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INVARIANCE PRINCIPLE FOR ASSOCIATED RANDOM FIELDS*

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In 1984, C. M. Newman posed the problem of proving the invariance principle in distribution for associated random fields (i.e., fields satisfying the so-called FKG-inequalities) $\mathbf{X} = \{X_j,\ j \in \mathbf{Z}^d\}$ *when* $d \geq 3$. *The solution of this problem for wide-sense stationary associated random fields is obtained here under slightly more restrictive conditions than those used by C. M. Newman and A. L. Wright for the strictly stationary case where* $d = 1$ *and* $d = 2$.

1. Introduction and the Main Result

The concept of association or positive dependence is widely used at present. This concept arose independently in reliability theory and statistical physics, where one prefers to say that the r.v.'s satisfy FKG-inequalities. Recall that a finite collection of real-valued r.v.'s $Y_1, \dots, Y_n$ is called associated if

$$\operatorname{cov}(f(Y_1, \dots, Y_n), g(Y_1, \dots, Y_n)) \geq 0$$

for any coordinatewise nondecreasing functions $f, g : \mathbf{R}^n \to \mathbf{R}$, whenever the covariance exists. An infinite family of r.v.'s is associated if every finite subfamily has this property. Note that any family of independent r.v.'s is always associated. The main advantage of dealing with positively dependent r.v.'s is not only the possibility of studying interesting models describing random fields which need not possess any mixing properties, but also the simplicity of conditions which guarantee the validity of many classical results of probability theory. One can refer in this vein to the beautiful CLT established by Newman [12] (see also [13, 5] for references therein).

In 1984, Newman [13, p. 138] posed the problem of proving the invariance principle in distribution (or FCLT) for strictly stationary associated random fields $\mathbf{X} = \{X_j,\ j \in \mathbf{Z}^d\}$ when $d \geq 3$. As far as we know, no progress was achieved in this direction. The solution of this problem is obtained in the present paper for wide-sense stationary associated random fields under slightly more restrictive conditions than were used by Newman and Wright [14] for the strictly stationary case when $d = 1$ and $d = 2$. For this purpose we establish a new maximal inequality and also prove the generalization of the Cox-Grimmett CLT [9]. We also mention in passing the Birkel [1] result for $d = 1$ in the nonstationary case under uniform integrability condition for squares of certain sums (see also [2, 8] and [10] for further development in the case $d = 1$). Scaling limits for associated random measures are studied in [7].

Without loss of generality, we can assume that $\mathbf{E}X_j = 0$ for all j. Define in the Skorokhod space $D([0,1]^d)$ the partial sum processes

$$W_n(t) = n^{-d/2} \sum_{j_1=1}^{[nt_1]} \dots \sum_{j_d=1}^{[nt_d]} X_j, \quad n \in \mathbf{N},$$

where $t = (t_1, \dots, t_d) \in [0,1]^d$, $(j_1, \dots, j_d) \in \mathbf{N}^d$, and $[\cdot]$ is the integer-part function.

Using the well-known Newman inequality (see the Appendix) for joint characteristic functions of associated r.v.'s, it is not difficult (see [12]) to prove, in the stationary case, for any $d \geq 1$, the weak convergence of finite-dimensional distributions of $W_n(\cdot)$ to the corresponding ones of $\sigma W(\cdot)$ as $n \to \infty$, provided that

$$\sigma^2 = \sum_j \operatorname{cov}(X_0, X_j) < \infty. \tag{1}$$

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Here $W(\cdot)$ stands for a standard d-parameter Wiener process (or Brownian sheet), i.e., mean zero Gaussian process with

$$\operatorname{cov}(W(t), W(t')) = \prod_{k=1}^{d} \min\{t_k, t'_k\}$$

for $t = (t_1, \ldots, t_d)$, $t' = (t'_1, \ldots, t'_d) \in [0, \infty)^d$.

The case $\sigma = 0$ means that $X_j = 0$ a.s. for all j. So we exclude this trivial case. Define the Cox–Grimmett [9] coefficient

$$u(m) = \sup_j \sum_{q:\ \|q-j\| \geq m} \operatorname{cov}(X_j, X_q), \qquad m \geq 0, \tag{2}$$

where $\|a\| = \max_{1 \leq k \leq d} |a_k|$ for $a = (a_1, \ldots, a_d) \in \mathbf{R}^d$.

