

ZAMM · Z. angew. Math. Mech. **77** (1997) 1, 59–74

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## The Use of Defect Correction for the Solution of Parabolic Singular Perturbation Problems

*Wir konstruieren diskrete Approximationen für eine Klasse von singular gestörten Randwertproblemen, wie etwa das Dirichlet-Problem für eine parabolische Differentialgleichung, für die die Koeffizienten der höchsten Ableitungen beliebig kleine Werte aus dem Intervall  $(0, 1]$  annehmen können. Die Diskretisierungsfehler für klassische Diskretisierungsmethoden hängen von diesem Parameter ab und können von einer mit der Lösung des Originalproblems vergleichbaren Größenordnung sein. Wir beschreiben, wie man spezielle Diskretisierungsmethoden konstruieren kann, für die die Genauigkeit der diskretisierten Lösung nicht vom Wert des Parameters, sondern nur von der Anzahl der verwendeten Gitterpunkte abhängt. Darüber hinaus konstruieren wir unter Verwendung von Defekt-Korrektur-Techniken eine Diskretisierungsmethode, die einen hohen Genauigkeitsgrad bezüglich der Zeitvariablen liefert. Die mit dieser speziellen Methode erhaltene Approximation konvergiert in der  $l^\infty$ -Norm unabhängig von dem kleinen Parameter gegen die exakte Lösung. Für ein Modell-Problem geben wir die Ergebnisse für unser Schema an und vergleichen diese mit den aus der klassischen Methode erhaltenen Resultaten.*

*We construct discrete approximations for a class of singularly perturbed boundary value problems, such as the Dirichlet problem for a parabolic differential equation, for which the coefficient multiplying the highest derivatives can take an arbitrarily small value from the interval  $(0, 1]$ . Discretisation errors for classical discrete methods depend on the value of this parameter and can be of a size comparable with the solution of the original problem. We describe how to construct special discrete methods for which the accuracy of the discrete solution does not depend on the value of the parameter, but only on the number of mesh points used. Moreover, using defect correction techniques, we construct a discrete method that yields a high order of accuracy with respect to the time variable. The approximation, obtained by this special method, converges in the discrete  $l^\infty$ -norm to the true solution, independent of the small parameter. For a model problem we show results for our scheme and we compare them with results obtained by the classical method.*

MSC (1991): 65N22, 35K20, 35B25, 35A40, 35K57

### 1. Introduction

For the solution of boundary value problems that have a smooth solution, well established methods can be used such as finite difference, finite element or finite volume schemes (see e.g. [3, 8, 9]). However, on a non-adapted grid the accuracy of the approximate solutions may deteriorate when the solution becomes less smooth.

In the case of singularly perturbed differential equations, when the coefficient of the highest derivatives,  $\varepsilon$ , takes smaller values, the solution of the boundary value problem may have a degrading smoothness. That means that the derivatives of such solutions may increase without bound when the small parameter tends to zero. When classical numerical methods are used, the error of the approximate solution depends on the value of the small parameter and it may be of a size comparable with, or even much larger than the solution of the original problem [4]. Therefore, for singularly perturbed boundary value problems it makes sense to construct special numerical methods for which the accuracy of the approximate solution does not depend on the parameter, and for which size of the error depends only on the number of nodal points used, i.e. methods which converge  $\varepsilon$ -uniformly.

In this paper we consider a class of singularly perturbed boundary value problems for parabolic equations. For these problems we construct special  $\varepsilon$ -uniformly convergent schemes. Moreover, using a defect correction technique [2], we construct a difference method which yields a high order of accuracy with respect to the time variable. For sufficiently regular problems with smooth solutions, we know that other methods also provide accuracy up to second order with respect to space and time variables, for example the Crank-Nicolson scheme. However, defect correction has the advantage that less stability requirements exist for the scheme (i.e. larger timesteps are possible) and that the order of accuracy can easily be adapted to the solution (to very high order if the solution is sufficiently smooth, cf. Section 8.1) during the computation.

In Section 7 we give numerical results and compare them on uniform and on specially adapted grids.

