenient form, as follows.

(1) When $|i-j|=1$, each i-line P contains a vertex r satisfying the following property: for any edge (u,v) in $P_i^{\rm in}(r)$, one holds $t_j(v)=t_j(u)-1$ and $h_j(v)=h_j(u)$, and for any edge (u',v') in $P_i^{\rm out}(r)$, one holds $t_j(v')=t_j(u')$ and $h_j(v')=h_j(u')+1$. When $|i-j|\ge 2$, for any edge (u,v) with color i, one holds $t_j(v)=t_j(u)$ and $h_j(v)=h_j(u)$.

Such a vertex r (which is unique) is called the *critical* vertex for P,i,j. In light of (A2), it is convenient to assign to each i-edge e *label* $\ell_{i,j}(e)$ taking value 0 if e occurs in the corresponding i-line *before* the critical vertex, and 1 otherwise. Emphasize that the critical vertex (and therefore, edge labels) on an i-line P depends on j, because the critical vertex on P with respect to the other color j' neighboring to i (i.e., $\{j,j'\}=\{i-1,i+1\}$ for $i \ne 1,n$) may be different.

The third axiom describes situations when for neighboring i,j, the operators F_i, F_j, as well as and their inverse ones, commute.

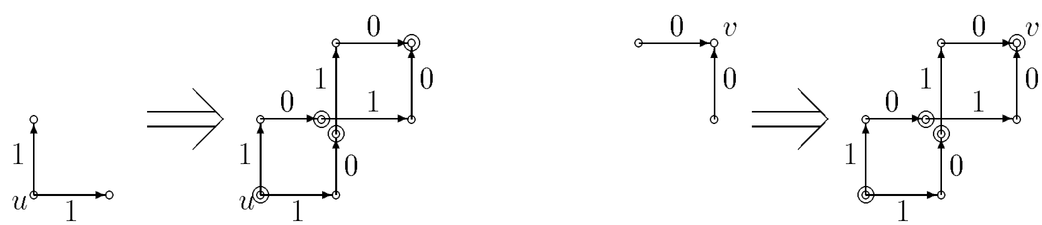
(A3) Let $|i-j|=1$. (a) If a vertex u has outgoing i-edge (u,v) and outgoing j-edge (u,v') and if $\ell_{i,j}(u,v)=0$, then $\ell_{j,i}(u,v')=1$ and $F_iF_j(u)=F_jF_i(u)$. Symmetrically: (b) if a vertex v has incoming i-edge (u,v) and incoming j-edge (u',v) and if $\ell_{i,j}(u,v)=1$, then $\ell_{j,i}(u',v)=0$ and $F_i^{-1}F_j^{-1}(v)=F_j^{-1}F_j^{-1}(v)$. (See the picture.)



Note that for each "square" u, v, v', w, where $v = F_i(u)$, $v' = F_j(u)$ and $w = F_j(v) = F_i(v')$, the trivial relations $h_j(u) = h(v') + 1$ and $h_j(v) = h_j(w) + 1$ imply that the opposite edge (u, v) and (v', w) have equal labels $\ell_{i,j}$; similarly $\ell_{j,i}(u, v') = \ell_{j,i}(v, w)$. A simple but important consequence of (A3) is that for neighboring i, j, if v is the critical vertex on an i-line w.r.t. color j, then v is also the critical vertex on the j-line passing through v w.r.t. color i, i.e., we can speak of common critical vertices for the pair $\{i, j\}$. (Indeed, if a vertex v has incoming i-edge (u, v) with $\ell_{i,j}(u, v) = 0$ and outgoing j-edge (v, w), then we have $h_j(u) = h_j(v) \geq 1$, and hence u has outgoing j-edge (u, v'). By (A3), $w = F_i(v')$ and $\ell_{j,i}(u, v') = 1$; the latter implies $\ell_{j,i}(v, w) = 1$. Symmetrically, if v has outgoing i-edge e with $\ell_{i,j}(e) = 1$ and incoming j-edge e', then $\ell_{j,i}(e') = 0$.)

The fourth axiom points out situations when for neighboring i, j, the operators F_i, F_j and their inverse ones "remotely commute" (one says that they satisfy the Verma relation of length 4).

(A4) Let $|i - j| = 1$. (i) If a vertex u has outgoing edges with color i and color j and if each edge is labeled 1 (w.r.t. the other color), then $F_i F_j^2 F_i(u) = F_j F_i^2 F_j(u)$. Symmetrically: (ii) if v has incoming edges with color i and color j and if both are labeled 0, then $F_i^{-1}(F_j^{-1})^2 F_i^{-1}(v) = F_j^{-1}(F_i^{-1})^2 F_j^{-1}(v)$. (See the picture.)



One can show that the labels w.r.t. i, j of all involved edges are determined uniquely, just as indicated in the above picture (where the circles indicate the critical vertices).

The final axiom says that the operators F_i, F_j, as well as their inverse ones, always commute for non-neighboring colors i, j.

(A5) Let $|i-j| \geq 2$. If a vertex v has outgoing i-edge and outgoing j-edge, then $F_iF_j(u) = F_jF_i(u)$. Symmetrically: if v has incoming i-edge and incoming j-edge, then $F_i^{-1}F_j^{-1}(v) = F_j^{-1}F_j^{-1}(v)$.

From (A5) and (A2) it easily follows that for $|i-j| \geq 2$, each component of the 2-colored subgraph $(V(K), E_i \cup E_j)$ is the Cartesian product of a path with color i and a path with color j, i.e., it is viewed as a rectangular grid.

Next we review some known properties of RAN-crystals.

A vertex v of a finite or infinite digraph G is called the *source* (resp. *sink*) if any inclusion-wise maximal path begins (resp. ends) at v; in particular, v has zero indegree (resp. zero outdegree). When such a vertex exists, we say that G *has source* (resp. *has sink*). The importance of simply-laced crystals with source is emphasized by a result of Stembridge in [13]; in the A_n case it reads as follows:

(2) For any n-tuple $\mathbf{c} = (c_1, \ldots, c_n)$ of nonnegative integers, there exists precisely one RAN-crystal K with source s such that $h_i(s) = c_i$ for $i = 1, \ldots, n$. This K is the crystal graph of the irreducible $U_q(sl_{n+1})$-module of highest weight $\mathbf{c}$.

(Hereinafter we prefer to denote n-tuples in bold.) We say that $\mathbf{c}$ is the tuple of *parameters* of such a K and denote K by $K(\mathbf{c})$. If we reverse the edges of K while preserving their colors, we again obtain a RAN-crystal (since (A1)–(A5) remain valid for it). It is called *dual* of K and denoted by K^*.

Another useful property, indicated in [13] for simply-laced crystals with a nonsingular Cartan matrix, is easy.

(3) A RAN-crystal K is *graded* for each color i, which means that for any cycle ignoring the orientation of edges, the number of i-edges in one direction is equal to the number of i-edges in the other direction. In particular, K is acyclic and has no parallel edges.

(Indeed, associate to each vertex v the n-vector $wt(v)$ whose j-th entry is equal to $h_j(v) - t_j(v)$, $j = 1, \ldots, n$. Then for each i-edge (u,v), the difference $wt(u) - wt(v)$ coincides with the i-th row vector m_i of the Cartan matrix M, in view of axiom (A2) and the obvious equality $h_i(u) - t_i(u) = h_i(v) - t_i(v) + 2$. So under the map $wt : V(K) \to \mathbb{R}^n$, the edges of each color i correspond to parallel translations of one and the same vector m_i, and now (3) follows from the fact that the vectors $m_1, \ldots, m_n$ are linearly independent.)

In general a regular simply-laced crystal need not have source and/or sink; it may be infinite and may contain directed cycles. One simple result on regular simply-laced crystals in [13] remains valid for more general digraphs, in particular, for a larger class of crystals of representations.

Proposition 2.1 *Let G be an (uncolored) connected and graded digraph with the following property ($*$): for any vertex v and any edges e, e' entering v, there exist two paths from some vertex w to v such that one path contains e and the other contains e'. Then either G has source or all maximal paths in G are infinite in backward direction.*

(A similar assertion concerns sinks and infinite paths in forward direction. Also for any RAN-crystal, condition ($*$) in the proposition follows from axioms (A3)–(A5).)

Proof Suppose this is not so. Then, since G is connected and acyclic (as it is graded), there exists a vertex v and two paths P, P' ending at v such that P begins at a zero-indegree vertex s, while P' is either infinite in backward direction or begins at a zero-indegree vertex different from s. Let such v, P, P' be chosen so that the length $|P|$ of P is minimum. Then the last edges $e = (u, v)$ and $e' = (u', v)$ of P and P', respectively, are different. By ($*$), there is a vertex w, a path Q from w to v containing e and a path Q' from w to v containing e'. Extend Q to a maximal path $\overline{Q}$ ending at v. Three cases are possible: (i) $\overline{Q}$ is infinite in backward direction; (ii) $\overline{Q}$ begins at a (zero-indegree) vertex different from s; and (iii) $\overline{Q}$ begins at s. In cases (i),(ii), we come to a contradiction with the minimality of P by taking the vertex u and the part of P from s to u. And in case (iii), there is a path $\overline{Q}'$ from s to v that contains e'. Since G is graded, $|\overline{Q}'| = |P|$. Then we again get a contradiction with the minimality of P by taking u' and the part of $\overline{Q}'$ from s to u'. ■

(The fact that G is graded is important. Indeed, let the vertices of G be s and u_i, v_i for all integer $i \geq 0$, and let the edges be (s, u_0) and $(u_i, u_{i+1}), (v_{i+1}, v_i), (u_i, v_i)$ for all i. This G satisfies ($*$), the vertex s has zero indegree, and the path on the vertices v_i is infinite in backward direction. One can also construct a locally finite graph satisfying ($*$) and having many zero-indegree vertices.)

Our crossing model will generate n-colored graphs satisfying axioms (A1)–(A5); moreover, it generates one RAN-crystal with source for each parameter tuple $\mathbf{c} \in \mathbb{Z}_+^n$. In light of (2) and Proposition 2.1, a reasonable question is whether *every* RAN-crystal has source and sink (or, equivalently, is finite). The question will be answered affirmatively in the next section, thus implying that the crossing model gives the whole set of RAN-crystals.

As a consequence of the crossing model, we will also observe the following anti-symmetrical property of a RAN-crystal K: if we reverse the numeration of colors (regarding each color i as $n - i + 1$) in the dual crystal K^*, then the resulting crystal is isomorphic to K. In other words, $h_i(s_K) = t_{n-i+1}(\overline{s}_K)$ for $i = 1, \ldots, n$, where s_K and $\overline{s}_K$ are the source and sink of K, respectively.

Finally, recall that a Gelfand-Tsetlin pattern [5], or a *GT-pattern* for short, is a triangular array $X = (x_{ij})_{1 \le j \le i \le n}$ of integers satisfying $x_{ij} \ge x_{i-1,j}, x_{i+1,j+1}$ for all i, j. Given a weakly decreasing n-tuple $\mathbf{a} = (a_1 \ge \cdots \ge a_n)$ of nonnegative integers, one says that X *is bounded* by $\mathbf{a}$ if $a_j \ge x_{n,j} \ge a_{j+1}$ for $j = 1, \ldots, n$, letting $a_{n+1} := 0$. It is known that GT-patterns, as well as the corresponding semi-standard Young tableaux, are closely related to crystals of representations for $U_q(sl_{n+1})$ (cf. [1, 7, 9, 12]). More precisely,

(4) for any $\mathbf{c} \in \mathbb{Z}_+^n$, there is a bijection between the vertex set of the RAN-crystal $K(\mathbf{c})$ and the set of GT-patterns bounded by the n-tuple $\mathbf{c}^\Sigma$, defined by $c_j^\Sigma := \sum_{k=j}^n c_k$ for $j = 1, \ldots, n$.

As mentioned in the Introduction, there is a correspondence between GT-patterns and feasible functions in the crossing model; it will be exposed in Proposition 4.1.

3 Properties of A_2-crystals

In this section we give a brief review of certain results from [2], important for us later on, for the simplest case $n = 2$, namely, for regular A_2-crystals, or *RA2-crystals* for short. They describe the combinatorial structure (formation) of such crystals and demonstrate some additional properties.

A RA2-crystal K is defined by axioms (A1)–(A4) with $\{i, j\} = \{1, 2\}$ (since (A5) becomes redundant). It turns out that these crystals can be produced from elementary 2-colored crystals by use of a certain operation of replicating and gluing together. This operation can be introduced for arbitrary finite or infinite graphs as follows.

Consider graphs $G = (V, E)$ and $H = (V', E')$ with distinguished vertex subsets $S \subseteq V$ and $T \subseteq V'$. Take $|T|$ disjoint copies of G, denoted as G_t ($t \in T$), and $|S|$ disjoint copies of H, denoted as H_s ($s \in S$). We glue these copies together in the following way: for each $s \in S$ and each $t \in T$, the vertex s in G_t is identified with the vertex t in H_s. The resulting graph, consisting of $|V||T| + |V'||S| - |S||T|$ vertices and $|E||T| + |E'||S|$ edges, is denoted by $(G, S) \bowtie (H, T)$.