Now we can state our main result.

THEOREM. *Let $\mathbf{X} = \{X_j,\ j \in \mathbf{Z}^d\}$ be an associated wide-sense stationary random field. Let*

$$M_s = \sup_j \mathbf{E}(X_j)^s < \infty \quad \text{for some} \quad s > 2 \tag{3}$$

and

$$u(m) = O(m^{-\nu}) \quad \text{for some} \quad \nu > 0 \quad (\text{as } m \to \infty). \tag{4}$$

Then the FCLT is valid, i.e.,

$$W_n(\cdot) \xrightarrow{D} \sigma W(\cdot) \quad \text{in} \quad D([0,1]^d) \quad \text{as} \quad n \to \infty. \tag{5}$$

Here σ is given by (1) *and as usual* $\xrightarrow{D}$ *denotes convergence in distribution.*

Remark 1. This statement is valid for all $d \geq 1$. Note also that the summability condition (1) implies $u(m) \to 0$ as $m \to \infty$.

Proof. To establish (5), we have to prove that the finite-dimensional distributions of $W_n(\cdot)$ converge weakly and establish the tightness of the distributions of $W_n(\cdot)$. The first part is rather simple and is contained in the Appendix for the sake of completeness. Note also that to prove tightness in the case $d \geq 3$, we cannot use the semimartingale approach of [14], which works only for $d = 1$ and $d = 2$.

Let $\mathcal{A}$ be a family of parallelepipeds in $\mathbf{R}_+^d$ of the form $V = (a, b]$, i.e., $V = (a_1, b_1] \times \cdots \times (a_d, b_d]$, where $0 \leq a_k \leq b_k < \infty$, $a_k, b_k \in \mathbf{N} \cup \{0\}$, $k = 1, \ldots, d$. For $V \in \mathcal{A}$, denote $|V| = \prod_{k=1}^{d} (b_k - a_k)$ and

$$S(V) = \sum_{j \in V} X_j, \qquad M(V) = \max\{|S(Q)| :\ Q = (a, q] \subset V\}. \tag{6}$$

LEMMA 1. *Let $\mathbf{X}$ be a field satisfying the conditions of the theorem stated above. Then there exists x_0 such that for all $V \in \mathcal{A}$ and $x \geq x_0$*

$$\mathbf{P}(M(V) \geq x|V|^{\frac{1}{2}}) \leq 2\mathbf{P}(|S(V)| \geq x(|V|^{\frac{1}{2}}/2)). \tag{7}$$

Here x_0 depends on d, s, M_s, $u(\cdot)$ (that is on c_0 and ν if $u(m) \leq c_0 m^{-\nu}$ for $m \in \mathbf{N}$) and does not depend on V.

Proof. Analogously to [14, p. 365], one has for $y, z > 0$,

$$\mathbf{P}(\overline{S}(V) \geq y) \leq \mathbf{P}(S(V) \geq z) + \mathbf{P}(\overline{S}(V) \geq y, \overline{S}(V) - S(V) \geq y - z)$$

$$\leq \mathbf{P}(S(V) \geq z) + \mathbf{P}(\overline{S}(V) \geq y)\mathbf{P}(\overline{S}(V) - S(V) \geq y - z), \tag{8}$$

where $\overline{S}(V) = \max\{S(Q) :\ Q = (a, q] \subset V\}$. We use here that $\overline{S}(V)$ and $S(V) - \overline{S}(V)$ are nondecreasing functions in X_j, $j \in V$, and that $\operatorname{cov}(\mathbf{1}_{[y,\infty)}(\xi), \mathbf{1}_{[t,\infty)}(-\eta)) \leq 0$ for any $y, t \in \mathbf{R}$ if ξ and η are associated ($\mathbf{1}_B(\cdot)$ is the indicator function of a set B).