### 2. The class of boundary value problems studied

#### 2.1 The mathematical problem

Here we describe a class of time-dependent diffusion problems in one space dimension. On the domain  $G = D \times (0, T]$ ,  $D = (0, d)$ , with the boundary  $S = \bar{G} \setminus G$  we consider the singularly perturbed parabolic equation

with Dirichlet boundary conditions

$$L_{(2.1)}u(x, t) \equiv \varepsilon^2 \frac{\partial}{\partial x} \left( a(x, t) \frac{\partial}{\partial x} u(x, t) \right) - c(x, t) u(x, t) - p(x, t) \frac{\partial}{\partial t} u(x, t) = f(x, t), \quad (x, t) \in G, \quad (2.1a)$$

$$u(x, t) = \varphi(x, t), \quad (x, t) \in S. \quad (2.1b)$$

Here  $a(x, t)$ ,  $c(x, t)$ ,  $p(x, t)$ ,  $f(x, t)$ ,  $(x, t) \in \bar{G}$ , and  $\varphi(x, t)$ ,  $(x, t) \in S$ , are sufficiently smooth and bounded functions which satisfy

$$0 < a_0 \leq a(x, t), \quad 0 < p_0 \leq p(x, t), \quad c(x, t) \geq 0, \quad (x, t) \in \bar{G}. \quad (2.1c)$$

The real parameter  $\varepsilon$  may take any small value:

$$\varepsilon \in (0, 1]. \quad (2.1d)$$

When the parameter  $\varepsilon$  tends to zero in (2.1a), in the neighbourhood of the lateral boundary  $S_1$ , where  $S = S_0 \cup S_1$ ,  $S_0 = \{(x, t) : x \in \bar{D}, t = 0\}$ , boundary layers appear, which are described by an equation of parabolic type (parabolic boundary layers). If an additional first order term,  $b(x, t) \partial u(x) / \partial x$ , would be present in (2.1a) then we would see a boundary layer at the outflow boundary, that would be described by an ordinary differential equation (an ordinary boundary layer).

## 2.2 The problem of constructing a parameter-uniformly convergent scheme

As we shall see in Section 3, for boundary value problem (2.1) classical difference approximations on a uniform grid [8, 9] converge to the solution of (2.1) only if the parameter  $\varepsilon$  is kept fixed (see Theorem 3.1). The accuracy of the numerical solution depends on the value of the parameter and it decreases with  $\varepsilon$ . All accuracy is lost when the value of  $\varepsilon$  is comparable with the space mesh-size of the uniform grid. We shall see that classical finite difference schemes do not converge  $\varepsilon$ -uniformly for small values of the parameter  $\varepsilon$ . Therefore, it is our concern to analyse an approach which can be successfully applied, and to construct special grids which make the method converge  $\varepsilon$ -uniformly. In addition, using defect correction techniques we shall construct a method of which the solution approximates the solution of problem (2.1) with a high order of accuracy with respect to the time variable.

## 3. An arbitrary non-uniform grid

### 3.1 Results for a fixed parameter $\varepsilon$

To solve problem (2.1) we first study a classical finite difference method on a (possibly) non-uniform grid. On the set  $\bar{G}$  we introduce the rectangular grid

$$\bar{G}_h = \bar{\omega} \times \bar{\omega}_0, \quad (3.1)$$

where  $\bar{\omega}$  is the (possibly) non-uniform grid on  $\bar{D}$ , with not too large a distance between the nodes;  $\bar{\omega}_0$  is a uniform grid on the interval  $[0, T]$ ;  $N+1$  and  $K+1$  are the numbers of nodes in grids  $\bar{\omega}$  and  $\bar{\omega}_0$ , respectively. Define  $\tau = TK^{-1}$ ,  $h^i = x^{i+1} - x^i$ ,  $h = \max_i h^i$ ,  $G_h = G \cap \bar{G}_h$ ,  $S_h = S \cap \bar{G}_h$ . For convenience, we suppose  $h \leq MN^{-1}$ .

Here and in the following we denote by  $M$  (or  $m$ ) sufficiently large (or small) positive constants which do not depend on the value of parameter  $\varepsilon$  or on the difference operators.

