NORTH-HOLLAND

Regulation as an Interpolation Problem

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ABSTRACT

The design of a controller such that the closed-loop system will track reference signals or reject disturbance signals from a specified class is known as the "servomechanism problem" or the "regulator problem." We show here that the regulator problem can be looked at as an interpolation problem for a subspace-valued function that can be viewed as a multivariable version of the Nyquist curve. The result is applied to obtain a simple parametrization of all solutions. © *Elsevier Science Inc., 1997*

1. INTRODUCTION

In a classical paper [13], Martin and Hermann introduced the idea of associating to a given observable and controllable linear system with m inputs

LINEAR ALGEBRA AND ITS APPLICATIONS 253:311–340 (1997)

0024-3795/97/$17.00
SSDI 0024-3795(95)00739-3

and p outputs a mapping from the extended complex plane into the Grassmannian manifold of m-dimensional subspaces of $(m + p)$-dimensional complex space. The idea was applied by Brockett and Byrnes [4] to study feedback stabilization and root loci. More recently, it was recognized that subspace-valued functions offer an excellent framework to define a distance measure between linear systems and to study robustness issues (see for instance [15] and [17]). In this paper, we use subspace-valued functions to study the *regulator problem* (sometimes also known as the *servo problem*), which is one of the most widely studied problems in control theory. A particular instance is the rejection of constant disturbances under closed-loop stability, the study of which dates back to Maxwell [14]. Instead of attempting to list the many contributions since, we refer the reader to [18] and [3] for entries into the literature. In this paper, we show that the regulator problem can be viewed as an interpolation problem for a subspace-valued function associated to the controller.

It turns out that in the study of the regulator problem it is necessary to extend the point of view of [13] in several ways. In the first place, since we will be interested in stability properties, it is natural to use the closed right half plane as a domain of definition for subspace-valued functions, rather than the extended complex plane as a whole; the same shift of focus also already occurred in for instance [15] and [17]. By taking the closed right half plane as the domain of definition, it becomes natural to consider systems that are stabilizable and detectable rather than controllable and observable. However, in the regulator problem one is dealing with nonstabilizable systems. We shall still associate subspace-valued functions to such systems; the price we pay is that the resulting functions will have singularities, in the sense that at certain points the dimension of the associated subspace "jumps up." Another new element is introduced by the interpolation conditions. We want to allow higher-multiplicity conditions, so that somehow derivatives should be involved. We deal with these by a concept that we call the "blowup."

The main results of the paper may be summarized as follows. First we introduce subspace-valued functions associated to linear systems with the extensions to the Martin-Hermann framework as mentioned above. Then we give conditions for the regulator problem in terms of these subspace-valued functions. The conditions are interpolation conditions, in the sense that they partly specify the values of a subspace-valued function associated to the controller at a finite number of points in the complex plane corresponding to the characteristic frequencies of the exogenous signals specified in the regulator problem. For the case of simple multiplicities, this partial specification is of the form

$$\mathscr{C}(\lambda) \cap \mathscr{M}(\lambda) \subset \mathscr{K} \tag{1.1}$$

where $\mathscr{C}(s)$ and $\mathscr{M}(s)$ are subspace-valued functions defined by the controller and by the problem data respectively, $\mathscr{X}$ is a given subspace, and λ is a characteristic frequency. The full version (including higher multiplicities) is given in Theorem 4.2. One important reason why one may want to write a given problem as an interpolation problem is to obtain a parametrization of all solutions, and we show that also in this case such a parametrization can be obtained (Theorem 5.5). In the companion paper [6], this parametrization is used to optimize robustness of closed-loop stability over the set of regulators.

The paper is organized as follows. A formulation of the regulator problem as it will be considered here is given in Section 2, where we also define the associated subspace-valued functions and discuss the description of closed-loop stability in terms of these. In Section 3, we introduce the "blowup" and obtain its basic properties. After these preliminaries, it is not difficult to interpret the regulator problem as an interpolation problem, and this is done in Section 4. The parametrization of all solutions to the regulator problem is derived under an extra condition in Section 5.

2. PROBLEM FORMULATION AND PRELIMINARIES

We shall freely use standard terminology from the linear systems literature; for explanation, see any textbook on linear systems such as [18, 5]. Consider a finite-dimensional linear time-invariant system of the following form:

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t), \tag{2.1}$$

$$\dot{x}_2(t) = A_{22}x_2(t), \tag{2.2}$$

$$y(t) = C_1x_1(t) + C_2x_2(t). \tag{2.3}$$

The interpretation is as follows: x_1 denotes the state of the plant, whereas x_2 is the state of an "exosystem" that generates signals which can be disturbances or references. Typically the matrix A_{22} has its eigenvalues on the imaginary axis, allowing the reference/disturbance signals to be steps, ramps, sinusoids, etc. The variable $y(t)$ should converge to zero, irrespective of the presence of the signals generated by the exosystem. This is to be achieved by a linear time-invariant compensator of the form

$$\dot{z}(t) = Fz(t) + Gy(t), \tag{2.4}$$

$$u(t) = Hz(t) + Jy(t). \tag{2.5}$$

The *closed-loop system* takes the form

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ z \\ x_2 \end{bmatrix}(t) = A_e \begin{bmatrix} x_1 \\ z \\ x_2 \end{bmatrix}(t), \tag{2.6}$$

$$y(t) = \begin{bmatrix} C_1 & 0 & C_2 \end{bmatrix}\begin{bmatrix} x_1 \\ z \\ x_2 \end{bmatrix}(t), \tag{2.7}$$

where

$$A_e = \begin{bmatrix} A_{11} + B_1 J C_1 & B_1 H & A_{12} + B_1 J C_2 \\ G C_1 & F & G C_2 \\ 0 & 0 & A_{22} \end{bmatrix}. \tag{2.8}$$

The compensator is said to satisfy the *internal stability requirement* if the closed-loop system is stable when $x_2(t) = 0$, that is, if all eigenvalues of the matrix

$$\begin{bmatrix} A_{11} + B_1 J C_1 & B_1 H \\ G C_1 & F \end{bmatrix}$$

are in the left half plane. It is said to satisfy the *regulation requirement* if $y(t)$ tends to zero for all initial values, that is, if

$$\mathscr{X}_+(A_e) \subset \ker\begin{bmatrix} C_1 & 0 & C_2 \end{bmatrix}, \tag{2.9}$$

where $\mathscr{X}_+(A_e)$ denotes the unstable subspace of A_e. A compensator (2.4)–(2.5) is called a *regulator* if it satisfies both the internal stability requirement and the regulation requirement (Maxwell's term was *governor* [14]). The regulator problem can now be formulated simply as the problem of finding a regulator for the given system (2.1)–(2.3). A number of variations and extensions of this problem have also been studied in the literature; the formulation above is referred to as the "autonomous regulator problem" in [3, p. 317].

The following will be standing assumptions throughout this paper. Recall that a matrix pair (A, B) ($A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$) is said to be *stabilizable* if there exists an $F \in \mathbb{R}^{m\times n}$ such that $A + BF$ has all its eigenvalues in the open left half plane, or equivalently if the matrix $[sI - A \;\; B]$ has full row rank for all s with $\operatorname{Re} s \geqslant 0$, and that a matrix pair (C, A) $(C \in \mathbb{R}^{p\times n}, A \in \mathbb{R}^{n\times n})$ is said to be *detectable* if (A^T, C^T) is stabilizable (see for instance [5, p. 259]).

ASSUMPTIONS. The system (2.1)–(2.3) satisfies:

(A1) the pair (A_{11}, B_1) is stabilizable;
(A2) the pair (C, A) given by

$$C = [C_1 \quad C_2], \qquad A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \tag{2.10}$$

is detectable;
(A3) all eigenvalues of A_{22} are in the closed right half plane.

Assumption (A1) is necessary for the plant to be stabilizable by a feedback compensator, and so this is a natural assumption to make. Detectability of the pair (C_1, A_{11}) is necessary as well for closed-loop stability to be achieved by a compensator of the form (2.4)–(2.5); assumption (A2) requires a bit more, however. It can be argued that (A2) may be assumed without essential loss of generality in the regulator problem (cf. [18, §8.1]). The final assumption (A3) is standard; it is not interesting to consider external signals that decay to zero (or alternatively, they may be considered as a noncontrollable but stabilizable part of the plant). Concerning the compensator (2.4)–(2.5), we shall only consider triples (F, G, H) that are controllable and observable, since there is nothing to be gained by not doing so.

