ELSEVIER

Theoretical Computer Science 151 (1995) 79–124

Theoretical Computer Science

Duality beyond sober spaces: Topological spaces and observation frames

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Abstract

We introduce observation frames as an extension of ordinary frames. The aim is to give an abstract representation of a mapping from observable predicates to all predicates of a specific system. A full subcategory of the category of observation frames is shown to be dual to the category of $\mathcal{T}_0$ topological spaces. The notions we use generalize those in the adjunction between frames and topological spaces in the sense that we generalize finite meets to infinite ones.

We also give a predicate logic of observation frames with both infinite conjunctions and disjunctions, just like there is a geometric logic for (ordinary) frames with infinite disjunctions but only finite conjunctions. This theory is then applied to two situations: firstly to upper power spaces, and secondly we restrict the adjunction between the categories of topological spaces and of observation frames in order to obtain dualities for various subcategories of $\mathcal{T}_0$ spaces. These involve nonsober spaces.

1. Introduction

A topological duality is a correspondence between two mathematical structures involving points and predicates such that isomorphic structures can be identified. Stone [31] first found such a duality between topology and logic. He considered ordered sets (representing the syntax of some logical system) and constructed from a boolean algebra a set of points using prime filters. Conversely, by using a topology on a set of points he was able to construct a Boolean algebra. For certain topological spaces (later called Stone spaces) these constructions give an isomorphism. In a later paper [32], he generalized this correspondence from Stone spaces to spectral spaces and from Boolean algebra's to distributive lattices. Hofmann and Keimel [11] described this duality in a categorical framework. Even further, Isbell [15] gives an adjunction between

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SSDI 0304-3975(95)00048-8

the category of topological spaces with continuous functions and the opposite category of frames with frame homomorphisms (which yields a duality between sober spaces and spatial frames). Stone duality in a mathematical context is studied in a book of Johnstone [19], and for the context of domain theory we refer the reader to [10]. In his thesis Abramsky [1] applied Stone duality to get logics of domains, as used in denotational semantics. He argues that Stone duality is the bridge between denotational and axiomatic semantics.

Smyth [29, 30] generalized the duality between sober spaces and spatial frames by allowing upper semi-continuous multifunctions as morphisms in the category of sober spaces, and by allowing Scott-continuous and finitely multiplicative morphisms on spatial frames. Smyth realized that by this restriction the morphisms on the spatial frames are generalizations of predicate transformers in the sense of Dijkstra [6, 7].

There are interesting spaces which are not sober. For example posets taken with the Alexandrov topology are not always sober. Johnstone shows [18] that not every dcpo with the Scott topology is sober. There are also spaces which have an exponent in the category of topological spaces that are not sober [15]. Also if one wishes to study fairness or countable nondeterminism then it seems that one has to go beyond sober spaces: one has to consider a category of ω_0 and ω_1 chain complete partial orders with functions preserving least upper bounds of ω_1-chains [26].

In [3] two of the authors introduced the notion of completely multiplicative predicate transformers. This notion was used in [4] for an extension of the results of Smyth [29] from sober to $\mathscr{T}_0$ spaces (considering also lower semi-continuous and Vietoris continuous multifunctions in addition to the upper semi-continuous ones). For this result frames were used whose elements are open sets of some $\mathscr{T}_0$ space. This forms the basis for the investigation below.

Usually subsets represent predicates and open sets [4, 22, 29] represent observable predicates. In a more abstract view, a complete lattice represents predicates while a frame represents observable predicates. This leads us to introduce observation frames: they map observable predicates to arbitrary predicates. They are mappings from frames to complete lattices preserving arbitrary joins and finite meets. An example of an observation frame is the embedding of open sets into the power set of points. Also Vickers [33] and Abramsky [1] view frames as collections of observable predicates.

Concerning liveness predicates, Abramsky [1] suggests that one has to look for structures more complicated than the simple lattice of open sets. Liveness predicates can be seen as arbitrary (but according to [29] only countable intersections should be considered) intersections of open sets. Our framework of observation frames has both arbitrary unions and intersections of open sets. We use an abstract framework where open sets are just elements of a frame. A different approach has been followed by Kwiatkowska et al. in [2]. In this approach liveness predicates can be interpreted as greatest fixed points of monotone operators on compact open sets. This interpretation, however, does not coincide with the classical interpretation of the infinitary conjunction since these fixed points are calculated using meets (or joins) which, in general, do not coincide with intersections (or unions).

The outline of the paper is as follows. First we introduce observation frames and turn them into a category. Then we construct topological spaces from observation frames by taking as points special kind of prime elements. In this way we obtain a duality between observation frames and topological spaces. Next we give a logic of observation frames with arbitrary conjunctions and disjunctions. This is done by the introduction of M-topological systems, which are a generalization of the topological systems of Vickers [33]. Subsequently we elaborate in two directions: firstly we apply the theory to upper power spaces of posets using filter theorems. Secondly we restrict our duality to some subcategories of $\mathscr{T}_0$ spaces that are in general nonsober. We consider $\mathscr{T}_1$ spaces, compact spaces, open and core compact spaces and posets. This leads us to a pointless version of the directed ideal completion of a poset (using the lattice side of the dualities of posets and algebraic directed complete partial orders (dcpo's)). Finally we study Galois connections in the context of observation frames.

2. Mathematical preliminaries

In this section we provide some basic notions and facts on lattices and topological spaces. For more detailed discussions consult [8–10]. We assume some familiarity with basic notions of category theory [23].

A *poset* $L = (L, \leqslant)$ is a set together with a reflexive, transitive and antisymmetric relation $\leqslant$ on L. A poset which has meets (greatest lower bounds) for every pair of elements is a *meet-semilattice*. A poset with joins (least upper bounds) for every pair of elements is a *join-semilattice*.

A *complete lattice* is a poset L in which every subset $A \subseteq L$ has a join $\bigvee A$ in L (and hence also a meet since $\bigwedge A = \bigvee\{x \in L \mid \forall a \in A.\ x \leqslant a\}$). Notice that L has a bottom and a top element given, respectively, by $\bot = \bigvee\emptyset$ and $\top = \bigwedge\emptyset$. A complete lattice L is called *completely distributive* if

$$\bigvee_{A\in\mathscr{A}} \bigwedge A = \bigwedge\left\{ \bigvee_{A\in\mathscr{A}} f(A) \mid f : \mathscr{A} \to \bigcup\mathscr{A} \text{ and } f(A) \in A\right\},$$

for every set of sets $\mathscr{A} \in \mathscr{P}(\mathscr{P}(L))$. Equivalently (see [27]), L is completely distributive if

$$\bigwedge_{A\in\mathscr{A}} \bigvee A = \bigvee\left\{ \bigwedge_{A\in\mathscr{A}} f(A) \mid f : \mathscr{A} \to \bigcup\mathscr{A} \text{ and } f(A) \in A\right\}$$

for every $\mathscr{A} \in \mathscr{P}(\mathscr{P}(L))$. A *frame* L is a complete lattice which satisfies the following infinite distributivity law, a restriction of the completely distributive law:

$$x \wedge \bigvee A = \bigvee\{x \wedge a \mid a \in A\}$$

for all $x \in L$ and $A \subseteq L$. For example given a set X the set $\mathscr{P}(X)$ of all the subsets of X is a completely distributive lattice when ordered by subset inclusion and hence also a frame. Given two frames F and G, a *frame morphism* is a function $\phi : F \to G$ that preserves arbitrary joins and finite meets, that is $\phi(\bigvee A) = \bigvee \phi(A)$ for every $A \subseteq F$ and $\phi(\bigwedge B) = \bigwedge \phi(B)$ for every finite $B \subseteq F$. Frames together with frame morphisms form the category **Frm**.

For a meet semilattice L, a nonempty subset $\mathscr{F}$ of L is called a *filter* if

(i) $\forall a \in \mathscr{F}.\ \forall b \in L.\ a \leqslant b \Rightarrow b \in \mathscr{F}$,

(ii) $\forall a, b \in \mathscr{F}.\ a \wedge b \in \mathscr{F}$.

The collection of all filters of L is denoted by $Fil(L)$. If L is a complete lattice then a filter $\mathscr{F} \subseteq L$ is *completely prime* if for every $S \subseteq L$ such that $\bigvee S \in \mathscr{F}$ there exists $s \in S$ such that $s \in \mathscr{F}$. There is an isomorphism between the completely prime filters and the prime elements of a complete lattice, where $p \in L$ is called *prime* if $p \neq \top$ and if $a \wedge b \leqslant p$ for some $a, b \in L$ then $a \leqslant p$ or $b \leqslant b$. The collection of all prime elements of L is denoted by $Spec(L)$.

A *topology* $\mathcal{O}(X)$ on a set X is a collection of subsets of X that is closed under finite intersections and arbitrary unions. Every topology $\mathcal{O}(X)$ can be ordered by subset inclusion and forms a frame with the empty set as bottom element and the whole set X as top element. The pair $(X, \mathcal{O}(X))$ is called *topological space* and every $o \in \mathcal{O}(X)$ is called an *open set* of the space X. Given an open set $o \in \mathcal{O}(X)$, its complement $c = X \setminus o$ is called a *closed set*. The collection of all closed sets of a topological space $(X, \mathcal{O}(X))$ is denoted by $\mathscr{C}(X)$ and, dually to the open sets, is closed under finite unions and arbitrary intersections. Closed sets are ordered by superset inclusion and form a complete lattice. A set $q \subseteq X$ is *saturated* if q is the intersection of all open sets $o \in \mathcal{O}(X)$ such that $q \subseteq o$. The collection of the saturated sets is denoted by $\mathscr{Q}(X)$. It is closed under arbitrary intersections and forms a completely distributive complete lattice.

A topological space is called $\mathscr{T}_0$ if for each pair of distinct points there exists an open set which contains one of the points and that does not contain the other; it is called $\mathscr{T}_1$ if for each pair of distinct points x, y there exists an open set o such that $x \in o$ and $y \notin o$; and it is called $\mathscr{T}_2$ if for each pair of distinct points x, y there exists two disjoint open sets o_1, o_2 such that $x \in o_1$ and $y \in o_2$. A topological space $(X, \mathcal{O}(X))$ is *sober* if for every completely prime filter $\mathscr{F} \subseteq \mathcal{O}(X)$ there is exactly one point $x \in X$ such that $\mathscr{F} = \{o \in \mathcal{O}(X) \mid x \in o\}$. Every sober space is $\mathscr{T}_0$ while every $\mathscr{T}_2$ space is sober. There are $\mathscr{T}_1$ spaces that are not sober and there are also sober spaces which are not $\mathscr{T}_1$.

Every topology $\mathcal{O}(X)$ on a set X induces a *specialization preorder* on X given by:

$$x \leqslant_{\mathcal{O}} y \Leftrightarrow (\forall o \in \mathcal{O}(X).\ x \in o \Rightarrow y \in o),$$

where $x, y \in X$. A topological space is $\mathscr{T}_0$ if and only if the specialization preorder is a partial order, while a topological space is $\mathscr{T}_1$ if and only if this preorder is equality (the discrete order).

Let $(X, \mathcal{O}(X))$ and $(Y, \mathcal{O}(Y))$ be two topological spaces. The *inverse* of a function $f : X \to Y$ is the function $f^{-1} : \mathscr{P}(Y) \to \mathscr{P}(X)$, defined by $f^{-1}(S) = \{x \mid f(x) \in S\}$. The function f is called *continuous* if $f^{-1}(o) \in \mathcal{O}(X)$ for every $o \in \mathcal{O}(Y)$ (or, equivalently, if the inverse of each closed set is closed). Topological spaces form a category **Sp** with the continuous functions as morphisms. We write $\mathbf{Sp}_0$ for the full subcategory of $\mathcal{T}_0$ spaces and **Sob** for the full subcategory of sober spaces.

3. Observation frames

Elements of a lattice can be thought of as predicates, where the meaning of the order $\leqslant$ is the entailment relation $\vdash$. The meet $\wedge$ corresponds to the logical "and" and the join $\vee$ corresponds to the logical "or". A complete lattice can be seen as being an (abstract) collection of predicates where equivalent predicates are identified.

It is closed under arbitrary conjunctions and disjunctions via $\bigwedge$ and $\bigvee$. Following [1, 29, 33] we see a frame as an (abstract) collection of finitely observable predicates where equivalent observations are identified. Finitely observable predicates are closed under arbitrary disjunctions and under finite conjunctions, represented by the arbitrary joins and finite meets. Even if a frame is closed under arbitrary meets, such meets need not represent the infinite conjunction. For example, in the poset of open sets of a topology on a set X we have the interior of the intersection $\bigcap S$ as the meet $\bigwedge S$ for every subset S of X. This leads us to consider a frame F (representing the observable predicates) together with a complete lattice L (representing all the predicates). Every observable predicate can be seen as a predicate, and hence we introduce a function that maps F to L. This map must preserve the logic of finite observations, that is, it preserves finite meets and arbitrary joins. This motivates the following definition.

Definition 3.1. An *observation frame* is a function $(-)^\vee : F \to L$ where $(F, \leqslant)$ is a frame and $(L, \sqsubseteq)$ is a complete lattice such that

$$(\bigvee A)^\vee = \bigsqcup A^\vee \text{ for all } A \subseteq F,$$
$$(\bigwedge B)^\vee = \bigsqcap B^\vee \text{ for all } \textit{finite } B \subseteq F$$

Notation 3.2. For an observation frame $(-)^\vee : F \to L$ as above, we will almost always write $(-)^\vee$ for the function involved. Hence we often omit it and simply write $F \rightarrowtriangle L$ for $(-)^\vee : F \to L$. For clarity, we use $(\leqslant, \bigvee, \bigwedge)$ in F and $(\sqsubseteq, \bigsqcup, \bigsqcap)$ in L. In case F (or L) is a subset of $\mathcal{P}(X)$ for some set X, we use the standard $\subseteq$, $\bigcup$ or $\bigcap$ whenever these coincide with the order, join or meet in F (or L).

For a function $f : A \to B$ and subset $S \subseteq A$, $f(S) = \{f(a) \mid a \in S\}$. In particular $S^\vee = \{a^\vee \mid a \in S\}$.

Note that in an observation frame $F \rightarrowtriangle L$ we have that the function $(-)^\vee$ preserves the top $\top = \bigwedge\emptyset$ and the bottom $\bot = \bigvee\emptyset$ elements.

Example 3.3. (i) Let $2 = \{\bot, \top\}$ be ordered by $\bot \leqslant \top$. It is a frame (and hence also a complete lattice). Therefore the identity function $id : 2 \to 2$ is an observation frame. We will refer to it as **2**. More generally, given a frame F, the identity function on F is an observation frame.

(ii) Let F be a frame and let $X = CPF(F)$ be the set of all the completely prime filters on F. The assignment

$$a \mapsto a^{\vee} = \{\mathscr{F} \in CPF(F) \mid a \in \mathscr{F}\}$$

yields a function $F \to \mathscr{P}(X)$ which preserves finite meets and arbitrary joins [19, 30, Proposition 4.3.5]. Hence we get an observation frame. In general this function $(-)^{\vee}$ is not injective, but it is so in case F is a spatial frame (see [19, 30] again).

(iii) Let $X = (X, \mathcal{O}(X))$ be a topological space. Since $\mathcal{O}(X)$ is a frame the inclusion $\mathcal{O}(X) \hookrightarrow \mathscr{P}(X)$ is an observation frame. We will denote it by $\Omega(X)$. Notice that $\mathbf{2}$ in (i) is $\Omega(1)$, where 1 is the one element (terminal) topological space.

(iv) Let X be a topological space and let $\mathscr{Q}(X)$ be the collection of saturated sets. The set $\mathscr{Q}(X)$ is closed under arbitrary intersections and hence it is a complete lattice. The inclusion $\mathcal{O}(X) \hookrightarrow \mathscr{Q}(X)$ forms an observation frame.

Next we organize observation frames into a category. We need an appropriate notion of morphism of observation frames.

Definition 3.4. Given two observation frames $F \rightarrowtail L$ and $G \rightarrowtail K$, a *morphism* ϕ *of observation frames* $(F \rightarrowtail L) \longrightarrow (G \rightarrowtail K)$ consists of a function $\phi : F \to G$ satisfying

(i) ϕ is a morphism of frames (i.e. ϕ preserves finite meets and arbitrary joins);

(ii) ϕ is (completely) multiplicative, that is, for $S, T \subseteq F$

$$\sqcap S^{\vee} \sqsubseteq \sqcap T^{\vee} \Rightarrow \sqcap \phi(S)^{\vee} \sqsubseteq \sqcap \phi(T)^{\vee}.$$

This gives a category (with composition as for ordinary functions) which is denoted by **OFrm**.

The idea is that a morphism between observation frames not only preserves the logic of observable properties, but also takes into account what happens to infinite conjunctions of these observable properties (which are usually outside the frame). A morphism $\phi : (F \rightarrowtail L) \longrightarrow (G \rightarrowtail K)$ in **OFrm** is clearly bottom and top preserving. Furthermore, if the function of the observation frame $G \rightarrowtail K$ is injective, then the multiplicativity condition (ii) of Definition 3.4 above implies that ϕ preserves all meets in F which are also preserved by $(-)^{\vee}$ (in particular all finite meets). Indeed let $A \subseteq F$ be such that $\sqcap A^{\vee} = (\bigwedge A)^{\vee}$. By multiplicativity of ϕ we have $\sqcap \phi(A)^{\vee} = \phi(\bigwedge A)^{\vee}$. Moreover by monotonicity of ϕ we have that $\phi(\bigwedge A) \leqslant \bigwedge \phi(A)$ and hence by monotonicity of $(-)^{\vee}$ we obtain $\phi(\bigwedge A)^{\vee} \sqsubseteq (\bigwedge \phi(A))^{\vee} \sqsubseteq \sqcap \phi(A)^{\vee}$. But $\sqcap \phi(A)^{\vee} = \phi(\bigwedge A)^{\vee}$, hence $\phi(\bigwedge A)^{\vee} = (\bigwedge \phi(A))^{\vee}$ and since $(-)^{\vee}$ is injective we have $\bigwedge \phi(A) = \phi(\bigwedge A)$.

Example 3.5. (i) For a topological space X there is an isomorphism in **OFrm** between the observation frame $\mathcal{O}(X) \hookrightarrow \mathscr{Q}(X)$ and $\Omega(X) = (\mathcal{O}(X) \hookrightarrow \mathscr{P}(X))$ given by the identity function on $\mathcal{O}(X)$.

(ii) Let X and Y be two topological spaces and $f : X \to Y$ be a continuous function (i.e. a map in the category of topological spaces **Sp**). Then f induces a morphism

$\Omega(f) : \Omega(Y) \to \Omega(X)$ in **OFrm** defined by its inverse image, i.e., $\Omega(f)(o) = f^{-1}(o) = \{x \in X \mid f(x) \in o\}$ for every $o \in \mathcal{O}(Y)$. We check the multiplicativity condition. Assume $S, T \subseteq \mathcal{O}(Y)$ with $\bigcap S \subseteq \bigcap T$. Then

$$\begin{aligned} x \in \bigcap\{\Omega(f)(o) \mid o \in S\} &\Leftrightarrow \forall o \in S.\ f(x) \in o \\ &\Leftrightarrow f(x) \in \bigcap S \\ &\Rightarrow f(x) \in \bigcap T \qquad \text{since } \bigcap S \subseteq \bigcap T \\ &\Leftrightarrow x \in \bigcap\{\Omega(f)(o) \mid o \in T\}. \end{aligned}$$

Thus we have a functor $\Omega : \mathbf{Sp} \to \mathbf{OFrm}^{op}$. Later it will be shown that Ω has a right adjoint.

The next definition introduces saturated elements of an observation frame.

Definition 3.6. Let $F \rhd\!\!\rightarrow L$ be an observation frame. The set of *saturated elements* of $F \rhd\!\!\rightarrow L$ is defined by

$$\mathcal{Q}(F \rhd\!\!\rightarrow L) = \{\sqcap A^{\vee} \mid A \subseteq F\}.$$

It is ordered by the restriction to $\mathcal{Q}(F \rhd\!\!\rightarrow L)$ of the order on L.

Saturated elements can be seen as observable specifications. Indeed, they are defined as the meets of a (possibly infinite) number of observable predicates. In Section 4 we will see that a point satisfies a specification (i.e. a saturated element) if and only if it satisfies all the observable predicates which constitute such specification. Sometimes specifications of this kind are referred to as liveness predicates [3, 22, 29].