For problem (2.1) we use the difference scheme [7, 8]

$$A_{(3.2)}z(x, t) = f(x, t), \quad (x, t) \in G_h, \quad (3.2a)$$

$$z(x, t) = \varphi(x, t), \quad (x, t) \in S_h. \quad (3.2b)$$

Here

$$\begin{aligned} A_{(3.2)}z(x, t) &\equiv \varepsilon^2 \delta_{\bar{x}}(a^h(x, t) \delta_{\bar{x}}z(x, t)) - c(x, t) z(x, t) - p(x, t) \delta_{\bar{t}}z(x, t), \\ \delta_{\bar{x}}(a^h(x^i, t) \delta_{\bar{x}}z(x^i, t)) &= 2(h^{i-1} + h^i)^{-1} (a^h(x^{i+1}, t) \delta_{\bar{x}}z(x^i, t) - a^h(x^i, t) \delta_{\bar{x}}z(x^i, t)), \\ a^h(x^i, t) &= a((x^{i-1} + x^i)/2, t), \quad \delta_{\bar{x}}z(x^i, t) = (h^{i-1})^{-1} (z(x^i, t) - z(x^{i-1}, t)), \\ \delta_{\bar{x}}z(x^i, t) &= (h^i)^{-1} (z(x^{i+1}, t) - z(x^i, t)), \quad \delta_{\bar{t}}z(x^i, t) = \tau^{-1} (z(x^i, t) - z(x^i, t - \tau)), \end{aligned}$$

$\delta_{\bar{x}}z(x, t)$  and  $\delta_{\bar{t}}z(x, t)$ ,  $\delta_{\bar{x}}z(x, t)$  are the forward and backward differences and the difference operator  $\delta_{\bar{x}}(a^h(x, t) \delta_{\bar{x}}z(x, t))$  is an approximation of the operator  $\frac{\partial}{\partial x} \left( a(x, t) \frac{\partial}{\partial x} u(x, t) \right)$  on the non-uniform grid.

The difference scheme (3.2), (3.1) is monotone (i.e., for this scheme the maximum principle holds).

By means of the maximum principle and taking into account estimates of the derivatives (see Theorem 8.1) we find that the solution of the difference scheme (3.2), (3.1) converges for a fixed value of the parameter  $\varepsilon$ :

$$|u(x, t) - z(x, t)| \leq M(\varepsilon^{-1}N^{-1} + \tau), \quad (x, t) \in \bar{G}_h. \quad (3.3)$$

The proof of (3.3) follows the lines of the classical proof of convergence for monotone difference schemes [9, 14]. Taking into account an a-priori estimate for the solution (which is found in the Appendix, Section 8.1), this results in the following theorem.

**Theorem 3.1:** *Let the estimate (8.2) hold for the solution of boundary value problem (2.1). Then, for a fixed value of parameter  $\varepsilon$ , the solution of (3.2), (3.1) converges to the solution of the boundary value problem (2.1) with an error bound given by (3.3).*

#### 4. The $\varepsilon$ -uniformly convergent method

In this section, by means of a special refined mesh in the neighbourhood of the boundary layer, we construct an  $\varepsilon$ -uniformly convergent method for (2.1). It is known that on uniform grids no fitted operators exist that converge  $\varepsilon$ -uniformly. Therefore we need a special mesh, condensed in the neighbourhood of boundary layers. The placement of the nodes is derived from a-priori estimates of the solution and its derivatives. The way to construct the mesh for problems (2.1) is the same as in [6, 12, 13, 14].