Markov's inequality yields for $y > z$ and $r \leq s$,

$$\mathbf{P}(\overline{S}(V) - S(V) > y - z) \le (y-z)^{-r}\mathbf{E}|\overline{S}(V) - S(V)|^r \le 2^r(y-z)^{-r}\mathbf{E}|\overline{S}(V)|^r. \tag{9}$$

If $d = 1$ or $d = 2$ and $r > 2$, $\delta > 0$, $r + \delta \le s$, then, due to [4], we have, under conditions (3) and (4), the following sharp bound:

$$\mathbf{E}|S(V)|^r \le c|V|^{\gamma(r,\delta,\nu)}, \tag{10}$$

where c does not depend on V and

$$\gamma(r,\delta,\nu) = \begin{cases} r - \delta(1+\nu d^{-1})(r+\delta-2)^{-1}, & \text{if } 0 \le \nu < d\nu_0, \\ r/2, & \text{if } \nu \ge d\nu_0, \end{cases} \tag{11}$$

and $\nu_0 = (r+\delta)(r-2)/(2\delta)$. According to Remark 3 [4], the estimate (10) (with the same γ as in (11)) applies also in the case $d \ge 3$ when $\nu_0 < (d-2)^{-1}$. Take $r > 2$ and $\delta > 0$ in such a way that $r + \delta \le s$, $d(r+\delta)(r-2)/(2\delta) \le \nu$ and $(r+\delta)(r-2)/(2\delta) < (d-2)^{-1}$ (for $d \ge 3$), e.g., one can take $r = 2 + \delta^2$ for positive δ small enough. Thus one has the "classical behavior" of the absolute moments of order r of partial sums $S(V)$ (i.e., the estimate of the type $O(|V|^{r/2})$ as in the case of i.i.d. r.v.'s having finite rth absolute moment).

The d-dimensional version of the Erdös–Stechkin inequality obtained by Moricz [11] (see Corollary 1a) states that if for some $\alpha > 1$, $r \ge 1$, $v_j \ge 0$, $j \in \mathbf{N}^d$, and every $V \in \mathcal{A}$, one has

$$\mathbf{E}|S(V)|^r \le \left(\sum_{j\in V} v_j\right)^\alpha, \tag{12}$$

then

$$\mathbf{E}(M(V))^r \le (5/2)^d(1 - 2^{(1-\alpha)r})^{-dr}\left(\sum_{j\in V} v_j\right)^\alpha, \tag{13}$$

where $M(V)$ is defined in (6). So, setting $v_j = c_0 > 0$, $j \in \mathbf{N}^d$, and $\alpha = r/2 > 1$, we get from (10)–(13) that, for some constant $C > 0$ and all $V \in \mathcal{A}$,

$$\mathbf{E}(M(V))^r \le C|V|^{r/2}. \tag{14}$$

From (8), (9), and (14) as $\overline{S}(V) \le M(V)$ one has for all $V \in \mathcal{A}$ and $y > z$,

$$(1 - 2^r C(y-z)^{-r}|V|^{r/2})\mathbf{P}(\overline{S}(V) \ge y) \le \mathbf{P}(S(V) \ge z).$$

Taking $z = y/2$, $y = x|V|^{\frac{1}{2}}$, one obtains

$$(1 - 4^r x^{-r} C)\mathbf{P}(\overline{S}(V) \ge x|V|^{\frac{1}{2}}) \le \mathbf{P}(S(V) \ge x|V|^{\frac{1}{2}}/2).$$

Then for all x large enough,

$$\mathbf{P}(\overline{S}(V) \ge x|V|^{\frac{1}{2}}) \le 2\mathbf{P}(S(V) \ge x|V|^{\frac{1}{2}}/2).$$

Taking into account that $\{-X_j,\ j \in \mathbf{N}^d\}$ is also an associated field satisfying the conditions of the theorem, we come to the inequality (7), and the lemma is proved