In our special case the role of G and H is played by 2-colored digraphs R and L viewed as triangular parts of square grids. More precisely, R depends on a parameter $c_1 \in \mathbb{Z}_+$ and its vertices v correspond to the integer points (i, j) in the plane such that

$0 \leq j \leq i \leq c_1$. The vertices v of L, depending on a parameter $c_2 \in \mathbb{Z}_+$, correspond to the integer points (i, j) such that $0 \leq i \leq j \leq c_2$. We say that v has the *coordinates* (i, j) in the sail. The edges with color 1 in these digraphs correspond to all possible pairs $((i, j), (i + 1, j))$, and the edges with color 2 to the pairs $((i, j), (i, j + 1))$. We call R the *right sail* of size c_1, and L the *left sail* of size c_2.

It is easy to check that R satisfies axioms (A1)–(A4) and is just the crystal $K(c_1, 0)$, and that the set of critical vertices in R coincides with the *diagonal* $D_R = \{(i, i) : i = 0, \ldots, c_1\}$. Similarly, $L = K(0, c_2)$, and the set of critical vertices in it coincides with the diagonal $D_L = \{(i, i) : i = 0, \ldots, c_2\}$. These diagonals are just taken as the distinguished subsets in these digraphs. The vertices in D_R (D_L) are ordered in a natural way, according to which (i, i) is referred as the i-th critical vertex in R (L).

We refer to the digraph obtained by use of operation $\bowtie$ in this case as the *diagonal-product* of R and L, and for brevity write $R \bowtie L$, omitting the distinguished subsets. The edge colors in the resulting graph are inherited from R and L. Using the above numeration in the diagonals, we may speak of p-th right sail in $R \bowtie L$, denoted by R_p. Here $0 \leq p \leq c_2$, and R_p is the copy of R corresponding to the vertex (p, p) of L. In a similar way, one defines q-th left sail L_q in $R \bowtie L$ for $q = 0, \ldots, c_1$. The common vertex of R_p and L_q is denoted by $v_{p,q}$.

One can check that $R \bowtie L$ has source and sink and satisfies axioms (A1)–(A4). Moreover, it is exactly the crystal $K(\mathbf{c})$. (The order of entries in $\mathbf{c}$ here is different from that in [2] but this is not important.) The critical vertices in it are just $v_{p,q}$ for all p, q, the source is $v_{0,0}$ and the sink is v_{c_1,c_2}. The case $c_1 = 1$ and $c_2 = 2$ is illustrated in the picture; here the critical vertices are indicated by circles, 1-edges by horizontal arrows, and 2-edges by vertical arrows.

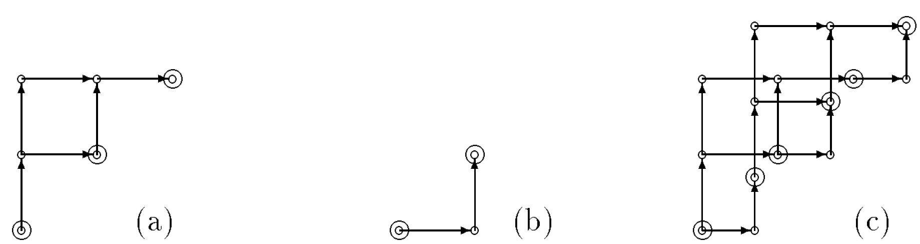


Figure 1: (a) $K(0, 2)$, (b) $K(1, 0)$, (c) $K(1, 0) \bowtie K(0, 2)$.

It is shown that the above construction gives *all* RA2-crystals.

Theorem 3.1 [2] *Any RA2-crystal K is representable as $K(a,0) \bowtie K(0,b)$ for some $a, b \in \mathbb{Z}_+$.*

So K is finite, and the set of RA2-crystals is exactly $\{K(\mathbf{c}) : c \in \mathbb{Z}^2_+\}$.

A useful consequence of the above construction is that the vertices v of K one-to-one correspond to the quadruples $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ of integers such that

(5) (i) $0 \le \alpha_2 \le \alpha_1 \le c_1$, (ii) $0 \le \beta_1 \le \beta_2 \le c_2$, and (iii) at least one of the equalities $\alpha_2 = \alpha_1$ and $\beta_1 = \beta_2$ takes place,

and each i-edge ($i = 1, 2$) corresponds to the increase by 1 of one of α_i, β_i, subject to maintaining (5).

Under this correspondence, if $\beta_1 = \beta_2$ then v occurs in the right sail with number β_1 and has the coordinates (α_1, α_2) in it, while if $\alpha_2 = \alpha_1$ then v occurs in the left sail with number α_1 and has the coordinates (β_1, β_2). In particular, a critical vertex $v_{p,q}$ corresponds to (q, q, p, p).

Remark 1. The representation of the vertices of K as the above quadruples satisfying (5) gives rise to constructing the crossing model for the simplest case $n = 2$, as we explain in the next section. A more general numerical representation (which is beyond our consideration in this paper) does not impose condition (iii) in (5). In this case the admissible transformations of quadruples $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ (giving the edges of a digraph on the quadruples) are assigned as follows. For $\Delta := \min\{\alpha_1 - \alpha_2, \beta_2 - \beta_1\}$, we choose one of $\alpha_1, \alpha_2, \beta_1, \beta_2$ and increase it by 1 unless this increase violates (i) or (ii) in (5) or changes Δ. One can see that the resulting digraph Q is the disjoint union of $1 + \min\{c_1, c_2\}$ RA2-crystals, namely, $K(c_1 - \Delta, c_2 - \Delta)$ for $\Delta = 0, \ldots, \min\{c_1, c_2\}$ (one can check that Q is the tensor product of crystals (sails) $K(c_1, 0)$ and $K(0, c_2)$).

One more useful result in [2] is the following.

Proposition 3.2 *Part (ii) of axiom (A4) for RAN-crystals is redundant. Furthermore, axiom (A4) itself follows from (A1)–(A3) if we add the condition that each component of $(V, E_i \cup E_j)$ with $|i - j| = 1$ has exactly one zero-indegree vertex.*

Finally, consider an arbitrary RAN-crystal K. For a color i, let H_i denote the operator on $V(K)$ that brings a vertex v to the end vertex of the path $P_i(v)$, i.e., $H_i(v) = F_i^{h_i(v)}(v)$ (letting $F_i^0 = \mathrm{id}$). We observe that

(6) for neighboring colors i, j and a vertex v, if $h_i(v) = 0$ then the vertex $w = H_i H_j(v)$ satisfies $h_i(w) = h_j(w) = 0$.

Indeed, the RA2-subcrystal with colors i,j in K that contains v is $K(c_i,c_j)$ for some $c_i,c_j\in\mathbb{Z}_+$. Represent v as quadruple $q=(\alpha_i,\alpha_j,\beta_i,\beta_j)$ in (5) (with (i,j) in place of (1,2)). Then $h_i(q)=0$ implies $\alpha_i=c_i$ and $\beta_i=\beta_j$. One can see that applying H_j to q results in the quadruple $q'=(c_i,c_i,\beta_i,c_j)$ and applying H_i to q' results in (c_i,c_i,c_j,c_j). This gives (6).

Using (6), we can show the following important property of RAN-crystals.

Proposition 3.3 *Any RAN-crystal K has a zero-outdegree vertex.*

Proof For a vertex u, let $p(u)$ be the maximum integer p such that $h_i(u)=0$ for $i=1,\dots,p-1$. Assuming $p(u)<n+1$, we claim that the vertex $w=H_1H_2\dots H_{p(u)}(u)$ satisfies $p(w)>p(u)$, whence the result will immediatelly follow. (In other words, by applying the operator $\overline{H}_n\overline{H}_{n-1}\dots\overline{H}_1$ to an arbitrary vertex, where $\overline{H}_i$ stands for $H_1H_2\dots H_i$, we get a zero-outdegree vertex.)

Indeed, let $p=p(u)$. For the vertex $v_p:=H_p(u)$, we have $h_p(v_p)=0$ and $h_i(v_p)=h_i(u)$ for all $i\ne p-1,p+1$ (since colors p,i commute), while $h_{p-1}(v_p)$ may differ from $h_{p-1}(u)$. So $h_i(v_p)=0$ for $i=1,\dots,p-2,p$. Similarly, the vertex $v_{p-1}:=H_{p-1}(v_p)$ satisfies $h_{p-1}(v_{p-1})=0$ and $h_i(v_{p-1})=h_i(v_p)$ for all $i\ne p-2,p$. Moreover, applying (6) to $v=u$, $i=p-1$ and $j=p$, we obtain $h_p(v_{p-1})=0$. So $h_i(v_{p-1})=0$ for $i=1,\dots,p-3,p-1,p$. On the next step, in a similar fashion one shows that $v_{p-2}:=H_{p-2}(v_{p-1})$ satisfies $h_i(v_{p-2})=0$ for all $i\in\{1,\dots,p\}\setminus\{p-3\}$, and so on. Then the final vertex $v_1:=H_1\dots H_p(u)$ in the process has the property $h_i(v_1)=0$ for $i=1,\dots,p$, as required in the claim. ∎

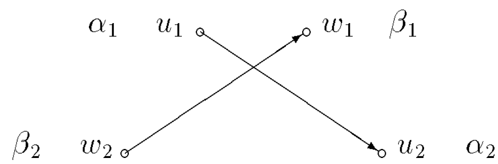
Also K has a zero-indegree vertex (since Proposition 3.3 can be applied to the dual crystal K^*). This together with (2) and Proposition 2.1 gives the following.

Corollary 3.4 *Every RAN-crystal K is finite and has source and sink. Therefore, $K=K(\mathbf{c})$ for some $c\in\mathbb{Z}^n_+$.*

4 Description of the crossing model

As mentioned in the Introduction, the *crossing model* $\mathcal{M}_n$ for crystals with n colors consists of three ingredients: (i) a certain digraph $G=(V(G),E(G)$ depending only on the number n of colors, called the *support-graph* (the *structural* part of $\mathcal{M}$); (ii) a certain set $\mathcal{F}=\mathcal{F}(\mathbf{c})$ of nonnegative integer-valued functions on $V(G)$, called *feasible functions*, depending on an n-tuple of parameters $\mathbf{c}\in\mathbb{Z}^n_+$ (the *numerical* part); and (iii) n partial operators acting on $\mathcal{F}$, called *moves* (the *operator* part). (The set $\mathcal{F}$ will correspond to the vertex set of the crystal with the parameters $\mathbf{c}$, and the moves to the edges of this crystal.) Parts (i) and (ii) are described in this section, and part (iii) in the next one. To avoid a possible mess when both a crystal and the support-graph are considered simultaneously, we will refer to a vertex of the latter graph as a *node*.

To explain the idea, we first consider the simplest case $n = 2$ and a 2-colored crystal $K = K(\mathbf{c})$. The model $\mathcal{M}_2$ is constructed by relying on encoding (5) of the vertices of K (moves in this model will be defined as in a general case). The support-graph G is formed by two disjoint edges (u_1, u_2) and (w_2, w_1) (which are related to the elementary crystals, or sails, $K(c_1, 0)$ and $K(0, c_2)$). A feasible function f on $V(G)$ takes values $f(u_1) = \alpha_1$, $f(u_2) = \alpha_2$, $f(w_1) = \beta_1$, $f(w_2) = \beta_2$ for α_i, β_i as in (5). So the direction of each edge e of G indicates the corresponding inequality to be imposed on the values of any feasible function f on the end nodes of e, and each f one-to-one corresponds to a vertex of K. The model is illustrated on the picture:



Note that each admissible quadruple $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ generates the GT-pattern X of size 2 (see the end of Section 3), defined by $x_{11} := \alpha_1 + \beta_1$, $x_{21} := \beta_2 + c_1$ and $x_{22} := \alpha_2$ (see the picture). This pattern is bounded by $\mathbf{c}^{\Sigma} = (c_1 + c_2, c_2)$.

$$\begin{array}{ccc} & \alpha_1 + \beta_1 & \\ & & \\ c_1 + \beta_2 & & \alpha_2 \end{array}$$

Next we start describing the model for an arbitrary n. The "simplest" case of an n-colored crystal $K = K(\mathbf{c})$ arises when all entries in $\mathbf{c} = (c_1, \ldots, c_n)$ are zero except for one entry c_k. In this case we say that K is the k-th *base crystal* of size c_k and denote it by $K_n^k(c_k)$.

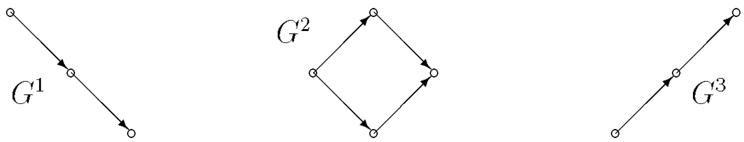
4.1. The support-graph of $\mathcal{M}_n$. This digraph G is formed as the disjoint union of digraphs $G^k = G_n^k$ that we construct as the support-graphs for base crystals K_n^k. Each G^k is viewed as a square grid of size $k - 1$ by $n - k$. More precisely, the node set of G^k consists of the nodes

(7) $\quad v_i^k(j)$, where j runs from 1 to $n - k + 1$, and i runs from j to $j + k - 1$,

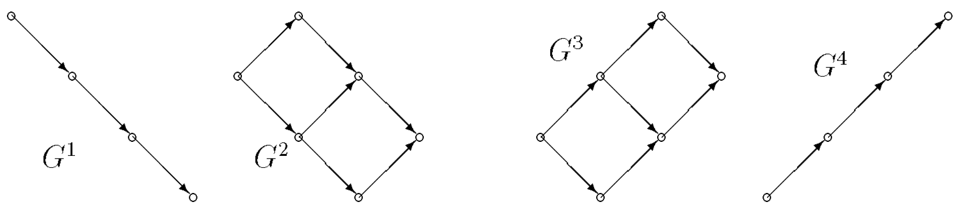
and its edges are all possible pairs of the form

(8) $\quad (v_i^k(j), v_{i+1}^k(j+1)) \quad \text{or} \quad (v_i^k(j), v_{i-1}^k(j))$.