Let us consider the special grid

$$\bar{G}_h^* = \bar{\omega}^*(\sigma) \times \bar{\omega}_0, \quad (4.1)$$

where  $\bar{\omega}_0$  is a uniform mesh with step-size  $\tau = T/K$ ,  $\bar{\omega}_0 = \bar{\omega}_{0(3.1)}$ , and  $\bar{\omega}^* = \bar{\omega}^*(\sigma)$  is a special piecewise uniform mesh depending on the parameter  $\sigma$ ;  $\sigma$  is a function of  $\varepsilon$  and  $N$ . The mesh  $\bar{\omega}^*(\sigma)$  is constructed as follows. The interval  $[0, d]$  is divided into three parts  $[0, \sigma]$ ,  $[\sigma, d - \sigma]$ ,  $[d - \sigma, d]$ ,  $0 < \sigma \leq d/4$ . We divide each interval  $[0, \sigma]$ ,  $[d - \sigma, d]$  into  $N/4$  equal parts and the interval  $[\sigma, d - \sigma]$  into  $N/2$  equal parts. We assume that  $\sigma = \sigma_{(4.1)}(\varepsilon, N) = \min[d/4, m\varepsilon \ln N]$ , where  $m = m_{(4.1)}$  is an arbitrary number.

**Theorem 4.1:** *If the solution of problem (2.1) satisfies the conditions of Theorem 8.1 (in the appendix), then the solution of the discrete problem (3.2), (4.1) converges  $\varepsilon$ -uniformly to the solution of the boundary value problem (2.1). In particular, for the solution of the discrete problem the following estimate holds:*

$$|u(x, t) - z(x, t)| \leq M(N^{-1} \ln N + \tau), \quad (x, t) \in \bar{G}_h^*. \quad (4.2)$$

The proof of this theorem can be found in [12, 13, 14].

Notice that the term  $N^{-1} \ln N$  in the estimate (4.2) can be replaced by  $(N^{-1} \ln N)^2$ , for example, see [10]. However, for simplicity, here we will not go into these details. Related results can also be found in a forthcoming book by MILLER, O'RIORDAN, and SHISHKIN.

#### 5. Numerical results

By the theory (Theorem 3.1) it is shown that the classical difference scheme (3.2) on the uniform grid (5.4) does not converge in the  $l^\infty$ -norm  $\varepsilon$ -uniformly to the solution of the boundary value problem (2.1). But it would be possible that the error  $\max_{\bar{G}_h} |z(x, t) - u(x, t)|$  is not too large. That would reduce the need for a special adapted mesh.

On the other hand, Theorem 4.1 shows that the special method (3.2), (4.1) converges  $\varepsilon$ -uniformly, but no indication is given about the value of the order constant  $M$  in (4.2). It might be possible that the error is rather large for any reasonable value of  $N$ ,  $K$ . This might reduce the practical value of the special mesh. To decide on the practical value of the mesh (4.1), numerical experiments should give the final answer.

##### 5.1 The model problem

To see the effect of the special mesh in practice, for the approximation of the model problem we study the singularly perturbed parabolic equation

$$\begin{aligned} L_{(5.1)} u(x, t) &\equiv \varepsilon^2 \frac{\partial^2}{\partial x^2} u(x, t) - \frac{\partial}{\partial t} u(x, t) = f(x, t), & (x, t) \in G, \\ u(x, t) &= \varphi(x, t), & (x, t) \in S, \end{aligned} \quad (5.1)$$

where  $f(x, t) = -4t^3$ ,  $(x, t) \in \bar{G}$ ,  $\varphi(x, t) = 0$ ,  $(x, t) \in S$ .

We compare the numerical results for the uniform mesh (3.2), (5.4) and the special mesh (3.2), (4.1). Here  $T = 1$  and  $G = \{(x, t) : 0 < x < 1, 0 < t \leq T\}$ .

For the solution of problem (5.1) we have the representation

$$u(x, t) = U(x, t) + W(x, t), \quad (x, t) \in \bar{G}, \quad (5.2)$$

where  $U(x, t) = t^4$ ,  $(x, t) \in \bar{G}$ , is the outer solution, and  $W(x, t)$  represents the parabolic boundary layer in the neighbourhood of the edges at  $x = 0$  and  $x = 1$ . The function  $W(x, t)$  is represented as a sum,