Now to prove the tightness of distributitions of $W_n(\cdot)$, it is sufficient (see, e.g., [15, Theorem 14]) to show that for every $\varepsilon > 0$,

$$\lim_{\delta\to 0+}\limsup_{n\to\infty}\mathbf{P}(M_n^\delta > \varepsilon) = 0, \tag{15}$$

where $M_n^\delta = \sup\{|W_n(t) - W_n(t')| :\ t, t' \in [0,1]^d,\ \|t - t'\| < \delta\}$. Evidently, instead of (15) it is enough to prove that for any $\varepsilon > 0$,

$$L := \lim_{\ell\to\infty}\sum_{0\le j\le \ell-1}\limsup_{n\to\infty}\mathbf{P}(\sup_{t\in Q_\ell(j)}|W_n(t) - W_n(j/\ell)| > \varepsilon) = 0. \tag{16}$$

where $0 = (0,\ldots,0)$, $1 = (1,\ldots,1)$, $\boldsymbol{\ell} = (\ell,\ldots,\ell)$, $Q_\ell(j) = (j/\ell,(j+1)/\ell]$, $\ell \in \mathbf{N}$, and $0 \le j \le \ell - 1$ means $0 \le j_k \le \ell - 1$ for $k = 1,\ldots,d$.

For $t \in Q_\ell(j)$, one has $(0,t] = \bigcup_{i\in I} B_\ell^i(t)$, where $i = (i_1,\ldots,i_d) \in I := \{0,1\}^d$ and

$$B_\ell^i(t) = \prod_{k=1}^{d}(i_k j_k/\ell, (i_k t_k - j_k(i_k-1))/\ell].$$

Thus,

$$\mathbf{P}(\sup_{t\in Q_\ell(j)} |W_n(t) - W_n(j/\ell)| > \varepsilon) \le \sum_{i\in I_0} \mathbf{P}(M(V_\ell^i(n)) \ge \varepsilon 2^{-d} n^{d/2}), \tag{17}$$

where $I_0 = I\backslash\{0\}$ and $V_\ell^i(n)$ is the largest $V \in \mathcal{A}$ such that $V \subset nB_\ell^i := nB_\ell^i((i+1)/\ell)$.

Applying (7) and the bound $|V_\ell^i(n)| \le n^d \ell^{-1}$ for $i \in I_0$, one gets for all ℓ large enough

$$\mathbf{P}(M(V_\ell^i(n)) \ge \varepsilon 2^{-d} n^{d/2}) \le 4\mathbf{P}(|S(V_\ell^i(n))| \ge \varepsilon 2^{-d} \ell^{\frac{1}{2}} |V_\ell^i(n)|^{\frac{1}{2}}). \tag{18}$$

Now we use the following generalization of the Cox-Grimmett theorem (Theorem 1.2 in [9]).

Lemma 2. *Let $\mathbf{X}(N) = \{X_j(N),\ j \in \mathbf{Z}^d\}$ be a sequence of associated (for each $N \in \mathbf{N}$) centered random fields such that*

$$\sup_N \sup_{j\in\mathbf{Z}^d} \mathbf{E}|X_j(N)|^s < \infty \quad \textit{for some} \quad s > 2 \tag{19}$$

and

$$\sup_N u_N(m) \to 0 \quad \textit{as} \quad m \to \infty, \tag{20}$$

where $u_N(m)$ is defined for every field $\mathbf{X}(N)$ according to (2). Let

$$V(N) = (a^{(N)}, b^{(N)}] \longrightarrow \infty, \tag{21}$$

that is, $\min_{1\le k\le d}(b_k^{(N)} - a_k^{(N)}) \to \infty$ as $N \to \infty$ and

$$\liminf_{N\to\infty} |V(N)|^{-1}\sigma(N)^2 > 0, \tag{22}$$

where $\sigma(N)^2 = \operatorname{var} S(N)$ and $S(N) = \sum_{j\in V(N)} X_j(N)$. Then

$$S(N)/\sigma(N) \xrightarrow{D} \xi \sim \mathcal{N}(0,1) \quad \textit{as} \quad N \to \infty. \tag{23}$$

Remark 2. Conditions (19) and (22) are less restrictive than their counterpart (i) in [9]. Namely, we do not suppose that $s = 3$ in (19); also if

$$\inf_N \inf_{j\in V(N)} \operatorname{var} X_j(N) > 0, \tag{24}$$

then (22) is fulfilled for associated fields $\mathbf{X}(N)$. If $\mathbf{X}(N)$, $N = 1,2,\ldots$, are wide-sense stationary associated random fields, then (24) is equivalent to the hypothesis $\inf_N \operatorname{var} X_0(N) > 0$.