To obtain a visualization more convenient for further use, we take vectors $\rho = (-\sqrt{2}/2, -\sqrt{2}/2)$ and $\sigma = (1,0)$ in the plane and dispose each vertex $v_i^k(j)$ at the point $(i-1)\rho + (j-1)\sigma$. For example, when $n=3$, the graphs G^1, G^2, G^3 are viewed as:

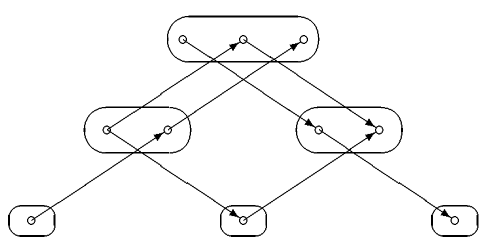


(where the topmost node $v_1^k(1)$ is disposed every time at the origin $(0,0)$), and when $n=4$, the graphs G^1, G^2, G^3, G^4 are viewed as:

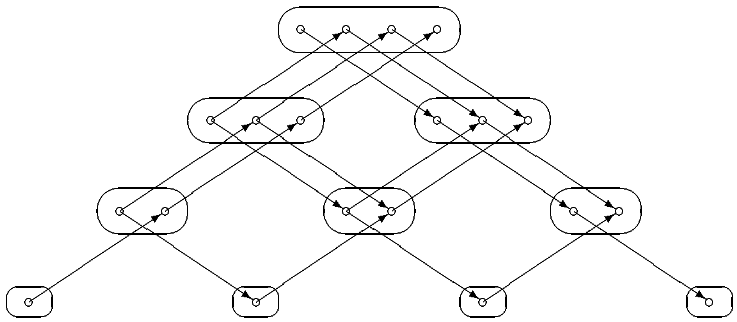


Under this visualization, (the image of) G^k looks like a rhombic grid (unless $k = 1, n$), with edges oriented from left to right, and we will refer to its interior faces as (little) *rhombi*. It has source at the leftmost node $v_k^k(1)$, denoted by left^k, and sink at the rightmost node $v_{n-k+1}^k(n-k+1)$, denoted by right^k. Each node $v = v_i^k(j)$ has at most four incident edges, namely, $(v_{i-1}^k(j-1), v)$, $(v_{i+1}^k(j), v)$, $(v, v_{i-1}^k(j))$, $(v, v_{i+1}^k(j+1))$, and when such edges do exist, we refer to them as the *NW-*, *SW-*, *NE-*, and *SE-edges* for v, and denote by $e^{\text{NW}}(v)$, $e^{\text{SW}}(v)$, $e^{\text{NE}}(v)$, and $e^{\text{SE}}(v)$, respectively.

To avoid overlapping edges in the visualization of the whole support-graph G, we slightly shift each G^k to the right, by adding to (the images of all nodes of) G^k the vector $(k\epsilon, 0)$ for a small real $\epsilon > 0$. Here is how G is viewed for $n=3$:



and for $n=4$:



In these pictures each group of "related" nodes is surrounded by an oval. It is formed by the nodes $v_i^k(j)$ with the same i and the same j, namely, by $v_i^{i-j+1}(j), \dots, v_i^{n-j+1}(j)$; we denote this group by $V_i(j)$ and call it a *multinode* of G. Sometimes we will refer to usual nodes of G as *ordinary* ones. Under the above visualization, for each $i = 1, \dots, n$, the multinodes $V_i(1), \dots, V_i(i)$ occur in the same horizontal line, forming the i-th *level* of G; the direction from left to right in this level determines the order on the multinodes $V_i(j)$ by increasing j, and the order on the ordinary nodes $v_i^k(j)$ in each $V_i(j)$ by increasing k.

4.2. Weights of nodes. We consider nonnegative integer-valued functions f on $V(G)$ and refer to the value $f(v)$ as the *weight* of a node v. As mentioned above, the numerical part of $\mathcal{M}_n$ consists of a certain set $\mathcal{F}$ of such functions, called feasible ones; this set depends on the parameter-tuple $\mathbf{c}$. A *feasible* function f is defined by three conditions. The first condition requires f be weakly decreasing on the edges (the *monotonicity condition*), and the second one requires f be bounded by the parameters (the *boundary condition*):

(9) $\quad f(u) \ge f(v)$ for each edge $e = (u, v)$ of G;

(10) $\quad f(v_i^k(j)) \le c_k$ for each node $v_i^k(j)$.

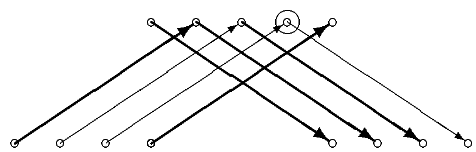
(In light of (9), condition (10) can be replaced by $f(\mathrm{left}^k) \le c_k$ for $k = 1, \dots, n$.)

To state the third condition, consider a multinode $V_i(j)$ in a level $i < n$. It is formed by nodes $v_i^p(j), \dots, v_i^q(j)$ (in this order) for some $1 \le p < q \le n$. A node $v = v_i^k(j)$ is connected with nodes of level $i+1$ by the incoming edge $e^{\mathrm{SW}}(v)$ when $p < k \le q$ and by the outgoing edge $e^{\mathrm{SE}}(v)$ when $p \le k < q$. Let us say that these edges form the *roof* at v. The roof consists of the single edge $e^{\mathrm{SW}}(v)$ for the first node v $(k = p)$, and of the single edge $e^{\mathrm{SE}}(v)$ for the last node v $(k = q)$.

For a function f on $V(G)$ and an edge (u, v), denote the difference $f(u) - f(u)$ by $\Delta f(u, v)$, and say that this edge is *tight* if $\Delta f(u, v) = 0$. Then the final condition on f to be feasible is:

(11) for each multinode $V_i(j)$ with $i < n$, there exists a node v in it such that the edge $e^{\rm SE}(u)$ is tight for all nodes $u \in V_i(j)$ preceeding v, while the edge $e^{\rm SW}(u')$ is tight for all nodes u' succeeding v.

We say that such a v satisfies the *switch condition*. There may be several such nodes in $V_i(j)$ (then they go in succession) and the *first* node $v = v_i^k(j)$ among them (i.e., with k minimum) is called the *switch-node* in $V_i(j)$. (We shall see later that the *forward moves* in the model, related to acting the operators F_i, handle just switch-nodes, while the *backward moves*, related to acting F_i^{-1}, use *last* nodes satisfying the switch condition.) We illustrate (11) on the picture where tight edges are drawn bold and only one node, marked by a circle, satisfy the switch condition.



So a weight function f is feasible if it satisfies (9)–(11).

That the feasible functions one-to-one correspond to the vertices of the crystal $K(\mathbf{c})$ can be shown by two methods. A direct proof of the assertion that $\mathcal{F}$ along with the moves obeys axioms (A1)–(A5) will be given in Section 6. Another way is to show a correspondence to GT-patterns and use property (4). For $p, q \in \{1, \ldots, n\}$ with $p \le q$, let $c[p:q]$ denote $c_p + \ldots + c_q$. As before, c_j^Σ stands for $c[j:n]$.

Proposition 4.1 *For $1 \le j \le i \le n$, define*

$$x_i(j) := \overline{f}_i(j) + c[1 : i-j], \tag{12}$$

where $\overline{f}_i(j)$ denotes the sum of values of f on all nodes in $V_i(j)$. This gives a bijection between the set of feasible functions f and the set of GT-patterns $X = (x_i(j))$ of size n bounded by $\mathbf{c}^\Sigma$.

(Note that this can also be regarded as an alternative proof of property (4), via the crossing model.)

Proof For a weight function f satisfying (9) and (10) (but not necessarily (11)), define X by (12). Each multinode $V_n(j)$ in the bottom level consists of the single node $v = v_n^{n-j+1}(j)$, and we have $0 \le f(v) \le c_{n-j+1}$ (since v is in G^{n-j+1}). Therefore, $x_n(j)$ is between $c[1:n-j]$ and $c[1:n-j+1]$.

The inequality $x_i(j) \geq x_{i+1}(j+1)$ is implied by non-increasing f on the edges from $V_i(j)$ to $V_{i+1}(j+1)$ and by the fact that the term in (12) concerning $\mathbf{c}$ is the same for (i,j) and $(i+1,j+1)$. The inequality $x_{i+1}(j) \geq x_i(j)$ follows from non-increasing f on the edges from $V_{i+1}(j)$ to $V_i(j)$ and from the inequality $c_{i-j+1} \geq f(v_i^{i-j+1}(j))$. Thus, X is a GT-pattern bounded by $\mathbf{c}^\Sigma$.

Conversely, let X be a GT-pattern bounded by $\mathbf{c}^\Sigma$. We construct the desired f step by step, starting from the bottom level. For each node $v = v_n^{n-j+1}(j)$ (forming $V_n(j)$), we define $f(v) := x_n(j) - c[1:n-j]$. This value is nonnegative, and (12) holds for $i = n$.

Now consider a multinode $V_i(j)$ with $i < n$, assuming that f is already defined for all levels below i and satisfies (9)–(12) for the nodes in these levels and the edges between them. We show that f can be properly extended to the nodes in $V_i(j)$ and that such an extension is unique. Consider an intermediate node v in $V_i(j)$ (existing when $i < n-1$). Its roof consists of two edges, say, $(u,v),(v,w)$. The weights of u and w (already defined) satisfy $f(u) \geq f(w)$ (since v,u,v are contained in a rhombus of some base subgraph, and therefore, $f(u) \geq f(v') \geq f(w)$, where v' is the node of this rhombus in the level $i+2$). The maximum possible weight of v not violating (9) is $f(u)$, while the minimum possible weight is $f(w)$. In its turn, the roof of the first node v in $V_i(j)$ consists only of the edge $e^{\mathrm{SE}}(v) = (v,w)$, and the maximum possible weight of v is c_{i-j+1} (since v,w belong to G^{i-j+1}), while the minimum one is $f(w)$. And the roof of the last node v in $V_i(j)$ consists of the edge $e^{\mathrm{SW}}(v) = (u,v)$, the maximum possible weight of v is $f(u)$, and the minimum one is zero.

Thus, the maximum assignment of weights in all nodes of $V_i(j))$ would give $\overline{f}_i(j) = \overline{f}_{i+1}(j) + c_{i-j+1}$, implying $x_i(j) \leq \overline{f}_i(j) + c[1:i-j]$, in view of $x_i(j) \leq x_{i+1}(j) = \overline{f}_{i+1}(j) + c[1:i-j+1]$. And the minimum assignment would give $\overline{f}_i(j) = \overline{f}_{i+1}(j+1)$, implying $x_i(j) \geq \overline{f}_i(j) + c[1:i-j]$, in view of $x_i(j) \geq x_{i+1}(j+1) = \overline{f}_{i+1}(j+1) + c[1:i-j]$. Therefore, starting with the maximum assignment, and decreasing step by step the weights of nodes in $V_i(j)$ according to the order there, one can always correct the weights so as to satisfy (11) and (12), while maintaining (9) and (10). Moreover, (11) implies that such weights within $V_i(j)$ are determined uniquely. Eventually, after handling level1 1, we obtain the desired function f on $V(G)$. ■

5 Moves in the model

So far, we have dealt with the case of nonnegative *upper* bounds (parameters) $c_1, \ldots, c_n$ and zero *lower* bounds, i.e., for any feasible function f, the weight $f(v)$ of each node v of a k-th base subgraph lies between 0 and c_k. However, it is advantageous to extend the setting, by admitting nonzero lower bounds (e.g., for purposes of Section 8 where the model is extended to produce crystals with possible infinite monochromatic paths).

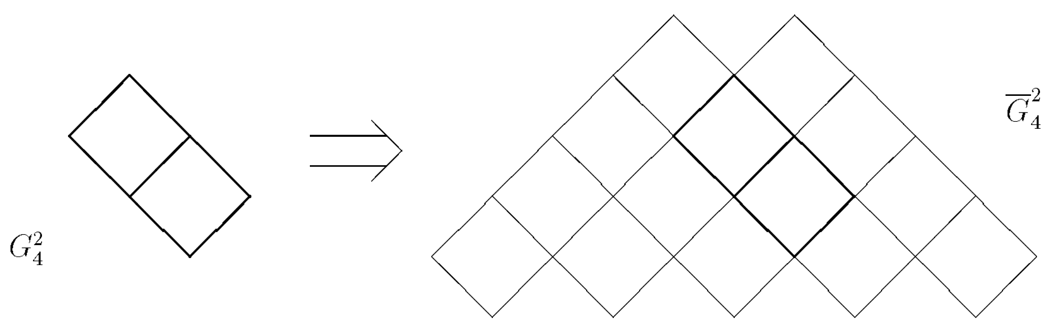
Formally: for $\mathbf{c}, \mathbf{d} \in \mathbb{Z}^n$ with $\mathbf{c} \geq \mathbf{d}$, we define a feasible function to be an integer function f on $V(G)$ satisfying (9), (11) and the relation

$$d_k \leq f(v_i^k(j)) \leq c_k \qquad \text{for all } k, i, j, \tag{13}$$

instead of (10). The set of feasible functions for $(\mathbf{c}, \mathbf{d})$ is denoted by $\mathcal{F}(\mathbf{c}, \mathbf{d})$. Clearly the numerical part of the model remains equivalent when for any k, we add a constant to both c_k and d_k and accordingly add this constant to any weight function for G^k. In particular, $\mathcal{F}(\mathbf{c}, \mathbf{d})$ is isomorphic to $\mathcal{F}(\mathbf{c} - \mathbf{d}, \mathbf{0})$, and when $\mathbf{d} = \mathbf{0}$, $\mathcal{F}(\mathbf{c}, \mathbf{d})$ coincides with $\mathcal{F}(\mathbf{c})$ as above.