The following notational conventions will be used. The input and output spaces of (2.1)–(2.3) will be denoted by $\mathscr{U}$ and $\mathscr{Y}$, with dimensions m and p respectively. The closed right half plane will be denoted by

$$\mathbb{C}^+ \stackrel{\text{def}}{=} \{s \in \mathbb{C} \mid \operatorname{Re} s \geqslant 0\} \cup \{\infty\}. \tag{2.11}$$

Finally, RH_∞ denotes the ring of rational functions that are analytic on $\mathbb{C}^+$, i.e., proper stable rational functions.

We now introduce the subspace-valued functions associated to plant and controller. With the plant given by the triple (A_{11}, B_1, C_1) we associated the function

$$\mathscr{P}(s) = \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \middle| \exists x \text{ s.t.} \begin{bmatrix} sI - A_{11} & 0 & -B_1 \\ C_1 & -I & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ u \end{bmatrix} = 0 \right\},$$

$$\mathscr{P}(\infty) = \operatorname{im} \begin{bmatrix} 0 \\ I \end{bmatrix}. \tag{2.12}$$

It follows from assumptions (A1) and (A2) that dim $\mathscr{P}(s)$ is equal to m for all s with Re $s \geqslant 0$. With the full system (2.1)–(2.3) we associate

$$\mathscr{M}(s) = \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \middle| \exists x_1, x_2 \text{ s.t.} \begin{bmatrix} sI - A_{11} & -A_{12} & 0 & -B_1 \\ 0 & sI - A_{22} & 0 & 0 \\ C_1 & C_2 & -I & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y \\ u \end{bmatrix} = 0 \right\},$$

$$\mathscr{M}(\infty) = \operatorname{im} \begin{bmatrix} 0 \\ I \end{bmatrix}. \tag{2.13}$$

The system (2.1)–(2.3) is detectable but not stabilizable, and so, although dim $\mathscr{M}(s) = m$ for most points in $\mathbb{C}^+$, at the eigenvalues λ of A_{22} we have dim $\mathscr{M}(\lambda) > m$. We finally associate to the controller the subspace-valued function

$$\mathscr{C}(s) = \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \middle| \exists z \text{ s.t.} \begin{bmatrix} sI - F & -G & 0 \\ H & J & -I \end{bmatrix} \begin{bmatrix} z \\ y \\ u \end{bmatrix} = 0 \right\}, \mathscr{C}(\infty) = \operatorname{im} \begin{bmatrix} I \\ J \end{bmatrix}, \tag{2.14}$$

which has constant dimension p on the entire extended complex plane. Note that all functions take values in the set of subspaces of the product space $\mathscr{Y} \times \mathscr{U}$, which is an $(m + p)$-dimensional space.

We used state-space terms above; other popular representations include, of course, matrix fraction descriptions and the transfer matrix. In fact, Martin and Hermann used polynomial coprime factorizations in their original paper [13]. In our present context, factorizations over RH_∞ are more appropriate. The following lemma gives the connections between various representations (see also [8, Lemma 2.4], where an alternative proof is given).

LEMMA 2.1. *Consider a set of state-space parameters* (A, B, C, D) *and assume that* (A, B) *is stabilizable and that* (C, A) *is detectable. Let* $N(s)D^{-1}(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$ *be respectively a right and a left coprime factorization over* RH_∞ *of the transfer matrix* $G(s) = C(sI - A)^{-1}B + D$. *Under these conditions, one has*

$$\operatorname{im} \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = \ker\begin{bmatrix} \tilde{D}(s) & -\tilde{N}(s) \end{bmatrix} = \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \ker\begin{bmatrix} sI - A & -B \end{bmatrix} \tag{2.15}$$

for all $s \in \mathbb{C}$ *with* Re $s \geqslant 0$, *and*

$$\operatorname{im}\begin{bmatrix} N(\infty) \\ D(\infty) \end{bmatrix} = \ker\begin{bmatrix} \tilde{D}(\infty) & -\tilde{N}(\infty) \end{bmatrix} = \operatorname{im}\begin{bmatrix} D \\ I \end{bmatrix}. \tag{2.16}$$

Proof. All functions appearing in (2.15) are continuous as mappings from $\{s \in \mathbb{C} \mid \operatorname{Re} s \geqslant 0\}$ to the Grassmannian manifold of m-dimensional subspaces of $\mathscr{Y} \times \mathscr{U}$; the extension indicated in (2.16) even makes all functions continuous as mappings from the extended right half plane (including the point at infinity) to the Grassmannian. For the state-space representation, this follows from the stabilizability and detectability assumptions (see [8]); concerning the image and kernel representations, see [13]. For all points s in the right half plane that are not eigenvalues of A, it is easily seen that all entries in (2.15) are just alternative ways of writing $\ker[I \quad -G(s)]$, so that equality holds in these points. But since A has only finitely many eigenvalues, equality must then by continuity hold everywhere in $\mathbb{C}^+$. ■

If $P(s)$ and $\tilde{P}(s)$ are any matrix functions of full generic column and row rank respectively, and

$$\mathscr{P}(s) = \operatorname{im} P(s) = \ker \tilde{P}(s), \tag{2.17}$$

then we shall call $P(s)$ an *image representation* and $\tilde{P}(s)$ a *kernel representation* of $\mathscr{P}(s)$. By way of convention, we use the tilde here and below to indicate kernel representations. As is seen from the above, kernel representations can be seen as left factorizations and image representations as right factorizations; coprimeness corresponds to the representations having full rank everywhere on their domains of definition. By putting the subspace-valued functions at center stage rather than their representations, we emphasize a geometric viewpoint.

REMARK 2.2. Note that the minimality assumptions in the lemma are essential; it is immediately clear from dimension considerations that a subspace-valued function associated to a nonstabilizable system, such as $\mathscr{M}(s)$ as defined in (2.13), cannot have an image representation. Below we do construct kernel representations for $\mathscr{M}(s)$, however, adding some extra requirements allowing to distinguish for instance $\tilde{M}_1(s) = s$ from $\tilde{M}_2(s) = s^2$ if necessary, even though $\ker \tilde{M}_1(s) = \ker \tilde{M}_2(s)$ for all s.

REMARK 2.3. Consider a subspace-valued function $\mathscr{P}(s) = \operatorname{im} P(s)$ on the closed right half plane, where $P(s)$ is an RH_∞ matrix having full rank everywhere on $\mathbb{C}^+$. It can readily be seen (cf. [9]) that it is actually sufficient to give the values of $\mathscr{P}(s)$ on the extended imaginary axis, by the uniqueness of analytic continuation into the right half plane. The curve $\mathscr{P}(i\omega)$ traced out as ω traverses the real line may reasonably be called the *Nyquist curve* of the system that gives rise to $\mathscr{P}(s)$. Indeed, the usual Nyquist curve for single-input, single-output systems is obtained via the standard identification of the Grassmannian manifold $G^1(\mathbb{C}^2)$ with the extended complex plane by the mapping

$$\operatorname{im}\begin{bmatrix} s \\ 1 \end{bmatrix} \mapsto s, \qquad \operatorname{im}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \infty.$$

Since we start in this paper from a state-space context, we insert a lemma about the characterization of closed-loop stability in terms of the subspace-valued functions associated to the plant and the compensator; compare [17] for a polynomial version. We first prove the lemma below, using the well-known fact (see for instance [11, p. 650]) that a square matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

in which the block A_{22} is invertible, is invertible itself if and only if the Schur complement $A_{11} - A_{12}A_{22}^{-1}A_{21}$ is invertible.

LEMMA 2.4. *The closed-loop connection of a linear system*

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{2.18}$$

$$y(t) = Cx(t) \tag{2.19}$$

with a compensator of the form (2.4)–(2.5) *is stable if and only if for each s in $\mathbb{C}$ with* Re $s \geqslant 0$ *the two subspaces*

$$\ker\begin{bmatrix} sI - A & 0 & 0 & -B \\ -C & 0 & I & 0 \end{bmatrix} \tag{2.20}$$

and

$$\ker\begin{bmatrix} 0 & sI - F & -G & 0 \\ 0 & -H & -J & I \end{bmatrix} \tag{2.21}$$

are complementary.