Notice that for an observation frame $F \rhd\!\!\rightarrow L$, we have that $q \in \mathcal{Q}(F \rhd\!\!\rightarrow L)$ if and only if $q = \sqcap\{a^{\vee} \mid a \in F \text{ and } q \sqsubseteq a^{\vee}\}$. From right to left is clear. For the converse, let $q \in \mathcal{Q}(F \rhd\!\!\rightarrow L)$, say $q = \sqcap A^{\vee}$ for some $A \subseteq F$. Then $\sqcap\{x^{\vee} \mid x \in F \text{ and } q \sqsubseteq x^{\vee}\} = \sqcap A^{\vee}$. Indeed $q \sqsubseteq \sqcap\{x^{\vee} \mid x \in F \text{ and } q \sqsubseteq x^{\vee}\}$ is clear. Conversely $q = \sqcap A^{\vee}$ implies that $q \sqsubseteq a^{\vee}$ for every $a \in A$. Hence $A^{\vee} \subseteq \{x^{\vee} \mid x \in F \text{ and } q \sqsubseteq x^{\vee}\}$ which implies $\sqcap\{x^{\vee} \mid x \in F \text{ and } q \sqsubseteq x^{\vee}\} \sqsubseteq \sqcap A^{\vee} = q$.

We define the saturation $\mathcal{Q}(p)$ of a $p \in L$ by $\mathcal{Q}(p) = \sqcap\{a^{\vee} \in L \mid p \sqsubseteq a^{\vee}\}$. The set of saturated elements $\mathcal{Q}(F \rhd\!\!\rightarrow L)$ equals $\{p \in L \mid \mathcal{Q}(p) = p\}$.

Example 3.7. Given a topological space X, the collection of saturated sets $\mathcal{Q}(X)$ is the collection of arbitrary intersections of open sets and hence is $\mathcal{Q}(\Omega(X))$.

The following lemma states that saturated elements form a complete lattice.

Lemma 3.8. *For every observation frame $F \rhd\!\!\rightarrow L$, the induced partial order on the saturated elements $\mathcal{Q}(F \rhd\!\!\rightarrow L) \subseteq L$ yields a complete lattice in which arbitrary meets coincide with those in L.*

Proof. Let $S \subseteq \mathcal{Q}(F \rhd\!\!\rightarrow L)$ and suppose $s = \sqcap A_s^{\vee}$ with $A_s \subseteq F$ for every $s \in S$. We prove $\sqcap(\bigcup_{s \in S} A_s^{\vee}) = \sqcap S$ and hence $\sqcap S \in \mathcal{Q}(F \rhd\!\!\rightarrow L)$ since $\bigcup_{s \in S} A_s \subseteq F$.

($\sqsubseteq$) For every $s \in S$ and $a^{\vee} \in A_s^{\vee}$ we have $\sqcap(\bigcup_{s \in S} A_s^{\vee}) \sqsubseteq a^{\vee}$ and hence $\sqcap(\bigcup_{s \in S} A_s^{\vee}) \sqsubseteq \sqcap A_s^{\vee} = s$ which implies $\sqcap(\bigcup_{s \in S} A_s^{\vee}) \sqsubseteq \sqcap S$.

($\sqsupseteq$) For every $s \in S$, $\sqcap S \sqsubseteq s = \sqcap A_s^{\vee}$. But $\sqcap A_s^{\vee} \sqsubseteq a^{\vee}$ for every $a \in A_s$ and for every $s \in S$. Hence $\sqcap S \sqsubseteq \sqcap(\bigcup_{s \in S} A_s^{\vee})$.

Therefore $\sqcap S \in \mathcal{Q}(F \rhd\!\!\rightarrow L)$. It is the meet of S. □

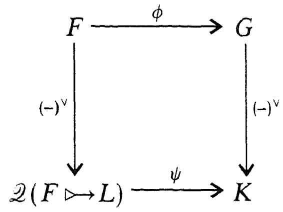
Given an observation frame $F \rhd\!\!\rightarrow L$, we have $a^{\vee} \in \mathcal{Q}(F \rhd\!\!\rightarrow L)$ for every $a \in F$; hence by Lemma 3.8 also the restriction $F \rhd\!\!\rightarrow \mathcal{Q}(F \rhd\!\!\rightarrow L)$ is an observation frame. Furthermore $F \rhd\!\!\rightarrow L$ is isomorphic to $F \rhd\!\!\rightarrow \mathcal{Q}(F \rhd\!\!\rightarrow L)$ in **OFrm**. This shows that in $F \rhd\!\!\rightarrow L$ we can distinguish only predicates which are finitely observable (i.e. in F) or can be deduced from the finite observations (i.e. in $\mathcal{Q}(F \rhd\!\!\rightarrow L)$). Predicates which are neither observable nor deducible are not captured in our definition of the category **OFrm**.

Next we provide a lemma that is a justification for our definition of the morphisms in **OFrm**. A morphism ϕ induces a unique meet-preserving function on the lattices.

Lemma 3.9. *Let $F \rhd\!\!\rightarrow L$ and $G \rhd\!\!\rightarrow K$ be two observation frames and $\phi : F \rightarrow G$ be a frame morphism. The following two statements are equivalent.*

(i) *ϕ is a morphism in* **OFrm**;

(ii) *there exists a unique function $\psi : \mathcal{Q}(F \rhd\!\!\rightarrow L) \rightarrow K$ preserving arbitrary meets for which the following diagram commutes*:



Proof. (i) $\Rightarrow$ (ii): Define for every $A \subseteq F$, $\psi(\sqcap A^{\vee}) = \sqcap \phi(A)^{\vee}$. It is well defined because if $\sqcap A^{\vee} = \sqcap B^{\vee}$ for $A, B \subseteq F$ then $\sqcap \phi(A)^{\vee} = \sqcap \phi(B)^{\vee}$ since ϕ is completely multiplicative. Furthermore, ψ makes the diagram commute and, by definition, it is meet-preserving. It is also the unique such, because for any meet-preserving function $\rho : \mathcal{Q}(F \rhd\!\!\rightarrow L) \rightarrow K$ which makes the diagram above commute, we have

$$\begin{aligned} \psi(\sqcap A^{\vee}) &= \sqcap \phi(A)^{\vee} \\ &= \sqcap \rho(A^{\vee}) && \text{commutativity of the diagram} \\ &= \rho(\sqcap A^{\vee}) && \rho \text{ is meet-preserving.} \end{aligned}$$

(ii) $\Rightarrow$ (i): It is enough to prove that ϕ is a completely multiplicative morphism. Let $S,T \subseteq F$ be such that $\sqcap S^{\vee} \sqsubseteq \sqcap T^{\vee}$. Then we have

$$\begin{aligned}
\sqcap \phi(S)^{\vee} &= \sqcap \psi(S^{\vee}) && \text{commutativity of the diagram}\\
&= \psi(\sqcap S^{\vee}) && \psi \text{ is meet preserving}\\
&\sqsubseteq \psi(\sqcap T^{\vee}) && \text{monotonicity of } \psi \text{ and } \sqcap S^{\vee} \sqsubseteq \sqcap T^{\vee}\\
&= \sqcap \psi(T^{\vee}) && \psi \text{ is meet preserving}\\
&= \sqcap \phi(T)^{\vee} && \text{commutativity of the diagram.} \qquad \square
\end{aligned}$$

Given an observation frame morphism $\phi : (F \rhd\!\!\rightarrow L) \to (G \rhd\!\!\rightarrow K)$ the induced meet preserving function $\psi : \mathcal{Q}(F \rhd\!\!\rightarrow L) \to K$ which makes the diagram of Lemma 3.9 commute preserves arbitrary joins of saturated elements in L which are images (under $(-)^{\vee}$) of elements in F. Indeed, for an arbitrary $S \subseteq F$ we have

$$\begin{aligned}
\psi(\sqcup S^{\vee}) &= \psi((\bigvee S)^{\vee}) && (-)^{\vee} \text{ is joins preserving}\\
&= (\phi(\bigvee S))^{\vee} && \text{commutativity of the diagram}\\
&= (\bigvee \phi(S))^{\vee} && \phi \text{ is joins preserving}\\
&= \sqcup \phi(S)^{\vee} && (-)^{\vee} \text{ is joins preserving}\\
&= \sqcup \psi(S^{\vee}) && \text{commutativity of the diagram.}
\end{aligned}$$

In the next remark the above is used to give a condition under which a meet-preserving function ψ on lattices induces a morphism in **OFrm**. Also, in the second part of the remark we will show that if all elements in L are saturated and both L and K are completely distributive, then ψ preserves both arbitrary meets and arbitrary joins.

Remark 3.10. (i) Let $F \rhd\!\!\rightarrow L$ and $G \rhd\!\!\rightarrow K$ be two observation frames where $G \rhd\!\!\rightarrow K$ is an isomorphism, and let $\psi : \mathcal{Q}(F \rhd\!\!\rightarrow L) \to K$ be a function such that

$$\psi(\sqcap S^{\vee}) = \sqcap \psi(S^{\vee})$$
$$\psi(\sqcup S^{\vee}) = \sqcup \psi(S^{\vee})$$

for all $S \subseteq F$. Then there is a unique morphism in **OFrm** $\phi : F \to G$ such that $\phi(a)^{\vee} = \psi(a^{\vee})$ for every $a \in F$. Since $G \rhd\!\!\rightarrow K$ is an isomorphism there is a unique way of defining ϕ in order to satisfy the conditions in Lemma 3.9: $\phi(a) = b$ for the unique $b \in G$ such that $b^{\vee} = \psi(a^{\vee})$. Clearly ϕ is a frame morphism and by Lemma 3.9 it is also a morphism in **OFrm**.

(ii) Let $F \rhd\!\!\rightarrow L$ and $G \rhd\!\!\rightarrow K$ be two observation frames such that $L = \mathcal{Q}(F \rhd\!\!\rightarrow L)$ and both L and K are completely distributive complete lattices. Let also $\phi : (F \rhd\!\!\rightarrow L) \to (G \rhd\!\!\rightarrow K)$ be an observation frame morphism. Then the induced meet preserving function

$\psi : L \to K$ given in Lemma 3.9 preserves arbitrary joins. Take an arbitrary $S \subseteq L$, then

$$\bigsqcup S = \bigsqcup_{q \in S} \bigsqcap \{a^{\vee} \in L \mid q \sqsubseteq a^{\vee}\}$$

because $L = \mathcal{Q}(F \rhd\!\!\rightarrowtail L)$. Since L is completely distributive we have

$$\bigsqcup_{q \in S} \bigsqcap \{a^{\vee} \in L \mid p \sqsubseteq a^{\vee}\} = \bigsqcap \{\bigsqcup f(q) | f : S \to L \text{ and } f(q) \in \{a^{\vee} \in L \mid q \sqsubseteq a^{\vee}\}\}.$$

By Lemma 3.9, $\psi : L \to K$ preserves all joins in L of the form $\bigsqcup S^{\vee}$ where $S \subseteq F$, because ψ makes the diagram of Lemma 3.9 commute. But ψ preserves also arbitrary meets in L, therefore we have

$$\begin{aligned}
\psi(\bigsqcup S) &= \psi(\bigsqcup_{q \in S} \bigsqcap \{a^{\vee} \in L \mid q \sqsubseteq a^{\vee}\}) \\
&= \psi(\bigsqcap \{\bigsqcup f(q) | f : S \to L \text{ and } f(q) \in \{a^{\vee} \in L \mid q \sqsubseteq a^{\vee}\}\}) \\
&= \bigsqcap \{\psi(\bigsqcup f(q)) | f : S \to L \text{ and } f(q) \in \{a^{\vee} \in L \mid q \sqsubseteq a^{\vee}\}\} \\
&= \bigsqcap \{\bigsqcup \psi(f(q)) | f : S \to L \text{ and } f(q) \in \{a^{\vee} \in L \mid q \sqsubseteq a^{\vee}\}\} \\
&\stackrel{*}{=} \bigsqcup_{q \in S} \bigsqcap \{\psi(a^{\vee}) \in K \mid q \sqsubseteq a^{\vee}\} \\
&= \bigsqcup_{q \in S} \psi(\bigsqcap \{a^{\vee} \in K \mid q \sqsubseteq a^{\vee}\}) \\
&= \bigsqcup_{q \in S} \psi(q) \\
&= \bigsqcup \psi(S).
\end{aligned}$$

where $\stackrel{*}{=}$ holds because K is a completely distributive complete lattice. Notice that for every topological space X, $\mathcal{O}(X) \hookrightarrow \mathcal{Q}(X)$ is an observation frame for which $\mathcal{Q}(X) = \mathcal{Q}(\mathcal{O}(X) \hookrightarrow \mathcal{Q}(X))$ is a completely distributive complete lattice.

3.1. M-filters and M-prime elements

In this subsection we introduce the notions of M-filter and of M-prime element of an observation frame. They will be used later to construct the points of a topological space associated with an observation frame. Furthermore we prove that completely prime M-filters and the M-prime elements of an observation frame $L \rhd\!\!\rightarrowtail F$ and morphisms from $L \rhd\!\!\rightarrowtail F$ to **2** in **OFrm** are essentially the same. Later in the paper (Lemmas 3.12 and Lemma 5.5) we state the relationship with the ordinary notions of filter and prime element of a frame.

Definition 3.11. Let $F \rhd\!\!\rightarrowtail L$ be an observation frame.

(i) *A (completely) multiplicative filter* (M-filter for short) is a set $\mathcal{U} \subseteq F$ such that

$$\bigsqcap \mathcal{U}^{\vee} \sqsubseteq a^{\vee} \text{ implies } a \in \mathcal{U} \text{ for every } a \in F$$

(ii) An M-filter $\mathcal{U} \subseteq F$ is called completely prime if for every $S \subseteq F$

$$\bigvee S \in \mathcal{U} \text{ implies } \exists s \in S.\, s \in \mathcal{U}.$$

The set of all completely prime M-filters is denoted $CPMF(F \rhd\!\!\rightarrowtail L)$ while the set of all M-filters is denoted $MF(F \rhd\!\!\rightarrowtail L)$. Both are posets under inclusion.

Notice that a completely prime M-filter $\mathscr{U}$ cannot contain $\bot_F$ because $\bot_F = \bigvee\emptyset$ and hence by the definition above there should be $s \in \emptyset$ such that $s \in \mathscr{U}$.

Lemma 3.12. *Every M-filter $\mathscr{U} \subseteq F$ in an observation frame $F \rhd\!\!\rightarrow L$ is a filter of F.*

Proof. Let $\mathscr{U} \subseteq F$ be an M-filter of the observation frame $F \rhd\!\!\rightarrow L$. It is nonempty, because $\bigsqcap \mathscr{U}^\vee \sqsubseteq \top_L = \top_F^\vee$ and hence $\top_F \in \mathscr{U}$. It is also upper closed with respect to the order in F because for every $a, b \in F$ with $a \in \mathscr{U}$, if $a \leqslant b$ then by monotonicity of $(-)^\vee$ we have $\bigsqcap \mathscr{U}^\vee \sqsubseteq a^\vee \sqsubseteq b^\vee$. Hence also $b \in \mathscr{U}$ because $\mathscr{U}$ is an M-filter.

Finally, suppose $a, b \in \mathscr{U}$. Then $\bigsqcap \mathscr{U}^\vee \sqsubseteq a^\vee$ and $\bigsqcap \mathscr{U}^\vee \sqsubseteq b^\vee$ implies $\bigsqcap \mathscr{U}^\vee \sqsubseteq a^\vee \sqcap b^\vee = (a \wedge b)^\vee$. Since $\mathscr{U}$ is an M-filter we have $a \wedge b \in \mathscr{U}$. □

More generally we have for every M-filter $\mathscr{U} \subseteq F$

$$\mathscr{U}^\vee = \uparrow(\textstyle\bigsqcap \mathscr{U}^\vee) \cap F^\vee$$

where $\uparrow$ denotes the upper closure with respect to the order of L. Indeed for every $a \in F$ such that $a^\vee \in \mathscr{U}^\vee$ we have $\bigsqcap \mathscr{U}^\vee \sqsubseteq a^\vee$ and hence $a^\vee \in \uparrow(\bigsqcap \mathscr{U}^\vee) \cap F^\vee$. On the other hand, if $x \in \uparrow(\bigsqcap \mathscr{U}^\vee) \cap F^\vee$, then there exists an $a \in F$ such that $a^\vee = x$ and $\bigsqcap \mathscr{U}^\vee \sqsubseteq a^\vee$. Hence $a \in \mathscr{U}$ because $\mathscr{U}$ is an M-filter.

Example 3.13. Let X be a topological space and consider the observation frame $\Omega(X)$.

(i) Every saturated set $q \in \mathscr{Q}(X)$ induces an M-filter $\mathscr{U}(q) = \{o \in \mathscr{O}(X) \mid q \subseteq o\}$. Indeed, if $\bigcap \mathscr{U}(q) \subseteq o$ then $q \subseteq o$ because by definition of saturated sets $q = \bigcap \mathscr{U}(q)$. Therefore $o \in \mathscr{U}(q)$.

(ii) For every $x \in X$ the set $\mathscr{U}_0(x) = \{o \in \mathscr{O}(X) \mid x \in o\}$ is a completely prime M-filter. It is an M-filter because for every $o \in \mathscr{O}(X)$, if $\bigcap \mathscr{U}_0(x) \subseteq o$ then $o \in \mathscr{U}_0(x)$ since by definition $x \in \bigcap \mathscr{U}_0(x) \subseteq o$. It is completely prime since for every $S \subseteq \mathscr{O}(X)$ and $o \in \mathscr{U}_0(x)$ such that $o \subseteq \bigcup S$ we have $x \in o \subseteq \bigcup S$. Hence there exists $s \in S$ such that $x \in s$. Therefore by definition $s \in \mathscr{U}_0(x)$.

Next we introduce appropriate prime elements for an observation frame.

Definition 3.14. For an observation frame $F \rhd\!\!\rightarrow L$, an element $p \in F$ is called *completely multiplicative prime* (M-prime for short) if for all $S \subseteq F$ it holds that

$$\textstyle\bigsqcap S^\vee \sqsubseteq p^\vee \text{ implies } \exists s \in S.\, s \leqslant p$$

The set of all M-prime elements of $F \rhd\!\!\rightarrow L$ is denoted by $MP(F \rhd\!\!\rightarrow L)$.

Notice that $\top \in F$ cannot be an M-prime element of $F \rhd\!\!\rightarrow L$ since $\bigsqcap S^\vee = \top$ for $S = \emptyset$.

Example 3.15. Consider the observation frame $\Omega(X)$ of a topological space X. Define for every $x \in X$ the open set

$$o_x = int(X \setminus \{x\}) = \bigcup\{o \in \mathscr{O}(X) \mid x \notin o\} \subseteq X \setminus \{x\},$$

where $int(\cdot)$ is the interior operator associated with the topology on X. By definition o_x is the greatest (with respect to subset inclusion) open set not containing x, that is, for an open o', $x \notin o'$ if and only if $o' \subseteq o_x$. It is also an M-prime element. Indeed, for every $S \subseteq \mathcal{O}(X)$ if $\bigcap S \subseteq o_x$ then $x \notin \bigcap S$ because otherwise one would have $x \in \bigcap S \subseteq o_x$ contradicting $x \notin o_x$. But then there exists an $s \in S$ such that $x \notin s$. But o_x is the greatest open set not containing x, hence $s \subseteq o_x$. Thus $o_x \in MP(\Omega(X))$ for every $x \in X$.

Notice that for every $o \in \mathcal{O}(X)$ we have $\bigcap\{o_x | x \notin o\} = o$:

($\subseteq$) If $y \in \bigcap\{o_x | x \notin o\}$ then $y \in o$ because otherwise $y \in (X \backslash o)$ and hence $o_y \in \{o_x | x \notin o\}$. But this yields $y \in \bigcap\{o_x | x \notin o\} \subseteq o_y$, a contradiction.

($\supseteq$) For every $x \in (X \setminus o)$ we have $o \subseteq (X \backslash \{x\})$. Hence by idempotency and monotonicity of the interior operator we obtain $o = int(o) \subseteq int(X \backslash \{x\}) = o_x$ for every $x \notin o$. Therefore $o \subseteq \bigcap\{o_x | x \notin o\}$.

(For the case when $o = X$ observe that $\{o_x | x \notin o\} = \emptyset$ and then $\bigcap \emptyset = X = o$). Next we show that every M-prime element in $\Omega(X)$ is of the form o_x for some $x \in X$. Indeed, let $p \in MP(\Omega(X))$. Since $p \in \mathcal{O}(X)$ we have just seen that $\bigcap\{o_x | x \notin p\} \subseteq p$. But then $o_x \subseteq p$ for some $x \notin p$. The latter yields $p \subseteq o_x$ and hence $p = o_x$. This fact will be crucial later on for obtaining our duality.