Now we start describing the desired transformations of functions in $\mathcal{F}(\mathbf{c}, \mathbf{d})$, or moves (that will correspond to edges of the crystal $K(\mathbf{c} - \mathbf{d})$). Each transformation is performed only within one level i, in which case it is called an *i-move*. We need some additional definitions, notation and construction.

First of all, to simplify our description and technical details later, we extend each G^k by adding *extra* nodes and edges. More precisely, in the extended digraph $\overline{G}^k$, the node set consists of elements $v_i^k(j)$ for $(i, j) = (0, 0)$ and for all i, j such that $0 \leq i, j \leq n + 1$ and $j \leq i + 1$, except for $(i, j) = (n + 1, 0)$. The edge set of $\overline{G}^k$ consists of all possible pairs of the form $(v_i^k(j), v_{i-1}^k(j))$ or $(v_i^k(j), v_{i+1}^k(j + 1))$ (as before). An instance is illustrates in the picture; here $n = 4$, $k = 2$, and the thick lines indicate the edges of the original graph.



The disjoint union of these $\overline{G}^k$ gives the *extended support-graph* $\overline{G}$. It possesses the property that the original multinodes becomes balanced, in the sense that for the set J of index pairs (i, j) satisfying $1 \leq j \leq i \leq n$, the extended multinodes $\overline{V}_i(j)$ contain the same number n of nodes (these are $v_i^1(j), \ldots, v_i^n(j)$). Also each node $v = v_i^k(j)$ of $\overline{G}$ with $(i, j) \in J$ has exactly four incident edges, namely, $e^{\mathrm{NW}}(v)$, $e^{\mathrm{SW}}(v)$, $e^{\mathrm{NE}}(v)$, and $e^{\mathrm{SE}}(v)$.

Each feasible function on $V(G)$ is extended to the extra nodes $v = v_i^k(j)$ in a natural way: we put $f(v) := c_k$ if there is a path from v to G^k (one may say that v lies on the left from G^k), and $f(v) := d_k$ otherwise (when v lies on the right from G^k). In particular, each edge of $\overline{G}$ not incident with a node of G is tight. One can see that such an extension maintains conditions (9)–(11) everywhere; moreover, for any $(i,j) \in J$, the sets of nodes satisfying the switch condition in $V_i(j)$ and in $\overline{V}_i(j)$ are equal.

Given a feasible function f on $V(\overline{G})$ and a node $v = v_i^k(j)$, define

$$\epsilon(v) := \Delta f(e^{\mathrm{NW}}(v)) \qquad \text{and} \qquad \delta(v) := \Delta f(e^{\mathrm{SE}}(v)),$$

when the corresponding NW- or SE-edge exists in $\overline{G}$ (i.e., $1 \le j \le i+1$ in the former case and $0 \le j \le i$ in the latter case). We call these the *upper slack* and the *lower slack* at v, respectively. Using these, we define the upper slack $\epsilon_i(j)$ and the lower slack $\delta_i(j)$ at a multinode $\overline{V}_i(j)$ by

$$\text{(14)} \qquad \epsilon_i(j) := \sum\nolimits_{k=1}^{n} \epsilon(v_i^k(j)) \qquad \text{and} \qquad \delta_i(j) := \sum\nolimits_{k=1}^{n} \delta(v_i^k(j))$$

(the former when $j = 1, \dots, i+1$, and the latter when $j = 0, \dots, i$). Note that

$$\text{(15)} \qquad \epsilon_i(1) \ge \delta_i(0) \qquad \text{for } i = 1, \dots, n$$

(since $f(v_{i-1}^k(0)) = f(v_i^k(0)) = c_k$ and $f(v_i^k(1)) \le f(v_{i+1}^k(1))$ for all k). Also $\epsilon_i(i+1) = 0$.

To define the move from f in a level $i \in \{1, \dots, n\}$, we first recursively reduce slacks $\epsilon_i(\cdot), \delta_i(\cdot)$ by the following rule:

(16) choose some $0 \le j' < j \le i$ such that $\epsilon_i(j) > 0$, $\delta_i(j') > 0$, and $\epsilon_i(q) = \delta_i(q) = 0$ for all $j' < q < j$, and then subtract from each of $\epsilon_i(j)$ and $\delta_i(j')$ their minimum.

Let $\widetilde{\epsilon}_i(j)$ ($j = 1, \dots, i+1$) and $\widetilde{\delta}_i(j)$ ($j = 0, \dots, i$) denote the numbers obtained upon termination of this *cancellation process*, called the *residual upper and lower slacks* at $V_i(j)$, respectively. It is not difficult to realise that the residual slacks do not depend on the order of reducing the current slacks in the process; also (15) implies $\widetilde{\delta}_i(0) = 0$. (The value $\widetilde{\epsilon}_i(j)$ can be determined directly as follows. Take $0 \le p < j$ minimizing $\sum_{q=p+1}^{j} \epsilon_i(q) - \sum_{q=p}^{j-1} \delta_i(q)$, and denote this minimum by μ. Then $\widetilde{\epsilon}_i(j) = \max\{0, \mu\}$. The values $\widetilde{\delta}_i(j)$ are determined in a similar way.) The residual slacks are integers and there exists $j \in \{1, \dots, i\}$ such that

$$\text{(17)} \qquad \widetilde{\delta}_i(0) = \ldots = \widetilde{\delta}_i(j-1) = 0 \quad \text{and} \quad \widetilde{\epsilon}_i(j+1) = \ldots = \widetilde{\epsilon}_i(i) = 0.$$

Take the minimum j satisfying (17) (if there are many). If $\widetilde{\epsilon}_i(j) > 0$, then we say that $V_i(j)$ is the *active* multinode in level i (otherwise $\widetilde{\epsilon}_i(1) = \ldots = \widetilde{\epsilon}_i(i) = 0$).

The *moving operator* ϕ_i in level i is applicable when the active multinode $V_i(j)$ does exist, and its action is simple: it increases by one the value of f on the switch-node in $V_i(j)$, preserving f on all other nodes of $\overline{G}$.

We have to show that ϕ_i is well-defined.

Proposition 5.1 *The function $f' := \phi_i(f)$ is feasible.*

Proof We have to check validity of (9) and (11) for f' (then (13) for the restriction of f' to $V(G)$ would follow automatically). Below, when speaking of switch-nodes or using extressions with $\epsilon, \widetilde{\epsilon}, \delta, \widetilde{\delta}$, we always mean the corresponding objects for f. Let $X = V_i(j)$ be the active multinode for f and i, and v the switch-node in it. We will use the simple observation that

(18) if z', u', v', w' are, respectively, the left, upper, right and lower nodes of a little rhombus, then $\Delta f(z', u') + \Delta f(u', v') = \Delta f(z', w') + \Delta(w', v')$; in particular, $\epsilon(v') - \delta(z') = \Delta f(w', v') - \Delta f(z', u')$.

Suppose $\Delta f'(e) < 0$ for some edge e. This is possible only if $\Delta f(e) = 0$ and e enters v, i.e., e is $e^{\mathrm{NW}}(v)$ or $e^{\mathrm{SW}}(v)$.

(a) Let $e = e^{\mathrm{SW}}$. If v is not the first node in X, then $\Delta f(e) > 0$ (otherwise the switch-node in X would occur before v). So v is the first node. Then the SW-edges of all nodes in X are tight for f, by (11). In view of (18), this implies $\epsilon_i(j) \le \delta_i(j-1)$, which is impossible since $\widetilde{\epsilon}_i(j) > 0$.

(b) Now let $e = e^{\mathrm{NW}}(v)$. The beginning node of e belongs to the multinode $V_{i-1}(j-1)$. Consider the nodes $v^1, \ldots, v^n$ in X (in this order) and the rhombi $\rho^1, \ldots, \rho^n$ containing them as right nodes, respectively. Let z^k, u^k, w^k denote, respectively, the left, upper and lower nodes in ρ^k. So $z^1, \ldots, z^n$ are the elements of $V_i(j-1)$, $u^1, \ldots, u^n$ are the elements of $V_{i-1}(j-1)$, and $w^1, \ldots, w^n$ are the elements of $V_{i+1}(j)$, and they follow in this order in these multinodes. Let $v = v^p$ and let u^q be the switch-node in $V_{i-1}(j-1)$. By (11), the edges (w^k, v^k) for $k = p+1, \ldots, n$ and the edges $(u^{k'}, v^{k'})$ for $k' = 1, \ldots, q-1$ are tight for f. This gives

$$\epsilon(v^k) \le \delta(z^k) \qquad \text{for } k = 1, \ldots, q-1 \text{ and for } k = p+1, \ldots, n.$$

(For $k > p+1$ this follows from (18).) Also the tightness of e gives $\epsilon(v^p) \le \delta(z^p)$. Suppose $q < p$. Then u^p occurs in $V_{i-1}(j-1)$ after the switch-node u^q, and therefore,

(z^p, u^p) is tight for f. We have $\Delta f(z^p, u^p) + \Delta f(u^p, v^p) = 0$, which implies the tightness for f of all edges in ρ^p. Then $\Delta f(e^{\rm SW}(v)) = 0$, contrary to shown in (a). Thus, $q \ge p$, implying $\epsilon(v^k) \le \delta(z^k)$ for all k, and therefore, $\epsilon_i(j) \le \delta_i(j)$; a contradiction.

So, (9) for f' is proven. Since $\Delta f'(e) \le \Delta f(e)$ for all SW- and SE-edges e of nodes in $V_{i-1}(j-1)$, (11) is valid for f' and this multinode. Also (11) is, obviously, valid for f' and $V_i(j)$. It remains to examine the multinode $Y = V_{i-1}(j)$ since for the edge $e = e^{\rm NE}(v)$, which is the SW-edge for the corresponding node u in Y, the value $\Delta f'(e)$ becomes greater than $\Delta f(e)$. If e is not tight for f or if the *last* node u' in Y satisfying the switch condition for f does not occur before u, then (11) follows automatically.

Suppose $\Delta f(e) = 0$ and u' occurs before u. We show that this is not the case by arguing in a way close to (b). For $k = 1, \ldots, n$, let z^k, u^k, v^k, w^k denote, respectively, the left, upper, right and lower nodes of the rhombus whose upper node (namely, u^k) is contained in Y. Then $v = z^p$ and $u' = u^q$ for some p, q with $q < p$. The fact that both v, u' satisfy the switch condition for f (in their multinodes), together with $q < p$, implies that for each $k = 1, \ldots, n$, at least one of $\Delta f(z^k, u^k)$ and $\Delta(z^k, w^k)$ is zero. This gives (cf. (18)):

$$\delta(z^k) \le \epsilon(v^k) \qquad \text{for all } k.$$

Moreover, this inequality is strict for $k = q$. Indeed, we have $\Delta f(z^q, w^q) = 0$ and $\Delta f(u^q, v^q) > 0$ (otherwise the node in Y next to u would satisfy the switch condition for f as well, but u' is the last of such nodes). So we obtain $\delta_i(j) < \epsilon_i(j+1)$. But, in view of $\widetilde{\epsilon}_i(j) > 0$ and $\widetilde{\delta}_i(j') = 0$ for $j' = 0, \ldots, j-1$, this implies $\widetilde{\delta}_i(j) = 0$ and $\widetilde{\epsilon}_i(j+1) > 0$, and therefore, the active multinode in level i should occur after $V_i(j)$; a contradiction.

This completes the proof of the proposition. ■

In conclusion of this section we discuss one more important aspect.

Backward moves. Besides the above description of partial operators ϕ_i increasing functions in $\mathcal{F}(\mathbf{c}, \mathbf{d})$, we can describe explicitly the corresponding decreasing operators, which make *backward moves*. For $i = 1, \ldots, n$, such an operator ψ_i acts on a feasible function f as follows (as before, we prefer to deal with extended functions on $V(\overline{G})$). We take the *first* multinode $V_i(j)$ (with j minimum) in level i for which $\widetilde{\delta}_i(j) > 0$; the operator does not act when $\widetilde{\delta}_i(j) = 0$ for all j. In view of (15), $1 \le j \le n$ takes place. In this multinode, called *active in backward direction*, we take the *last* node v possessing the switch condition in (11), called the *switch-node in backward direction*. Then the action of ψ_i consists in decreasing the weight $f(v)$ by one, preserving the weights of all other nodes of G.

Proposition 5.2 *The function $f' := \psi_i(f)$ is feasible. Moreover, ϕ_i is applicable to f', and $\phi_i(f') = f$.*

Proof One can prove this by arguing in a similar spirit as in the proof of Proposition 5.1. Instead, we can directly apply that proposition to the reversed model. This is based on a simple observation, as follows.