Proof. The closed-loop system matrix is

$$\begin{aligned} A_e &= \begin{bmatrix} A + BJC & BH \\ GC & F \end{bmatrix} \\ &= \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} + \begin{bmatrix} 0 & -B \\ -G & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ -J & I \end{bmatrix}^{-1}\begin{bmatrix} -C & 0 \\ 0 & -H \end{bmatrix}, \end{aligned}$$

so that $sI - A_e$ is invertible for all s in the closed right half plane if and only if the matrix

$$\begin{bmatrix} sI - A & 0 & 0 & -B \\ 0 & sI - F & -G & 0 \\ -C & 0 & I & 0 \\ 0 & -H & -J & I \end{bmatrix}$$

has the same property. This in turn is equivalent to the condition in the statement of the lemma. ∎

The subspaces $\mathscr{P}(s)$ defined analogously to (2.12) and $\mathscr{C}(s)$ defined as in (2.14) are simply the projections of the two subspaces (2.20) and (2.21) above on the product of the input space $\mathscr{U}$ and the output space $\mathscr{Y}$. The characterization of closed-loop stability in terms of complementarity is now proved as follows.

LEMMA 2.5. *Let a plant* (2.18)–(2.19) *and a compensator* (2.4)–(2.5) *be given, and suppose that both are stabilizable and detectable. Let $\mathscr{P}(s)$ and $\mathscr{C}(s)$ denote the associated subspace-valued functions. Then the closed-loop system is stable if and only if the subspaces $\mathscr{P}(s)$ and $\mathscr{C}(s)$ are complementary for all s in $\mathbb{C}$ with* Re $s \geqslant 0$.

Proof. It follows from Lemma 2.3 in [8] that $\dim \mathscr{P}(s) = \dim \mathscr{U}$ and $\dim \mathscr{C}(s) = \dim \mathscr{Y}$ for all s in the closed right half plane. To prove complementarity of the two subspaces, it therefore suffices to show that they intersect only in zero. Suppose to the contrary that, for some λ with Re $\lambda \geqslant 0$, the intersection $\mathscr{P}(\lambda) \cap \mathscr{C}(\lambda)$ contains a nonzero vector $\begin{bmatrix} y \\ u \end{bmatrix}$. By definition, this means that there exists an x such that

$$\begin{bmatrix} \lambda I - A & 0 & -B \\ C & -I & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ u \end{bmatrix} = 0 \tag{2.22}$$

and a z such that

$$\begin{bmatrix} \lambda I - F & -G & 0 \\ H & J & -I \end{bmatrix} \begin{bmatrix} z \\ y \\ u \end{bmatrix} = 0. \tag{2.23}$$

But then obviously

$$\begin{bmatrix} x \\ z \\ y \\ u \end{bmatrix} \in \ker \begin{bmatrix} \lambda I - A & 0 & 0 & -B \\ -C & 0 & I & 0 \end{bmatrix} \cap \ker \begin{bmatrix} 0 & \lambda I - F & -G & 0 \\ 0 & -H & -J & I \end{bmatrix}, \tag{2.24}$$

which shows, by the previous lemma, that the closed-loop system is not stable. The converse part of the proof is obtained by reversing this reasoning. ∎

REMARK 2.6. If the plant is not strictly proper and is given by state-space parameters (A, B, C, D), then the description of $\mathscr{P}(s)$ is modified in the obvious way, and $\mathscr{P}(\infty)$ is given by $\ker[-I \;\; D]$. The statement of the above lemma is then changed to: the closed-loop system is stable *and well-posed* if and only if the subspaces $\mathscr{P}(s)$ and $\mathscr{C}(s)$ are complementary for all s in the *extended* closed right half plane (cf. [17]).

3. THE BLOWUP

In order to handle higher-order interpolation conditions, it is convenient to introduce the concept of the *blowup* of a subspace-valued function. We begin by defining blowups of matrix-valued functions. Let an analytic function $M(s)$ be given that is defined on some domain Ω of the complex plane and that takes values in the set of linear mappings from a linear space $\mathscr{X}$ to a linear space $\mathscr{Y}$. If $x(s)$ is an analytic vector-valued function taking values in $\mathscr{X}$, then the first r coefficients in the Taylor series development of $M(s)x(s)$ around any point $\lambda \in \Omega$ are determined by the first r coefficients in the Taylor series development of $x(s)$ around λ. The dependence is of course linear, and we denote the associated mapping by $M^{[r]}(\lambda)$, which is a linear mapping from the r-fold product $\mathscr{X}^r$ to the r-fold product $\mathscr{Y}^r$. By repeating this construction at every $\lambda \in \Omega$ we obtain a new operator-valued function $M^{[r]}(s)$, which we shall call the *r-fold blowup* of $M(s)$. An explicit expression for $M^{[r]}(s)$ in terms of $M(s)$ is given by

$$M^{[r]}(s) = \begin{bmatrix} M(s) & 0 & \cdots & \cdots & 0 \\ M'(s) & M(s) & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \frac{1}{(r-1)!}M^{(r-1)}(s) & \cdots & \cdots & M'(s) & M(s) \end{bmatrix}. \tag{3.1}$$

This clearly shows that $M^{[r]}(s)$ will again be an analytic operator-valued function. We shall sometimes use the notation $[M(s)]^{[r]}$ instead of $M^{[r]}(s)$, in particular when $M(s)$ is a partitioned matrix, and in such cases even write $[M(s)]^{[r]}(\lambda)$ instead of $M^{[r]}(\lambda)$.

Now we come to defining blown-up versions of the various subspace-valued functions that were introduced above. For the functions $\mathscr{P}(s)$ and $\mathscr{C}(s)$ defined in (2.12) and (2.14) respectively, these can be defined via either image or kernel representations as follows:

$$\mathscr{P}^{[r]}(s) = \ker \tilde{P}^{[r]}(s) = \operatorname{im} P^{[r]}(s), \tag{3.2}$$

and similarly for $\mathscr{C}(s)$. It follows from Lemmas 3.3 and 3.4 below that this definition is unambiguous. The subspace-valued function $\mathscr{M}(s)$ defined in (2.13) requires more care because it has singularities. Note that we may write

$$\mathscr{M}(s) = \Pi \ker \begin{bmatrix} sI - A & 0 & -B \\ C & -I & 0 \end{bmatrix}, \tag{3.3}$$

where Π denotes the natural projection from $\mathscr{X} \times \mathscr{Y} \times \mathscr{U}$ to $\mathscr{Y} \times \mathscr{U}$. We now define $\mathscr{M}^{[r]}(s)$ by

$$\mathscr{M}^{[r]}(s) = \Pi^{[r]} \ker \begin{bmatrix} sI - A & 0 & -B \\ C & -I & 0 \end{bmatrix}^{[r]}, \qquad \mathscr{M}^{[r]}(\infty) = \operatorname{im} \begin{bmatrix} 0 \\ I \end{bmatrix}^{[r]}. \tag{3.4}$$

A matrix function $\tilde{M}(s)$ will be called a *kernel representation* of the *sequence* of subspace-valued functions $\mathscr{M}^{[r]}(s)$ if $\ker \tilde{M}^{[r]}(s) = \mathscr{M}^{[r]}(s)$ for all s in the considered domain. It has to be shown that such representations do indeed exist; this will be done in Lemma 3.9 below.

We start the description of the properties of blowups with a simple but crucial product formula.

LEMMA 3.1. *For any matrix functions $T(s) \in \mathbb{R}^{p \times m}(s)$ and $S(s) \in \mathbb{R}^{m \times l}(s)$ and any $r = 1, 2, \dots,$ one has*

$$(TS)^{[r]}(s) = T^{[r]}(s) S^{[r]}(s). \tag{3.5}$$

Proof. This is immediate from the definition, since $T(s)(S(s)x(s)) = (TS)(s)x(s)$. One may also give a more computational proof based on the expression (3.1), using the Leibniz rule for derivatives of products:

$$\begin{aligned} \frac{1}{k!}(TS)^{(k)}(s) &= \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} T^{(j)}(s) S^{(k-j)}(s) \\ &= \sum_{j=0}^{k} \frac{1}{j!} T^{(j)}(s) \frac{1}{(k-j)!} S^{(k-j)}(s). \end{aligned} \tag{3.6}$$

■

The blowup does not commute with matrix partitioning; indeed, if A and B are linear mappings from $\mathscr{X}$ to $\mathscr{Z}$ and from $\mathscr{Y}$ to $\mathscr{Z}$ respectively, then $[A \;\, B]^{[r]}$ is a mapping from $(\mathscr{X} \times \mathscr{Y})^r$ to $\mathscr{Z}^r$, but $[A^{[r]} \;\, B^{[r]}]$ is a mapping from $\mathscr{X}^r \times \mathscr{Y}^r$ to $\mathscr{Z}^r$. To get a proper correspondence we need an operator from $\mathscr{X}^r \times \mathscr{Y}^r$ to $(\mathscr{X} \times \mathscr{Y})^r$ that we shall call the *mingling* operator. It is defined by

$$\mathrm{Mi}: (x_1, \dots, x_r, y_1, \dots, y_r) \mapsto (x_1, y_1, \dots, x_r, y_r). \tag{3.7}$$

We shall use the mingling operator between various spaces and even use its obvious generalization to products of more than two factors, employing the same symbol Mi every time; this rather severe abuse of notation should cause no confusion. The following lemma is given without proof.