Finally, if X is a $\mathscr{T}_0$ topological space then clearly every M-prime element of $\Omega(X)$ is of the form o_x for a unique $x \in X$.

The next lemma is the main result of this subsection. It gives isomorphisms between M-filters, M-prime elements of an observation frame $F \rhd\!\!\to L$ and also M-morphisms from $F \rhd\!\!\to L$ to **2**.

Lemma 3.16. *For an observation frame $F \rhd\!\!\to L$ there are bijective correspondences between*

(i) *morphisms $\phi : (F \rhd\!\!\to L) \to \mathbf{2}$ in* **OFrm**,

(ii) *completely prime M-filters $\mathscr{U} \subseteq F$,*

(iii) *M-prime elements $p \in F$.*

The correspondences are given by

$$\begin{array}{lll}
\text{(i)} \Rightarrow \text{(ii)} & \phi \mapsto \mathscr{U}_\phi & = \{a \in F \mid \phi(a) = \top\}; \\
\text{(ii)} \Rightarrow \text{(i)} & \mathscr{U} \mapsto \phi_{\mathscr{U}} & = \lambda a \in F. \begin{cases} \top & \textit{if } a \in \mathscr{U} \\ \bot & \textit{otherwise;} \end{cases} \\
\text{(ii)} \Rightarrow \text{(iii)} & \mathscr{U} \mapsto p_{\mathscr{U}} & = \bigvee\{a \in F \mid a \notin \mathscr{U}\}; \\
\text{(iii)} \Rightarrow \text{(ii)} & p \mapsto \mathscr{U}_p & = F \backslash (\downarrow p); \\
\text{(iii)} \Rightarrow \text{(i)} & p \mapsto \phi_p & = \lambda a \in F. \begin{cases} \bot & \textit{if } a \leqslant p \\ \top & \textit{otherwise;} \end{cases} \\
\text{(i)} \Rightarrow \text{(iii)} & \phi \mapsto p_\phi & = \bigvee\{a \in F \mid \phi(a) = \bot\}.
\end{array}$$

Proof. Let $F \rhd\!\!\to L$ be an observation frame, $\phi : (F \rhd\!\!\to L) \to \mathbf{2}$ be a morphism in **OFrm**, $\mathscr{U} \subseteq F$ be a completely prime M-filter and $p \in F$ be an M-prime element. We prove only (i) ⇒ (ii) ⇒ (iii) ⇒ (i). The verification of the other correspondences is left to the reader.

(i) ⇒ (ii) We have to prove that $\mathscr{U}_\phi$ is a completely prime M-filter. We start by proving that $\mathscr{U}_\phi$ is an M-filter. For every $x \in F$ such that $\sqcap \mathscr{U}_\phi^\vee \sqsubseteq x^\vee$ we have $\bigwedge\{\phi(a) | a \in \mathscr{U}_\phi\} \leqslant \phi(x)$ since ϕ is a morphism in **OFrm**. But $a \in \mathscr{U}_\phi$ if and only if $\phi(a) = \top$ by definition, hence also $\phi(x) = \top$. Therefore $x \in \mathscr{U}_\phi$.

It remains to show that $\mathscr{U}_\phi$ is completely prime. Let $S \subseteq F$ and $a \in \mathscr{U}_\phi$ be such that $a \leqslant \bigvee S$. Then $\phi(\bigvee S) = \top$ because ϕ is a frame morphism and $\top = \phi(a) \leqslant \phi(\bigvee S) = \bigvee \phi(S)$. Therefore there is $s \in S$ such that $\phi(s) = \top$, that is, there is $s \in S$ such that $s \in \mathscr{U}_\phi$.

(ii) ⇒ (iii): We have to prove that $p_\mathscr{U}$ is an M-prime element. Let $S \subseteq F$ be such that $\sqcap S^\vee \sqsubseteq p_\mathscr{U}^\vee$. Then $\sqcap S^\vee \sqsubseteq (\bigvee\{a \in F \,|\, a \notin \mathscr{U}\})^\vee = \bigsqcup\{a \in F \,|\, a \notin \mathscr{U}\}^\vee$. There must exist a $s \in S$ such that $s \notin \mathscr{U}$ because if not, $S \subseteq \mathscr{U}$ would imply $\sqcap \mathscr{U}^\vee \sqsubseteq \sqcap S^\vee \sqsubseteq (\bigvee\{a \in F \,|\, a \notin \mathscr{U}\})^\vee$ and hence $\bigvee\{a \in F \,|\, a \notin \mathscr{U}\} \in \mathscr{U}$ as $\mathscr{U}$ is an M-filter. But it is also completely prime, hence contradicting that there exists $a \notin \mathscr{U}$ such that $a \in \mathscr{U}$.

(iii) ⇒ (i): We have to prove that ϕ_p is a morphism in **OFrm**. It is easily verified that it is a frame morphism and hence we concentrate on the proof that ϕ_p is completely multiplicative. Let $S, T \subseteq F$ be such that $\sqcap S^\vee \sqsubseteq \sqcap T^\vee$. Assume $\bigwedge \phi_p(S) = \top$ but suppose $\bigwedge \phi_p(T) = \bot$. Then there exists $t \in T$ such that $\phi_p(t) = \bot$ and hence $t \leqslant p$. Since p is an M-prime element, we have that $\sqcap S^\vee \sqsubseteq \sqcap T^\vee \sqsubseteq t^\vee \sqsubseteq p^\vee$ implies there exists $s \in S$ such that $s \leqslant p$. Hence $\phi_p(s) = \bot$ contradicting $\bigwedge \phi_p(S) = \top$. □

Notice that applying the inverse image of a morphism in **OFrm** to a completely prime M-filter yields again a completely prime M-filter. Indeed let $\phi: (F \rhd\!\!\!\rightarrow L) \to (G \rhd\!\!\!\rightarrow K)$ be a morphism in **OFrm** and let $\mathscr{U} \subseteq G$ be a completely prime M-filter. Then $\phi_\mathscr{U}$: $(G \rhd\!\!\!\rightarrow K) \to \mathbf{2}$ is also a morphism in **OFrm** which hence yields by composition a morphism from $F \rhd\!\!\!\rightarrow L$ to **2**, or, equivalently, a completely prime M-filter $\phi^{-1}(\mathscr{U}) \subseteq F$.

3.2. *Observation frames and topological spaces*

In this section we define a point functor *Pt* from the opposite of the category of observation frames to the category **Sp** of topological spaces by topologizing the M-prime elements. We show that *Pt* is right adjoint to the functor Ω (for its definition see Example 3.5).

Definition 3.17. Given an observation frame $F \rhd\!\!\!\rightarrow L$ define for every $a \in F$ the *complete multiplicative hull* of a (the M-hull of a) as the set

$$\nabla(a) = \{p \in MP(F \rhd\!\!\!\rightarrow L) \,|\, a \leqslant p\} = \uparrow a \cap MP(F \rhd\!\!\!\rightarrow L).$$

Define also the complement $\Delta(a)$ in $MP(F \rhd\!\!\!\rightarrow L)$ of the M-hull of a by

$$\Delta(a) = MP(F \rhd\!\!\!\rightarrow L) \backslash \nabla(a) = MP(F \rhd\!\!\!\rightarrow L) \backslash \uparrow a = \{p \in MP(F \rhd\!\!\!\rightarrow L) \,|\, a \not\leqslant p\}.$$

Lemma 3.18. *Let $F \rhd\!\!\rightarrow L$ be an observation frame. Then*

$$\Delta(\bot) = \emptyset;$$
$$\Delta(\top) = MP(F \rhd\!\!\rightarrow L);$$
$$\Delta(\bigvee A) = \bigcup\{\Delta(a) \mid a \in A\} \quad \textit{for all } A \subseteq F;$$
$$\Delta(\bigwedge B) = \bigcap\{\Delta(b) \mid b \in B\} \quad \textit{for finite } B \subseteq F;$$

Proof. We prove only the last item. The other ones are trivial.

$$\begin{aligned} p \in \bigcap\{\Delta(b) \mid b \in B\} &\Leftrightarrow p \in MP(F \rhd\!\!\rightarrow L) \text{ and } \forall b \in B.\ b \not\leqslant p \\ &\overset{*}{\Leftrightarrow} p \in MP(F \rhd\!\!\rightarrow L) \text{ and } \textstyle\bigwedge B \not\leqslant p \\ &\Leftrightarrow p \in \Delta(\textstyle\bigwedge B), \end{aligned}$$

where the implication ($\overset{*}{\Leftarrow}$) is trivial and for ($\overset{*}{\Rightarrow}$) we use that p is an M-prime element: if $\bigwedge B \leqslant p$ then also $\sqcap B^\vee \sqsubseteq p^\vee$ and hence $b \leqslant p$ for some $b \in B$. □

Corollary 3.19. *Let $F \rhd\!\!\rightarrow L$ be an observation frame. The collection of sets of the form $\Delta(a)$ for every $a \in F$ forms a topology on $MP(F \rhd\!\!\rightarrow L)$ which is called the M-hull topology (denoted by $\mathcal{O}_\Delta(MP(F \rhd\!\!\rightarrow L))$).*

Clearly the sets of the form $\nabla(a)$ are closed sets of the M-hull topology on $MP(F \rhd\!\!\rightarrow L)$ for every $a \in F$.

Definition 3.20. For every observation frame $F \rhd\!\!\rightarrow L$ define $Pt(F \rhd\!\!\rightarrow L)$ to be the topological space $MP(F \rhd\!\!\rightarrow L)$ endowed with the M-hull topology.

Remark 3.21. For every observation frame $F \rhd\!\!\rightarrow L$, the topological space $Pt(F \rhd\!\!\rightarrow L)$ is $\mathcal{T}_0$. Indeed, let $p, q \in MP(F \rhd\!\!\rightarrow L)$ be such that $p \leqslant_{\mathcal{O}} q$ and $q \leqslant_{\mathcal{O}} p$ in the specialization preorder induced by the M-hull topology on $MP(F \rhd\!\!\rightarrow L)$. Then $p \in \Delta(a)$ if and only if $q \in \Delta(a)$ for every $a \in F$. Hence for every $a \in F$ we have $a \not\leqslant p$ if and only if $a \not\leqslant q$, that is, $p = q$. Hence the specialization preorder is a partial order, or, equivalently, the topological space $Pt(F \rhd\!\!\rightarrow L)$ is $\mathcal{T}_0$.

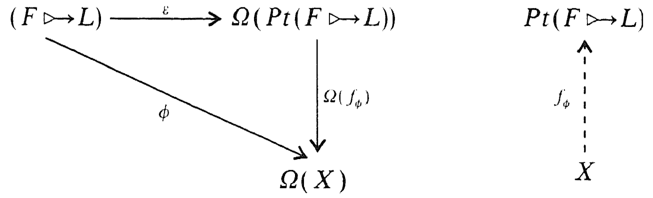
Lemma 3.22. *Let $F \rhd\!\!\rightarrow L$ be an observation frame. The map $\varepsilon : F \rightarrow \mathcal{O}_\Delta(MP(F \rhd\!\!\rightarrow L))$ defined by $\varepsilon(a) = \Delta(a)$ for every $a \in F$ is a morphism in* **OFrm** *from $F \rhd\!\!\rightarrow L$ to the observation frame $\Omega(Pt(F \rhd\!\!\rightarrow L))$. It is clearly surjective as a function.*

Proof. By Lemma 3.18, ε is a frame morphism. Also, by definition of the M-hull topology, ε is surjective. Thus it remains to prove that it is completely multiplicative. Let $S, T \subseteq F$ be such that $\sqcap S^\vee \sqsubseteq \sqcap T^\vee$ and take $p \in \bigcap \varepsilon(S)$. From the definition of ε and of the M-hull we have that $p \in MP(F \rhd\!\!\rightarrow L)$ and $s \not\leqslant p$ for every $s \in S$. We claim that also $t \not\leqslant p$ for every $t \in T$. If not, then there exists $t \in T$ such that $t \leqslant p$ and hence $t^\vee \sqsubseteq p^\vee$. But then $\sqcap S^\vee \sqsubseteq \sqcap T^\vee \sqsubseteq t^\vee \sqsubseteq p^\vee$ implies that there exists $s \in S$

such that $s \leqslant p$ since p is an M-prime element contradicting the hypothesis. Therefore $p \in \Delta(t)$ for every $t \in T$ and hence by definition of ε we get $p \in \bigcap \varepsilon(T)$. $\square$

Theorem 3.23. *Let X be a topological space, $F \rhd\!\!\rightarrow L$ be an observation frame and $\phi : (F \rhd\!\!\rightarrow L) \rightarrow \Omega(X)$ be a morphism in* **OFrm**. *Then there is a unique continuous function $f_\phi : X \rightarrow Pt(F \rhd\!\!\rightarrow L)$ in* **Sp** *such that $\Omega(f_\phi) \circ \varepsilon = \phi$;*

This extends Pt to a functor from **OFrm**^{op} *to* **Sp** *which is right adjoint of Ω.*



Proof. Let $a \in F$. In order to obtain the required commutativity we have to prove

$$\Omega(f_\phi)(\varepsilon(a)) = \{x \in X \mid f_\phi(x) \in \varepsilon(a)\} = \phi(a)$$

or, equivalently,

$$\forall x \in X.\ f_\phi(x) \in \varepsilon(a) \Leftrightarrow x \in \phi(a)$$

that is, by definition of $\varepsilon(a)$

$$\forall x \in X.\ f_\phi(x) \in \Delta(a) \Leftrightarrow x \in \phi(a)$$

that is, by definition of $\Delta(a)$

$$\forall x \in X.\ a \not\leqslant f_\phi(x) \Leftrightarrow x \in \phi(a).$$

This determines $f_\phi(x)$ uniquely as $\bigvee\{b \in F \mid x \notin \phi(b)\}$. Indeed for all $x \in X$ if $a \not\leqslant f_\phi(x)$ then $x \in \phi(a)$ because otherwise we would have $a \in \{b \in F \mid x \notin \phi(b)\}$ and hence the contradiction $a \leqslant \bigvee\{b \in F \mid x \notin \phi(b)\} = f_\phi(x)$.

Conversely, if $x \in \phi(a)$ then $a \not\leqslant f_\phi(x)$ because otherwise $a \leqslant f_\phi(x) = \bigvee\{b \in F \mid x \notin \phi(b)\}$ would imply, upon applying ϕ,

$$\phi(a) \leqslant \phi(\bigvee\{b \in F \mid x \notin \phi(b)\}) = \bigcup\{\phi(b) \in \mathcal{O}(X) \mid x \notin \phi(b)\}.$$

Since $x \in \phi(a)$ we would get that there exists $b \in F$ such that $x \in \phi(b)$ and $x \notin \phi(b)$.

Next we show that $f_\phi(x)$ is an M-prime element, i.e. $f_\phi(x) \in MP(F \rhd\!\!\rightarrow L)$. Let $S \subseteq F$ be such that $\bigsqcap S^\vee \sqsubseteq f_\phi(x)^\vee$. Then from the definition of $f_\phi(x)$ and upon applying (multiplicative) ϕ we obtain

$$\bigcap \phi(S) \subseteq \phi(\bigvee\{a \in F \mid x \notin \phi(a)\}) = \bigcup\{\phi(a) \in \mathcal{O}(X) \mid x \notin \phi(a)\}.$$

Hence there exists $s \in S$ such that $s \leqslant f_\phi(x)$ because otherwise for all $s \in S$ we would have $s \not\leqslant f_\phi(x)$ and hence by the above, $x \in \phi(s)$ for every $s \in S$. But then $x \in \bigcap \phi(S)$ which implies there exists $a \in F$ such that $x \in \phi(a)$ and $x \notin \phi(a)$.

The function f_ϕ is also continuous. Let $a \in F$ and consider the open set in the M-hull topology $\Delta(a)$. Then we have

$$\begin{aligned} f_\phi^{-1}(\Delta(a)) &= \{x \in X \mid f_\phi(x) \in \Delta(a)\} \\ &= \{x \in X \mid a \not\leq f_\phi(x)\} && \text{definition of } \Delta(a) \\ & && \text{and } f_\phi(x) \text{ is an M-prime element} \\ &= \{x \in X \mid x \in \phi(a)\} \\ &= \phi(a). \end{aligned}$$

But $\phi(a) \in \mathcal{O}(X)$ is open, therefore f_ϕ is continuous. □

The unit of the adjunction is given by the function η defined in the following Lemma.

Lemma 3.24. *Let X be a topological space. Then the unit of the adjunction between* **OFrm**^{op} *and* **Sp** *is given by function $\eta : X \to Pt(\Omega(X))$ defined by $\eta(x) = int(X \setminus \{x\}) = o_x$. It is a continuous surjective function in* **Sp**. *Moreover, η is injective and preserves open sets if and only if X is $\mathscr{T}_0$.*

Proof. By Theorem 3.23 the unit of the adjunction between **OFrm**^{op} and **Sp** is uniquely determined by the function f_ϕ, where $\phi : \Omega(X) \to \Omega(X)$ is the identity morphism in **OFrm**^{op}. Therefore for every space X, the unit $\eta : X \to Pt(\Omega(X))$ is defined as $\eta(x) = \bigcup\{o \in \mathcal{O}(X) \mid x \notin o\} = o_x$. Next we show η is a continuous surjective function in **Sp**.

We have already seen in Example 3.15 that the M-prime elements of $\Omega(X)$ are exactly those of the form $o_x = int(X \setminus \{x\})$ in a topological space X. Hence η is clearly onto. Let us now check it is also continuous. For $o \in \mathcal{O}(X)$ we have

$$\begin{aligned} \eta^{-1}(\Delta(o)) &= \{x \in X \mid \eta(x) \in \Delta(o)\} \\ &= \{x \in X \mid o \not\subseteq \eta(x)\} \\ &= \{x \in X \mid x \in o\} \\ &= o. \end{aligned}$$

If X is a $\mathscr{T}_0$ space, then we have seen in Example 3.15 that the M-prime elements of $\Omega(X)$ are exactly those of the form $o_x = int(X \setminus \{x\})$ for a unique $x \in X$. Therefore η is injective and since it is also onto, it is an isomorphism between X and $MP(\Omega(X))$. It remains to prove it is also an open map, i.e. preserves open sets. For $o \in \mathcal{O}(X)$ we have

$$\begin{aligned} \eta(o) &= \{\eta(x) \in MP(\Omega(X)) \mid x \in o\} \\ &= \{\eta(x) \in MP(\Omega(X)) \mid o \not\subseteq \eta(x)\} && \text{by definition of } \eta(x) \\ &= \{\eta(x) \in MP(\Omega(X)) \mid \eta(x) \in \Delta(o)\} && \text{by definition } \Delta(o) \end{aligned}$$

$= \{p \in MP(\Omega(X)) \mid p \in \Delta(o)\}$ $\quad \eta$ is an isomorphism

$= \Delta(o)$.

which is open in the M-hull topology. Therefore, if X is a $\mathscr{T}_0$ space then η is an isomorphism in **Sp**.

Finally, if η is injective and open then it forms an isomorphism in **Sp** between X and $Pt(\Omega(X))$. But for every observation frame $F \rightarrowtail L$ the space $Pt(F \rightarrowtail L)$ is $\mathscr{T}_0$, hence also X is $\mathscr{T}_0$. □

Recall now that an adjunction $\langle F, G, \eta, \varepsilon \rangle : \mathbf{A} \to \mathbf{B}$ is called *Galois* if it restricts to an equivalence between the categories $F(\mathbf{A})$ and $G(\mathbf{B})$ (here $F(\mathbf{A})$ denotes the full sub-category of B whose objects are in the image of F and $G(\mathbf{B})$ denotes the full subcategory of A whose objects are in the image of G). In [16] it is shown that an adjunction $\langle F, G, \eta, \varepsilon \rangle : \mathbf{A} \to \mathbf{B}$ is Galois if and only if it restricts to a reflection of $\mathbf{A}$ into $F(\mathbf{A})$.

By Lemma 3.24 and Remark 3.21 we have that the adjunction of Theorem 3.23 restricts to a reflection of $\mathbf{Sp}_0$ into the full image of the functor $\Omega : \mathbf{Sp}_0 \to \mathbf{OFrm}$. Therefore the adjunction between $\mathbf{Sp}_0$ and **OFrm** is Galois. In the next section we will characterize a full sub-category of **OFrm** and hence we will prove directly that the adjunction of Theorem 3.23 restricts to an equivalence.