For a node $v \in V(\overline{G})$, define $\mu(v) := \Delta f(e^{\mathrm{NE}}(v))$ and $\nu(v) := \Delta f(e^{\mathrm{SW}}(v))$, when such an NE- or SW-edge exists in $\overline{G}$. The *alternative* upper and lower slacks at a multinode $V_i(j)$ are defined to be, respectively, the sum of numbers $\mu(v)$ and the sum of numbers $\nu(v)$ for the nodes v in this miltinode (the former is defined for $j = 0, \ldots, i$, and the latter for $j = 1, \ldots, i+1$). Compare (14). Considering the little rhombus containing nodes $u = v_i^k(j-1)$ and $v = v_i^k(j)$, we have $\nu(v) - \mu(u) = \epsilon(v) - \delta(u)$ (cf. (18)). This gives

$$\nu_i(j) - \mu_i(j-1) = \epsilon_i(j) - \delta_i(j-1). \tag{19}$$

The *reversed model* $\mathcal{M}^r$ is obtained by reversing the edges of G, by replacing the upper bound $\mathbf{c}$ by $-\mathbf{d}$, and by replacing the lower bound $\mathbf{d}$ by $-\mathbf{c}$ (one may think that we now read the original model from right to left). Accordingly, a feasible function f in M is replaced by $f^r := -f$. One can see that f^r is feasible for M^r and that the last node satisfying the switch condition for f in an original multinode $V_i(j)$ turns into the switch-node for f^r in the corresponding multinode $V_i^r(j')$ in M^r. Also $\epsilon_i^r(j') = \mu_i(j)$ and $\delta_i^r(j') = \nu_i(j)$ (where ϵ^r, δ^r stand for ϵ, δ in the reversed model). In view of (19), the cancellation process (see (16)) with f^r in the level i of M^r will give $\widetilde{\epsilon}_i^r(j') = \widetilde{\delta}_i(j)$ and $\widetilde{\delta}_i^r(j') = \epsilon_i(j)$ for all j.

These observations enable us to conclude that the function $(f^r)'$ obtained by the forward move from f^r in M^r generates the function $f' = \psi_i(f)$ in M. Therefore, f' is feasible. To see the second part of the proposition, let v be the node of the active multinode $V_i(j)$ where f decreases (by one) to produce f'. The edge $e^{\mathrm{SW}}(v)$ is non-tight for f', which implies that v is the unique node in $V_i(j)$ satisfying the switch condition for f', and therefore, v becomes the switch-node there. Also decreasing f by one at v results in increasing $\epsilon(v)$, and one can see that the residual slack $\widetilde{\epsilon}_i(j)$ for f' is greater by one compared with f. This and (17) imply that $V_i(j)$ is just the active multinode for f' in the level i. Hence the forward move from f' increases it by one at v, and we obtain $\phi\psi(f) = f$, as required. ■

Clearly both operators ϕ_i and ψ_i are injective. Also the "double reversed" model coincides with the original one, and therefore, Proposition 5.2 implies that $\psi_i\phi_i(f) = f$

for each f to which ϕ_i is applicable. So ϕ and ψ are inverse to each other and we may denote ψ_i by ϕ_i^{-1}.

6 The relation of the model to RAN-crystals

We have seen that the feasible functions in the model one-to-one correspond to the vertices of a crystal, by using the GT-pattern model for the latter, see Proposition 4.1. In this section we directly verify that the set $\mathcal{F}$ of these functions along with the set of (forward) moves satisfies axioms (A1)–(A5), and therefore, does constitute a RAN-crystal. One may assume that the lower bounds are zero, i.e., $\mathcal{F} = \mathcal{F}(\mathbf{c})$ for $\mathbf{c} \in \mathbb{Z}_+^n$. When the operator ϕ_i is applicable to an $f \in \mathcal{F}$, we say that f and $f' := \phi_i(f)$ are connected by the directed edge (f, f') with color i; the set of these edges is denoted by $\mathcal{E}_i$. This produces the n-colored digraph $\mathcal{K}(\mathbf{c}) = (\mathcal{F}, \mathcal{E})$ in which $\mathcal{E}$ is partitioned into the color classes $\mathcal{E}_1, \ldots, \mathcal{E}_n$. So we are going to show the following.

Theorem 6.1 *$\mathcal{K}(\mathbf{c})$ is a RAN-crystal.*

Proof As before, it is more convenient to operate with the extended support-graph $\overline{G}$ and regard the functions in $\mathcal{F}$ as being properly extended to the nodes in $V(\overline{G}) - V(G)$.

Axiom (A1) immediately follows from properties of operators ϕ_i and ψ_i. Next we observe the following. For $f \in \mathcal{F}$ and a color i, if $V_i(j)$ is the active multinode, then the action of ϕ_i decreases $\widetilde{\epsilon}_i(j)$ by 1, increases $\widetilde{\delta}_i(j)$ by 1, and does not change the residual slacks $\widetilde{\epsilon}$ and $\widetilde{\delta}$ for the other multinodes in level i. This easily follows from (17) and the fact that under increasing f by 1 at the switch-node v in $V_i(j)$, $\epsilon(v)$ decreases by 1 and $\delta(v)$ increases by 1. Similarly, if $V_i(j')$ is the active multinode in backward direction, then ψ_i decreases $\widetilde{\delta}_i(j')$ by 1, increases $\widetilde{\epsilon}_i(j')$ by 1, and preserves the residual slacks for the other multinodes in level i. This implies

$$h_i(f) = \sum\nolimits_{j=1}^{i} \widetilde{\epsilon}_i(j) \qquad \text{and} \qquad t_i(f) = \sum\nolimits_{j=1}^{i} \widetilde{\delta}_i(j), \tag{20}$$

regarding f as a vertex of $\mathcal{K}$.

If i, i' are two colors with $|i - i'| \geq 2$, then any changes of f in level i do not affect the numbers $\epsilon(v)$ and $\delta(v)$ for nodes v in level i'. So $h_{i'}(f) = h_{i'}(f')$ and $t_{i'}(f) = t_{i'}(f')$ for $f' = \phi_i(f)$, as required in (A2) for non-neighboring colors. Validity of axiom (A5) is immediate as well.

In order to verify axioms (A2),(A3) and (especially) (A4) for neighboring colors we need a more careful analysis of the behavior of residual slacks. The following interpretation for the cancellation process (see (16)) is of help.

For $f \in \mathcal{F}$ and a fixed level i'', we may think of $V(j)$ as a *box* where $\epsilon(j)$ *white balls* and $\delta(j)$ *black balls* are contained (we omit the subindex i'' hereinafter). There is a set C of *couples*, each involving one black ball b from a box $V(j)$ and one white ball w from a box $V(j')$ such that $j < j'$ (each ball occurs in at most one couple). We associate with a couple (b, w) the integer *interval* $[j(b), j(w)]$, where $j(q)$ denotes the number of the box containing q. The set $\mathcal{I}$ of these intervals (with possible multiplicities) is required to form an *interval family*, which means that there are no two intervals $[\alpha, \beta]$, $[\alpha', \beta']$ such that $\alpha < \alpha' < \beta < \beta'$ (i.e., no crossing pairs). In particular, the set of maximal intervals in $\mathcal{I}$, not counting multiplicities, forms a linear order in a natural way. Also it is required that: (i) C is maximal, in the sense that there are no uncoupled, or *free*, a black ball b and a white ball w such that $j(b) < j(w)$; and (ii) no free ball lies in the interior of an interval in $\mathcal{I}$.

It is easy to realize that such a C exists and unique, up to recombining couples with equal intervals. We denote the set of free white (free black) balls by W (resp. B) and call (C, W, B) the *arrangement* for the given collection of black and white balls. Furthermore, for each j, the number of free white balls (free black balls) in $V(j)$ is precisely $\widetilde{\epsilon}(j)$ (resp. $\widetilde{\delta}(j)$).

Let p denote the maximal number $j(w)$ among $w \in W$ (letting $p = -\infty$ if $W = \emptyset$), and q the minimal number $j(b)$ among $b \in B$ (letting $q = \infty$ if $B = \emptyset$). Then $p \leq q$. One can see that if some *black* ball b is *removed*, then the arrangement changes as follows (we indicate only the changes important for us).

(21) If b is free, it is simply deleted from B. And if b is coupled and occurs in a maximal interval $\sigma = [\alpha, \beta]$, then: (a) if $\beta \leq q$ then one of the previously coupled white balls w with $j(w) = \beta$ becomes free (and σ is replaced by a maximal interval $[\alpha, \beta']$ for some $j(b) \leq \beta' \leq \beta$); and (b) if $q < \beta$ (and therefore, $q \leq \alpha$), then one free black ball b' whose number $j(b')$ is maximal provided that $j(b') \leq \alpha$ becomes coupled and generates the maximal interval $[j(b'), \beta]$.

On the other hand, when a new *white* ball w is *added*, the changes are as follows.

(22) In case $j(w) \leq q$: (a) if $j(w)$ is in the interior of some maximal interval $[\alpha, \beta]$, then w becomes coupled and one previously coupled white ball w' with $j(w') = \beta$ becomes free; (b) otherwise w is simply added to W. And in case $j(w) > q$: (c) w becomes coupled and one free black ball b with $j(b)$ maximum provided that $j(b) < j(w)$ becomes coupled as well.

Using this interpretation, we now check axioms (A2)–(A4) for neighboring levels (viz. colors) i and $i-1$ in the model. Here for $f \in \mathcal{F}$ in question, the number of the active multinode (the active multinode in backward direction) in level i is denoted by $p = p(f)$ (resp. $q = q(f)$), and $p' = p'(f)$ and $q' = q'(f)$ stand for the analogous numbers in level $i-1$ (as before, using the sign $-\infty$ or ∞ if such a multinode does not exist).

Verification of (A2). When ϕ_i applies to f (at $V_i(p)$), the value $\delta_{i-1}(p-1)$ decreases by 1. (Recall that for $v \in V_i(j)$ and $(u,v) = e^{\rm NW}(v)$, u belongs to $V_{i-1}(j-1)$.) In the above interpretation, this means that one black ball is removed from the arrangement for level $i-1$. Then (21) implies that in case $p-1 < q'$, the sum of values $\widetilde{\epsilon}_{i-1}(j)$ (equal to $h_{i-1}(f)$) increases by 1, while all $\widetilde{\delta}_{i-1}(j)$ preserve. And if $p-1 \geq q'$, then the sum of values $\widetilde{\delta}_{i-1}(j)$ (equal to $t_{i-1}(f)$) decreases by 1, while all $\widetilde{\epsilon}_{i-1}(j)$ preserve. Also in the former case, we obtain $p(f') \leq p(f)$ and $q'(f') = q'(f)$, where $f' := \phi_i(f)$, and therefore, the next application of ϕ_i will fall in the former case as well (further increasing h_{i-1}). Next, when ϕ_{i-1} applies to f, we observe from (22) that: in case $p' \leq q-1$, the sum of $\widetilde{\epsilon}_i(j)$ increases by one, while all $\widetilde{\delta}_i(j)$ preserve, and in case $p' \geq q$, the sum of $\widetilde{\delta}_i(j)$ decreases by one, while all $\widetilde{\epsilon}_i(j)$ preserve. Also in the former case, $p'(f') \leq p'(f)$ and $q(f') = q(f)$, where $f' := \phi_{i-1}(f)$, so the next application of ϕ_{i-1} increases h_i as well.

Verification of (A3). This is also easy. Let $f' := \phi_i(f)$ and $f'' := \phi_{i-1}(f)$. Suppose (f, f') has label 0. Then $p-1 \geq q'$ and $p'(f') = p'(f)$ (see the previous verification). Moreover, the switch-node u in $V_{i-1}(p')$ for f remains the switch-node for f'. (Indeed, since $p' \leq p-1$, the slacks of the SW-edges of all nodes in $V_{i-1}(p')$ preserve, and the slacks of their SE-edges do not increase.) In its turn, $p' \leq p-1 \leq q-1$ implies that (f, f'') has label 1, as required in the axiom. Also neither the active multinode in level i nor the switch-node v in it can change when ϕ_{i-1} applies to f. Thus, both $\phi_{i-1}\phi_i$ and $\phi_i\phi_{i-1}$ increase the original function f by 1 on the same elements u, v. A verification of the relation $\phi_{i-1}\phi_i(f) = \phi_i\phi_{i-1}(f)$ in the case when (f, f'') has label 0 is similar.

Verification of (A4). This is somewhat more involved. Assuming that both ϕ_i and ϕ_{i-1} are applicable to a feasible function f, define $f_1 := \phi_i(f)$ and $g_1 := \phi_{i-1}(f)$, and let both (f, f_1) and (f, g_1) have label 1. Then $p-1 < q'$ and $p'+1 \leq q$ (where $p = p(f)$, and similarly for q, p', q').

Since $\ell(f, f_1) = 1$, we have $h_{i-1}(f_1) = h_{i-1}(f) + 1 \geq 2$. Therefore, we can define $f_2 := \phi_{i-1}(f_1)$ and $f_3 := \phi_{i-1}(f_2)$. Similarly, we can define $g_2 := \phi_i(g_1)$ and $g_3 := \phi_i(g_2)$. Our aim is to show that ϕ_i is applicable to f_3, that ϕ_{i-1} is applicable to g_3, and that $\phi_i(f_3) = \phi_{i-1}(g_3)$. Two cases are possible: $p' \leq p-1$ and $p' \geq p$.