LEMMA 3.2. *For matrix functions $A(s)$ and $B(s)$ with the same domain space, we have*

$$\begin{bmatrix} A(s) \\ B(s) \end{bmatrix}^{[r]} = \mathrm{Mi}\begin{bmatrix} A^{[r]}(s) \\ B^{[r]}(s) \end{bmatrix}. \tag{3.8}$$

For matrix functions $A(s)$ and $B(s)$ with the same codomain space, we have

$$[A(s) \quad B(s)]^{[r]} = [A^{[r]}(s) \quad B^{[r]}(s)]\,\mathrm{Mi}^{-1}. \tag{3.9}$$

LEMMA 3.3. *Consider matrix functions $T(s)$ and $\tilde{T}(s)$ that are analytic on a neigborhood of a given point $\lambda \in \mathbb{C} \cup \{\infty\}$. Let r be any positive integer. If $T(\lambda)$ has full column rank, then the same holds for $T^{[r]}(\lambda)$, and if $\tilde{T}(\lambda)$ has full row rank, then the same is true for $\tilde{T}^{[r]}(\lambda)$. If moreover* ker $\tilde{T}(s) = \mathrm{im}\, T(s)$ *for all s in a neighborhood of λ, then* ker $\tilde{T}^{[r]}(\lambda) = \mathrm{im}\, T^{[r]}(\lambda)$ *for all $r \in \mathbb{N}$.*

Proof. The first claim is immediate from the matrix form of $T^{[r]}(s)$ and $\tilde{T}^{[r]}(s)$ [see (3.1)]. If now ker $\tilde{T}(s) = \mathrm{im}\, T(s)$ for all s in a neighborhood of λ, then $\tilde{T}(s)T(s) = 0$ so that $\tilde{T}^{[r]}(s)T^{[r]}(s) = 0$ which implies that im $T^{[r]}(\lambda) \subset \ker \tilde{T}^{[r]}(\lambda)$. By the full-rank assumptions and because dim ker $\tilde{T}(\lambda) = \dim \mathrm{im}\, T(\lambda)$, we also have dim ker $\tilde{T}^{[r]}(\lambda) = \dim \mathrm{im}\, T^{[r]}(\lambda)$, so that actually equality must hold. ■

LEMMA 3.4. *Let $T_1(s)$ and $T_2(s)$ be RH_∞ matrices. If* im $T_1(s) = \mathrm{im}\, T_2(s)$ *for $s \in \mathbb{C}^+$ and both $T_1(s)$ and $T_2(s)$ have full column rank everywhere on $\mathbb{C}^+$, then* im $T_1^{[r]}(s) = \mathrm{im}\, T_2^{[r]}(s)$ *for all $s \in \mathbb{C}^+$. An analogous statement is true for kernel representations.*

Proof. Under the stated conditions, there exists an RH_∞-unimodular matrix $U(s)$ such that $T_1(s) = T_2(s)U(s)$ for all $s \in \mathbb{C}^+$ (this is essentially the standard uniqueness theorem for right-coprime factorizations). From this we get $T_1^{[r]}(s) = T_2^{[r]}(s)U^{[r]}(s)$, where $U^{[r]}(s)$ is nonsingular for all $s \in \mathbb{C}^+$ by the previous lemma, and the claim follows. ■

It is well known that interpolation conditions for matrix-valued functions can often be expressed as divisibility conditions (cf. for instance [2, Chapter 10]). The connection between blown-up matrix functions and divisibility is brought out by the following proposition.

PROPOSITION 3.5. *Let $Q(s) \in \mathrm{RH}_\infty^{m\times p}$ and $H(s) \in \mathrm{RH}_\infty^{p\times p}$, and suppose that $H(s)$ is nonsingular. Under these conditions, $Q(s)$ is right divisible by $H(s)$, in the sense that the matrix function $Q(s)H^{-1}(s)$ belongs to $\mathrm{RH}_\infty^{m\times p}$, if and only if*

$$\ker Q^{[r]}(s) \supset \ker H^{[r]}(s) \tag{3.10}$$

for all $s \in \mathbb{C}^+$ and all $r \in \mathbb{N}$. The conclusion in fact already holds if the inclusion (3.10) is satisfied at each zero λ of $H(s)$ in $\mathbb{C}^+$, and with r equal to the multiplicity of that zero.

For the proof it is convenient to introduce the ring $A(\lambda)$ of functions analytic in a neighborhood of $\lambda \in \mathbb{C} \cup \{\infty\}$, and the $A(\lambda)$-module $Z_r(H;\lambda)$ defined by

$$Z_r(H;\lambda) = \{f \in A^p(\lambda) \mid (s-\lambda)^{-r}H(s)f(s) \in A^p(\lambda)\}, \tag{3.11}$$

where s^r should be read instead of $(s-\lambda)^{-r}$ if $\lambda = \infty$; the same convention will be used below. We now first prove the following lemma.

LEMMA 3.6. *In the situation of the above proposition, $Q(s)$ is right-divisible by $H(s)$ if and only if*

$$Z_r(Q;\lambda) \supset Z_r(H;\lambda) \qquad \forall\lambda \in \mathbb{C}^+,\ r \in \mathbb{N}. \tag{3.12}$$

Proof. It is clear that the condition is necessary. Assume now that (3.12) holds. We shall show that $Q(s)H^{-1}(s)f(s)$ belongs to RH_∞^m for every $f \in \mathrm{RH}_\infty^p$. Take such an f, and suppose to the contrary that $Q(s)H^{-1}(s)f(s)$ would have a pole at some point $\lambda \in \mathbb{C}^+$. We can write $H^{-1}(s)f(s) = (s-\lambda)^{-r}g(s)$ for some $r \in \mathbb{N}$ and some $g \in \mathrm{RH}_\infty^p$. Then $H(s)(s-\lambda)^{-r}g(s) = f(s)$ so that g belongs to $Z_r(H;\lambda)$ and hence to $Z_r(Q;\lambda)$ by (3.12). But then $Q(s)H^{-1}(s)f(s) = Q(s)(s-\lambda)^{-r}g(s)$ cannot have a pole at λ, and we have a contradiction. ∎

The proof shows that it is sufficient to consider only the zeros of $H(s)$, and to take r equal to the multiplicity of the zero. The proof of the proposition is now easy.

Proof (of Proposition 3.5) Given a matrix function $M(s)$, direct calculation shows that

$$Z_r(M;\lambda) = \left\{ f = \sum_{j=0}^{\infty} f_j(s-\lambda)^j \in A(\lambda) \,\middle|\, M^{[r]}(\lambda)\operatorname{col}(f_0, f_1, \dots, f_{r-1}) = 0 \right\}. \tag{3.13}$$

So the claim in the proposition is immediate from the above lemma. ■

We note the following corollaries of the proposition.

COROLLARY 3.7. *Let $Q_1(s) \in \mathrm{RH}_\infty^{m\times p}(s)$ and $Q_2(s) \in \mathrm{RH}_\infty^{l\times p}(s)$, and suppose that $Q_2(s)$ has full generic row rank. Under these conditions, there exists a matrix function $F(s) \in \mathrm{RH}_\infty^{p\times l}(s)$ such that $Q_1(s) = F(s)Q_2(s)$ if and only if*

$$\ker Q_1^{[r]}(s) \supset \ker Q_2^{[r]}(s) \tag{3.14}$$

for all $s \in \mathbb{C}^+$ and $r \in \mathbb{N}$.