The proof of Lemma 2 follows the main lines of the proof of Theorem 1.2 in [9], so we indicate only what one has to change. We recall some notations from [9, pp. 515, 518]. Let ℓ be a positive integer. Define $m = m(N,\ell) = (m_1,\ldots,m_d)$, where $m_k = m_k(N,\ell) = [(b_k^{(N)} - a_k^{(N)})/\ell]$, $V_j(N,\ell) = \{q:\ a^{(N)} + (j-1)/\ell < q \le a^{(N)} + j/\ell\}$, $Y_j(N) = Y_j(N,\ell) = \sum_{q\in V_j(N,\ell)} X_q(N)$, $S(N,\ell) = \sum_{1\le j\le m} Y_j(N)$, $\sigma(N,\ell)^2 = \operatorname{var} S(N,\ell)$, $s(N,\ell)^2 = \sum_{1\le j\le m} \operatorname{var} Y_j(N)$.

For any finite sets $V_1, V_2 \subset \mathbf{Z}^d$, one has

$$\operatorname{cov}(\sum_{j\in V_1} X_j(N), \sum_{q\in V_2} X_q(N)) \le \min\{|V_1|,|V_2|\} u_N(0), \tag{25}$$

where $|V_i|$ denotes the cardinality of V_i, $i = 1,2$. So, from the relation (2.8) in [9], applying (21), (22), and (25), we immediately come for each $\ell \ge 1$ to the relation

$$\sigma(N)^2 = \sigma(N,\ell)^2 + o(|V(N)|) \quad \text{as} \quad N \longrightarrow \infty.$$

Using (22),we get

$$\sigma(N)^2 \sim \sigma(N,\ell)^2 \quad \text{as} \quad N \longrightarrow \infty \quad (\text{for every } \ \ell \geq 1). \tag{26}$$

Now from (26) and relations (2.11), (2.12) of [9] we conclude that for some constant $c > 0$ and each ℓ large enough,

$$1 \leq \limsup_{N\to\infty} \sigma(N)^2/s(N,\ell)^2 \leq 1 + cd\ell^{-1}\sum_{k=1}^{\ell} u(k). \tag{27}$$

For independent copies $Y_j(N)$, one applies Lyapunov's theorem using absolute moments of order r. We choose $r > 2$ and $\delta > 0$ in such a way that $r + \delta \leq \min\{s, 3\}$, and we are able to use the estimate of the type (10) with $\gamma = r/2$. Thus

$$\mathbf{E}|Y_j(N)|^r \leq c|V_j(N,\ell)|^{r/2}, \tag{28}$$

where $c > 0$ does not depend on N and $V_j(N,\ell)$ (we take into account the uniform conditions (19), (20)). So from (22), (27), and (21) one has

$$\frac{1}{s(N,\ell)^r}\sum_{1\leq j\leq m}\mathbf{E}|Y_j(N)|^r \leq \underline{\underline{c}}\frac{m_1\dots m_d\ell^{3d/2}}{(m_1\dots m_d\ell^d)^{r/2}} \longrightarrow 0 \quad \text{as} \quad N \to \infty,$$

where $\underline{\underline{c}} > 0$ is some constant. Lemma 2 is proved.

Remark 3. Convergence rates in the CLT for associated random fields, when sums are taken over finite subsets V_N having arbitrary configurations, are studied in [5, 6].

Now we return to the proof of our main result. Note that

$$\operatorname{var} S(V_\ell^i(n)) \sim \sigma^2|V_\ell^i(n)| \quad \text{as} \quad n \to \infty \tag{29}$$

for every $\ell \in \mathbf{N}$ and all $i \in I$, where σ^2 is given by (1). One can deduce (29) from a more general result describing the growth of variances of partial sums $\sum_{j\in V_N} X_j$, where $\mathbf{X} = \{X_j,\ j \in \mathbf{Z}^d\}$ is a wide-sense stationary random field and $V_N \to \infty$ in a "regular manner"; see, e.g., [3].