Let $p' \le p-1$. For $k=1,2,3$, we denote $p(f_k), q(f_k), p'(f_k), q'(f_k)$ by p_k, q_k, p'_k, q'_k, respectively; similar numbers for g_k are denoted by $\overline{p}_k, \overline{q}_k, \overline{p}'_k, \overline{q}'_k$. We use the above interpretation and associate to each current function f' the corresponding arrangement $(C=C(f'), W=W(f'), B=B(f'))$ in level i and the corresponding arrangement $(C'=C'(f'), W'=W'(f'), B'=B'(f'))$ in level $i-1$.

Since $\widetilde{\epsilon}_i(p) > 0$, there is a white ball $w \in W(f)$ with $j(w)=p$. In view of $p-1<q'$, w corresponds to a coupled black ball b' with $j(b')=p-1$ in level $i-1$; let $[\alpha', \beta']$ be the maximal interval for $C'(f)$ that contains b'. Then $p-1<\beta' \le q'$. We also define the number β as follows: if the point $p'+1$ lies in the interior of some maximal interval $[\widetilde{\alpha}, \widetilde{\beta}]$ for $C(f)$, put $\beta := \widetilde{\beta}$; otherwise put $\beta := p'+1$. (The meaning of β is that, in view of $p'+1 \le q$, if a new white ball w with $j(w)=p'+1$ is added in level i, then the arrangement in this level changes so that there appears a free ball w' with $j(w')=\beta$; see (22).) Appealing to the interpretation, we can precisely characterize the changes of $\widetilde{\epsilon}_i, \widetilde{\delta}_i, \widetilde{\epsilon}_{i-1}, \widetilde{\delta}_{i-1}$ when the above transformations of our functions are fulfilled.

(i) The transformation $f \to f_1$ decreases $\widetilde{\epsilon}_i(p)$ by 1 and increases $\widetilde{\delta}_i(p)$ by 1. Also $\widetilde{\epsilon}_{i-1}(\beta')$ becomes equal to 1; cf. (21)(a).

In particular, $p'_1 = \beta'$, i.e., $V_{i-1}(\beta')$ becomes the active multinode in level $i-1$.

(ii) The transformation $f_1 \to f_2$ reduces $\widetilde{\epsilon}_{i-1}(\beta')$ to 0 and increases $\widetilde{\delta}_{i-1}(\beta')$ by 1. Also $\widetilde{\delta}_i(r)$ decreases by 1 for some $r \ge p = q_1$; cf. (22)(c).

This gives $p'_2 = p'$ and $q_2 \ge p$ and preserves all intervals for C that lie before p.

(iii) The transformation $f_2 \to f_3$ decreases $\widetilde{\epsilon}_{i-1}(p')$ by 1 and makes $\widetilde{\delta}_{i-1}(p')$ be equal to 1. Also $\widetilde{\epsilon}_i(\beta)$ increases by 1; cf. (22)(a),(b).

The latter property implies $p'+1 \le \beta \le p_3 \le p$. Then ϕ_i is applicable to f_3; define $f_4 := \phi_i(f_3)$. (Furthermore, one can see that $V_i(p_3)$ is the active multinode in level i for the function $\phi_i\phi_{i-1}(f)$ as well.)

Thus, the combined transformation $\phi_i\phi_{i-1}\phi_{i-1}\phi_i$ consecutively increases f by 1 in the switch-nodes v_0, v_1, v_2, v_3 of $V_i(p), V_{i-1}(\beta'), V_{i-1}(p'), V_i(p_3)$, respectively, where each switch-node is defined for the current function at the moment of the corresponding transformation. (Note that p' and β' are different, while p and p_3 may coincide.)

Next we consider the other chain of transformations.

(iv) The transformation $f \to g_1$ decreases $\widetilde{\epsilon}_{i-1}(p')$ by 1 and increases $\widetilde{\delta}_{i-1}(p')$ by 1. Also $\widetilde{\epsilon}_i(\beta)$ increases by 1.

From (22)(a),(b) it follows that $\beta \le p$, implying $\overline{p}_1 = p$.

(v) The transformation $g_1 \to g_2$ decreases $\widetilde{\epsilon}_i(p)$ by 1 and increases $\widetilde{\delta}_i(p)$ by 1. Also $\widetilde{\delta}_{i-1}(p')$ reduces to 0.

Moreover, (21)(b) implies the following important property $(*)$: $[p', \beta']$ becomes a maximal interval in the new arrangement in level $i-1$. Also (as mentioned after (iii)) $\overline{p}_2$ coincides with p_3.

(vi) The transformation $g_2 \to g_3$ decreases $\widetilde{\epsilon}_i(\overline{p}_2)$ by 1 and increases $\widetilde{\delta}_i(\overline{p}_2)$ by 1. Also, in view of $p'+1 \le \overline{p}_2 \le p$, the interval $[p', \beta']$ in level $i-1$ (see $(*)$ above) is destroyed and $\widetilde{\epsilon}_{i-1}(\beta')$ becomes equal to 1; cf. (21)(a).

So $\overline{p}'_3 = \beta'$ and we can apply ϕ_{i-1} to g_3; let $g_4 := \phi_{i-1}(g_3)$. We assert that $g_4 = f_4$.

To see this, notice that the combined transformation $\phi_{i-1}\phi_i\phi_i\phi_{i-1}$ increases the initial f within the same multinodes as those in the transformation $\phi_i\phi_{i-1}\phi_{i-1}\phi_i$, namely, $V_{i-1}(p'), V_i(p), V_i(\overline{p}_2 = p_3), V_{i-1}(\beta')$ (but now the order is different). Let $\overline{v}_0, \overline{v}_1, \overline{v}_2, \overline{v}_3$ be the switch-nodes in these multinodes, respectively, at the moment of the corresponding transformations. Since no change in level $i-1$ affects the slacks of SW- and SE-edges in level i, we have $\overline{v}_1 = v_0$ and $\overline{v}_2 = v_3$. Also $p'+1 \le \beta, p$ implies that the transformations in level i do not decrease the slacks of the SE-edges of nodes in $V_{i-1}(p')$ and do not change the slacks of their SW-edges, whence $\overline{v}_0 = v_2$.

It remains to check that $\overline{v}_3 = v_1$. Let u be the switch-node in $V_{i-1}(\beta') =: X$ for the initial function f. We have $p \le \beta'$. Therefore, the increase at $v_0 = \overline{v}_1$ can change the switch-node in X only if $p = \beta'$ and if the end u' of the edge $e^{\rm NE}(v_0)$ succeedes u in the ordering on X. If this is the case, then under each of the transformations $f \to f_1$ and $g_1 \to g_2$ (concerning $V_i(p)$) the switch-node u in X is replaced by u'. Besides these, there is only one transformation in level i that preceedes the transformation within X, namely, $g_2 \to g_3$. We know that $\overline{p}_2 \le p$ and that if $\overline{p}_2 = p$ then $\overline{v}_2$ coincides with or preceedes v_0 (taking into account that the transformation $g_1 \to g_2$ concerning $V_i(p)$ was applied earlier). This easily implies that $g_2 \to g_3$ can never change the switch-node in X. Thus, $\overline{v}_3 = v_1$.

The case $p' \ge p$ is examined in a similar fashion, and we leave it to the reader.

Finally, due to Proposition 3.2, verifying the second part of axiom (A4) (concerning the operators ϕ_i^{-1} and ϕ_{i-1}^{-1}) is not necessary.

This completes the proof of Theorem 6.1. ∎

Remark 2. In light of the second claim in Proposition 3.2, instead of the tiresome verification of axiom (A4) in the above proof, one may attempt to show that a maximal connected subgraph with colors i and $i-1$ in $\mathcal{K}$ has only one zero-indegree vertex. However, no direct method to show this is known to us.

Clearly the source of the crystal $\mathcal{K}(\mathbf{c})$ is the identically zero function f_0 on $V(G)$, and the sink is the function $f_{\mathbf{c}}$ taking the constant value c_k within each subgraph G^k, $k = 1, \ldots, n$. In particular, this implies that

(23) the distance (viz. the number of edges of a path) from the source to the sink, or the *length* of $\mathcal{K}(\mathbf{c})$, is equal to $\sum_{k=1}^{n} c_k |V(G^k)|$, or $\sum_{k=1}^{n} c_k k(n-k+1)$.

Also one can see that for the source function f_0 and a level i, one has $\widetilde{\epsilon}_i(1) = c_i$ and $\widetilde{\epsilon}_i(j) = 0$ for $j = 2, \ldots, i$ (moreover: starting from f_0, each application of ϕ_i increases the weight of $v_i^i(1)$ by 1 until the weight becomes c_i). So $h_i(f_0) = c_i$ for each color i. This means that $\mathcal{K}(\mathbf{c})$ is the crystal $K(\mathbf{c})$, and now the result due to Stembridge [13] that there exists exactly one RAN-crystal with source having a prescribed n-tuple $\mathbf{c}$ of parameters (see (2)) and Corollary 3.4 enable us to conclude with the following.

Theorem 6.2 *The crossing model $\mathcal{M}_n$ generates precisely the set of regular A_n-crystals.*

7 Principal lattice and skeleton

In this section we apply the crossing model to establish certain structural properties of RAN-crystals. We consider the initial setting for the crossing model, i.e., when the upper bounds are nonnegative integers and the lower bounds are zeros. So we deal with a tuple $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{Z}_+^n$ of parameters, the set $\mathcal{F} = \mathcal{F}(\mathbf{c})$ of feasible functions in the model, and the crystal $K = K(\mathbf{c}) = (V, E)$. As before, $G = (V(G), E(G))$ is the support-graph, $G^k = (V(G^k), E(G^k)$ is k-th base subgraph (component) in G, $K^k = K_n^k(c_k)$ is k-th base crystal of size c_k, and F_i is i-th partial operator on V. The latter corresponds to the partial operator ϕ_i on $\mathcal{F}$. Also we will use the following notation:

$v(f)$ denotes the vertex of K corresponding to a feasible function f;

$f^1 \circ f^2 \circ \ldots \circ f^n$, where $f^k : V(G^k) \to \mathbb{Z}$ $(i = 1, \ldots, n)$, denotes the function on $V(G)$ coinciding with f^k within each G^k;

$v^k(f^k)$ denotes the vertex of K^k corresponding to a feasible function f^k on $V(G^k)$;

$C^k a$ denotes the function on $V(G^k)$ taking a constant value $a \in \mathbb{Z}$.

Among the variety of feasible functions, certain functions are of most interest to us. These are functions f of the form $C^1 a_1 \circ \ldots \circ C^n a_n$, where each a_k is an integer satisfying $0 \le a_k \le c_k$. Such an f is feasible (since all edges of G are tight) and we call it a *principal function* and denote by $f[\mathbf{a}]$, where $\mathbf{a} = (a_1, \ldots, a_n)$. The corresponding vertex $v(f)$ is called a *principal vertex* of the crystal, and denoted by $v[\mathbf{a}]$. In particular, the source and sink of K are the principal vertices $v[\mathbf{0}]$ and $v[\mathbf{c}]$, respectively. So there are $(c_1+1)\times\ldots\times(c_n+1)$ principal vertices, and the set of these is denoted by $\Pi=\Pi(\mathbf{c})$ and called the *principal lattice* in K (it can be identified with the box in the lattice $\mathbb{Z}^n$ formed by the points $\mathbf{a}$ with $\mathbf{0}\le\mathbf{a}\le\mathbf{c}$).

Besides, we consider certain *1-relaxations* of principal functions. We use notation $\mathbf{a}^{(-k)}$ for an $(n-1)$-tuple of integers a_i where the index i ranges $1,\ldots,k-1,k+1,\ldots,n$. For a fixed $\mathbf{a}^{(-k)}$ satisfying $\mathbf{0}^{(-k)}\le\mathbf{a}^{(-k)}\le\mathbf{c}^{(-k)}$, define $\mathcal{F}[\mathbf{a}^{(-k)}]$ to be the set of all feasible functions $f=f^1\circ\ldots\circ f^n$ on $V(G)$ such that $f^i=C^ia_i$ for $i\ne k$. In other words, the non-fixed part f^k of f is any feasible function for G^k. (The latter is an arbitrary nonnegative integer function g on $V(G^k)$ bounded from above by c_k and satisfying the monotonicity condition $g(u)\ge g(v)$ for each edge $(u,v)\in E(G^k)$. Since the switch condition becomes redundant for G^k taken separately, just all these functions g generate the vertices v of K^k: $v=v^k(g)$.)

Let $K[\mathbf{a}^{(-k)}]$ denote the subgraph of K induced by the set of vertices $v(f)$ for all $f\in\mathcal{F}[\mathbf{a}^{(-k)}]$. For any $f\in\mathcal{F}[\mathbf{a}^{(-k)}]$, all edges in the subgraphs G^i with $i\ne k$ are tight. Also each multinode X of G contains at most one node of G^k. These facts imply that the moves from f do not depend on the entries of $\mathbf{a}^{(-k)}$, except for possible moves that transform f within leftmost multinodes $V_i(1)$ not intersecting G^k (i.e., with $i\ne k$). This leads to the following important property.

Proposition 7.1 *For any nonnegative $\mathbf{a}^{(-k)}\le\mathbf{c}^{(-k)}$, the subgraph $K[\mathbf{a}^{(-k)}]$ is isomorphic to the base crystal $K^k(c_k)$.*

The union $\mathcal{C}$ of these subgraphs $K[\mathbf{a}^{(-k)}]$ for all $\mathbf{a}^{(-k)}\le\mathbf{c}^{(-k)}$ and all k constitutes the object that we call the *skeleton* of K. Each $K[\mathbf{a}^{(-k)}]$ contains c_k+1 principal vertices $v(\mathbf{a}')$, namely, $a'_i=a_i$ for $i\ne k$ and a'_k ranges the interval $[0,\ldots,c_k]$ of integers. The corresponding set of c_k+1 vertices in the base crystal $K^k(c_k)$ is called the *axis* in it and denoted by $S^k=S^k(c_k)$. (In case $n=2$, [2] uses the name "diagonal" rather than "axis".)