Proof. The necessity of the condition is immediate from Lemma 3.1. To show the sufficiency, write (after a column permutation, if necessary) $Q_2(s) = [Q_{21}(s)\ Q_{22}(s)]$ where $Q_{21}(s)$ is nonsingular, and partition $Q_1(s)$ correspondingly as $[Q_{11}(s)\ Q_{12}(s)]$. From (3.14) it follows that $\ker Q_{11}^{[r]}(s) \supset \ker Q_{21}^{[r]}(s)$. By the proposition, this implies that there exists a matrix function $F(s) \in \mathrm{RH}_\infty^{m\times l}$ such that $Q_{11}(s) = F(s)Q_{12}(s)$; it remains to prove that also $Q_{12}(s) = F(s)Q_{22}(s)$. Take a rational vector $x_2(s)$ of length $p - l$, and define $x_1(s) = -Q_{21}^{-1}(s)Q_{22}(s)x_2(s)$. Applying (3.14) with $r = 1$, we then have $Q_{12}(s)x_2(s) = -Q_{11}(s)x_1(s) = -F(s)Q_{21}(s)x_1(s) = F(s)Q_{22}(s)x_2(s)$. Because $x_2(s)$ was arbitrary, the desired conclusion follows. ■

COROLLARY 3.8. *Let $Q_1(s) \in \mathrm{RH}_\infty^{m\times p}(s)$ and $Q_2(s) \in \mathrm{RH}_\infty^{l\times p}(s)$, and suppose that both matrix functions have full generic row rank. Under these* *conditions, there exists an* RH_∞*-unimodular matrix function* $U(s)$ *such that* $Q_1(s) = U(s)Q_2(s)$ *if and only if* $m = l$ *and*

$$\ker Q_1^{[r]}(s) = \ker Q_2^{[r]}(s) \tag{3.15}$$

for all $s \in \mathbb{C}^+$ *and* $r \in \mathbb{N}$.

Proof. The necessity follows from Lemma 3.1 and Lemma 3.3. Assume now that (3.15) holds. From the previous corollary it follows that there exist RH_∞-matrix functions $F_1(s)$ and $F_2(s)$ such that $Q_1(s) = F_2(s)Q_2(s)$ and $Q_2(s) = F_1(s)Q_1(s)$. We get $Q_2(s) = F_1(s)F_2(s)Q_2(s)$, and since $Q_2(s)$ is surjective as a mapping from $\mathbb{C}^p(s)$ to $\mathbb{C}^m(s)$, this implies that $F_1(s)F_2(s) = I$. In the same way we have $F_2(s)F_1(s) = I$ and it follows that both $F_1(s)$ and $F_2(s)$ are unimodular. ■

LEMMA 3.9. *Consider a set of state-space parameters* $(\mathscr{X}, \mathscr{Y}, \mathscr{U};$ $A, B, C, D)$ *and suppose that the pair* (C, A) *is detectable. Let* Π *denote the natural projection from* $\mathscr{X} \times \mathscr{Y} \times \mathscr{U}$ *to* $\mathscr{Y} \times \mathscr{U}$. *For each* $r = 1, 2, \ldots,$ *define a subspace-valued function* $\mathscr{M}^{[r]}(s)$ *by*

$$\mathscr{M}^{[r]}(s) = \Pi^{[r]} \ker \begin{bmatrix} sI - A & 0 & -B \\ C & -I & D \end{bmatrix}^{[r]}, \qquad \mathscr{M}^{[r]}(\infty) = \operatorname{im} \begin{bmatrix} D \\ I \end{bmatrix}^{[r]}. \tag{3.16}$$

Then we can find an RH_∞ *function* $\tilde{M}(s)$ *such that*

$$\mathscr{M}^{[r]}(s) = \ker \tilde{M}^{[r]}(s) \qquad \forall s \in \mathbb{C}^+, r \in \mathbb{N}. \tag{3.17}$$

Moreover, if $\tilde{M}_1(s)$ *and* $\tilde{M}_2(s)$ *are both matrix functions of full generic row rank satisfying* (3.17), *then there exists an* RH_∞*-unimodular matrix* $U(s)$ *such that* $\tilde{M}_2(s) = U(s)\tilde{M}_1(s)$.

Proof. Write $C(sI - A)^{-1} = \tilde{D}^{-1}(s)\tilde{N}(s)$ where $\tilde{D}(s)$ and $\tilde{N}(s)$ are left-coprime matrices over RH_∞. By the coprimeness and the detectability assumption, we have

$$\operatorname{im} \begin{bmatrix} sI - A \\ C \end{bmatrix} = \ker \begin{bmatrix} -\tilde{N}(s) & \tilde{D}(s) \end{bmatrix} \tag{3.18}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s \geqslant 0$. Now define

$$\tilde{M}(s) = \begin{bmatrix} -\tilde{N}(s) & \tilde{D}(s) \end{bmatrix} \begin{bmatrix} 0 & B \\ -I & D \end{bmatrix} = \begin{bmatrix} -\tilde{D}(s) & -\tilde{N}(s)B + \tilde{D}(s)D \end{bmatrix}. \tag{3.19}$$

Note that we may write

$$\mathscr{M}^{[r]}(s) = \left(\begin{bmatrix} 0 & B \\ -I & D \end{bmatrix}^{[r]} \right)^{-1} \operatorname{im} \begin{bmatrix} sI - A \\ C \end{bmatrix}^{[r]}, \tag{3.20}$$

whereas it follows from (3.18) by Lemma 3.4 that

$$\operatorname{im} \begin{bmatrix} sI - A \\ C \end{bmatrix}^{[r]} = \ker \begin{bmatrix} -\tilde{N}(s) & \tilde{D}(s) \end{bmatrix}^{[r]}. \tag{3.21}$$

Therefore, we have

$$\begin{aligned} \mathscr{M}^{[r]}(s) &= \left(\begin{bmatrix} 0 & B \\ -I & D \end{bmatrix}^{[r]} \right)^{-1} \ker \begin{bmatrix} -\tilde{N}(s) & \tilde{D}(s) \end{bmatrix}^{[r]} \\ &= \ker \left(\begin{bmatrix} -\tilde{N}(s) & \tilde{D}(s) \end{bmatrix}^{[r]} \begin{bmatrix} 0 & B \\ -I & D \end{bmatrix}^{[r]} \right) \\ &= \ker \left(\begin{bmatrix} -\tilde{N}(s) & \tilde{D}(s) \end{bmatrix} \begin{bmatrix} 0 & B \\ -I & D \end{bmatrix} \right)^{[r]} = \ker \tilde{M}^{[r]}(s) \end{aligned} \tag{3.22}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s \geqslant 0$. Concerning the point at infinity, we have

$$\ker \begin{bmatrix} -\tilde{N}(\infty) & \tilde{D}(\infty) \end{bmatrix}^{[r]} = \operatorname{im} \begin{bmatrix} I \\ 0 \end{bmatrix}^{[r]}. \tag{3.23}$$

This equality follows by taking limits in both sides of (3.21); note that the matrix $[-\tilde{N}(s) \;\; \tilde{D}(s)]^{[r]}$ has full row rank for all $s \in \mathbb{C}^+$ by Lemma 3.3, so that the subspace-valued function $\ker[-\tilde{N}(s) \;\; \tilde{D}(s)]^{[r]}$ is continuous on $\mathbb{C}^+$. It is now immediate from the definition (3.19) that the equality (3.17) also holds at $s = \infty$. The final claim about the uniqueness of solutions is immediate from (3.17) by Corollary 3.8. ■

4. INTERPOLATION CONDITIONS FOR THE REGULATOR PROBLEM

In this section we shall show how the regulator problem can be viewed as an interpolation problem. An important role is played by the relation between the subspace-valued functions $\mathscr{M}(s)$ and $\mathscr{P}(s)$ that were introduced in (2.13) and (2.12). Note that $\mathscr{M}(\lambda) = \mathscr{P}(\lambda)$ for all λ that are not eigenvalues of A_{22} (i.e. poles of the exosystem), and that in general we have $\mathscr{P}(s) \subset \mathscr{M}(s)$. Unlike $\mathscr{P}(s)$, the function $\mathscr{M}(s)$ has singularities, in the sense that it is not of constant dimension on the complex plane. In particular it can therefore not be considered as a mapping from the complex plane to any Grassmannian. The way in which $\mathscr{M}(s)$ plays a role in describing the regulation property is most easily seen in the case in which the eigenvalues of A_{22} are simple (i.e. when A_{22} is diagonalizable). We shall treat this case first in a proposition, and then make the necessary adjustments to handle the general case.