$$W(x, t) = V(x, t) + V(1 - x, t) + v(x, t), \quad (x, t) \in \bar{G},$$

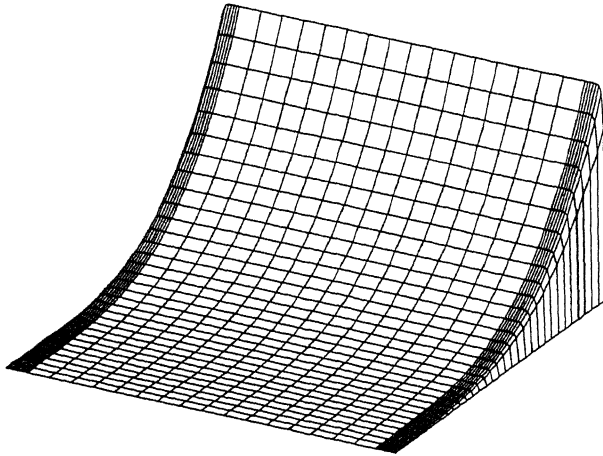


Fig. 1: The solution computed with the adapted mesh,  $\varepsilon = 2^{-7}$ ,  $N = K = 32$ ,  $m_{(4,1)} = 2.0$ .

where  $V(x, t)$  is the solution of the problem

$$\left. \begin{aligned} L_{(5.1)}V(x, t) &= 0, & 0 < x < \infty, & 0 < t \leq T, \\ V(0, t) &= t^4, & 0 < t \leq T, & V(x, 0) = 0, & 0 \leq x < \infty, \end{aligned} \right\} \quad (5.3)$$

the function  $v(x, t)$  is the remainder term. For the functions  $u(x, t)$ ,  $V(x, t)$ ,  $v(x, t)$  we have relations

$$\begin{aligned} 0 \leq u(x, t) \leq 1, & \quad (x, t) \in \bar{G}, & 0 \leq V(x, t) \leq 1, & \quad 0 \leq x < \infty, & 0 \leq t \leq T, \\ V(x, t) \rightarrow 0 & \text{ for } x \rightarrow \infty, t \in (0, T], & |v(x, t)| \leq O(\varepsilon^M), & \quad (x, t) \in \bar{G}, \end{aligned}$$

where  $M$  can be chosen arbitrarily large.

Due to Theorem 4.1 the solution of the discrete problem with the adapted mesh converges  $\varepsilon$ -uniformly to the solution of our model problem (5.1). The function  $z(x, t)$ , which is the solution of the special scheme (3.2), (4.1) is shown in Figure 1.

## 5.2 The behaviour of the numerical solution on a uniform mesh

To see the difference between the use of the uniform and the adapted grid, for the approximation of (5.1) we first use the classical scheme (3.2) on the grid,

$$\bar{G}_h^u = \{\bar{G}_{h(3.1)}, \text{ where } \bar{\omega} = \bar{\omega}^u \text{ is a uniform grid}\}. \quad (5.4)$$

This grid is determined by two numbers, its meshwidth in space and time. We solve the problem for different values of the meshwidth  $h = N^{-1}$ ,  $\tau = K^{-1}$  and for different values of the parameter  $\varepsilon$ . The results for a set of numerical experiments are given in Table 1.

Table 1. Table of errors  $E(N, N, \varepsilon)$  for the classical scheme (3.2), (5.4)

$\varepsilon$	$N: 8$	16	32	64	128	256
1.0	1.339e-02	7.038e-03	3.520e-03	1.670e-03	7.220e-04	2.420e-04
$2^{-1}$	9.478e-02	4.906e-02	2.433e-02	1.149e-02	4.955e-03	1.657e-03
$2^{-2}$	2.177e-01	1.081e-01	5.244e-02	2.449e-02	1.050e-02	3.499e-03
$2^{-3}$	2.593e-01	1.246e-01	5.946e-02	2.755e-02	1.176e-02	3.914e-03
$2^{-4}$	2.617e-01	1.250e-01	5.957e-02	2.758e-02	1.178e-02	3.918e-03
$2^{-5}$	2.617e-01	1.250e-01	5.957e-02	2.758e-02	1.178e-02	3.918e-03
$2^{-6}$	2.617e-01	1.250e-01	5.957e-02	2.758e-02	1.178e-02	3.918e-03
$2^{-7}$	2.617e-01	1.250e-01	5.957e-02	2.758e-02	3.075e-02	1.175e-02
$2^{-8}$	2.617e-01	1.250e-01	5.957e-02	2.758e-02	2.645e-02	3.616e-02
$2^{-9}$	2.617e-01	1.250e-01	5.957e-02	2.758e-02	1.178e-02	3.344e-02
$\bar{E}(N)$	2.617e-01	1.250e-01	5.957e-02	2.758e-02	3.075e-02	3.616e-02