Hence for all $i \in I_0$, according to (29) and Lemma 2, one has as $n \to \infty$

$$\mathbf{P}(|S(V_\ell^i(n))| > \varepsilon 2^{-d}\ell^{\frac{1}{2}}|V_\ell^i(n)|^{\frac{1}{2}}) \longrightarrow \mathbf{P}(\xi > \varepsilon 2^{-d}\ell^{\frac{1}{2}}\sigma^{-1}), \tag{30}$$

where $\xi \sim \mathcal{N}(0,1)$. Using the well-known inequality

$$\mathbf{P}(\xi > x) \leq \frac{1}{x\sqrt{2\pi}}e^{-\frac{x^2}{2}}, \quad x > 0, \tag{31}$$

one infers from (17), (18), (30), and (31) that the l.h.s. of (16)

$$L \leq \lim_{\ell\to\infty} \ell^d(\varepsilon 2^{-d}\ell^{\frac{1}{2}}\sigma^{-1}\sqrt{2\pi})^{-1}\exp\{-(\varepsilon^2 2^{-2d}\sigma^{-2}\ell)/2\} = 0.$$

This completes the proof of the theorem.

Appendix

We prove that for any $m \in \mathbf{N}$ and all $t_1,\dots,t_m \in [0,1]^d$,

$$(W_n(t_1),\dots,W_n(t_m)) \xrightarrow{D} (W(t_1),\dots,W(t_m)) \quad \text{as} \quad n \to \infty. \tag{32}$$

It is sufficient to establish that if $V_1,\dots,V_m \subset [0,1]^d$ are disjoint (nonempty) parallelepipeds of the form $V_\ell = (a^{(\ell)}, b^{(\ell)}]$, $\ell = 1,\dots,m$, then

$$\zeta^{(n)} := n^{-d/2}(S(nV_1),\dots,S(nV_m)) \xrightarrow{D} \zeta := \sigma(W(V_1),\dots,W(V_m)) \quad \text{as} \quad n \to \infty; \tag{33}$$

here $W(V_1),\dots,W(V_m)$ are independent and $W(V_\ell) \sim \mathcal{N}(0,|V_\ell|)$, where $|V_\ell| = \prod_{k=1}^d(b_k^{(\ell)} - a_k^{(\ell)})$, $\ell = 1,\dots,m$.

For all $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbf{R}^m$, according to Newman's inequality,

$$\left| \mathbf{E}e^{i(\alpha,\zeta^{(n)})} - \prod_{\ell=1}^{m} \mathbf{E}e^{i\alpha_\ell \zeta_\ell^{(n)}} \right| \le \sum_{k,\ell=1}^{m} |\alpha_k||\alpha_\ell| \operatorname{cov}(\zeta_k^{(n)}, \zeta_\ell^{(n)}), \tag{34}$$

where $(\cdot,\cdot)$ stands for the inner product in $\mathbf{R}^m$ and $\zeta_k^{(n)} = n^{-d/2} S(nV_k)$, $k = 1, \ldots, m$.

Assume that $\rho(V_k, V_\ell) = \inf\{\|x - y\| : x \in V_k, y \in V_\ell\} \ge \tau > 0$ for all $k, \ell = 1, \ldots, m (k \ne \ell)$. Then applying (25) we have

$$\operatorname{cov}(\zeta_k^{(n)}, \zeta_\ell^{(n)}) \le n^{-d} \min\{|nV_k|, |nV_\ell|\} u(\tau n) \le u(\tau n) \longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{35}$$

By the analogue of the relation (29) and Lemma 2, one gets, for each $\alpha_k \in \mathbf{R}$, $k = 1, \ldots, m$,

$$\mathbf{E}e^{i\alpha_k \zeta_k^{(n)}} \longrightarrow e^{(\alpha_k^2 \sigma^2 |V_k|)/2} \quad \text{as} \quad n \to \infty. \tag{36}$$

Thus (34)–(36) imply (33).

The general case can be reduced to the previous one by means of standard arguments. Namely, let $V_\ell = \prod_{k=1}^{d} (a_k^{(\ell)}, b_k^{(\ell)}]$ and $I_\ell = \{k:\ b_k^{(\ell)} - a_k^{(\ell)} > 0\}$. Define for all $\delta > 0$ small enough $V_\ell(\delta) = \prod_{k=1}^{d} (a_k^{(\ell)}(\delta), b_k^{(\ell)}(\delta)]$, where $a_k^{(\ell)}(\delta) = a_k^{(\ell)} + \delta < b_k^{(\ell)} - \delta = b_k^{(\ell)}(\delta)$ if $k \in I_\ell$ and $a_k^{(\ell)}(\delta) = a_k^{(\ell)} = b_k^{(\ell)} = a_k^{(\ell)}(\delta)$ if $k \in \{1, \ldots, d\} \setminus I_\ell$. Denote $\zeta^{(n)}(\delta) = n^{-d/2}(S(nV_1(\delta)), \ldots, S(nV_m(\delta)))$. Using (25), we have for all $n \in \mathbf{N}$ and some $c > 0$,