PROPOSITION 4.1. *In the regulator problem as defined in Section 2, assume that A_{22} is diagonalizable. A controller is then a solution to the regulator problem with internal stability if and only if the associated subspace-valued function $\mathscr{C}(s)$ is such that the interpolation condition*

$$\mathscr{C}(\lambda) \cap \mathscr{M}(\lambda) \subset \{0\} \times \mathscr{U} \tag{4.1}$$

holds for all eigenvalues λ of A_{22}, and the complementarity condition

$$\mathscr{C}(\lambda) \oplus \mathscr{P}(\lambda) = \mathscr{Y} \times \mathscr{U} \tag{4.2}$$

holds for all $\lambda \in \mathbb{C}^+$.

Proof. By Lemma 2.5, the complementarity condition is equivalent to internal stability of the combination of plant and compensator. If internal stability holds, the unstable eigenvalues of the closed-loop system matrix A_e must coincide with the eigenvalues of A_{22}. The regulation property will be satisfied if and only if the characteristic modes corresponding to these eigenvalues have zero output values associated to them. Because of the assumption that A_{22} has only simple eigenvalues, it suffices to consider solutions of the form $x(t) = x_0 e^{\lambda t}$, $z(t) = z_0 e^{\lambda t}$, $y(t) = y_0 e^{\lambda t}$, $u(t) = u_0 e^{\lambda t}$.

Substituting the assumed solutions in (2.1)–(2.5) and equating the coefficients of $e^{\lambda t}$ results in the equations

$$\begin{bmatrix} \lambda I - A & 0 & -B \\ C & -I & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ u_0 \end{bmatrix} = 0, \tag{4.3}$$

$$\begin{bmatrix} \lambda I - F & -G & 0 \\ H & J & -I \end{bmatrix} \begin{bmatrix} z_0 \\ y_0 \\ u_0 \end{bmatrix} = 0, \tag{4.4}$$

where A and C are as in (2.10) and

$$B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}.$$

So the regulation property holds if and only if the equations (4.3)–(4.4) only allow solutions with $y_0 = 0$. But this in turn is equivalent to (4.1). ■

We now proceed to the general (higher-multiplicity) version of the above proposition. For ease of notation, we introduce

$$\mathscr{K} = \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \middle| \, y = 0 \right\} \tag{4.5}$$

and denote the natural projection from $\mathscr{Y} \times \mathscr{U}$ to $\mathscr{Y}$ by $\tilde{K} = [I \;\; 0]$, so that

$$\mathscr{K} = \ker \tilde{K} = \operatorname{im} \begin{bmatrix} 0 \\ I \end{bmatrix}. \tag{4.6}$$

Regarding $\tilde{K}$ as a constant matrix-valued function, we can also consider $\tilde{K}^{[r]}$ which is simply a block-diagonal matrix with $\tilde{K}$ on the diagonal entries, and $\mathscr{K}^{[r]} = \ker \tilde{K}^{[r]}$. By the *multiplicity* of an eigenvalue of a matrix we mean the length of the longest Jordan chain associated with that eigenvalue.

THEOREM 4.2. *A controller of the form* (2.4)–(2.5) *is a solution to the regulator problem with internal stability as formulated in section* 2 *if and* *only if the associated subspace-valued function* $\mathscr{C}(s)$ *is such that the higher-order interpolation condition*

$$\mathscr{C}^{[r]}(\lambda) \cap \mathscr{M}^{[r]}(\lambda) \subset \mathscr{K}^{[r]} \tag{4.7}$$

holds for all eigenvalues λ *of* A_{22} *of multiplicity* r, *and the complementarity condition* (4.2) *holds for all* $\lambda \in \mathbb{C}^{+}$.

Proof. The analysis is the same as in the proposition above, except that we now have to take into account (for an eigenvalue λ of A_{22} of multiplicity r) solutions of the form

$$x(t) = \left(x_0 + x_1 t + \cdots + x_{r-1} t^{r-1}\right) e^{\lambda t}$$

and similarly for $z(t)$, $y(t)$, and $u(t)$. Substituting these solutions in (2.1)–(2.5) and equating the coefficients of $t^k e^{\lambda t}$ for $k = 0, 1, \ldots, r-1$ results in the following equations, where $x^r = \operatorname{col}(x_{r-1}, \ldots, x_0)$ and y^r and u^r are defined likewise, and where we use the mingling operator of (3.7):

$$\begin{bmatrix} sI - A & 0 & -B \\ C & -I & 0 \end{bmatrix}^{[r]} (\lambda) \, \mathrm{Mi} \begin{bmatrix} x^r \\ y^r \\ u^r \end{bmatrix} = 0, \tag{4.8}$$

$$\begin{bmatrix} sI - F & -G & 0 \\ H & J & -I \end{bmatrix}^{[r]} (\lambda) \, \mathrm{Mi} \begin{bmatrix} z^r \\ y^r \\ u^r \end{bmatrix} = 0. \tag{4.9}$$

The regulation property holds if the above equations imply that $y_0 = \cdots = y_{r-1} = 0$, that is, if (4.7) holds. Conversely, if (4.7) is not satisfied, then it follows as in the proof of Proposition 4.1 that the given controller does not solve the regulator problem. ∎

The above formulation of the regulator problem shows that a necessary condition for the problem to be solvable is that at each exosystem pole λ, there should exist a subspace $\mathscr{C}$ complementary to $\mathscr{P}(\lambda)$, which moreover should be such that $\mathscr{C} \cap \mathscr{M} \subset \mathscr{K}$. This observation can be used to derive "local necessary conditions" for the solvability of the regulator problem.

5. PARAMETRIZATION OF ALL REGULATORS

As an application of the interpolation conditions found in the previous section, we shall here consider the parametrization of regulators. We shall do this under the following assumption, additional to the standing assumptions (A1)–(A3):

(A4) For every eigenvalue λ of A_{22}, the matrix

$$\begin{bmatrix} \lambda I - A_{11} & -B_1 \\ C_1 & 0 \end{bmatrix}$$

has full column rank.

This assumption implies that the number of outputs is at least equal to the number of inputs, whereas it is well known [18, Chapter 8] that the regulator problem can only be well posed if the number of outputs is at most equal to the number of inputs. One may therefore say that (A4) essentially limits one to the case in which the number of control inputs is equal to the number of regulated outputs. The assumption requires that the plant zeros do not coincide with the exosystem poles, which is a well-known condition in connection with the regulator problem [18, Theorem 8.3; 3, Corollary 5.2-2]. A geometric interpretation can be given as follows.

LEMMA 5.1. *Consider the system* (2.1)–(2.3), *with associated subspace-valued function* $\mathscr{P}(s)$ *and under the standing assumptions* $(A1)$–$(A3)$. *Assumption* $(A4)$ *then holds if and only if*

$$\mathscr{P}(\lambda) \cap \mathscr{K} = \{0\} \tag{5.1}$$

for each eigenvalue λ *of* A_{22}.

Proof. Take an eigenvalue λ of A_{22}. First suppose that (5.1) holds, and let x and u be such that

$$\begin{bmatrix} \lambda I - A_{11} & -B_1 \\ C_1 & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0. \tag{5.2}$$

We then obviously have

$$\begin{bmatrix} \lambda I - A_{11} & 0 & -B_1 \\ C_1 & -I & 0 \end{bmatrix} \begin{bmatrix} x \\ 0 \\ u \end{bmatrix} = 0 \tag{5.3}$$

which implies that

$$\begin{bmatrix} 0 \\ u \end{bmatrix} \in \mathscr{P}(\lambda).$$

By (5.1) it then follows that $u = 0$, and by the detectability assumption (A2) we then also have $x = 0$ from (5.3). The converse is proved by reversing this reasoning. ∎