3.3. Duality for $\mathscr{T}_0$ topological spaces

In this subsection we characterize a subcategory of **OFrm** which is the dual of the category of $\mathscr{T}_0$ topological spaces using the adjunction of Theorem 3.23. The next definition and the subsequent proposition are standard and can be found for example in [10, I, Definition 3.8 and Proposition 3.9].

Definition 3.25. A subset X of a complete lattice L is said to be *order generating* in L (or equivalently L is said to be order generated by X) if

$$x = \bigwedge(\uparrow x \cap X) = \bigwedge\{y \in X \mid x \leqslant y\}$$

for every $x \in L$.

Proposition 3.26. *For $X \subseteq L$ where L is a complete lattice the following statements are equivalent.*

(i) *X is order generating in L;*
(ii) *every element of L can be written as a (possibly infinite) meet of a subset of X;*
(iii) *L is the smallest subset containing X closed under arbitrary meets;*
(iv) *whenever $y \nleqslant x$, then there is a $p \in X$ with $x \leqslant p$ but $y \nleqslant p$.*

Example 3.27. (i) Let X be a topological space and consider the observation frame $\Omega(X)$. Then $\mathscr{O}(X)$ is order generated by $MP(\Omega(X))$. In Example 3.15 we have already seen that every M-prime element of $\Omega(X)$ is of the form $o_x = int(X \setminus \{x\})$ for some $x \in X$. Therefore we have to show $o = \bigwedge\{o_x | o \subseteq o_x\}$ for every $o \in \mathcal{O}(X)$.

Clearly, $o \subseteq \bigwedge\{o_x | o \subseteq o_x\}$. To prove the other direction of the inclusion, consider $y \in \bigwedge\{o_x | o \subseteq o_x\}$ and suppose towards a contradiction that $y \notin o$. Then $o_y = int(X \setminus \{y\})$ is the greatest open set not containing y, so $o \subseteq o_y$. But then $o_y \in \{o_x | o \subseteq o_x\}$ and hence $y \in o_y$ because, obviously, $y \in \bigwedge\{o_x | o \subseteq o_x\} \subseteq o_y$.

(ii) Let $F \rhd\!\!\rightarrow L$ be an observation frame. The complete lattice $\mathcal{Q}(F \rhd\!\!\rightarrow L)$ is the smallest subset closed under arbitrary meets of L which contains $F^\vee$ by Lemma 3.9. Therefore by Proposition 3.26, $\mathcal{Q}(F \rhd\!\!\rightarrow L)$ is order generated by $F^\vee$. Moreover $F \rhd\!\!\rightarrow L$ is isomorphic in **OFrm** to $F \rhd\!\!\rightarrow \mathcal{Q}(F \rhd\!\!\rightarrow L)$, hence if $F \rhd\!\!\rightarrow L$ is such that F is order generated by $MP(F \rhd\!\!\rightarrow L)$ we have that every $q \in \mathcal{Q}(F \rhd\!\!\rightarrow L)$ is the meet in L of elements which are the image under $(-)^\vee$ of meets in F of M-prime elements.

The next definition gives the full subcategory which is used later in our duality.

Definition 3.28. Define $\mathbf{OFrm}_M$ to be the full subcategory of **OFrm** with the observation frames $F \rhd\!\!\rightarrow L$ in which F is order generated by the set $MP(F \rhd\!\!\rightarrow L)$ of M-prime elements, as objects.

We have seen in Example 3.27 (i) that the functor Ω maps every topological space to an object of $\mathbf{OFrm}_M$. Also, Remark 3.21 shows that the functor Pt maps every observation frame to an object of $\mathbf{Sp}_0$. Moreover for every $\mathcal{T}_0$ topological space X the unit of the adjunction is an isomorphism by Lemma 3.24. The following lemma gives a similar result for the counit.

Lemma 3.29. *Let $F \rhd\!\!\rightarrow L$ be an observation frame. The counit morphism $\varepsilon : (F \rhd\!\!\rightarrow L) \rightarrow \Omega(Pt((F \rhd\!\!\rightarrow L))$ is an order isomorphism if and only if $F \rhd\!\!\rightarrow L$ is order generated by its M-primes (i.e. it is in $\mathbf{OFrm}_M$).*

Proof. (*only if*) Assume $a \not\leq b$ for some $a, b \in F$. Since ε is an order isomorphism (and hence order reflecting) we have that also $\varepsilon(a) = \Delta(a) \not\subseteq \Delta(b) = \varepsilon(b)$ and hence, by definition of $\Delta(-)$, there exists $p \in MP(F \rhd\!\!\rightarrow L)$ such that $a \not\leq p$ but $b \leq p$. But Proposition 3.26 then implies F is order generated by $MP(F \rhd\!\!\rightarrow L)$ and hence $F \rhd\!\!\rightarrow L$ is an object in $\mathbf{OFrm}_M$.

(*if*) Define $\varepsilon^{-1}(\Delta(a)) = \bigwedge(MP(F \rhd\!\!\rightarrow L) \setminus \Delta(a))$ for every $a \in F$. Then we have

$$
\begin{aligned}
\varepsilon^{-1}(\varepsilon(a)) &= \varepsilon^{-1}(\Delta(a)) \\
&= \bigwedge(MP(F \rhd\!\!\rightarrow L) \setminus \Delta(a)) \\
&= \bigwedge(MP(F \rhd\!\!\rightarrow L) \setminus (MP(F \rhd\!\!\rightarrow L) \setminus \uparrow a)) \\
&= \bigwedge(MP(F \rhd\!\!\rightarrow L) \cap \uparrow a) \\
&= a.
\end{aligned}
$$

Therefore ε is injective. Since we have already seen in Lemma 3.22 that ε is onto, we have that ε is an isomorphism with inverse ε^{-1}. It is also order reflecting because if $a \not\leqslant b$ for $a, b \in F$, then by Proposition 3.26 there is a $p \in MP(F \rhd\!\!\rightarrow L)$ such that $a \not\leqslant p$ but $b \leqslant a$. Therefore $\varepsilon(a) = \Delta(a) \nsubseteq \Delta(b) = \varepsilon(b)$. □

Now our main result follows.

Corollary 3.30. *The adjunction* $\mathbf{Sp} \leftrightarrows \mathbf{OFrm}^{op}$ *restricts to an equivalence of categories* $\mathbf{Sp}_0 \simeq \mathbf{OFrm}_M^{op}$. *Hence* $\mathbf{Sp}_0$ *and* $\mathbf{OFrm}_M$ *are each others duals and the adjunction is Galois.*

We constructed the duality with M-prime elements. Using Lemma 3.16 we can see that our duality comes from the "schizophrenic" object $\mathbf{2} = \Omega(1)$ in **OFrm**.

Remark 3.31. If $F \rhd\!\!\rightarrow L$ is an observation frame such that F is order generated by the M-prime elements, then $(-)^\vee : F \rhd\!\!\rightarrow L$ is order-reflecting (and hence injective). Indeed by Lemma 3.29 there is an order isomorphism $\varepsilon : (F \rhd\!\!\rightarrow L) \rightarrow \Omega(Pt(F \rhd\!\!\rightarrow L))$. Furthermore, by Lemma 3.9, there exists a meet-preserving function $\psi : \mathscr{Q}(F \rhd\!\!\rightarrow L) \rightarrow \mathscr{P}(MP(F \rhd\!\!\rightarrow L))$ such that $\psi(a^\vee) = \varepsilon(a)$ (recall that $\Omega(Pt(F \rhd\!\!\rightarrow L)) = \mathscr{O}_\Delta(MP(F \rhd\!\!\rightarrow L)) \hookrightarrow \mathscr{P}(MP(F \rhd\!\!\rightarrow L))$ is simply the inclusion) for all $a \in F$. Hence if $a^\vee \sqsubseteq b^\vee$ for some $a, b \in F$, then by monotonicity of ψ we have $\varepsilon(a) = \psi(a^\vee) \sqsubseteq \psi(b^\vee) = \varepsilon(b)$. But ε is order-reflecting, thus $a \leqslant b$. Since $(-)^\vee$ is also monotone we obtain $a \leqslant b$ if and only if $a^\vee \sqsubseteq b^\vee$.

4. M-topological systems

Topological systems were introduced by Vickers [33] in order to subsume both topological spaces and (ordinary) frames. In a topological system we have a set of subjects (or points) and a set of predicates (or opens) and a satisfaction relation matching the geometric propositional logic (or logic of finite observations). In this section we generalize these topological systems in order to obtain a satisfaction relation of propositional logic for observation frames (with both infinite conjunctions and disjunctions). Our interest in M-topological systems is justified since they clarify the connections between the infinitary operations of an observation frame $F \rhd\!\!\rightarrow L$ (the arbitrary joins $\bigsqcup$ and the arbitrary meets $\bigsqcap$ living in L) and the points of $F \rhd\!\!\rightarrow L$.

Definition 4.1. Let X be a set, let $F \rhd\!\!\rightarrow L$ be an observation frame, and let $\models \subseteq X \times L$ be a relation. Then $(X, \models, F \rhd\!\!\rightarrow L)$ is called a *completely multiplicative topological system* (M-topological system for short) if and only if $\models$ satisfies

$$x \models \bigsqcup S^\vee \;\Leftrightarrow\; \exists s \in S.\, x \models s^\vee \text{ for all } S \subseteq F,$$

$$x \models \bigsqcap S^\vee \;\Leftrightarrow\; \forall s \in S.\, x \models s^\vee \text{ for all } S \subseteq F.$$

The next two examples show that both topological spaces and observation frames give rise to M-topological systems.

Example 4.2. (i) Let X be a topological space and let $\Omega(X)$ be the induced observation frame. Define for every $x \in X$ and $s \in \mathscr{P}(X)$, $x \models s$ if and only if $x \in s$. Then $\mathscr{T}(X) = (X, \models, \Omega(X))$ is obviously an M-topological system.

(ii) Let $F \rhd\!\!\rightarrow L$ be an observation frame and let $MP(F \rhd\!\!\rightarrow L)$ be the set of its M-prime elements (or points). Define a relation $\models \subseteq MP(F \rhd\!\!\rightarrow L) \times L$ by

$$p \models q \quad \Leftrightarrow \quad \forall a \in F.\ q \sqsubseteq a^{\vee} \quad \Rightarrow \quad a \not\leqslant p.$$

By Lemma 3.16 we have that $p \models q$ if and only if $\phi_p(a) = \top$ for all $a \in F$ such that $q \sqsubseteq a^{\vee}$, where $\phi_p : (F \rhd\!\!\rightarrow L) \rightarrow \mathbf{2}$ is the morphism in **OFrm** corresponding to the M-prime element $p \in F$.

Next we show that $\mathscr{S}(F \rhd\!\!\rightarrow L) = (MP(F \rhd\!\!\rightarrow L), \models, F \rhd\!\!\rightarrow L)$ is an M-topological system. We have to prove $p \models \bigsqcap S^{\vee}$ if and only if $p \models s^{\vee}$ for all $s \in S \subseteq F$.

From right to left, if $p \models s^{\vee}$ then $\phi_p(s) = \top$ for all $s \in S$, that is $\bigsqcap \phi_p(S) = \top$. Hence, if $a \in F$ is such that $\bigsqcap S^{\vee} \sqsubseteq a^{\vee}$ then $\bigsqcap \phi_p(S) \sqsubseteq \phi_p(a)$ as ϕ_p is a morphism in **OFrm**. But $\bigsqcap \phi_p(S) = \top$, hence $\phi_p(a) = \top$. Therefore $p \models \bigsqcap S^{\vee}$.

Conversely, if $p \models \bigsqcap S^{\vee}$ then $\phi_p(a) = \top$ for all $a \in F$ such that $\bigsqcap S^{\vee} \sqsubseteq a^{\vee}$. But $\bigsqcap S^{\vee} \sqsubseteq s^{\vee}$ for all $s \in S$, hence for all $s \in S$ and for all $a \in F$ such that $s^{\vee} \sqsubseteq a^{\vee}$ we have that $\bigsqcap S^{\vee} \sqsubseteq a^{\vee}$ and hence $\phi_p(a) = \top$. Therefore $p \models s^{\vee}$ for all $s \in S$.

It remains to prove that $p \models \bigsqcup S^{\vee}$ if and only if there exists $s \in S \subseteq F$ such that $p \models s^{\vee}$. From right to left we have that $s^{\vee} \sqsubseteq (\bigvee S)^{\vee} = \bigsqcup S^{\vee}$ for $s \in S$. Hence if $a \in F$ is such that $\bigsqcup S^{\vee} \sqsubseteq a^{\vee}$ then $\phi_p(a) = \top$ because by hypothesis there exists $s \in S$ such that for all $a \in F$ if $s^{\vee} \sqsubseteq a^{\vee}$. Therefore $p \models \bigsqcup S^{\vee}$.

Conversely, suppose that for all $s \in S$, $p \not\models s^{\vee}$. Then for all $s \in S$ there exists $a_s \in F$ such that $s^{\vee} \sqsubseteq a_s^{\vee}$ but $\phi_p(a_s) = \bot$. Hence $\bigsqcup S^{\vee} \sqsubseteq \bigsqcup \{a_s^{\vee} | s \in S\} = (\bigvee \{a_s | s \in S\})^{\vee}$ because $(-)^{\vee}$ is preserving joins. But $p \models \bigsqcup S^{\vee}$ hence $\top = \phi_p(\bigvee \{a_s | s \in S\}) = \bigsqcup_{s \in S} \phi_p(a_s)$, which implies there exists a_s such that $\phi_p(a_s) = \top$ contradicting the assumption.

Let $(X, \models, F \rhd\!\!\rightarrow L)$ be an M-topological system. Directly from its definition we can deduce that

(i) $x \models \top$ for all $x \in X$;
(ii) $x \models \bot$ for no $x \in X$;
(iii) $x \models a^{\vee}$ and $a \leqslant b$ implies $x \models b^{\vee}$ for every $a, b \in F$.

Furthermore, for every $q, q' \in \mathscr{Q}(F \rhd\!\!\rightarrow L)$ if $x \models q$ and $q \sqsubseteq q'$ then $x \models q'$. Indeed, if we assume $q = \bigsqcap S^{\vee}$ and $q' = \bigsqcap T^{\vee}$ for $S, T \subseteq F$ then $q \sqsubseteq q'$ implies $q = \bigsqcap S^{\vee} = \bigsqcap S^{\vee} \sqcap \bigsqcap T^{\vee} = \bigsqcap (S \cup T)^{\vee}$ using Lemma 3.8. Hence $x \models q$ if and only if $x \models \bigsqcap (S \cup T)^{\vee}$. But then, by definition of $\models$, $x \models t^{\vee}$ for all $t \in T$ and hence $x \models \bigsqcap T^{\vee} = q'$.

The next remark shows how to derive from a satisfaction relation in an M-topological system a morphism in the category **OFrm**. The converse holds only for a certain kind of morphisms in **OFrm**. This will be used later in order to show that the category of M-topological systems is equivalent to a comma category.

Remark 4.3. (i) Let $(X, \models, F \rhd\!\!\to L)$ be an M-topological system. Recall that every saturated element $q \in \mathscr{Q}(F \rhd\!\!\to L)$ is of the form $q = \sqcap S^\vee$ for some $S \subseteq F$. The relation $\models\ \subseteq X \times L$ induces a function $\psi_\models : \mathscr{Q}(F \rhd\!\!\to L) \to \mathscr{P}(X)$ defined by $\psi_\models(q) = \{x \in X \mid x \models q\}$. We have that

$$\begin{aligned}\psi_\models(\sqcup S^\vee) &= \{x \in X \mid x \models \sqcup S^\vee\}\\ &= \bigcup_{s \in S}\{x \in X \mid x \models s\} \qquad \text{Definition 4.1}\\ &= \bigcup_{s \in S}\psi_\models(s^\vee)\end{aligned}$$

for every $S \subseteq F$. Similarly $\psi_\models(\sqcap S^\vee) = \bigcap_{s \in S}\psi_\models(s^\vee)$ for all $S \subseteq F$. But then by Remark 3.10 and by considering the observation frame $id : \mathscr{P}(X) \to \mathscr{P}(X)$, there exists a unique morphism $\phi_\models : F \to \mathscr{P}(X)$ in **OFrm** such that $\phi_\models(a) = \psi_\models(a^\vee)$ for every $a \in F$.

(ii) Conversely, by Lemma 3.9, every morphism $\phi : F \to \mathscr{P}(X)$ in **OFrm** induces an M-topological system $(X, \models_\phi, F \rhd\!\!\to \mathscr{Q}(F \rhd\!\!\to L))$, where for every $x \in X$ and $q \in \mathscr{Q}(F \rhd\!\!\to L)$, $x \models_\phi q$ if and only if for every $a \in F$ if $q \sqsubseteq a^\vee$ then $x \in \phi(a)$.

(iii) These constructions are each other's inverse in the following sense: for every morphism $\phi : F \to \mathscr{P}(X)$ in **OFrm** and $a \in F$ we have

$$\begin{aligned}\phi_{\models_\phi}(a) &= \psi_{\models_\phi}(a^\vee) && \text{definition of } \phi_{\models_\phi}\\ &= \{x \in X \mid x \models_\phi a^\vee\} && \text{definition of } \psi_{\models_\phi}\\ &= \{x \in X \mid \forall b \in F.\ a^\vee \sqsubseteq b^\vee \Rightarrow x \in \phi(b)\} && \text{definition of } \models_\phi\\ &= \bigcap\{\phi(b) \mid a^\vee \sqsubseteq b^\vee\}\\ &= \phi(a)\end{aligned}$$

where the last equality holds because ϕ is completely multiplicative and hence $a^\vee = \sqcap\{b^\vee \mid b \in F \text{ and } a^\vee \sqsubseteq b^\vee\}$ implies $\phi(a) = \bigcap\{\phi(b) \mid b \in F \text{ and } a^\vee \sqsubseteq b^\vee\}$.

For every M-topological system $(X, \models, F \rhd\!\!\to L)$, and for every $x \in X$ and $q \in \mathscr{Q}(F \rhd\!\!\to L)$ we have

$$\begin{aligned}x \models_{\phi_\models} q &\Leftrightarrow \forall a \in F.\ q \sqsubseteq a^\vee \Rightarrow x \in \phi_\models(a) && \text{definition of } \models_{\phi_\models}\\ &\Leftrightarrow \forall a \in F.\ q \sqsubseteq a^\vee \Rightarrow x \in \psi_\models(a^\vee) && \text{definition of } \phi_\models\\ &\Leftrightarrow \forall a \in F.\ q \sqsubseteq a^\vee \Rightarrow x \models a^\vee && \text{definition of } \psi_\models\\ &\Leftrightarrow x \models \sqcap\{a^\vee \mid q \sqsubseteq a^\vee\} && \text{definition of } \models\\ &\Leftrightarrow x \models q && q \text{ is a saturated element.}\end{aligned}$$

Using the above remark, Lemma 3.9, and Remarks 3.10, we have that for an observation frame $F \rhd\!\!\rightarrow L$ such that

(i) L is a completely distributive lattice, and

(ii) $q = \sqcap\{a^\vee \in L \mid q \sqsubseteq a^\vee \text{ and } a \in F\}$ for all $q \in L$,

a triple $(X, \models, F \rhd\!\!\rightarrow L)$ is an M-topological system if and only if the relation $\models \subseteq X \times L$ satisfies for all $S \subseteq L$

$$x \models \bigsqcup S \;\Leftrightarrow\; \exists s \in S.\, x \models s,$$
$$x \models \bigsqcap S \;\Leftrightarrow\; \forall s \in S.\; x \models s.$$

Note that in this definition $S \subseteq L$, while in Definition 4.1 $S \subseteq F$.

Next we organize M-topological systems in a category for which we introduce the following morphisms.

Definition 4.4. Let $D = (X, \models, F \rhd\!\!\rightarrow L)$ and $E = (Y, \models, G \rhd\!\!\rightarrow K)$ be two M-topological systems. A *morphism* from D and E consist of a pair (f, ϕ) where $f : X \to Y$ is a function, $\phi : (G \rhd\!\!\rightarrow K) \to (F \rhd\!\!\rightarrow L)$ is a morphism in **OFrm** (note the reverse direction of the arrow), that satisfies for every $x \in X$ and $a \in G$

$$x \models \phi(a)^\vee \quad \Leftrightarrow \quad f(x) \models a^\vee.$$

It is straightforward to check that composition of two morphisms defined as the usual element wise composition is again a morphism. Hence M-topological systems together with these- morphisms form a category which we refer to as **MTS**.