The theorem below asserts that the skeleton of $K(\mathbf{c})$ is obtained from the base crystals $K^k(c_k)$ by use of a construction which is a natural generalization of the diagonal-product construction for RA2-crystals (see Theorem 3.1) to the case of n colors.

Again (like for $n=2$) we can describe such a construction for arbitrary (disjoint) graphs $H^1,\ldots,H^n$ in which subsets $S^1,\ldots,S^n$ of vertices (respectively) are distinguished. Let $\mathcal{V}$ be the collection of all n-element sets containing exactly one vertex from each S^i. For $k=1,\ldots,n$, let $\mathcal{V}^{(-k)}$ be the collection of all $(n-1)$-element sets containing exactly one vertex from each S^i with $i \neq k$. For each k, take $|\mathcal{V}^{(-k)}|$ copies of H^k, each being indexed as H^k_q for $q \in \mathcal{V}^{(-k)}$. We glue these copies together by identifying, for each $q = \{v_1, \ldots, v_n\} \in \mathcal{V}$ (where $v_k \in S^k$), the copies of vertices v_k in $H^k_{q \setminus \{v_k\}}$, $i = 1, \ldots, n$, into one vertex. The resulting graph is denoted by $(H^1, S^1) \bowtie \cdots \bowtie (H^n, S^n)$, or by $\bowtie ((H^k, S^k) \colon k = 1, \ldots, n)$ (clearly the order of original graphs here is not important).

In our case the role of each H^k is played by the base crystal K^k, and the axis S^k is taken as the distinguished subset. We call $\bowtie ((K^k, S^k) \colon k = 1, \ldots, n)$ the *axis-product* of these base crystals and denote it by $\mathcal{A}(\mathbf{c})$; this is an n-colored digraph where the edge colors are inherited from the base crystals. A princinal vertex in $\mathcal{A}(\mathbf{c})$ is defined to be a vertex obtained by gluing together vertices from axes of graphs K^k. So the principal vertices of $\mathcal{A}(\mathbf{c})$ one-to-one correspond to the principal functions in the model, or to the n-tuples $\mathbf{a} \in \mathbb{Z}^n_+$ with $\mathbf{a} \leq \mathbf{c}$. Also each vertex of a copy of K^k involved in $\mathcal{A}(\mathbf{c})$ is associated, in a natural way, with a feasible function f which is constant within each G^i with $i \neq k$.

Summing up the above explanations, we come to the following.

Theorem 7.2 *$K(\mathbf{c})$ contains an induced subgraph K' isomorphic to $\mathcal{A}(\mathbf{c}) = \bowtie ((K^k_n(c_k), S^k) \colon k = 1, \ldots, n)$ (respecting edge colors). Moreover, K' is determined uniquely and its vertices correspond to the feasible functions $f^1 \circ \ldots \circ f^k$ for $(G, \mathbf{c})$ such that each f^i is a constant function on $V(G^i)$, except for possibly one function f^k, which is an arbitrary feasible function for (G^k, c_k).*

Here the uniqueness can be shown as follows. The length of a path in K from the source $v[\mathbf{0}]$ to the sink $v[\mathbf{c}]$ is equal to $\sum_{k=1}^n c_k |V(G^k)|)$ (see (23)). So is the length of a path from the source to the sink in $\mathcal{A}(\mathbf{c})$ as well. Therefore (since K is graded), the source of K' must be at $v[\mathbf{0}]$ and the sink of K' must be at $v[\mathbf{c}]$. Now it is easy to realize that K' is reconstructed in K in a unique way.

Next, for two principal vertices $v[\mathbf{a}]$ and $v[\mathbf{b}]$, let us say that the latter is the k-th *successor* of the former if $b_k = a_k + 1$ and $a_i = b_i$ for all $i \neq k$. One can see that every possible transformation of the function $f[\mathbf{a}]$ into $f[\mathbf{b}]$ (by use of forward moves in the model) consists of a sequence of $|V(G^k)|$ moves, and the corresponding sequence of nodes where the current function changes forms a *linear order* on $V(G^k)$ (agreeable with the poset structure of G^k). In other words, this is an ordering $(u_1, \ldots, u_d)$ of the nodes of G^k such that for each $p = 1, \ldots, d$, the set $U_p = \{u_1, \ldots, u_p\}$ is an ideal in G^k. Each U_p determines the function g_p on $V(G^k)$ taking the value $a_k + 1$ within U_p, and a_k on the rest. Let $q(p)$ denote the number of the level in G^k that contains u_p, and let f_p denote the function on $V(G)$ formed from f by replacing $C^k a_k$ by g_p.

One can check that f_p coincides with the function obtained from f_{p-1} by the move in level $q(p)$ of G (which just increases the weight of u_p by one). Thus, we have the following.

Proposition 7.3 *For $k = 1, \dots, n$ and a principal vertex v of K, if the k-th successor w of v exists, then each paths from v to w one-to-one corresponds to a linear order $(u_1, \dots, u_d)$ $(d = |V(G^k)|)$ for G^k. Under this correspondence, the node w can be expressed as $F_{q(d)} \cdots F_{q(1)}(v)$, where $q(p)$ is the level number for u_p.* ■

So we can associate with each $k = 1, \dots, n$ the set $FS(k)$ of strings $q(d) \cdots q(1)$ as above, which is invariant for all principal vertices having the k-th successor. We refer to such a string as a *fundamental* one. For example, $FS(k)$ contains the string $w_{n-k+1} \cdots w_2 w_1$, where w_i is $i(i+1) \cdots (i+k-1)$.

Example. Let $n = 3$. Since the subgraph G^1 forms a path, there is only one fundamental string for $k = 1$, namely, 321. Similarly, $FS(3)$ consists of a unique string, namely, 123. The set $FS(2)$ for the subgraph (rhombus) G^2 consists already of two strings: 2132 and 2312.

8 Shifting the bounds and infinite crystals

So far, we have dealt with crystals having a finite set of vertices, or finite crystals. However, by use of the crossing model one can generate infinite analogs of RAN-crystals, in which some or all maximal monochromatic paths are infinite. (Infinite analogs of RA2-crystals were introduced in [2].) "Crystals" of this sort, which are interesting in their own right, find applications to modified quantized enveloping algebras [8] and also help in studying and better understanding the formation of finite crystals (e.g., they are of extensive use in [4]). The largest "crystal" obtained in this way possesses the property that it contains all other (finite and infinite) ones as intervals of a special form.

We start with finite RAN-crystals and consider the model with tuples $\mathbf{c}, \mathbf{d} \in \mathbb{Z}^n$ of upper and lower bounds, $\mathbf{c} \geq \mathbf{d}$. This gives the crystal $K(\mathbf{c} - \mathbf{d})$, also denoted by $K(\mathbf{c}, \mathbf{d})$. Let $\mathbf{c}', \mathbf{d}' \in \mathbb{Z}^n$ be such that $\mathbf{c}' \geq \mathbf{c}$ and $\mathbf{d}' \leq \mathbf{d}$. Clearly

(24) any feasible function f for $(\mathbf{c}, \mathbf{d})$ is feasible for $(\mathbf{c}', \mathbf{d}')$ as well.

This associates to each vertex $v(f)$ of $K = K(\mathbf{c}, \mathbf{d})$ the corresponding vertex of the crystal $K' = K(\mathbf{c}', \mathbf{d}')$, yielding an injective map $\gamma : V(K) \to V(K')$. Comparing the residual slacks $\widetilde{\epsilon}_i(j)$ and $\widetilde{\delta}_i(j)$ for the function f in the model with the bounds $\mathbf{c}, \mathbf{d}$ and the residual slacks $\widetilde{\epsilon}'_i(j)$ and $\widetilde{\delta}'_i(j)$ for f in the model with the bounds $\mathbf{c}', \mathbf{d}'$, one can see that

$$\begin{aligned}(25)\qquad & \widetilde{\epsilon}'_i(1) = \widetilde{\epsilon}_i(1) + c'_i - c_i \quad \text{and} \quad \widetilde{\epsilon}'_i(j) = \widetilde{\epsilon}_i(j) \qquad \text{for} \quad j = 2, \dots, i; \\ & \widetilde{\delta}'_i(i) = \widetilde{\delta}_i(i) + d_i - d'_i \quad \text{and} \quad \widetilde{\delta}'_i(j') = \widetilde{\delta}_i(j') \qquad \text{for} \quad j' = 1, \dots, i-1.\end{aligned}$$

Moreover, for each multinode, the switch-nodes concerning f in both models are the same, and similarly for the switch-nodes in backward direction. Also the situation when an active multinode $V_i(j)$ for $f, \mathbf{c}', \mathbf{d}'$ is not active for $f, \mathbf{c}, \mathbf{d}$ can arise only if $j = 1$, the switch-node in $V_i(j)$ is $v^i_i(1)$, and $f(v^i_i(1)) = c_i$; and symmetrically for the active multinodes in backward direction. These observations show that γ is extendable to the edges of K, and moreover, we have the following.

Proposition 8.1 *The image of $K = K(\mathbf{c}, \mathbf{d})$ by γ is a subcrystal of $K' = K(\mathbf{c}', \mathbf{d}')$ isomorphic to K, and any path in K' connecting vertices of $\gamma(K)$ is entirely contained in $\gamma(K)$. Therefore, $\gamma(K)$ is the interval* $\mathrm{Int}(\gamma(s_K), \gamma(\overline{s}_K))$ *of K', where s_K and $\overline{s}_K$ are the source and sink of K, respectively.*

Here for vertices u, v in an (acyclic) digraph, the *interval* $\mathrm{Int}(u, v)$ is the subgraph formed by the vertices and edges lying on paths from u to v. Since $s_K = v[\mathbf{d}]$ and $\overline{s}_K = v[\mathbf{c}]$ (using notation from Section 7), we obtain that

(26) $K(\mathbf{c}, \mathbf{d})$ is isomorphic to the interval of $K(\mathbf{c}' - \mathbf{d}')$ between the principal vertices $v[\mathbf{d} - \mathbf{d}']$ and $v[\mathbf{c} - \mathbf{d}']$ in the latter.

Now we are ready to introduce the above-mentioned "infinite crystals". They arise when we admit infinite bounds in the model, i.e., consider $\mathbf{c} \in (\mathbb{Z} \cup \{\infty\})^n$ and $\mathbf{d} \in (\mathbb{Z} \cup \{-\infty\})^n$ with $\mathbf{c} \ge \mathbf{d}$. More strictly: for a variable $M \in \mathbb{Z}_+$ and each color i, define c^M_i to be c_i if $c_i < \infty$, and $\max\{M, d_i\}$ otherwise, and define d^M_i to be d_i if $d_i > -\infty$, and $\min\{-M, c_i\}$ otherwise. When M grows, there appears a sequence of finite crystals $K(\mathbf{c}^M, \mathbf{d}^M)$, each containing the previous crystal $K(\mathbf{c}^{M-1}, \mathbf{d}^{M-1})$ as an interval, by Proposition 8.1. At infinity we obtain the desired (well-defined) "infinite crystal" $K(\mathbf{c}, \mathbf{d})$ (when $\mathbf{c}$ or/and $\mathbf{d}$ is not finite).

Some trivial consequences of this construction are as follows. The largest "infinite crystal", denoted by $K^{\infty}_{-\infty}$, arises when $c_i = \infty$ and $d_i = -\infty$ for all i. Among the variety of "crystals" constructed as above, $K^{\infty}_{-\infty}$ is determined by the property that any monochromatic path is fully infinite, i.e., infinite in both forward and backward directions. Equivalently: the principal lattice of $K^{\infty}_{-\infty}$ is formed by the vertices $v[\mathbf{a}]$ for all $\mathbf{a} \in \mathbb{Z}^n$. Also this crystal has the following property:

(27) each maximal connected subgraph (component) with colors i, i' is the Cartesian product of the fully infinite i-path and i'-path if $|i-i'| \geq 2$, and is the largest infinite RA2-crystal if $|i-i'| = 1$.

Here, following [2], the largest infinite RA2-crystal is defined to be the diagonal-product of fully infinite right and left sails (in the former, the vertices are the pairs $(p,q) \in \mathbb{Z}^2$ with $p \geq q$, and in the latter, the pairs (p,q) with $p \leq q$; the edges are defined as in Section 3).

Taking corresponding finite or "infinite" intervals in $K_{-\infty}^{\infty}$, one can obtain any finite or "infinite" crystal $K(\mathbf{c}, \mathbf{d})$.

Remark 3. At the first glance, it may seem likely that $K_{-\infty}^{\infty}$ is the unique graph possessing property (27). This is so for the graphs generated by the crossing model (and for the case $n = 2$ as well). However, already for $n = 3$ one can construct several non-isomorphic graphs satisfying (27). Four such graphs are demonstrated in [4]; they admit embeddings in lattices $\mathbb{Z}^4, \mathbb{Z}^5, \mathbb{Z}^5, \mathbb{Z}^6$, respectively, with each edge being a parallel translation of a unit base vector.