The parametrization of regulators will be given through an image representation for $\mathscr{C}(s)$. First, let $\tilde{P}(s)$ be a kernel representation for $\mathscr{P}(s)$. Since $\tilde{P}(s)$ has full row rank everywhere on $\mathbb{C}^+$, we can find a matrix $\tilde{P}_1(s)$ such that $\begin{bmatrix} \tilde{P}(s) \\ \tilde{P}_1(s) \end{bmatrix}$ is RH_∞-unimodular. Write

$$\begin{bmatrix} \tilde{P}(s) \\ \tilde{P}_1(s) \end{bmatrix}^{-1} = \begin{bmatrix} P_1(s) & P(s) \end{bmatrix}; \tag{5.4}$$

then $P(s)$ is an image representation of $\mathscr{P}(s)$. A matrix $C(s)$ is an image representation for a stabilizing compensator $\mathscr{C}(s)$ if and only if $\tilde{P}(s)C(s)$ is RH_∞-unimodular; indeed, this is equivalent to $\mathscr{P}(s) = \ker \tilde{P}(s)$ and $\mathscr{C}(s) =$ im $C(s)$ being complementary for all $s \in \mathbb{C}^+$. Since an image representation is only determined up to right multiplication by unimodular matrices, we may without loss of generality even require that $\tilde{P}(s)C(s) = I$. Let $C_0(s)$ be a particular solution to this equation, and let $C(s)$ be any solution; then $\tilde{P}(s)\{C(s) - C_0(s)\} = 0$, so

$$C(s) - C_0(s) = \begin{bmatrix} P_1(s) & P(s) \end{bmatrix} \begin{bmatrix} \tilde{P}(s) \\ \tilde{P}_1(s) \end{bmatrix} \{C(s) - C_0(s)\}$$

$$= P(s)\tilde{P}_1(s)\{C(s) - C_0(s)\}, \tag{5.5}$$

which shows that $C(s)$ is of the form

$$C(s) = C_0(s) - P(s)Q(s) \tag{5.6}$$

for some RH_∞ matrix $Q(s)$. Conversely we see that any matrix of this form satisfies the equation $\tilde{P}(s)C(s) = I$. Here we have, of course, the Kučera-Youla parametrization of all stabilizing compensators [12, 19]. We now want to refine this parametrization in order to find all stabilizing compensators that solve the regulation problem. For this we need the following lemma.

LEMMA 5.2. *Let $\mathcal{W}$ be a vector space, and let $\mathcal{C}$, $\mathcal{P}$, and $\mathcal{M}$ be subspaces of $\mathcal{W}$ such that $\mathcal{P} \oplus \mathcal{C} = \mathcal{W}$ and $\mathcal{P} \subset \mathcal{M}$. Denote the projection onto $\mathcal{C}$ along $\mathcal{P}$ by $\Pi_{\mathcal{C}}^{\mathcal{P}}$. We then have*

$$\mathcal{C} \cap \mathcal{M} = \Pi_{\mathcal{C}}^{\mathcal{P}} \mathcal{M}. \tag{5.7}$$

Proof. If $w \in \mathcal{C} \cap \mathcal{M}$, then $w = \Pi_{\mathcal{C}}^{\mathcal{P}} w \in \Pi_{\mathcal{C}}^{\mathcal{P}} \mathcal{M}$. Conversely, suppose that $w \in \Pi_{\mathcal{C}}^{\mathcal{P}} \mathcal{M}$. Then certainly $w \in \mathcal{C}$, and also there is an $x \in \mathcal{M}$ such that $w = \Pi_{\mathcal{C}}^{\mathcal{P}} x$. Because $(I - \Pi_{\mathcal{C}}^{\mathcal{P}})x \in \mathcal{P} \subset \mathcal{M}$, we have $w = x - (I - \Pi_{\mathcal{C}}^{\mathcal{P}})x \in \mathcal{M}$. ∎

In view of the lemma, the regulation requirement (4.7) may be written in the form

$$\Pi_{\mathcal{C}^{[r]}}^{\mathcal{P}^{[r]}} \mathcal{M}^{[r]} \subset \mathcal{K}^{[r]}, \tag{5.8}$$

where $\Pi_{\mathcal{C}^{[r]}}^{\mathcal{P}^{[r]}}$ denotes the projection along $\mathcal{P}^{[r]}$ onto $\mathcal{C}^{[r]}$. If $C(s)$ is chosen such that $\tilde{P}(s)C(s) = I$, then

$$\Pi_{\mathcal{C}^{[r]}}^{\mathcal{P}^{[r]}} = [C\tilde{P}]^{[r]} \tag{5.9}$$

and so we can write Equation (5.8) as

$$[C\tilde{P}]^{[r]} \ker \tilde{M}^{[r]} \subset \ker \tilde{K}^{[r]}. \tag{5.10}$$

At this point we need a more precise description of the relation between $\tilde{M}(s)$ and $\tilde{P}(s)$. Such a description can be given on the basis of the lemma below.

LEMMA 5.3. *Suppose that the matrix function* $Q(s) \in \mathrm{RH}_\infty^{(k+m)\times(l+m)}$ *is of the form*

$$Q(s) = \begin{bmatrix} Q_{11}(s) & Q_{12}(s) \\ 0 & Q_{22}(s) \end{bmatrix} \tag{5.11}$$

and has full column rank for all $s \in \mathbb{C}^+$, *so that in particular the matrix* $Q_{11}(s)$ *has full column rank for all* $s \in \mathbb{C}^+$. *Let* $P(s) = [P_1(s) \;\; P_2(s)] \in \mathrm{RH}_\infty^{(k-l)\times(k+m)}$ *and* $P_{11}(s) \in \mathrm{RH}_\infty^{(k-l)\times k}$ *be kernel representations for the subspace-valued functions given by* im $Q(s)$ *and* im $Q_{11}(s)$ *respectively. Under these conditions, there exists a square and nonsingular matrix function* $H(s) \in \mathrm{RH}_\infty^{(k-l)\times(k-l)}$ *such that*

$$P_1(s) = H(s)P_{11}(s). \tag{5.12}$$

Moreover, the nontrivial elementary divisors of $H(s)$ *are the same as those of* $Q_{22}(s)$.

Proof. Because of the full-column-rank assumption on $Q(s)$, there exists a unimodular matrix $U(s)$ of size $k + m$ such that

$$\begin{bmatrix} U_{11}(s) & U_{12}(s) \\ U_{21}(s) & U_{22}(s) \end{bmatrix} \begin{bmatrix} Q_{11}(s) & Q_{12}(s) \\ 0 & Q_{22}(s) \end{bmatrix} = \begin{bmatrix} I_{l+m} \\ 0 \end{bmatrix}. \tag{5.13}$$

Note that, in this partitioning, $U_{21}(s)$ has size $(k - l) \times k$. Because the matrix $[P_1(s) \;\; P_2(s)]$ is determined only up to left multiplication by an RH_∞-unimodular matrix, we may for the purposes of the proof set

$$[P_1(s) \quad P_2(s)] = [U_{21}(s) \quad U_{22}(s)]. \tag{5.14}$$

Now, let $Q_0(s)$ be such that $[Q_0(s) \;\; Q_{11}(s)]$ is unimodular. Then there exists a unimodular matrix $V(s)$ such that

$$\begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} [Q_0(s) \quad Q_{11}(s)] = \begin{bmatrix} I_l & 0 \\ 0 & I_{k-l} \end{bmatrix}, \tag{5.15}$$

and we may set

$$P_{11}(s) = V_1(s). \tag{5.16}$$

Define

$$H(s) = U_{21}(s)Q_0(s). \tag{5.17}$$

Because $U_{21}(s)Q_{11}(s) = 0$ by (5.13), we then have

$$\begin{aligned} P_1(s) &= U_{21}(s) = U_{21}(s)\begin{bmatrix} Q_0(s) & Q_{11}(s) \end{bmatrix}\begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} \\ &= U_{21}(s)Q_0(s)V_1(s) = H(s)P_{11}(s). \end{aligned} \tag{5.18}$$

Finally note that

$$\begin{bmatrix} U_{11}(s) & U_{12}(s) \\ U_{21}(s) & U_{22}(s) \end{bmatrix}\begin{bmatrix} Q_0(s) & Q_{11}(s) & Q_{12}(s) \\ 0 & 0 & Q_{22}(s) \end{bmatrix} = \begin{bmatrix} U_{11}(s)Q_0(s) & I_{l+m} \\ U_{21}(s)Q_0(s) & 0 \end{bmatrix}. \tag{5.19}$$

The nontrivial elementary divisors of the left-hand side are equal to those of $Q_{22}(s)$, since $[Q_0(s)\ Q_{11}(s)]$ is unimodular, whereas on the right-hand side they are equal to those of $U_{21}(s)Q_0(s) = H(s)$. ■

In the context of the regulation problem, this leads to the following.

LEMMA 5.4. *Let $\tilde{P}(s)$ be a kernel representation of the subspace-valued function $\mathscr{P}(s)$ defined in* (2.12), *and let $\tilde{M}(s)$ be a kernel representation of the sequence of subspace-valued functions $\mathscr{M}^{[r]}(s)$ defined in* (3.4). *Then there exists a square and nonsingular* RH_∞*-matrix function $\tilde{H}(s)$ such that*

$$\tilde{M}(s) = \tilde{H}(s)\tilde{P}(s). \tag{5.20}$$

Moreover, the nontrivial elementary divisors of $\tilde{H}(s)$ are the same as those of $sI - A_{22}$.