Example 4.5. Let $f : X \to Y$ be a continuous function in **Sp**. Then $\mathscr{T}(f) = (f, \Omega(f))$ is a morphism from $\mathscr{T}(X) = (X, \models, \Omega(X))$ to $\mathscr{T}(Y) = (Y, \models, \Omega(Y))$ in **MTS** since $\Omega(f) : \Omega(Y) \to \Omega(X)$ is a morphism in **OFrm** and we have that

$$\begin{aligned} x \models \Omega(f)(o) &\Leftrightarrow x \in \Omega(f)(o) && \text{definition of } \models \\ &\Leftrightarrow f(x) \in o && \text{definition of } \Omega(f) \\ &\Leftrightarrow f(x) \models o && \text{definition of } \models. \end{aligned}$$

It is easy to check that $\mathscr{T}$ is a functor from **Sp** to **MTS**.

Remark 4.6. (i) Let $(X, \models, F \rhd\!\!\rightarrow L)$ be an M-topological system. Since the observation frames $F \rhd\!\!\rightarrow L$ and $F \rhd\!\!\rightarrow \mathscr{Q}(F \rhd\!\!\rightarrow L)$ are isomorphic in the category **OFrm** we have also that the topological system $(X, \models, F \rhd\!\!\rightarrow L)$ is isomorphic in **MTS** to the topological system $(X, \models, F \rhd\!\!\rightarrow \mathscr{Q}(F \rhd\!\!\rightarrow L))$.

(ii) Let $\mathscr{P}^{-1} : \mathbf{Set} \to \mathbf{OFrm}^{op}$ be the contravariant functor which maps every set X to the observation frame $id : \mathscr{P}(X) \to \mathscr{P}(X)$ and every function $f : X \to Y$ to its inverse. Consider the comma category (for its definition see [23]) $(\mathscr{P}^{-1} \downarrow \mathbf{OFrm}^{op})$ given by the functors $\mathscr{P}^{-1}$ and the identity functor on $\mathbf{OFrm}^{op}$. Remark 4.3 and the isomorphism above imply that we have an equivalence of categories between **MTS** and

($\mathscr{P}^{-1} \downarrow \mathbf{OFrm}^{op}$). This shows that our category of M-topological systems is obtained like the category of (ordinary) topological systems as used by Vickers. The latter is obtained as comma category ($\mathscr{P}^{-1} \downarrow \mathbf{Frm}^{op}$).

Next we show that the adjunction of Theorem 3.23 can be split in two parts: one from topological spaces to M-topological systems and one from M-topological systems to observation frames. We thus have a situation as in [33]. We start with the first adjunction.

Every M-topological system $D = (X, \models, F \rightarrowtail L)$ induces a topology on X by taking as open sets the *extent* of all $a \in F$:

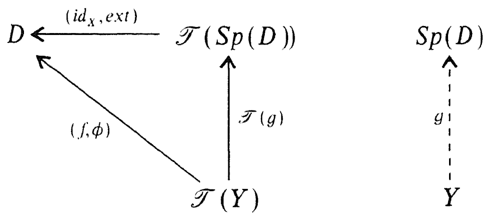
$$ext(a) = \{x \in X \mid x \models a^{\vee}\}.$$

By definition of $\models$ and since $(-)^{\vee}$ preserves finite meets and arbitrary joins we have that the collection of all extents forms a topology on X. We denote this topological space by $Sp(D)$. Furthermore the function $ext(-) : F \to \mathscr{P}(X)$ is a morphism from $F \rightarrowtail L$ to $id : \mathscr{P}(X) \to \mathscr{P}(X)$ in **OFrm**. Indeed, it is a frame morphism as the collection of all extents forms a topology and it is completely multiplicative because if $\bigsqcap S^{\vee} \sqsubseteq \bigsqcap T^{\vee}$ for some $S, T \subseteq F$ then

$$\begin{aligned} x \in \bigcap ext(S) &\Leftrightarrow \forall s \in S.\ x \models s^{\vee} && \text{definition of } ext(-) \\ &\Leftrightarrow x \models \bigsqcap S^{\vee} && \text{definition of } \models \\ &\Rightarrow x \models \bigsqcap T^{\vee} && \bigsqcap S^{\vee} \sqsubseteq \bigsqcap T^{\vee} \\ &\Leftrightarrow \forall t \in T.\ x \models t^{\vee} && \text{definition of } \models \\ &\Leftrightarrow x \in \bigcap ext(T) && \text{definition of } ext(-). \end{aligned}$$

This shows also that the pair $(id_X, ext) : \mathscr{T}(Sp(D)) \to D$ is a morphism in **MTS**.

Theorem 4.7. *Let $D = (X, \models, F \rightarrowtail L)$ be an M-topological system and let Y be a topological space such that there is a morphism $(f, \phi) : \mathscr{T}(Y) \to D$ in* **MTS**. *Then there exists a unique continuous function $g : Y \to Sp(D)$ in* **Sp** *such that the following diagram commutes:*



This extends Sp to a functor from **MTS** *to* **Sp** *which is right adjoint of $\mathscr{T}$.*

Proof. (sketch) Take $g = f$. It is clearly continuous and the unique one such that $\mathscr{T}(g)$ makes the diagram commute. □

A M-topological system is called *spatial* if it is isomorphic in **MTS** to $\mathcal{T}(X)$ for some topological space X.

Next we give the second adjunction between M-topological systems and observation frames. There is an obvious forgetful functor Fr : **MTS** $\to$ **OFrm**^{op} which maps every M-topological system $(X, \models, F \rightarrowtail L)$ to $F \rightarrowtail L$ and every morphism (f, ϕ): $(X, \models, F \rightarrowtail L) \to (Y, \models, H \rightarrowtail K)$ to $\phi : (H \rightarrowtail K) \to (F \rightarrowtail L)$.

Lemma 4.8. *Let $(X, \models, F \rightarrowtail L)$ be an M-topological system and $p : X \to MP(F \rightarrowtail L)$ be a function which assigns to a concrete point $x \in X$ the abstract point $p(x) = \bigvee\{a \in F \mid x \not\models a^\vee\}$. Then $p(x)$ is an M-prime element for every $x \in X$. Furthermore, the pair $(p, id_F) : D \to \mathcal{S}(Fr(D))$ forms a morphism in* **MTS** *where $\mathcal{S}(-)$ is as in Example 4.2*(ii).

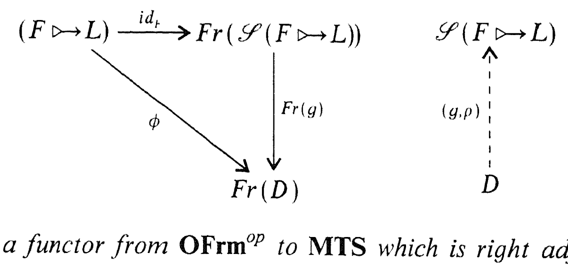
Proof. We begin by showing that $p(x)$ is M-prime. Let $S \subseteq F$ be such that $\sqcap S^\vee \sqsubseteq p(x)^\vee$. We prove that there exists $s \in S$ such that $s \leqslant p(x)$. Indeed there exists an $s \in S$ such that $x \not\models s^\vee$ because otherwise $x \models s^\vee$ for all $s \in S$. But this means $x \models \sqcap S^\vee$ and hence $x \models p(x)^\vee$ since $\sqcap S^\vee \sqsubseteq p(x)^\vee$. But then $x \models (\bigvee\{a \in F \mid x \not\models a^\vee\})^\vee = \bigsqcup\{a \in F \mid x \not\models a^\vee\}^\vee$. From the definition of $\models$ this holds if and only if there exists $a \in F$ such that $x \not\models a^\vee$ and $x \models a^\vee$. Contradiction.

Consider now the pair $(p, id_F) : D \to \mathcal{S}(Fr(D))$ where $\mathcal{S}$ is as in Example 4.2(ii). We show it forms a morphism in **MTS**. By the above it is enough to prove $x \models a^\vee$ if and only if $p(x) \models a^\vee$ where $p(x) \models a^\vee$ means $\forall b \in F.\ a^\vee \sqsubseteq b^\vee \Rightarrow b \not\leqslant p(x)$.

($\Rightarrow$) Let $b \in F$ be such that $a^\vee \sqsubseteq b^\vee$. Then $b \not\leqslant p(x) = \bigvee\{c \in F \mid x \not\models c^\vee\}$ because otherwise $x \models a^\vee$ and $a^\vee \sqsubseteq b^\vee$ implies $x \models b^\vee$ and hence also $x \models p(x)^\vee$. But this leads us to the contradiction that there exists a $c \in F$ such that $x \not\models c^\vee$ and $x \models c^\vee$.

($\Leftarrow$) If $p(x) \models a^\vee$ then $a \not\leqslant p(x) = \bigvee\{b \in F \mid x \not\models b^\vee\}$. Hence $x \models a^\vee$ because otherwise $a \in \{b \in F \mid x \not\models b^\vee\}$ and hence the contradiction $a \leqslant p(x)$. □

Theorem 4.9. *Let $F \rightarrowtail L$ be an observation frame and let $D = (Y, \models, H \rightarrowtail K)$ be an M-topological system such that there is a morphism $\phi : (F \rightarrowtail L) \to Fr(D)$ in* **OFrm**. *Then there exists a unique morphism $(g, p) : D \to \mathcal{S}(F \rightarrowtail L)$ in* **MTS** *such that the following diagram commutes*



This extends $\mathcal{S}$ to a functor from **OFrm**^{op} *to* **MTS** *which is right adjoint of Fr.*

Proof. Define $g(y) = \bigvee\{b \in F \mid x \not\models \phi(b)^\vee\}$ for all $y \in Y$ and $\rho(a) = \phi(a)$ for every $a \in F$. It is not hard to see that $g(y) \in MP(F \rhd\!\!\rightarrow L)$ for all $y \in Y$. We show $y \models \rho(a)^\vee$ if and only if $g(y) \models a^\vee$ for all $y \in Y$ and $a \in F$.

($\Rightarrow$) If $y \models \rho(a)^\vee$ then also $g(y) \models a^\vee$ because otherwise by definition of $\models$ in $\mathcal{S}(F \rhd\!\!\rightarrow L)$ there exists a $b \in F$ such that $a^\vee \sqsubseteq b^\vee$ and $b \leqslant g(y) = \bigvee\{c \in F \mid y \not\models \phi(c)^\vee\}$. Hence by the complete multiplicativity of ϕ we have

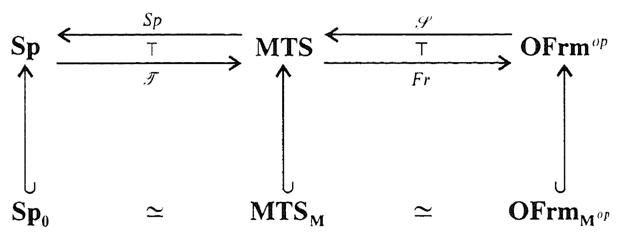
$$\phi(a)^\vee \sqsubseteq \phi(b)^\vee \sqsubseteq \phi(\bigvee\{c \in F \mid y \not\models \phi(c)^\vee\})^\vee = \bigsqcup\{\phi(c) | y \not\models \phi(c)^\vee\}^\vee.$$

But $y \models \rho(a)^\vee$ and hence also $y \models \bigsqcup\{c \in F \mid y \not\models \phi(c)^\vee\}^\vee$. Therefore, by definition of $\models$, we get the contradiction that there exists $c \in F$ such that $y \not\models \phi(c)^\vee$ and $y \models \phi(c)^\vee$.

($\Leftarrow$) If $g(y) \models a^\vee$ then $a \not\leqslant g(y) = \bigvee\{b \in F \mid x \not\models \phi(b)^\vee\}$ by definition of $\models$ in $\mathcal{S}(F \rhd\!\!\rightarrow L)$. But then $y \models \rho(a)^\vee = \phi(a)^\vee$ because otherwise $a \in \{b \in F \mid x \not\models \phi(b)^\vee\}$ and hence $a \leqslant \bigvee\{b \in F \mid x \not\models \phi(b)^\vee\}$ contradicting $a \not\leqslant g(y)$.

Since $Fr(g, \rho) = \rho$ obviously $id_F \circ Fr(g, \rho) = \phi$. Moreover $(g, \rho) : D \to \mathcal{S}(F \rhd\!\!\rightarrow L)$ is the unique morphism in **MTS** having this property. □

A M-topological system is called *observational* (or possibly M-localic) if it is isomorphic in **MTS** to $\mathcal{S}(H \rhd\!\!\rightarrow K)$ for some observation frame $H \rhd\!\!\rightarrow K$. Clearly the full subcategory of spatial observational M-topological systems is equivalent to the full subcategory of $\mathcal{T}_0$ topological spaces and is equivalent to the full subcategory of observation frames $F \rhd\!\!\rightarrow L$ where F is order generated by the M-prime elements. The following diagram summarizes the situation.



Next we show that spatiality of a localic M-topological frame corresponds to the completeness of the logic. To show this we need the definition of semantic entailment.

Definition 4.10. For an observation frame $F \rhd\!\!\rightarrow L$ define the relation of *semantic entailment* on F by for all $a, b \in F$, $a \vdash_F b$ if and only if for every M-topological system $(X, \models, F \rhd\!\!\rightarrow L)$ and $x \in X$ if $x \models a^\vee$ then $x \models b^\vee$.

We also define the relation of semantic entailment on $\mathcal{Q}(F \rhd\!\!\rightarrow L)$ for all $q, r \in \mathcal{Q}(F \rhd\!\!\rightarrow L)$ by putting $q \vdash_{\mathcal{Q}} r$ if and only if for every M-topological system $(X, \models, F \rhd\!\!\rightarrow L)$ and $x \in X$ if $x \models q$ then $x \models r$.

We could also define this relation on L, but in the category **OFrm** we are interested only in the observable predicates F and the predicates $\mathcal{Q}(F \rhd\!\!\rightarrow L)$ that can be deduced from the observable ones.

The next lemma says that for every observation frame $F \rhd\!\!\to L$ the order on F is contained in the entailment relation and hence we get soundness. Similarly we have soundness for $\mathcal{Q}(F \rhd\!\!\to L)$.

Lemma 4.11 (soundness). *Let $F \rhd\!\!\to L$ be an observation frame. Then*
(i) *$a \leqslant b$ implies $a \vdash_F b$ for all $a, b \in F$,*
(ii) *$q \sqsubseteq r$ implies $q \vdash_{\mathcal{Q}} r$ for all $q, r \in \mathcal{Q}(F \rhd\!\!\to L)$.*

Proof. We prove only the first item. If $a \leqslant b$ for $a, b \in F$ then $a^{\vee} \sqsubseteq b^{\vee}$, that is $a^{\vee} = \sqcap\{a^{\vee}, b^{\vee}\}$. Therefore, in every M-topological system $(X, \models, F \rhd\!\!\to L)$ if $x \models a^{\vee}$ then $x \models b^{\vee}$. Hence $a \vdash_F b$. The proof of the second item is equally simple. □

The next lemma states that the entailment relation is included in the order if and only if the observation frame $F \rhd\!\!\to L$ is such that F is order generated by the M-prime elements. This is equivalent to stating that completeness both for F and $\mathcal{Q}(F \rhd\!\!\to L)$ holds if and only if F is order generated by the M-prime elements.

Lemma 4.12 (completeness). *Let $F \rhd\!\!\to L$ be an observation frame. The following statements are equivalent.*
(i) *F is order generated by the M-prime elements of $F \rhd\!\!\to L$*
(ii) *$a \vdash_F b$ implies $a \leqslant b$ for all $a, b \in F$*
(iii) *$q \vdash_{\mathcal{Q}} r$ implies $q \sqsubseteq r$ for all $q, r \in \mathcal{Q}(F \rhd\!\!\to L)$ and $(\cdot)^{\vee}$ is order reflecting.*

Proof. We shall prove (i) $\Leftrightarrow$ (ii) and (ii) $\Leftrightarrow$ (iii).

(i) $\Rightarrow$ (ii). Suppose F is order generated by the M-prime elements of $F \rhd\!\!\to L$ and let $a \vdash_F b$ for some $a, b \in F$. Hence for all M-topological systems $(X, \models, F \rhd\!\!\to L)$ and $x \in X$ if $x \models a^{\vee}$ then $x \models b^{\vee}$. In particular consider the M-topological system $\mathcal{S}(F \rhd\!\!\to L)$ and the isomorphism $(id_{MP}, \varepsilon) : \mathcal{S}(F \rhd\!\!\to L) \to \mathcal{T}(Pt(F \rhd\!\!\to L))$ in **MTS**. We have

$$
\begin{array}{rcll}
p \in \Delta(a) & \Leftrightarrow & p \models \Delta(a) & \text{definition of } \models \text{ in } \mathcal{T}(Pt(F \rhd\!\!\to L)) \\
 & \Leftrightarrow & p \models \varepsilon(a) & \text{definition of } \varepsilon \\
 & \Leftrightarrow & p \models a^{\vee} & (id_{MP}, \varepsilon) \text{ is a morphism in } \mathbf{MTS} \\
 & \Rightarrow & p \models b^{\vee} & \text{because } a \vdash_F b \\
 & \Leftrightarrow & p \in \Delta(b). &
\end{array}
$$

Hence $\varepsilon(a) = \Delta(a) \subseteq \Delta(b) = \varepsilon(b)$. But ε is an order preserving isomorphism by Lemma 3.29, therefore $a \leqslant b$.

(ii) $\Rightarrow$ (i): We use the formulation of Proposition 3.26(iv). Let $a, b \in F$ be such that $a \not\leqslant b$. Then $a \not\vdash_F b$, that is, there exists a M-topological system $D = (X, \models, F \rhd\!\!\to L)$ and a $x \in X$ such that $x \models a^{\vee}$ but $x \not\models b^{\vee}$. Consider the morphism $(p, id_F) : D \to \mathcal{S}(Fr(D))$ in **MTS**. Then $x \models a^{\vee}$ if and only if $p(x) \models a^{\vee}$, where $p(x) \in MP(F \rhd\!\!\to L)$. Hence, by definition of $\models$ in $\mathcal{S}(Fr(D))$ (Example 4.2(ii)) we have found a M-prime element $p(x)$ such that $a \not\leqslant p(x)$ but $b \leqslant p(x)$. Therefore, by Proposition 3.26 we have that F is order generated by $MP(F \rhd\!\!\to L)$.

(ii) $\Rightarrow$ (iii): Suppose $a \vdash_F b$ implies $a \leqslant b$ for all $a,b \in F$ and let $q,r \in \mathcal{Q}(F \rhd\!\!\rightarrow L)$ be such that $q \vdash_{\mathcal{Q}} r$. Then for all M-topological systems $(X, \models, F \rhd\!\!\rightarrow L)$ and $x \in X$ we have that $x \models q \Rightarrow x \models r$. But $q = \bigsqcap\{a^\vee | a \in F \text{ and } q \sqsubseteq a^\vee\}$ and also $r = \bigsqcap\{b^\vee | b \in F \text{ and } r \sqsubseteq b^\vee\}$, hence, by definition of $\models$, $x \models a^\vee$ for all $a \in F$ such that $q \sqsubseteq a^\vee$, implies $x \models b^\vee$ for all $b \in F$ such that $r \sqsubseteq b^\vee$. But this means that $a \vdash_F b$ for $a,b \in F$ such that $q \sqsubseteq a^\vee$ and $r \sqsubseteq b^\vee$. Hence $a \leqslant b$ (which implies $a^\vee \sqsubseteq b^\vee$) for all $a,b \in F$ such that $q \sqsubseteq a^\vee$ and $r \sqsubseteq b^\vee$. Therefore $q = \bigsqcap\{a^\vee | a \in F \text{ and } q \sqsubseteq a^\vee\} \sqsubseteq \bigsqcap\{b^\vee | b \in F \text{ and } r \sqsubseteq b^\vee\} = r$. It is easy to see that $(\cdot)^\vee$ reflects the order. Assume $a^\vee \sqsubseteq b^\vee$, but $a \not\sqsubseteq b$. Then $a \not\vdash_F b$ and hence $a^\vee \not\vdash_{\mathcal{Q}} b^\vee$. This contradicts what we just proved.