9 Subcrystals with $n-1$ colors in $K(\mathbf{c})$

In this section we demonstrate one more application of the crossing model. For a subset $J \subset \{1, \ldots, n\}$ of colors, let $\mathcal{K}(J)$ denote the set of maximal connected subgraphs of $K = K(\mathbf{c})$ whose edges have colors from J, i.e., the components of the graph $(V, \cup(E_i \colon i \in J))$. When the colors in J go in succession, i.e., J is an interval of $(1, \ldots, n)$, each member K' of $\mathcal{K}(J)$ is a regular $A_{|J|}$-crystal. (When J has a gap, K' becomes the Cartesian product of several regular crystals. For example, for $J = \{1, 3\}$, K' is the Cartesian product of two paths, with color 1 and with color 3, or a regular $A_1 \times A_1$-crystal.)

We are interested in the case when J is either $\{1, \ldots, n-1\}$ or $\{2, \ldots, n\}$, denoting $\mathcal{K}(J)$ by $\mathcal{K}^{(-n)}$ in the former case, and by $\mathcal{K}^{(-1)}$ in the latter case. In other words, $\mathcal{K}^{(-n)}$ (resp. $\mathcal{K}^{(-1)}$) is the set of $(n-1)$-crystals arising when the edges with color n (resp. color 1) are removed from K.

Consider $K' \in \mathcal{K}^{(-n)}$ and let $\mathcal{F}(K')$ denote the set of feasible functions corresponding to the vertices of K'. Since K' is connected, any $f \in \mathcal{F}(K')$ can be obtained from any other $f' \in \mathcal{F}(K')$ by a series of forward and backward moves in levels $1, \ldots, n-1$. So all functions in $\mathcal{F}(K')$ have equal tuples of values within level n of G. This level consists of nodes $v_n^1(n), v_n^2(n-1), \ldots, v_n^n(1)$ (from right to left), and we denote the n-tuple $(f(v_n^1(n)), \ldots, f(v_n^n(1)))$, where $f \in \mathcal{F}(K')$, by $\mathbf{a}(K')$. Thus, we have the following property: each subcrystal $K' \in \mathcal{K}^{(-n)}$ contains at most one principal vertex v of K, in which case $v = v[\mathbf{a}]$ for $\mathbf{a} = \mathbf{a}(K')$. (On the other hand, the members of $\mathcal{K}^{(-n)}$ cover all principal vertices of K.)

Similarly, for $K'' \in \mathcal{K}^{(-1)}$ and for the set $\mathcal{F}(K'')$ of feasible functions corresponding to the vertices of K'', the tuple $\mathbf{a}(K'') := (f(v_1^1(1)), \ldots, f(v_1^n(1)))$ (where the nodes follow from left to right in level 1) is the same for all $f \in \mathcal{F}(K'')$. So each subcrystal $K'' \in \mathcal{K}^{(-1)}$ contains at most one principal vertex of K as well (and the members of $\mathcal{K}^{(-1)}$ cover all principal vertices of K).

We show a sharper property.

Proposition 9.1 *Each subcrystal in $\mathcal{K}^{(-n)}$ contains precisely one principal vertex of $K(\mathbf{c})$, and similarly for the subcrystals in $\mathcal{K}^{(-1)}$. In particular, $|\mathcal{K}^{(-n)}| = |\mathcal{K}^{(-1)}| = (c_1+1) \times \ldots \times (c_n+1)$.*

(This property need not hold when an $(n-1)$-element subset J of colors differs from $\{1, \ldots, n-1\}$ and $\{2, \ldots, n\}$.)

Proof Let $K' \in \mathcal{K}^{(-n)}$ and let $\mathbf{a} = (a_1, \ldots, a_n)$ stand for $\mathbf{a}(K')$. Consider an arbitrary function $f \in \mathcal{F}(K')$. We show that the principal function $f[\mathbf{a}]$ can be reached from f by a series of forward moves (in levels $\neq n$), followed by a series of backward moves, whence the desired inclusion $f[\mathbf{a}] \in \mathcal{F}(K')$ will follow.

To show this, let $\mathcal{F}_0$ be the set of functions $f' \in \mathcal{F}(K')$ that can be obtained by (a series of) forward moves from f and such that $f'(v_k^k(1)) = f(v_k^k(1)) =: b_k$ for $k = 1, \ldots, n$. Take a maximal function f_0 in $\mathcal{F}_0$. We assert that

(28) the SW-edges of all nodes in G (where such an edge exists) are tight for f_0.

Suppose this is not so for some node, and among such nodes choose a node $v = v_i^k(j)$ with i minimum. Acting as in Section 5, extend G to the graph $\overline{G}$ and extend f_0 to the corresponding function $\overline{f}_0$ on $V(\overline{G})$ by setting the upper bound $\mathbf{b}$ and the lower bound $\mathbf{0}$ (then $\overline{f}_0$ satisfies both the monotonicity condition and the switch condition at each multinode and its values within each subgraph G^k lie between 0 and b_k).

Consider an arbitrary node $v' = v_i^{k'}(j')$ with $1 \le j' \le n$ in level i and take the rhombus ρ containing v' as the right node; let z', u', w' be the left, upper and lower nodes of ρ, respectively. Then $\Delta\overline{f}_0(z', u') = 0$ (this is obvious when $j' = 1$ and follows from the minimality of i when $j' > 1$, in view of $(z', u') = e^{\mathrm{SW}}(u')$). This implies $\epsilon(v') \ge \delta(z')$ (where these numbers concern the bound $\mathbf{b}$); cf. (18). Moreover, this inequality is strict when $v' = v$ (since $(w', v') = e^{\mathrm{SW}}(v')$ and $e^{\mathrm{SW}}(v)$ is not tight).

These observations imply $\widetilde{\epsilon}_i(j) > 0$, where $\widetilde{\epsilon}$ concerns the bound $\mathbf{b}$. So level i contains an active multinode, and therefore, $\overline{f}_0$ can be increased by a forward move in this level. This move remains applicable when the bound changes to $\mathbf{c}$; cf. (25). Thus, f_0 is not maximal, and this contradiction proves (28).

From (28) it follows that for each k, all edges of the path in G^k from the bottommost node $v_n^k(n-k+1)$ to the sink right^k are tight for f_0. Hence $f(\text{right}^k) = a_k$.

Now we apply backward moves from f_0 in levels $\neq n$. Let $\mathcal{F}_1$ be the set of functions $f' \in \mathcal{F}(K')$ that can be obtained by such moves and such that $f'(\text{right}^k) = a_k$ for $k = 1, \dots, n$. Let f_1 be a minimal function in $\mathcal{F}$. Arguing in a similar fashion, one shows that

(29) the NW-edges of all nodes in G (where such an edge exists) are tight for f_1.

Now (29) implies that f_1 is constant within each G^k, i.e., $f_1 = f[\mathbf{a}]$, as required.

To show the second part of the proposition concerning $\mathcal{K}^{(-1)}$, we can simply renumber the colors, by counting color i as $n-i+1$, and apply the model for this numeration. Clearly the set of principal vertices preserves under this renumbering, and now the result for $\mathcal{K}^{(-1)}$ follows from that for $\mathcal{K}^{(-n)}$. ■

Remark 4. Renumbering the colors as above causes the "turn-over" of the original model, so that level i turns into level $n-i+1$. (Note that the model is not maintained by this transformation since the switch condition (11) is imposed on SW- and SE-edges of nodes, but not on NW- and NW-ones). A feasible function f in the original model corresponds to a feasible function f' in the new model, so that f and f' determine the same vertex of the crystal. It seems to be a nontrivial task to explicitly express f' via f (for $n = 2$ an explicit piece-wise linear relation is pointed out in [2]).

We denote the subcrystal in $\mathcal{K}^{(-n)}$ ($\mathcal{K}^{(-1)}$) containing a principal vertex $v[\mathbf{a}]$ by $K^{(-n)}[\mathbf{a}]$ (resp. $K^{(-1)}[\mathbf{a}]$). One can compute the parameters of $K' = K^{(-n)}[\mathbf{a}]$ and indicate some other features of it. Its source and sink correspond to the minimal funstion $f_{\min}(K')$ and the maximum function $f_{\max}(K')$ in $\mathcal{F}(K')$, respectively. One can see that in each G^k, $f_{\min}(K')$ takes value 0 on all nodes, except for those on the path from the source left^k to the bottommost node $v_n^k(n-k+1)$ where the value is identically a_k. And $f_{\max}(K')$ takes value c_k on all nodes, except for those on the path from $v_n^k(n-k+1)$ to the sink right^k where the value is a_k. Symmetrically: the source and sink of a subcrystal $K'' = K^{(-1)}[\mathbf{a}]$ correspond to the minimal and maximum functions in $\mathcal{F}(K'')$, respectively, and in each G^k, the former takes value 0 on all nodes, except for those on the path from left^k to the topmost node $v_1^k(1)$ where the value is a_k, while the latter takes value c_k on all nodes, except for those on the path from $v_1^k(1)$ to right^k where the value is a_k.

Proposition 9.2 *$K^{(-n)}[\mathbf{a}]$ is isomorphic to the crystal $K_{n-1}(\mathbf{q})$ with colors $1,\dots,n-1$, where $q_i = c_i - a_i + a_{i+1}$ for each i. In its turn, $K^{(-1)}[\mathbf{a}]$ is isomorphic to the crystal $K_{n-1}(\mathbf{q}')$ with colors $2,\dots,n$, where $q'_i = c_i - a_i + a_{i-1}$.*

Proof One can check that for $f_{\min}(K^{(-n)}[\mathbf{a}])$, the active multinode in a level i is the first multinode $V_i(1)$ and the switch-node in it is the second node $v = v_i^{i+1}(1)$ (unless $a_{i+1} = 0$). The operator ϕ_i can be applied a_{i+1} times to this node, making the edge $e^{\mathrm{SW}}(v)$ tight, after which the switch-node becomes the first node $v_i^i(1)$ and ϕ_i can be applied $c_i - a_i$ times to it. As to $f_{\min}(K^{(-1)}[\mathbf{a}])$, the active multinode in a level i is the second multinode $V_i(2)$ and the switch-node in it is the first node $v' = v_i^{i-1}(2)$ (unless $a_{i-1} = 0$). Then ϕ_i can be applied a_{i-1} times to this node, making $e^{\mathrm{NW}}(v')$ tight, after which the active miltinode becomes $V_i(1)$ and ϕ_i can be applied $c_i - a_i$ to the switch-node $v_i^1(1)$. A verification is left to the reader. ■

Remark 5. This proposition implies that the set of parameter-tuples $\mathbf{q}$ of crystals in $\mathcal{K}^{(-n)}$ is formed by the integer points of a polytope in $\mathbb{R}^{n-1}$. Note also that for corresponding tuples $\mathbf{q}$ and $\mathbf{a}$, the numbers $a_1,\dots,a_{n-1}$ are determined by $\mathbf{q}$ and a_n, namely: $a_i = c[i:n-1] - q[i:n-1] + a_n$ for $i < n$. This enables us to compute the number $\eta(q)$ of crystals in $\mathcal{K}^{(-n)}$ having a prescribed parameter-tuple $\mathbf{q}$: it is equal to the number of $a_n \in \mathbb{Z}$ for which $0 \le a_i \le c_i$ holds for all $i = 1,\dots,n$. (One can express $\eta(q)$ as the difference between $\min\{c_n,\ q[i:n-1] - c[i+1:n-1] \colon i = 1,\dots,n-1\}$ and $\max\{0,\ q[i:n-1] - c[i:n-1] \colon i = 1,\dots,n-1\}$. In particular, $c_i = 0$ for some i implies that all crystals in $\mathcal{K}^{(-n)}$ are different.) This gives the branching rule for decomposing an irreducible sl_{n+1}-module into the sum of irreducible sl_n-modules.

Next, using the crossing model, one can easily compute the lengths of maximal monochromatic paths in $K^{(-n)}[\mathbf{a}]$ (or in $K^{(-1)}[\mathbf{a}]$) that go through the principal vertex $v[\mathbf{a}]$ of K (the length concerning color i expresses the "i-width" of the subcrystal at this vertex).

A more difficult question is how the subcrystals $K^{(-n)}[\mathbf{a}]$ and $K^{(-1)}[\mathbf{b}]$ intersect depending on tuples $\mathbf{a}$ and $\mathbf{b}$. One can show (a proof is omitted here) that such subcrystals are disjoint if $a_i \ne b_i$ for all i. An exhaustive analysis of these intersections is given in [4] for $n = 3$, which results in a complete description of the structure of RA3-crystals.

Finally, we can associate with a RAN-crystal $K(\mathbf{c})$ a "hierarchy of lattices", as follows. At the top level we put the lattice Π of principal vertices of the whole crystal. Each principal vertex $v[\mathbf{a}]$ gives rise to two lattices, namely, the principal lattices of the "upper" and "lower" $(n-1)$-colored subcrystals $K^{(-n)}[\mathbf{a}]$ and $K^{(-1)}[\mathbf{a}]$. These lattices, for all $\mathbf{0} \le \mathbf{a} \le \mathbf{c}$, constitute the second level in the hierarchy. The third level arises when one considers the principal lattices of the "upper" and "lower" $(n-2)$-colored subcrystals in the subcrystals related to the second level, and so on. Repeated members, if any, can be ignored. Then the bottom level is formed by the lattices of critical vertices of 2-colored subcrystals (with neighboring colors), and we also can attach to them the corresponding right and left sails as described in Section 3. This formalism might be of use if we were able to tell more about pairwise intersections of 2-colored subcrystals at the bottom level.

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