Proof. A kernel representation for the sequence $\mathscr{M}^{[r]}(s)$ is constructed as follows (cf. the proof of Lemma 3.9). By the detectability assumption (A2), we can find RH_∞ matrices $\tilde{N}_1(s)$, $\tilde{N}_2(s)$, and $\tilde{D}(s)$ such that

$$\ker\begin{bmatrix} -\tilde{N}_1(s) & \tilde{D}(s) & -\tilde{N}_2(s) \end{bmatrix} = \mathrm{im}\begin{bmatrix} sI - A_{11} & -A_{12} \\ -C_1 & -C_2 \\ 0 & sI - A_{22} \end{bmatrix} \qquad \forall s \in \mathbb{C}^+. \tag{5.21}$$

We then set

$$\begin{aligned} \tilde{M}(s) &= \begin{bmatrix} -\tilde{N}_1(s) & \tilde{D}(s) & -\tilde{N}_2(s) \end{bmatrix}\begin{bmatrix} 0 & -B_1 \\ I & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -\tilde{N}_1(s) & \tilde{D}(s) \end{bmatrix}\begin{bmatrix} 0 & -B_1 \\ I & 0 \end{bmatrix}. \end{aligned} \tag{5.22}$$

On the other hand, a kernel representation $\tilde{P}(s)$ is constructed by finding $\tilde{N}_0(s)$ and $\tilde{D}_0(s)$ such that

$$\ker\begin{bmatrix} -\tilde{N}_0(s) & \tilde{D}_0(s) \end{bmatrix} = \mathrm{im}\begin{bmatrix} sI - A_{11} \\ -C_1 \end{bmatrix} \qquad \forall s \in \mathbb{C}^+ \tag{5.23}$$

and setting

$$\tilde{P}(s) = \begin{bmatrix} -\tilde{N}_0(s) & \tilde{D}_0(s) \end{bmatrix}\begin{bmatrix} 0 & -B_1 \\ I & 0 \end{bmatrix}. \tag{5.24}$$

It follows from Lemma 5.3 that there exists an RH_∞ matrix $\tilde{H}(s)$ with the properties as stated in the lemma such that

$$\begin{bmatrix} -\tilde{N}_1(s) & \tilde{D}(s) \end{bmatrix} = \tilde{H}(s)\begin{bmatrix} -\tilde{N}_0(s) & \tilde{D}_0(s) \end{bmatrix}. \tag{5.25}$$

From this together with (5.22) and (5.24), the claim in the lemma follows for the matrix functions $\tilde{M}(s)$ and $\tilde{P}(s)$ constructed above. Lemma 3.9 shows that the same conclusion must hold for any representations $\tilde{M}(s)$ and $\tilde{P}(s)$ that satisfy the specified conditions. ■

Using this lemma, we can rewrite (5.10) as

$$\left[C\tilde{P}\right]^{[r]}\ker\left[\tilde{H}\tilde{P}\right]^{[r]} \subset \ker \tilde{K}^{[r]}. \tag{5.26}$$

Because $\tilde{P}(s)$ has full row rank everywhere on $\mathbb{C}^+$, the same holds for $\tilde{P}^{[r]}(s)$ (Lemma 3.4) and so (5.26) is equivalent to

$$C^{[r]} \ker \tilde{H}^{[r]} \subset \ker \tilde{K}^{[r]} \tag{5.27}$$

which is the same as

$$\left(\tilde{K}C\right)^{[r]} \mid_{\ker \tilde{H}^{[r]}} = 0. \tag{5.28}$$

Because the matrix function $\tilde{H}(s)$ is nonsingular, the same holds for $\tilde{H}^{[r]}(s)$, and so the subspace-valued function $\ker \tilde{H}^{[r]}(s)$ takes the value $\{0\}$ almost everywhere on $\mathbb{C}^+$. Consequently, the inclusion (5.27) is trivial almost everywhere. The only interesting points are those at which $\tilde{H}(s)$ has a zero, which by the lemma above are exactly the exosystem poles. The lemma also guarantees that the multiplicities of the zeros of $\tilde{H}(s)$ are the same as the multiplicities of the exosystem poles, so that we may reformulate the condition (5.28) as follows:

$$\left(\tilde{K}C\right)^{[r]}(\lambda) \mid_{\ker \tilde{H}^{[r]}(\lambda)} = 0 \qquad \text{for all } \lambda \text{ in } \sigma(A_{22}) \text{ of multiplicity } r. \tag{5.29}$$

Now, assume that the regulator problem with internal stability is solvable, and let $C_0(s)$ be an image representation of the subspace-valued function associated to a particular solution. We know from the Kučera-Youla parametrization that any controller achieving internal stability can be represented by $C(s) = C_0(s) - P(s)Q(s)$ where $Q(s)$ is an arbitrary RH_∞ matrix of the appropriate size. It is clear from (5.29) that such a controller will also be a solution to the regulator problem if and only if

$$\left(\tilde{K}PQ\right)^{[r]}(\lambda) \mid_{\ker \tilde{H}^{[r]}(\lambda)} = 0 \qquad \text{for all } \lambda \text{ in } \sigma(A_{22}) \text{ of multiplicity } r. \tag{5.30}$$

If we assume now that assumption (A4) holds, so that $\tilde{K}P(\lambda)$ is injective (cf. Lemma 5.1), then the same holds for $(\tilde{K}P)^{[r]}(\lambda)$, and the condition (5.30) simplifies to

$$Q^{[r]}(\lambda)\mid_{\ker \tilde{H}^{[r]}(\lambda)} = 0 \qquad \text{for all } \lambda \text{ in } \sigma(A_{22}) \text{ of multiplicity } r. \quad (5.31)$$

But then we also have

$$\ker \tilde{H}^{[r]}(s) \subset \ker Q^{[r]}(s) \qquad \forall s \in \mathbb{C}^{+}, \qquad (5.32)$$

since the inclusion is trivial for those s that are not eigenvalues of A_{22}. By Lemma 3.5, (5.32) implies that

$$Q(s) = \Psi(s)\tilde{H}(s) \qquad (5.33)$$

for some RH_∞ matrix $\Psi(s)$. Conversely, it is clear that any matrix of the form $C_0(s) - P(s)\Psi(s)\tilde{H}(s)$ provides a solution to the regulator problem. Therefore, we have proved the main result of this section, which gives a parametrization of all controllers of the form (2.4)–(2.5) that achieve regulation with internal stability.

THEOREM 5.5. *Consider the system* (2.1)–(2.3) *under assumptions* (A1)–(A4). *Let* $P(s)$ *and* $\tilde{P}(s)$ *denote image and kernel representations respectively for the subspace-valued function* $\mathscr{P}(s)$ *associated to the plant as defined by* (2.12). *Assume that the regulator problem with internal stability is solvable, and let* $C_0(s)$ *be an image representation of the function* $\mathscr{C}(s)$ *associated as in* (2.14) *to a particular solution, normalized such that* $\tilde{P}(s)C_0(s) = I$. *Let* $\tilde{H}(s)$ *be as in Lemma* 5.4. *Under these conditions, the general form of an image representation* $C(s)$ *of a solution of the regulator problem with internal stability is given by*

$$C(s) = C_0(s) - P(s)\Psi(s)\tilde{H}(s), \qquad (5.34)$$

where $\Psi(s)$ *is an arbitrary element of* $\mathrm{RH}_\infty^{m\times p}$.

Comparing this with the Kučera-Youla parametrization (5.6), we see that the parametrization of regulators comes down to constraining the "central" compensator $C_0(s)$ to be a regulator, and requiring that the parameter $Q(s)$ be right-divisible by the square matrix function $\tilde{H}(s)$, which can be constructed from the problem data. Taking into consideration that the nontrivial elementary divisors of $\tilde{H}(s)$ coincide with those of the exosystem $sI - A_{22}$, this result may be viewed as an instance of the internal model principle (see in particular the version of [10]). For other parametrizations of all solutions to the regulator problem, see for instance [7, 16, 1]. The parametrization given above turns out to be particularly useful in connection with the robust stabilization problem [6].

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Received 26 July 1995; final manuscript accepted 16 October 1995