(iii) $\Rightarrow$ (ii): If $a \vdash_F b$, then $a^\vee \vdash_{\mathcal{Q}} b^\vee$, so $a^\vee \sqsubseteq b^\vee$ and thus $a \leqslant b$ since $(\cdot)^\vee$ reflects the order. □

5. Saturated elements and upper power spaces

In this section we investigate the relationships between saturated elements of an observation frame $F \rhd\!\!\rightarrow L$ and its M-filters. We subsequently discuss their importance for so-called "filter theorems" and give three applications of upper power spaces on (ordinary) posets, continuous dcpo's and algebraic dcpo's.

In Definition 3.6 we described the *saturated elements* of an observation frame $F \rhd\!\!\rightarrow L$ as the elements $q \in \mathcal{Q}(F \rhd\!\!\rightarrow L)$ (equivalently, if $q = \bigsqcap\{a^\vee | a \in F \text{ and } q \sqsubseteq a^\vee\}$). One often is interested in the subset $\mathcal{K}\mathcal{Q}(F \rhd\!\!\rightarrow L) \subseteq \mathcal{Q}(F \rhd\!\!\rightarrow L)$ of *compact saturated elements*, where an element $s \in L$ is called *compact* if for every $A \subseteq F$ such that $s \sqsubseteq \bigsqcup A^\vee$ there exists a finite subset $B \subseteq A$ such that $s \sqsubseteq \bigsqcup B^\vee$. For example, if X is a topological space then the saturated elements of $\Omega(X)$ are the ordinary saturated sets of X, and the compact elements of $\Omega(X)$ are the ordinary compact sets. For a topological space X the saturation $\mathcal{Q}(\{x\})$ is the principal upper set $\uparrow x$, for every $x \in X$, where the order used in $\uparrow(\cdot)$ is the specialization preorder on X.

Given an observation frame $F \rhd\!\!\rightarrow L$, the set $\mathcal{Q}(F \rhd\!\!\rightarrow L)$ of saturated elements can be made into a topological space by taking as opens the sets $U_a = \{q \in \mathcal{Q}(F \rhd\!\!\rightarrow L) \,|\, q \sqsubseteq a^\vee\}$ for every $a \in F$. They form a topology since $U_a \cap U_b = U_{a \wedge b}$ and $\bigcup_{a \in A} U_a = U_{\vee A}$ for every $a,b \in F$ and $A \subseteq F$. The induced specialization order on $\mathcal{Q}(F \rhd\!\!\rightarrow L)$ is denoted by $\leqslant_U$. It is not hard to show that $q \leqslant_U r$ if and only if $r \sqsubseteq q$, for $q,r \in \mathcal{Q}(F \rhd\!\!\rightarrow L)$. Sometimes we extend this order to elements $p, p' \in L$ by $p \leqslant_U p'$ if and only if $\mathcal{Q}(p) \leqslant_U \mathcal{Q}(p')$ if and only if $p' \sqsubseteq \mathcal{Q}(p)$, where $\mathcal{Q}((\cdot))$ is the saturated closure described in Section 3. The set $\mathcal{Q}(F \rhd\!\!\rightarrow L)$ with this topology is the *upper power space* associated with $F \rhd\!\!\rightarrow L$.

For topological spaces, the assignment $X \mapsto \mathcal{Q}(X)$ extends to a functor $\mathbf{Sp}_0 \to \mathbf{Sp}_0$: for $f: X \to Y$ one gets a function $\mathcal{Q}(X) \to \mathcal{Q}(Y)$ by $A \mapsto \mathcal{Q}(f(A))$. Even more, one gets a monad with unit $X \to \mathcal{Q}(X)$ given by $x \mapsto \uparrow x$ and multiplication $\mathcal{Q}(\mathcal{Q}(X)) \to \mathcal{Q}(X)$ by $\mathcal{A} \mapsto \mathcal{Q}(\bigcup \mathcal{A})$. We thus extend the definition of upper power space in [29] from topological spaces to observation frames.

The following three lemmas were partly present in [2]. They establish the fundamental role of M-filters in this setting. The first and fundamental step is that upper power spaces can also be described in terms of M-filters. Subsequently this correspondence is refined to compact and completely prime saturated elements.

Lemma 5.1. *Let $F \rightarrowtail L$ be an observation frame. There is an order isomorphism between the collection of saturated elements $(\mathscr{Q}(F \rightarrowtail L), \leqslant_U)$ and the collection of M-filters $MF(F \rightarrowtail L)$ ordered by subset inclusion.*

Proof. Let $q \in \mathscr{Q}(F \rightarrowtail L)$ and define the set $\mathscr{U}(q) = \{a \in F \mid q \sqsubseteq a^{\vee}\}$. It is an M-filter since if $\sqcap \mathscr{U}(q)^{\vee} \sqsubseteq x^{\vee}$ for some $x \in F$, then $x \in \mathscr{U}(q)$ because $q = \sqcap \mathscr{U}(q)^{\vee}$ as it is a saturated element. Conversely, for every M-filter $\mathscr{V} \subseteq F$ we have from the definition of saturated elements that $\sqcap \mathscr{V}^{\vee} \in \mathscr{Q}(F \rightarrowtail L)$.

Furthermore, for every saturated element q we have $\sqcap \mathscr{U}(q)^{\vee} = q$ because q is saturated. Conversely, for every M-filter $\mathscr{V} \subseteq F$ we have

$$\begin{aligned}\mathscr{U}(\sqcap \mathscr{V}^{\vee}) &= \{a \in F \mid \sqcap \mathscr{V}^{\vee} \sqsubseteq a^{\vee}\} \\ &= \{a \in F \mid a \in \mathscr{V}\} \qquad \mathscr{U} \text{ is an M-filter} \\ &= \mathscr{V}.\end{aligned}$$

The isomorphism is clearly order preserving. □

Next we restrict the order-isomorphism of Lemma 5.1 to compact saturated elements and Scott open M-filters, where an M-filter $\mathscr{U} \subseteq F$ of the observation frame $F \rightarrowtail L$ is said to be Scott open if $\mathscr{U}$ is an open set in the Scott topology on F. Recall that a subset $o \subseteq F$ is open in the *Scott topology* if it is upper closed and for all directed set $D \subseteq F$ if $\bigvee D \in o$ then there exists $d \in D \cap o$.

Lemma 5.2. *Let $F \rightarrowtail L$ be an observation frame. Then $q \in \mathscr{Q}(F \rightarrowtail L)$ is a compact saturated element if and only if $\mathscr{U}(q) \in MF(F \rightarrowtail L)$ is a Scott open M-filter. Equivalently, $\mathscr{U}(q) \in MF(F \rightarrowtail L)$ is a Scott open M-filter if and only if $\sqcap \mathscr{U}^{\vee} \in \mathscr{Q}(F \rightarrowtail L)$ is a compact saturated element.*

Proof. Let $q \in \mathscr{Q}(F \rightarrowtail L)$ be a compact saturated element. Then $\mathscr{U}(q) = \{a \in F \mid q \sqsubseteq a^{\vee}\}$ is a M-filter by Lemma 5.1. It is also open in the Scott topology of F because if $S \subseteq F$ is a directed set such that $\bigvee S \in \mathscr{U}(q)$ then $q \sqsubseteq (\bigvee S)^{\vee} = \bigsqcup S^{\vee}$. But q is compact, hence there exists $s \in S$ such that $q \sqsubseteq s^{\vee}$. Therefore there exists $s \in S$ such that $s \in \mathscr{U}(q)$.

Conversely, suppose $\mathscr{U}(q)$ is open in the Scott topology of F and let $S \subseteq F$ be a directed set such that $q = \sqcap \mathscr{U}(q)^{\vee} \sqsubseteq \bigsqcup S^{\vee} = (\bigvee S)^{\vee}$. Since $\mathscr{U}(q)$ is an M-filter we have $\bigvee S \in \mathscr{U}(q)$. But it is also Scott open, thus there exists $s \in S$ such that $s \in \mathscr{U}(q)$, that is $q \sqsubseteq s^{\vee}$. Therefore, q is compact. □

For an observation frame $F \rhd\!\!\rightarrow L$ we have in a next step a relationship between completely prime M-filters and completely prime saturated elements in the lattice $(\mathcal{Q}(F \rhd\!\!\rightarrow L), \leqslant_U)$. Recall that $q \in \mathcal{Q}(F \rhd\!\!\rightarrow L)$ is said to be completely prime if for every $S \subseteq F$ such that $q \sqsubseteq \bigsqcup S^\vee$ there exists a $s \in S$ such that $q \sqsubseteq s^\vee$. The proof is as before.

Lemma 5.3. *Let $F \rhd\!\!\rightarrow L$ be a topological space. Then $q \in \mathcal{Q}(F \rhd\!\!\rightarrow L)$ is a completely prime saturated element if and only if $\mathcal{U}(q) \in MF(F \rhd\!\!\rightarrow L)$ is a completely prime M-filter. Equivalently, $\mathcal{V} \in MF(F \rhd\!\!\rightarrow L)$ is a completely prime M-filter if and only if $\bigsqcap \mathcal{V}^\vee \in \mathcal{Q}(F \rhd\!\!\rightarrow L)$ is a completely prime saturated element.*

Remark 5.4. (i) Let X be a $\mathcal{T}_0$ topological space. Then X is isomorphic to its M-prime elements and hence by Lemma 3.16 also to its completely prime M-filters. But completely prime M-filters are isomorphic to the completely prime saturated elements and saturated elements are upper closed sets (with respect to the specialization pre-order). Therefore the completely prime saturated elements are sets of the form $\uparrow x$ for a unique $x \in X$.

(ii) The bijective correspondence between completely prime M-filters of an observation frame $F \rhd\!\!\rightarrow L$ and completely multiplicative frame morphisms from F to **2** extends to a correspondence between Scott open M-filters and completely multiplicative, Scott continuous and finite meet preserving functions from F to **2**. Also M-filters are in bijective correspondence with completely multiplicative finite meet preserving maps from F to **2**.

Lemma 5.2 is of a more fundamental nature than what is normally called the Hofmann–Mislove theorem (also known as Scott-open filter theorem) given in Corollary 5.6 below. The latter is about Scott-open sets $F \subseteq \mathcal{O}(X)$ of a (sober) space X, which are *ordinary* filters. This theorem is due to Hofmann and Mislove [12], and can in our present setting be obtained from the following result. It identifies where one uses the Axiom of Choice and the fact that the space is sober. We sketch the proof for reasons of completeness. It is very similar to Lemma 8.2.2 in [33]. From now on we will label with 'AC' the results which make use of the Axiom of Choice.

Lemma 5.5 (AC). *For a sober space X, a Scott-open set $F \subseteq \mathcal{O}(X)$ is an M-filter if and only if it is an ordinary filter.*

Proof. The (only-if) part is obvious, so we concentrate on the (if) part. Assume a Scott-open filter $F \subseteq \mathcal{O}(X)$. We have to show

$$\bigcap F \subseteq o' \quad \Rightarrow \quad o' \in F.$$

Towards a contradiction, suppose $o' \notin F$. Then we have to produce an element $x \in X$ with $x \in \bigcap F$ but $x \notin o'$. Because X is sober it suffices to give a prime-open $p \in \mathcal{O}(X)$

with $p \notin F$ and $o' \subseteq p$, where we think of p as the directed union $\bigcup\{o \in \mathcal{O}(X) \mid x \notin o\}$. Hence one considers the poset

$$P = \{a \in \mathcal{O}(X) \mid o' \subseteq a \text{ and } a \notin F\}, \quad \text{ordered by inclusion.}$$

Every chain in P has an upper bound, so by Zorn's Lemma we get a maximal element $p \in P$. It remains to show that p is prime-open. Towards the contrary, assume

$$o_1 \cap o_2 \subseteq p \quad \text{but} \quad o_1 \nsubseteq p \text{ and } o_2 \nsubseteq p.$$

Then by maximality of p, both the open sets $o_1' = p \cup o_1$ and $o_2' = p \cup o_2$ are in F and hence, because F is a filter, also $o_1' \cap o_2' \in F$. But $o_1' \cap o_2' = p \cup (o_1 \cap o_2) = p$. Contradiction. □

Finally we obtain the result of Hofmann and Mislove [12] as a direct consequence of Lemmas 5.5 and 5.2.

Corollary 5.6 (Hofmann–Mislove theorem [12]). *For a sober space X, there is an order isomorphism between the the poset $(\mathscr{K}\mathscr{Q}(X), \leqslant_U)$ of compact saturated sets and the poset of Scott-open filters $F \subseteq \mathcal{O}(X)$, ordered by inclusion.*

The following result is a direct consequence of Lemma 5.5 and can be found in [2].

Lemma 5.7. *Let X be a sober space, Y be a topological space and $\phi : \Omega(X) \to \Omega(Y)$ be a Scott continuous function. Then ϕ is finite multiplicative (i.e. preserves finite meets) if and only if it is completely multiplicative.*

In the remainder of this section we will have a brief look at upper power spaces on posets, continuous dcpo's and algebraic dcpo's. In the latter two examples we make essential use of the above Corollary 5.6. The first example is more elementary and can be described without it.

5.1. Upper power space on posets

Let **PoSets** be the category of posets and monotone functions. There is a full and faithful functor **PoSets** $\to$ $\mathbf{Sp}_0$ which maps a poset $(X, \leqslant)$ to the underlying set X equipped with the Alexandrov topology (in which all upper sets $A = \uparrow A$ (for $A \subseteq X$) are open). The upper space monad $X \mapsto \mathscr{Q}(X)$ described in the beginning of this section restricts to a monad on **PoSets**. Our aim is to describe this monad in terms of certain ideals of subsets of a poset X.

Since Alexandrov open sets are closed under arbitrary intersections (the topology can even be characterized in such a way, see [19, Ch. II, Exercise 1.7], we get that the saturation $\mathscr{Q}(A)$ is the upper closure $\uparrow A$. Hence for a poset X, $\mathscr{Q}(X) = \mathcal{O}(X)$ and multiplicative functions $\Omega(X) \to \mathbf{2}$ are functions $\mathcal{O}(X) \to \mathbf{2}$ that preserve all intersections. The upper order $\leqslant_U$ is extended to the power set $\mathscr{P}(X)$ by $a \leqslant_U b \Leftrightarrow b \subseteq \uparrow a$. Below we write $\mathscr{P}_{\mathrm{f}}(X)$ for the set of finite subsets of X.

A *principal ideal* of a preorder $(X, \sqsubseteq)$ is a set of the form $\downarrow x$ for $x \in X$. Clearly, they can be identified with subsets $I \subseteq X$ such that

(i) I is a lower set: $a \sqsubseteq b \in I \Rightarrow a \in I$,

(ii) for each collection $S \subseteq I$ there is an element $b \in I$ with $\forall a \in S.\ a \sqsubseteq b$.

We find this second characterization of principal ideals more convenient because it is similar to that of directed ideals.

Remark 5.8. Given a poset $(X, \sqsubseteq)$ notice that the poset of principal ideals in $(X, \sqsubseteq)$ is order-isomorphic to $(X, \sqsubseteq)$. However, if $(X, \sqsubseteq)$ is only a preorder, then the poset of principal ideal is order isomorphic with the anti-symmetrization (i.e. $\mathcal{T}_0$-fication) of X.

These principal ideals capture the saturated (= open) sets.

Lemma 5.9. *Let X be a poset, taken with its Alexandrov topology.*

(i) *The poset of principal ideals in $(\mathcal{P}_{\mathrm{f}}(X), \leqslant_U)$, ordered by inclusion, is order isomorphic to the poset $(\mathcal{K}\mathcal{Q}(X), \leqslant_U)$.*

(ii) *The poset of principal ideals in $(\mathcal{P}(X), \leqslant_U)$, ordered by inclusion, is order isomorphic to the poset $(\mathcal{Q}(X), \leqslant_U)$.*

Proof. (i) We use Lemma 5.2. For every ideal I in $(\mathcal{P}_{\mathrm{f}}(X), \leqslant_U)$ and Scott open M-filter $\mathcal{U} \subseteq \mathcal{O}(X)$ define maps $\mathcal{U} \mapsto I_{\mathcal{U}}$ and $I \mapsto \mathcal{U}_I$ by

$$I_{\mathcal{U}} = \{a \in \mathcal{P}_{\mathrm{f}}(X) \mid \uparrow a \in \mathcal{U}\} \quad \text{and} \quad \mathcal{U}_I = \{o \in \mathcal{O}(X) \mid \exists a \in I.\ a \subseteq o\}$$

First we show that $I_{\mathcal{U}}$ is a principal ideal. Obviously $a \leqslant_U b \in I_{\mathcal{U}}$ implies $a \in I_{\mathcal{U}}$. And if a subset $S \subseteq I_{\mathcal{U}}$ is given, then $b = \bigcap_{a \in S} \uparrow a$ is an open set, so $\bigcap \mathcal{U} \subseteq \bigcap_{a \in S} \uparrow a = b$ implies $b \in \mathcal{U}$ because $\mathcal{U}$ is an M-filter. Hence $b \in I_{\mathcal{U}}$ and $a \supseteq b = \uparrow b$, so $a \leqslant_U b$ for all $a \in S$.

Next we show that $\mathcal{U}_I$ is a Scott-open M-filter. Let $o \in \mathcal{O}(X)$ be such that $\bigcap \mathcal{U}_I \subseteq o$ and assume $o \notin \mathcal{U}_I$. Then $a \not\subseteq o$ for all $a \in I$. Since I is a principal ideal, there exists $b \in I$ such that $a \leqslant_U b$ for all $a \in I$, that is $b \subseteq \uparrow a$. Hence we get the contradiction $b \subseteq \bigcap \mathcal{U}_I \subseteq o$. Further, $\mathcal{U}_I$ is also Scott-open because for every directed set $S \subseteq \mathcal{O}(X)$ such that $\bigcup S \in \mathcal{U}_I$, there exists a finite set $a \in I$ such that $a \subseteq \bigcup S$. Hence by directness of S there exists $o \in S$ such that $a \subseteq o$, that is $o \in \mathcal{U}_I$.

Finally we prove the isomorphism (preservation of the orders is immediate):

$$\begin{aligned} I_{\mathcal{U}_I} &= \{a \in \mathcal{P}_f(X) \mid \uparrow a \in \mathcal{U}_I\} \\ &= \{a \in \mathcal{P}_{\mathrm{f}}(X) \mid \exists b \in I.\ b \subseteq \uparrow a\} \\ &= \{a \in \mathcal{P}_{\mathrm{f}}(X) \mid \exists b \in I.\ a \leqslant_U b\} \\ &= I; \\ \mathcal{U}_{I_{\mathcal{U}}} &= \{o \in \mathcal{O}(X) \mid \exists a \in I_{\mathcal{U}}.\ a \subseteq o\} \\ &= \{o \in \mathcal{O}(X) \mid \exists\} \in a.\ \mathcal{P}_{\mathrm{f}}(X).\ \uparrow a \in \mathcal{U} \text{ and } a \subseteq o\} \\ &\stackrel{*}{=} \mathcal{U}. \end{aligned}$$

where the inclusion ($\overset{*}{\subseteq}$) holds because $\uparrow a \subseteq o$. Conversely, ($\overset{*}{\supseteq}$) follows from the fact that $\mathscr{U}$ is Scott-open and that every Alexandrov open set o is equal to the directed union $\bigcup\{\uparrow a | a \in \mathscr{P}_f(X) \text{ and } a \subseteq o\}$.

(ii) We use Lemma 5.1 and the fact that M-filters are closed under arbitrary unions. The latter correspond to ideals I in $(\mathscr{P}(X), \leqslant_U)$. The correspondence $\mathscr{U} \mapsto I_{\mathscr{U}}$ and $I \mapsto \mathscr{U}_I$ are, just as before, given by

$$I_{\mathscr{U}} = \{a \in \mathscr{P}(X) \mid \uparrow a \in \mathscr{U}\} \quad \text{and} \quad \mathscr{U}_I = \{o \in \mathscr{O}(X) \mid \exists a \in I.\, a \subseteq o\}.$$

Again $I_{\mathscr{U}}$ is a principal ideal and $\mathscr{U}_I$ is an M-filter. Like before the two constructions form an order isomorphism (notice that for proving $\mathscr{U}_{I_{\mathscr{U}}} = \mathscr{U}$ we use $o \in \mathscr{P}(X)$ and $\uparrow o = o$). □

5.2. *Upper power space on continuous dcpo*

Recall that for a directed complete partial order (dcpo) $(X, \leqslant)$ an element y is way-below x, written $y \ll x$, if for every directed set $S \subseteq X$, if $x \leqslant \bigvee S$ then there exists $s \in S$ such that $y \leqslant s$. A continuous poset $(X, \leqslant)$ is then a dcpo in which for every element $x \in X$ the set $\twoheaddownarrow x = \{y \in X \mid y \ll x\}$ of elements way-below x is directed and has x as join. We write **CPos** for the category of continuous dcpo's and continuous (directed join preserving) functions.