$$\mathbf{E}(\zeta_k^{(n)} - \zeta_k^{(n)}(\delta))^2 \le cn^d |V_k \setminus V_k(\delta)| n^{-d} \le 2c\delta, \quad k = 1, \ldots, m,$$

where $\zeta^{(n)} = (\zeta_1^{(n)}, \ldots, \zeta_m^{(n)})$ is given by (33). Consequently, $\zeta^{(n)}(\delta) \xrightarrow{D} \zeta^{(n)}$ for each n as $\delta \to 0+$. Now it is easy to show that, for each $\alpha \in \mathbf{R}^m$,

$$\mathbf{E}e^{i(\alpha,\zeta^{(n)})} - \prod_{\ell=1}^{m} \exp(-\sigma^2 \alpha_\ell^2 |V_\ell|/2) \longrightarrow 0 \quad \text{as} \quad n \to \infty,$$

which is equivalent to (33).

Note that we can also prove (32) applying the Cramer–Wold device and using Lemma 2 with the analogue of relation (29).

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REFERENCES

1. T. Birkel. "An invariance principle for associated processes," *Stoch. Proc. Appl.*, **27**, 57–71 (1988).
2. T. Birkel. "A functional central limit theorem for positively dependent random variables," *J. Multivar. Anal.*, **44**, 314–320 (1993).
3. A. V. Bulinski, "CLT for shot-noise fields," in: *Problems of Probability Distributions* [in Russian], V. N. Sudakov ed., Nauka, Moscow (1990).
4. A. V. Bulinsky, "Inequalities for the moments of sums of associated multi indexed random variables," *Theory Probab. Appl.*, **38**, No. 2, 215–224 (1993).
5. A. V. Bulinski, "Limit theorems for positively and negatively dependent random fields," in: *Proceedings of the Kolmogorov Semester*, Int. Euler Math. Inst., March 1993 (to appear).
6. A. V. Bulinski, "Estimate of the convergence rate in the CLT for associated random fields," *Teor. Veroyatn. Primen.*, **40**, No. 1, 165–174 (1995).
7. R. Burton and E. Waymir, "Scaling limits for associated random measures," *Ann. Probab.*, **13**, No. 4, 1267–1278 (1985).
8. R. M. Burton, A. R. Dabrowski, and H. Dehling, "An invariance principle for weakly associated random vectors," *Stoch. Proc. Appl.*, **23**, 301–306 (1986).
9. J. T. Cox and G. Grimmett, "Central limit theorem for associated random variables and the percolation model," *Ann. Probab.*, **12**, No. 2, 514–528 (1984).

10. P. Matula and Z. Rychlik, "The invariance principle for nonstationary sequences of associated random variables," *Ann. Inst. H. Poincare*, **26**, No. 3, 387–397 (1990).
11. F. Moricz, "A general moment inequality for the maximum of the rectangular partial sums of multiple series," *Acta. Math. Hung.*, **41**, 3-4, 337-346 (1983).
12. C. M. Newman, "Normal fluctuations and the FKG-inequalities," *Commun. Math. Phys.*, **74**, 119–128 (1980).
13. C. M. Newman, "Asymptotic independence and limit theorems for positively and negatively dependent random variables," in: *Inequalities in Probability and Statistics*, Y. L. Tong, ed., IMS Lect. Notes-Monograph Series, Hayward, CA, Vol. 5 (1984), pp. 127–140.
14. C. M. Newman and A. L. Wright, "Associated random variables and martingale inequalities," *Z. Wahrsch. Verw. Geb.*, **59**, 361–371 (1982).
15. M. Wichura, "Inequalities with applications to the weak convergence of random processes with multi-dimensional time parameters," *Ann. Math. Statist.*, **40**, 681–687 (1969).

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