These continuous dcpo's are considered with the Scott topology. There are then full and faithful functors **CPos** $\to$ **Sob** $\to$ **Sp** (for a proof that each continuous dcpo is sober, see e.g. [10]). The sets $\twoheaduparrow x = \{y \in X \mid x \ll y\}$ form a basis for the Scott topology on a continuous dcpo X and a subset $A \subseteq X$ is open if and only if $A = \bigcup_{x \in A} \twoheaduparrow x$. The latter expression is also written as $\twoheaduparrow A$. The way below relation $\ll$ satisfies a certain interpolation property, which is axiomatized as (INT) below.

An *abstract basis* (according to Jung [21] consists of a set B with a transitive relation $\prec$, such that for finite subsets $S \subseteq B$ and $x \in B$,

$$\text{(INT)} \qquad S \prec x \;\Rightarrow\; \exists y \in B.\, S \prec y \prec x.$$

A *directed ideal* in such an abstract basis $(B, \prec)$ is a subset $I \subseteq B$ satisfying

(i) I is a lower set: $x \prec y \in I \;\Rightarrow\; x \in I$,

(ii) I is directed: I is not empty and for $x, y \in I$ there is a $z \in I$ with $x, y \prec z$.

A useful property of such an ideal I is that if $x \in I$, then by directness there is a $y \in I$ with $x \prec y$. It is not hard to verify that the set of ideals in $(B, \prec)$, ordered by inclusion, is a continuous dcpo, with

$$I \ll J \quad\Leftrightarrow\quad \exists x, y \in B.\, x \prec y \text{ and } I \subseteq \downarrow x \subseteq \downarrow y \subseteq J.$$

And if $\prec$ is a reflexive relation on a set B, then the condition (INT) obviously holds and the set of ideals ordered by the subset inclusion forms an algebraic dcpo (see below) denoted by *Idl*(B).

Let X be a continuous dcpo. The set of finite subsets of X can be made into an abstract basis by putting for $a, b \in \mathscr{P}_f(X)$,

$$a \prec b \iff b \subseteq \Uparrow a \iff \forall y \in b.\ \exists x \in a.\, x \ll y$$

see [13] where this approach stems from (but where it is used for convex power domains). With the same definition of the approximation relation, also the set of all subsets of X can be made in an abstract basis. Recall that the collection of all filters of $\mathcal{O}(X)$ is denoted by $Fil(\mathcal{O}(X))$.

Lemma 5.10 (AC). *Let X be a continuous dcpo, taken with its Scott topology.*

(i) *The continuous dcpo of ideals in $(\mathscr{P}_f(X), \prec)$, ordered by inclusion, is order isomorphic to the dcpo $(\mathscr{K}\mathscr{Q}(X), \leqslant_U)$ of compact saturated sets. The latter can be identified with the Scott open filters of $(\mathcal{O}(X), \subseteq)$ (see Corollary 5.6).*

(ii) *The continuous dcpo of ideals in $(\mathscr{P}(X), \prec)$, ordered by inclusion, is order isomorphic to the poset $(Fil(\mathcal{O}(X)), \subseteq)$ of (ordinary) filters of $(\mathcal{O}(X), \subseteq)$.*

Proof. (i) We use Corollary 5.6 and establish a bijective correspondence between ideals I in $(\mathscr{P}_f(X), \prec)$ and Scott-open filters $\mathscr{F} \subseteq \mathcal{O}(X)$. The correspondence $\mathscr{F} \mapsto I_{\mathscr{F}}$ and $I \mapsto \mathscr{F}_I$ are:

$$I_{\mathscr{F}} = \{a \in \mathscr{P}_f(X) \mid \Uparrow a \in \mathscr{F}\} \quad \text{and} \quad \mathscr{F}_I = \{o \in \mathcal{O}(X) \mid \exists a \in I.\, a \subseteq o\}.$$

It is not hard to see that $a \prec b \in I_{\mathscr{F}}$ implies $a \in I_{\mathscr{F}}$. And $I_{\mathscr{F}}$ is non-empty because $X \in \mathscr{F}$, and X is the directed union,

$$X = \bigcup\{\Uparrow a \mid a \in \mathscr{P}_f(X)\}$$

where the inclusion ($\subseteq$) is obtained from the fact that for $x \in X$ the set $\Downarrow x$ is directed and hence nonempty. Since $\mathscr{F}$ is Scott-open, we get $\Uparrow a \in \mathscr{F}$ for some $a \in \mathscr{P}_f(X)$, and thus $I_{\mathscr{F}} \neq \emptyset$. To show that $I_{\mathscr{F}}$ is upward directed, we need that

$$\Uparrow a \cap \Uparrow b = \bigcup\{\Uparrow c \mid c \in \mathscr{P}_f(X) \text{ and } c \subseteq \Uparrow a \cap \Uparrow b\}.$$

Then, if $a, b \in I_{\mathscr{F}}$, we have $\Uparrow a, \Uparrow b \in \mathscr{F}$ and hence as $\mathscr{F}$ is a filter we obtain $\Uparrow a \cap \Uparrow b \in \mathscr{F}$. Since $\mathscr{F}$ is Scott-open, we get $\Uparrow c \in \mathscr{F}$ for some finite $c \subseteq \Uparrow a \cap \Uparrow b$. But then $c \in I_{\mathscr{F}}$ and $a, b \prec c$. It is almost immediate that $\mathscr{F}_I$ is a Scott-open subset of $\mathcal{O}(X)$. It contains the top element X, since every ideal is nonempty. And if $o, o' \in \mathscr{F}_I$, say via $a, a' \in I$ with $a \subseteq o$ and $a' \subseteq o'$, then there is a $b \in I$ with $a, a' \prec b$. But then $b \subseteq \Uparrow a \cap \Uparrow a' \subseteq o \cap o'$ and so also $o \cap o' \in \mathscr{F}_I$. Therefore $\mathscr{F}_I$ is a Scott-open filter.

Finally we prove the isomorphism (preservation of the orders is immediate)

$$\begin{aligned} I_{\mathscr{F}_I} &= \{a \in \mathscr{P}_f(X) \mid \Uparrow a \in \mathscr{F}_I\} \\ &= \{a \in \mathscr{P}_f(X) \mid \exists b \in I.\, b \subseteq \Uparrow a\} \\ &= \{a \in \mathscr{P}_f(X) \mid \exists b \in I.\, a \prec b\} \\ &= I; \end{aligned}$$

$$\begin{aligned}\mathscr{F}_{I_{\mathscr{F}}} &= \{o \in \mathcal{O}(X) \mid \exists a \in I_{\mathscr{F}}.\, a \subseteq o\}\\ &= \{o \in \mathcal{O}(X) \mid \exists a \in \mathscr{P}_{\mathrm{f}}(X).\ \Uparrow a \in \mathscr{F} \text{ and } a \subseteq o\}\\ &\stackrel{*}{=} \mathscr{F},\end{aligned}$$

where the inclusion ($\stackrel{*}{\subseteq}$) holds because $\Uparrow a \subseteq o$. Conversely, ($\stackrel{*}{\supseteq}$) follows from the fact that $\mathscr{F}$ is Scott-open and that o is equal to the directed union $\bigcup\{\Uparrow a | a \in \mathscr{P}_{\mathrm{f}}(X)$ and $a \subseteq o\}$.

(ii) The correspondence $\mathscr{F} \mapsto I_{\mathscr{F}}$ and $I \mapsto \mathscr{F}_I$ are, like before, given by

$$I_{\mathscr{F}} = \{a \in \mathscr{P}(X) \mid \ \Uparrow a \in \mathscr{F}\} \qquad \text{and} \qquad \mathscr{F}_I = \{o \in \mathcal{O}(X) \mid \exists a \in I.\, a \subseteq o\}.$$

Again $I_{\mathscr{F}}$ is an ideal and $\mathscr{F}_I$ is a filter. Moreover, the two constructions form an order isomorphism (notice that for proving $\mathscr{F}_{I_{\mathscr{F}}} = \mathscr{F}$ we use the fact that every open set $o \in \mathcal{O}(X)$ is an element of $\mathscr{P}(X)$ and moreover $\Uparrow o = o$). □

A further investigation of power domains on continuous dcpo's can be found in [13, 14] and also in [21].

5.3. Upper power space on algebraic dcpo

An element x in a dcpo X is called compact if $x \ll x$. We write $\mathscr{K}(X)$ for the set of compact elements in X. One calls X an algebraic dcpo if it is a dcpo in which for each element x the set $\downarrow x \cap \mathscr{K}(X)$ of compact elements below x is directed and has x as join. The principal upper sets $\uparrow x$ for $x \in \mathscr{K}(X)$ form a basis for the Scott topology on X. Since an algebraic dcpo is continuous, it is in particular sober as a topological space.

Lemma 5.11 (AC). *Let X be an algebraic dcpo, taken with its Scott topology.*

(i) *The algebraic dcpo of ideals in $(\mathscr{P}_{\mathrm{f}}(\mathscr{K}(X)), \leqslant_U)$, ordered by inclusion is order isomorphic to the poset $(\mathscr{K}\mathscr{Q}(X), \leqslant_U)$ of compact saturated sets. The latter correspond to Scott open filters of $(\mathcal{O}(X), \subseteq)$ (see Corollary 5.6).*

(ii) *The algebraic dcpo of ideals in $(\mathscr{P}(\mathscr{K}(X)), \leqslant_U)$, ordered by inclusion is order isomorphic to the poset $(Fil(\mathcal{O}(X)), \subseteq)$ of (ordinary) filters of $(\mathcal{O}(X), \subseteq)$.*

Proof. (i) We proceed as in the previous subsection and use Corollary 5.6 to get a bijective correspondence between ideals I in $(\mathscr{P}_{\mathrm{f}}(\mathscr{K}(X)), \leqslant_U)$ and Scott-open filters $\mathscr{F} \subseteq \mathcal{O}(X)$. The correspondence $\mathscr{F} \mapsto I_{\mathscr{F}}$ and $I \mapsto \mathscr{F}_I$ are given by

$$I_{\mathscr{F}} = \{a \in \mathscr{P}(X) \mid \uparrow a \in \mathscr{F}\} \qquad \text{and} \qquad \mathscr{F}_I = \{o \in \mathcal{O}(X) \mid \exists a \in I.\ a \subseteq o\}.$$

The rest of the proof is as before, and hence left to the reader.

(ii) Similarly, the correspondence $\mathscr{F} \mapsto I_{\mathscr{F}}$ and $I \mapsto \mathscr{F}_I$ are given by

$$I_{\mathscr{F}} = \{a \in \mathscr{P}(\mathscr{K}(X)) \mid \uparrow a \in \mathscr{F}\} \qquad \text{and} \qquad \mathscr{F}_I = \{o \in \mathcal{O}(X) \mid \exists a \in I.\ a \subseteq o\}.$$

As in the previous lemma we have that $I_{\mathscr{F}}$ is an ideal and that $\mathscr{F}_I$ is a filter. Moreover they form an order isomorphism (notice that for proving $\mathscr{F}_{I_{\mathscr{F}}} = \mathscr{F}$ we use the fact that every open set $o \in \mathcal{O}(X)$ is equal to $\uparrow a$ for some $a \subseteq \mathscr{K}(X)$). □

The power domains on algebraic dcpo's in (i) were first studied by Plotkin [24] and Smyth [28].

Remark 5.12. The compact elements of the algebraic dcpo $(Fil(\mathcal{O}(X)), \subseteq)$ are isomorphic to the principal ideals of the poset $(\mathscr{P}(\mathscr{K}(X)), \leqslant_U)$, which are isomorphic, by Lemma 5.9, to the M-filters of the space $\mathscr{K}(X)$ taken with the Alexandrov topology. Hence we can say that every filter $\mathscr{F}$ of the Scott topology of an algebraic dcpo X is the directed union of all the M-filters $\mathscr{U}$ of the Alexandrov topology of $K(X)$ such that $\mathscr{U} \subseteq \mathscr{F}$. Similarly, every Scott open filter $\mathscr{F}$ of the Scott topology of an algebraic dcpo X is the directed union of all the Scott open M-filters $\mathscr{U}$ of the Alexandrov topology of $K(X)$ such that $\mathscr{U} \subseteq \mathscr{F}$.

6. Some further equivalences

In this section we restrict our attention to subcategories of **Sp**. In the first four subsections we consider topological spaces which are not, in general, sober. For these spaces we give a duality by restricting the adjunction of Theorem 3.23. Of special interest is a duality for the category **PoSet**. We derive a pointless version of the (directed) ideal completion of posets. Finally, in the last two subsections we study Galois connections in the context of observation frames and consider the relationship between frames, observation frames and sober spaces.

6.1. $\mathscr{T}_1$ Spaces and atomic observation frames

Recall that a space X is $\mathscr{T}_1$ if for every $x, y \in X$ with $x \neq y$ there exists an open set $o \in \mathcal{O}(X)$ such that $x \in o$ but $y \notin o$. For an example of a $\mathscr{T}_1$ space which is not sober and an example of a sober space which is not $\mathscr{T}_1$ see [30, Ch. IV, Example 4.1.4]. The full subcategory of **Sp** whose objects are $\mathscr{T}_1$ spaces is denoted by $\mathbf{Sp}_1$.

An observation frame $F \rhd\!\!\rightarrow L$ will be called *atomic* if for every $p, q \in MP(F \rhd\!\!\rightarrow L)$ if $p \leqslant q$ then $p = q$. The full sub-category of **OFrm** whose objects are atomic observation frames is denoted by **AOFrm**.

Lemma 6.1. *The functors Ω : **Sp** $\to$ **OFrm**^{op} and Pt : **OFrm**^{op} $\to$ **Sp** restrict to an adjunction between $\mathbf{Sp}_1$ and **AOFrm**^{op} and hence to a duality between $\mathbf{Sp}_1$ and $\mathbf{AOFrm}_M$.*

Proof. If a space X is $\mathscr{T}_1$ then the specialization preorder is the equality. Moreover, since every $\mathscr{T}_1$ space is $\mathscr{T}_0$, we have that points are M-prime elements $o_x = \bigcup\{o \in \mathcal{O}(X) \mid x \notin o\}$. Therefore, for every $o_x, o_y \in MP(\Omega(X))$ of a given $\mathscr{T}_1$ space X, if $o_x \subseteq o_y$ then $x \leqslant y$ and hence $x = y$, i.e. $o_x = o_y$.

Conversely, let $F \rightarrowtail L$ be an atomic observation frame and take $p, q \in MP(F \rightarrowtail L)$ with $p \neq q$. This implies $p \not\leq q$ or $q \not\leq p$. Suppose $p \not\leq q$ then clearly q is in the open $\Delta(p) = \{r \in MP(F \rightarrowtail L) \mid p \not\leq r\}$ but p is not. The other case can be treated similarly. Hence $Pt(F \rightarrowtail L)$ is a $\mathscr{T}_1$ space. □

Notice that for an atomic observation frame $F \rightarrowtail L$ with F order generated by its M-prime elements there can be no element different from the $\top$ which is above some other M-prime element. This means that the M-prime elements of $F \rightarrowtail L$ are exactly the co-atoms of F (that is, maximal elements which differ from the top).

6.2. *Open compact spaces and algebraic observation frames*

A space X is called *open compact*, if for every $x \in X$ and open set $o \in \mathcal{O}(X)$ such that $x \in o$ there exists a compact open set $u \in \mathcal{O}(X)$ such that $x \in u \subseteq o$. For example every poset taken with the Alexandrov topology is open compact.

Denote by **oKSp** the full subcategory of **Sp** whose objects are open compact spaces. Let $\mathbf{OAFrm}_M$ denote the full subcategory of **OFrm** whose objects are observation frames $F \rightarrowtail L$ such that F is an algebraic lattice and is order generated by the M-prime elements.

Lemma 6.2. *The functors $\Omega : \mathbf{Sp} \to \mathbf{OFrm}^{op}$ and $Pt : \mathbf{OFrm}^{op} \to \mathbf{Sp}$ restrict to a duality between $\mathbf{oKSp}_0$ and $\mathbf{OAFrm}_M$.*

Proof. It is enough to prove that a space X is open compact if and only if $\mathcal{O}(X)$ is an algebraic complete lattice. Let X be a open compact space and let $o \in \mathcal{O}(X)$. For every $x \in o$, since X is open compact, there exists a compact open u such that $x \in u \subseteq o$. Hence $o \subseteq \bigcup\{u \in \mathscr{K}\mathcal{O}(X) \mid u \subseteq o\}$. The reverse inclusion is clear, hence $\mathcal{O}(X)$ is algebraic.

Conversely, if $\mathcal{O}(X)$ is algebraic then for every open set o we have $o = \bigcup\{u \in \mathscr{K}\mathcal{O}(X) \mid u \subseteq o\}$. Hence for every $x \in X$, if $x \in o$ then there exists a compact open $u \in \mathscr{K}\mathcal{O}(X)$ such that $x \in u \subseteq o$, that is X is open compact. □

6.3. *Core compact spaces and continuous observation frames*

Recall that a space X is called *core compact*, or quasi-locally compact, if for every $x \in X$ and open set $o \in \mathcal{O}(X)$ such that $x \in o$ there exists a compact set $A \subseteq X$ and an open set $o' \in \mathcal{O}(X)$ such that $x \in o' \subseteq A \subseteq o$. Core compact spaces are important because they are exponentiable in **Sp** [17] (the converse also holds, as shown in [5]). Isbell [17] gives an example of a core compact space which is not sober.

Denote by $\mathbf{cKSp}_0$ the full subcategory of $\mathbf{Sp}_0$ whose objects are core compact $\mathscr{T}_0$ spaces and let $\mathbf{OCFrm}_M$ denote the full subcategory of $\mathbf{OFrm}_M$ whose objects are observation frames $F \rightarrowtail L$ such that F is a continuous lattice and is order generated by the M-prime elements. The proof of the following lemma is left to the reader as it consists only of some verification steps along the lines of Lemma 6.2.

Lemma 6.3. *The functors $\Omega : \mathbf{Sp} \to \mathbf{OFrm}^{op}$ and $Pt : \mathbf{OFrm}^{op} \to \mathbf{Sp}$ restrict to a to a duality between $\mathbf{cKSp}_0$ and $\mathbf{OCFrm}_M$.*

6.4. Posets and complete lattices

Let **AlSp** denote the full subcategory of **Sp** whose objects are topological spaces X in which open sets are closed under arbitrary intersection (i.e. they form the Alexandrov topology). The full and faithful functor from the category **PoSet** (posets and monotone functions) to $\mathbf{Sp}_0$ which maps a poset $(X, \leqslant)$ to the underlying set X equipped with the Alexandrov topology, determines an equivalence of categories between **PoSet** and $\mathbf{AlSp}_0$.

Lemma 6.4. *The functors $\Omega : \mathbf{Sp} \to \mathbf{OFrm}^{op}$ and $Pt : \mathbf{OFrm}^{op} \to \mathbf{Sp}$ restrict to an adjunction between* **AlSp** *and $\bigwedge$-$\mathbf{OFrm}^{op}$, the full subcategory of $\mathbf{OFrm}^{op}$ whose objects are observation frames $F \rhd\!\!\to L$ for which $(\bigwedge A)^{\vee} = \bigsqcap A^{\vee}$ for all $A \subseteq F$. Moreover this adjunction restricts to a duality between $\mathbf{AlSp}_0$ and $\bigwedge$-$\mathbf{OFrm}_M$.*

Proof. It is enough to prove for every $F \rhd\!\!\to L$ in $\bigwedge$-**OFrm** that $\bigcap \Delta(A) = \Delta(\bigwedge A)$ for every $A \subseteq F$.

$$\begin{aligned} p \in \bigcap\{\Delta(a) \mid a \in A\} &\Leftrightarrow p \in MP(F \rhd\!\!\to L) \text{ and } \forall a \in A.\ a \not\leqslant p \\ &\overset{*}{\Leftrightarrow} p \in MP(F \rhd\!\!\to L) \text{ and } \textstyle\bigwedge A \not\leqslant p \\ &\Leftrightarrow p \in \Delta(\textstyle\bigwedge A), \end{aligned}$$

where the implication $(\overset{*}{\Leftarrow})$ is trivial and for $(\overset{*}{\Rightarrow})$ we use that $p \in MP(F \rhd\!\!\to L)$ and the following contradiction: if $\bigwedge A \leqslant p$ then also $(\bigwedge A)^{\vee} = \bigsqcap A^{\vee} \sqsubseteq p^{\vee}$ and hence $a \leqslant p$ for some $a \in A$. □

Let now **CLat** be the category whose objects are complete lattices and whose morphisms are complete lattice homomorphisms (functions preserving both arbitrary joins and arbitrary meets). Given a complete lattice L, an element $p \in L$ is called M-prime if $\bigwedge A \leqslant p$ for $A \subseteq L$ implies there exists $a \in A$ such that $a \leqslant p$. The set of all M-prime elements of L is denoted $MP(L)$.

Lemma 6.5. *The category $\bigwedge$-$\mathbf{OFrm}_M$ is equivalent to $\mathbf{CLat}_M$, the full sub-category of* **CLat** *whose objects are order generated by the M-prime elements.*

Proof. Let $\phi : (F \rhd\!\!\to L) \to (G \rhd\!\!\to H)$ be a morphism between observation frames. Then $\phi : F \to G$ preserves arbitrary joins since it is a frame morphism, but preserves also arbitrary meets because $(\bigwedge A)^{\vee} = \bigsqcap A^{\vee}$ implies $\phi(\bigwedge A)^{\vee} = \bigsqcap \phi(A)^{\vee} = (\bigwedge \phi(A))^{\vee}$ and hence $\phi(\bigwedge A) = \bigwedge \phi(A)$ since $(-)^{\vee} : G \rhd\!\!\to H$ is order-reflecting by Remark 3.31. Therefore we have a forgetful functor $\bigwedge$-$\mathbf{OFrm}_M \to \mathbf{CLat}_M$ which maps an observation frame $F \rhd\!\!\to L$ to the underlying frame F and a morphism between observation frames $\phi : (F \rhd\!\!\to L) \to (G \rhd\!\!\to H)$ to the underlying frame morphism $\phi : F \to G$.

Proof. Let us at first notice that the inclusion functor i^{op} is naturally isomorphic to the functor given by the composition $\mathcal{O}_{\mathrm{Sc}} \circ Idl \circ MP(-)$. Indeed for every algebraic complete lattice L we have $\mathcal{O}_{\mathrm{Sc}}(Idl(MP(L))) \cong \mathcal{O}_{\mathrm{Al}}(MP(L)) \cong L = i^{op}(L)$, where the latter isomorphism holds by Corollary 6.7. Naturality follows from the fact that the functor Idl : **PoSet** $\to$ **AlgPos** is faithful.

Since the functor $\mathcal{O}_{\mathrm{Sc}} \circ Idl \circ MP(-)$ has a right adjoint, namely $\mathcal{O}_{\mathrm{Al}} \circ U \circ Spec(-) = \mathcal{O}_{\mathrm{Al}} \circ Spec(-) = j^{op}(-)$, we have that $j^{op}(-)$ is also right adjoint of $i^{op}(-)$. Therefore j : **AlgCDFrm** $\to$ **AlgCDLat** is left adjoint of $i(-)$. Commutativity of the diagram is immediate from the definition of $j(-)$. $\square$

Finally, we just mention without proof the following dualities which can be obtained by combining the results of the previous subsections with Corollary 6.7.

(i) Posets with bottom element vs. algebraic complete lattices with a completely coprime top element;

(ii) sets vs. atomic algebraic complete lattices;

(iii) finite sets vs. compact atomic algebraic complete lattices.

6.5. *Frames and observation frames*

Let **Frm** be the category of frames whose objects are frames and whose morphisms are functions preserving finite meets and arbitrary joins. Recall that for a meet semilattice F an element $p \in F$ is called *prime* if for all finite $S \subseteq F$ such that $\bigwedge S \leqslant p$ there exists $s \in S$ with $s \leqslant p$. The set of all prime elements of a meet semilattice F is denoted by $Spec(F)$.

Given a frame F, we write $Pt_\omega(F)$ for the set $Spec(F)$ together with the collection of open sets $\Delta_\omega(a) = Spec(F) \setminus \uparrow a$ for every $a \in F$. Adapting the proof of Corollary 3.19 we see that this collection forms a topology. Let now **Frm** be the category of frames whose objects are frames and whose morphisms are functions that preserve finite meets and arbitrary joins. The full subcategory of **Frm** whose objects are frames F order generated by $Spec(F)$ is denoted by **SFrm** (spatial frames). The following lemma can be found in [19, Ch. II, Corollary 1.7].

Lemma 6.10. *The assignment $F \mapsto Pt_\omega(F)$ defines a functor* $\mathbf{Frm}^{op} \to \mathbf{Sp}$ *which is right adjoint of $\mathcal{O}(-)$:* $\mathbf{Sp} \to \mathbf{Frm}^{op}$ *(the functor which maps every topological space to its lattice of open set and every continuous function to its inverse restricted to the open sets). Furthermore we have*

(i) *the adjunction restricts to a duality between the categories* **SFrm** *and* **Sob**;

(ii) *the inclusion* $\mathbf{Sob} \hookrightarrow \mathbf{Sp}_0$ *has a left adjoint, namely the composite $Pt_\omega \circ \mathcal{O}(-)$;*

(iii) *the inclusion* $\mathbf{SFrm} \hookrightarrow \mathbf{Frm}$ *has a left adjoint, namely the composite $\mathcal{O}(-)^{op} \circ Pt_\omega^{op}$.*

We have as an immediate consequence the following relation between the category of observation frames and that of frames.

Corollary 6.11. *The functor* $\Omega^{op} \circ Pt_{\omega}^{op}$: $\mathbf{Frm} \to \mathbf{OFrm}_M$ *has right adjoint, namely the composite* $\mathcal{O}^{op} \circ Pt_{\omega}^{op} \circ \mathcal{O}^{op} \circ Pt^{op}$.

Proof. By composition of the following adjoints and taking the opposite:

$$\mathbf{OFrm}_M^{op} \simeq \mathbf{Sp}_0 \overset{\hookleftarrow}{\rightarrow} \mathbf{Sob}- \simeq \mathbf{SFrm}^{op} \overset{\leftarrow}{\hookrightarrow} \mathbf{Frm}^{op} \quad \square$$

Since the forgetful functor $\mathbf{Frm} \to \mathbf{Set}$ has left adjoint (see [19]), we have, by composition, an adjunction also between $\mathbf{Set}$ and $\mathbf{OFrm}_M$.

6.6. *Galois connections*

In this section we take a closer look at Galois connections between posets. Galois connections play an important role in spectral theory (see for example [10]) and in general in lattice theory. In particular we are interested in those posets which constitute the frame part of an observation frame. This will allow us to give a necessary and sufficient condition for a pair of maps to form a Galois connection such that the lower adjoint is an observation frame morphism.

Definition 6.12. Let F, G be two posets and $f : F \to G$, $g : G \to F$ be two functions. We say the pair (f, g) is a *Galois connection* between F and G if

(i) both f and g are monotone, and

(ii) $f(x) \leqslant y$ if and only if $x \leqslant g(y)$ for all $x \in F$ and $y \in G$.

For a Galois connection (f, g) the function g is called upper (or left) adjoint and the function f is called lower (or right) adjoint. A Galois connection is a very special case of adjoint functors, where the posets F and G are seen as categories (see for example [23, Ch. IV]). Any upper adjoint g preserves all meets in G, while any lower adjoint f preserves all joins in F. More generally we have the following characterization of Galois connections (cf. e.g. [10, Corollary 0-3.5, Theorem 0-3.6]).

Lemma 6.13. *Let F, G be two complete lattices.*

(i) *A function $g: G \to F$ preserves all meets in G if and only if g is monotone and has lower adjoint $f: F \to G$ given by $f(x) = \bigwedge\{y \in G \mid x \leqslant g(y)\}$.*

(ii) *A function $f: F \to G$ preserves all joins in F if and only if f is monotone and has upper adjoint $g: G \to F$ given by $g(y) = \bigvee\{x \in F \mid f(x) \leqslant y\}$.*

(iii) *A pair of monotone functions (f, g) with $f: F \to G$ and $g: G \to F$ is a Galois connection if and only if $f(g(y)) \leqslant y$ and $x \leqslant g(f(x))$ for all $x \in F$ and $y \in G$.*

If F and G are frames and $\phi : F \to G$ is a frame morphism then, since ϕ preserves arbitrary joins, it has an upper adjoint, say $g: G \to F$, which preserves arbitrary meets by Lemma 6.13 (ii). Also, the upper adjoint g preserves prime elements because ϕ

preserves finite meets [10, Lemma IV-4.5]. If $\phi : F \to G$ is also an observation frame morphism from $F \rhd\!\!\to L$ to $G \rhd\!\!\to H$ then we have the following.

Lemma 6.14. *Let $\phi : (F \rhd\!\!\to L) \to (G \rhd\!\!\to H)$ be an observation frame morphism. Then $\phi : F \to G$ has upper adjoint $g : G \to F$ which preserves arbitrary meets of G, prime elements and also the M-prime elements of $G \rhd\!\!\to H$.*

Proof. Since an observation frame morphism $\phi : (F \rhd\!\!\to L) \to (G \rhd\!\!\to H)$ is a frame morphism from F to G it has upper adjoint $g : G \to F$ which preserves arbitrary meets of G and prime elements in $Spec(G)$. Let now $p \in MP(G \rhd\!\!\to H)$ and $S \subseteq F$, then

$$\begin{aligned}
\sqcap S^{\vee} \sqsubseteq g(p)^{\vee} &\Rightarrow \sqcap \phi(S)^{\vee} \sqsubseteq \phi(g(p))^{\vee} && \text{M-multiplicativity}\\
&\Rightarrow \sqcap \phi(S)^{\vee} \sqsubseteq p^{\vee} && \text{Lemma 6.13(iii) and } (-)^{\vee} \text{ is monotone}\\
&\Rightarrow \exists s \in S.\ \phi(s) \leqslant p\ p && p \text{ is M-prime}\\
&\Leftrightarrow \exists s \in S.\ s \leqslant g(p) && (\phi, g) \text{ is a Galois connection}
\end{aligned}$$

that is, $g(p) \in MP(F \rhd\!\!\to L)$. $\square$

For a Galois connection (f, g) there does not seem to be any condition on g alone which implies f preserving finite meets (see [19]). Hence in general the converse of the above Lemma does not hold. But if we restrict our attention to observation frames $G \rhd\!\!\to H$ order generated by M-primes, then we have the following.

Theorem 6.15. *Let $F \rhd\!\!\to L$ and $G \rhd\!\!\to H$ be two observation frames with $G \rhd\!\!\to H$ order generated by its M-prime elements. Let also $\phi : F \to G$ and $g : G \to F$ be two functions forming a Galois connection. Then ϕ is an observation frame morphism if and only if g preserves the (arbitrary) meets of G, the prime elements in $Spec(G)$ and also the M-prime elements in $MP(G \rhd\!\!\to H)$.*

Proof. By Lemma 6.14 we only need to prove that if $g : G \to F$ is a function preserving the (arbitrary) meets of G, the prime elements and the M-prime elements then g has lower adjoint ϕ which is also an observation frame morphism. Since g preserves arbitrary meets of G it has lower adjoint $\phi : F \to G$ which preserves arbitrary joins of F by Lemma 6.13(i). We will prove ϕ preserves also finite meets of F and that is M-multiplicative.

By abuse of notation, let $g : MP(G \rhd\!\!\to H) \to MP(F \rhd\!\!\to L)$ be the restriction of g to the M-prime elements (since they are preserved this is well defined). Consider now the $\mathscr{T}_0$ topological spaces $Pt(G \rhd\!\!\to H)$ and $Pt(F \rhd\!\!\to L)$. We show that $g : MP(G \rhd\!\!\to H) \to MP(F \rhd\!\!\to L)$ is a continuous function. Indeed for every $\Delta(x) \in \mathcal{O}_{\Delta}(MP(F \rhd\!\!\to L)))$ with

$x \in F$ we have

$$\begin{aligned} g^{-1}(\Delta(x)) &= \{p \in MP(G \rhd\!\!\!\rightarrow H) \mid g(p) \in \Delta(x)\} \\ &= \{p \in MP(G \rhd\!\!\!\rightarrow H) \mid x \not\leq g(p)\} \\ &= \{p \in MP(G \rhd\!\!\!\rightarrow H) \mid \phi(x) \not\leq p\} \quad \text{Galois connection} \\ &= \{p \in MP(G \rhd\!\!\!\rightarrow H) \mid p \in \Delta(\phi(x))\} \\ &= \Delta(\phi(x)) \end{aligned}$$

which is open in $\mathcal{O}_\Delta(MP(G \rhd\!\!\!\rightarrow H)))$. Consider now the following diagram:

$$\begin{array}{ccccc} (F \rhd\!\!\!\rightarrow L) & \xrightarrow{\ \varepsilon\ } & \Omega(Pt(F \rhd\!\!\!\rightarrow L)) & \qquad & Pt(F \rhd\!\!\!\rightarrow L) \\ g \uparrow \ \downarrow \phi & & \downarrow \Omega(g) & & \uparrow g \\ (G \rhd\!\!\!\rightarrow H) & \xleftarrow{\ \varepsilon^{-1}\ } & \Omega(Pt(G \rhd\!\!\!\rightarrow H)) & & Pt(G \rhd\!\!\!\rightarrow H) \end{array}$$

where ε and $\Omega(g)$ are observation frame morphisms as defined in the previous section and ε^{-1} is an observation frame morphism because $G \rhd\!\!\!\rightarrow H$ is order generated by its M-primes (Lemma 3.29). Next we prove $\phi = \varepsilon^{-1} \circ \Omega(g) \circ \varepsilon$. For all $x \in F$ we have

$$\begin{aligned} \varepsilon^{-1}(\Omega(g)(\varepsilon(x))) &= \varepsilon^{-1}(\Omega(g)(\Delta(x))) \\ &= \varepsilon^{-1}(g^{-1}(\Delta(x))) \\ &= \varepsilon^{-1}(\Delta(\phi(x))) \\ &= \phi(x) \qquad \text{Lemma 3.29.} \end{aligned}$$

Therefore ϕ is an observation frame morphism from $F \rhd\!\!\!\rightarrow L$ to $G \rhd\!\!\!\rightarrow H$. □

Let us denote by **SOLoc** the category of "spatial observation locales" objects of which are observation frames order generated by M-primes and morphisms of which from $F \rhd\!\!\!\rightarrow L$ to $G \rhd\!\!\!\rightarrow H$ are functions $f : F \rightarrow G$ such that

(i) $f(\bigwedge S) = \bigwedge f(S)$ for all $S \subseteq F$;
(ii) $f(p) \in Spec(G)$ for all $p \in Spec(F)$;
(iii) $f(p) \in MP(G \rhd\!\!\!\rightarrow H)$ for all $p \in MP(F \rhd\!\!\!\rightarrow L)$.

As a direct consequence of Corollary 3.30 and Theorem 6.15 we have the following.

Corollary 6.16. *The categories* $\mathbf{OFrm}_M^{op}, \mathbf{Sp}_0$ *and* **SOLoc** *are equivalent.*

7. Discussion

We introduced the category of observation frames as a pointless counterpart of topological spaces. Since our main interest is the duality between topological spaces and observation frames and an infinitary logic for the latter, we did not look at various constructions in the category of observation frames. Further investigations are necessary to describe limits, colimits, monos and epis in this category. A related question is whether the category observation frames is any good for doing some form of pointless topology, as in [20].

Finally, we mention two more points which need to be explored: a representation of general (nonsober) directed complete partial orders with Scott topology and the question whether the category of observation frames is monadic over some base category or not. Regarding the first point, the category of dcpo's is fully and faithfully embedded into $\mathbf{Sp}_0$, and hence into some full subcategory of **OFrm**. Another interesting point of further study could be a generalization of "sheaves" over locales to suitable sheaves over observation frames.

Acknowledgements

We wish to thank the Amsterdam Concurrency Group especially Jaco de Bakker, Franck van Breugel, Jan Rutten and Daniele Turi, for discussions and suggestions about the contents of this paper. Comments from Vincent van Oostrom, Jaap van Oosten, and Fer-Jan de Vries were really useful. We finally wish to thank the two anonymous referees and Paul Johnson for their careful reading of the preliminary version of the paper and for their valuable suggestions to improve and correct some of the results.

Appendix A: Complete observation lattices

In this appendix we show that for an observation frame $F \rhd\!\!\rightarrow L$, the condition of F being a frame is natural in the following sense. If we require that a function between two complete lattices G and K preserves arbitrary joins and finite meets and that G is order generated by the M-prime elements, then G is a frame. This motivates our requirement that the lattice of observable predicates is a frame, in particular it motivates the infinite distributivity law.

Definition A.1. A *complete observation lattice* is a function $(-)^\vee : L \to R$ between two complete lattices $(L, \leqslant)$ and $(R, \sqsubseteq)$ which preserves arbitrary joins and finite meets.

Define a morphism between the complete observation lattices $L \rhd\!\!\rightarrow R$ and $G \rhd\!\!\rightarrow K$ similarly to Definition 3.4, i.e. a morphism ϕ between $(L \rhd\!\!\rightarrow R)$ and $(G \rhd\!\!\rightarrow K)$ is a function $\phi : L \to G$ that preserves finite meets and arbitrary joins and, for all $S, T \subseteq L$, if $\sqcap S^\vee \sqsubseteq \sqcap T^\vee$ then $\sqcap \phi(S)^\vee \sqsubseteq \sqcap \phi(T)^\vee$.

Define the M-prime elements of an observation lattice $L \rhd\!\!\rightarrow R$ as in Definition 3.14: an element $p \in L$ is called M-prime if for all $S \subseteq L$ such that $\sqcap S^\vee \sqsubseteq p^\vee$ there exists $s \in S$ such that $s \leqslant p$. The set of all M-prime elements of $L \rhd\!\!\rightarrow R$ is denoted by $MP(L \rhd\!\!\rightarrow R)$.

Remark A.2. Adapting the proof of Lemma 3.16 we have that, for a complete observation lattice $L\rhd\!\!\rightarrow R$, $p \in L$ is M-prime if and only if the function $\phi_p : L \to \mathbf{2}$ is a morphism between the observation lattice $L\rhd\!\!\rightarrow R$ and $\mathbf{2}$, where ϕ_p maps $x \in L$ to $\perp$ if and only if $x \leqslant p$.

Lemma A.3. *Take a complete observation lattice $L\rhd\!\!\rightarrow R$. If $MP(L\rhd\!\!\rightarrow R)$ is order generating the lattice L, then L is a frame.*

Proof. By Lemma 3.26 and Remark A.2 the morphisms $\phi_p : (L\rhd\!\!\rightarrow R) \to \mathbf{2}$ determined by the M-prime elements $p \in L$ separate the points of L. If H is the set of all these morphisms, then $\gamma : L \to \mathscr{P}(H)$ defined by $\gamma(x) = \{\phi_p \in H \mid \phi_p(x) = \top\}$ is a function preserving arbitrary joins and finite meets. Indeed for $S \subseteq L$ we have

$$
\begin{aligned}
\gamma(\textstyle\bigvee S) &= \{\phi_p \in H \mid \phi_p(\textstyle\bigvee S) = \top\} \\
&= \{\phi_p \in H \mid \textstyle\bigvee \phi_p(S) = \top\} \qquad \phi_p \text{ preserves arbitrary joins} \\
&= \textstyle\bigcup_{s \in S}\{\phi_p \in H \mid \phi_p(s) = \top\} \\
&= \textstyle\bigcup \gamma(S).
\end{aligned}
$$

Similarly for every finite $S \subseteq L$ we have

$$
\begin{aligned}
\gamma(\textstyle\bigwedge S) &= \{\phi_p \in H \mid \phi_p(\textstyle\bigwedge S) = \top\} \\
&= \{\phi_p \in H \mid \textstyle\bigwedge \phi_p(S) = \top\} \qquad \phi_p \text{ preserves finite meets} \\
&= \textstyle\bigcap_{s \in S}\{\phi_p \in H \mid \phi_p(s) = \top\} \\
&= \textstyle\bigcap \gamma(S).
\end{aligned}
$$

Furthermore γ is injective because if $x \neq y$ then $x \nleq y$ or $y \nleq x$. Hence by the special choice of H and applying Lemma 3.26 we obtain $\gamma(x) \neq \gamma(y)$. Since $\mathscr{P}(H)$ is a frame, so is also L. □

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