In this table the error  $E(N, K, \varepsilon)$  is defined by

$$E(N, K, \varepsilon) = \max_{(x,t) \in \bar{G}_h} |z(x, t) - u^*(x, t)|, \quad (5.5)$$

where  $u^*(x, t)$  is the piecewise linear interpolation obtained from the numerical solution  $z(x, t)$  on an adapted grid with parameters  $m = 2$ ,  $N = N_0 = 512$ . Here  $\bar{G}_h = \bar{G}_h^u$ . Notice that no special interpolation is needed for the time variable.

From Table 1 we can see that the solution of scheme (3.2), (5.4) does not converge  $\varepsilon$ -uniformly. The errors, for a fixed value of  $N$  at  $K = N$ , depend on the parameter  $\varepsilon$ . For  $\varepsilon \geq 2^{-6}$  the error behaviour is regular: when  $N$  increases, the error decreases. For  $\varepsilon \leq 2^{-7}$  for some values of  $N$  the error increases with increasing  $N$  and for  $N = K \geq 128$  the error is about 3%. Thus, the numerical results illustrate that the lack of  $\varepsilon$ -uniform convergence leads to large errors indeed.

**5.3 The behaviour of the numerical solution of the special scheme**

In Table 2 we show the behaviour of (3.2), (4.1), with  $m = 2$ , applied to the model problem (5.1). The grid  $\bar{G}_h^*$  is determined by the two numbers  $N, K$  and the parameter  $\varepsilon$ , i.e.  $\bar{G}_h^* = \bar{G}^*(N, K, \varepsilon)$ . The results for a set of numerical experiments are given in Table 2. We solve the problem for different values of  $N = K$  and for different values of the parameter  $\varepsilon$ .

Table 2. Table of errors  $E(N, N, \varepsilon)$  for the special method (3.2), (4.1)

$\varepsilon$	$N: 8$	16	32	64	128	256
1.0	1.339e-02	7.038e-03	3.520e-03	1.670e-03	7.220e-04	2.420e-04
$2^{-1}$	9.478e-02	4.906e-02	2.433e-02	1.149e-02	4.955e-03	1.657e-03
$2^{-2}$	2.177e-01	1.081e-01	5.244e-02	2.449e-02	1.050e-02	3.499e-03
$2^{-3}$	2.593e-01	1.246e-01	5.946e-02	2.755e-02	1.176e-02	3.914e-03
$2^{-4}$	2.617e-01	1.250e-01	5.957e-02	2.758e-02	1.178e-02	3.918e-03
$2^{-5}$	2.617e-01	1.250e-01	5.957e-02	2.758e-02	1.178e-02	3.918e-03
$2^{-6}$	2.617e-01	1.250e-01	5.957e-02	2.758e-02	1.178e-02	3.918e-03
$2^{-7}$	2.617e-01	1.250e-01	5.957e-02	2.758e-02	1.178e-02	3.918e-03
$2^{-8}$	2.617e-01	1.250e-01	5.957e-02	2.758e-02	1.178e-02	3.918e-03
$2^{-9}$	2.617e-01	1.250e-01	5.957e-02	2.758e-02	1.178e-02	3.918e-03
$\bar{E}(N)$	2.617e-01	1.250e-01	5.957e-02	2.758e-02	1.178e-02	3.918e-03

In this table the function  $E(N, K, \varepsilon)$  is defined by (5.5), but now  $z(x, t)$  in (5.5) is the solution of (3.2), (4.1) with  $m = 2$  and  $\bar{G}_h = \bar{G}_h^*$ .

From Table 2 we see that the solution of the scheme (3.2), (4.1) converges  $\varepsilon$ -uniformly indeed. The errors for a fixed value of  $\varepsilon$  from  $1.0$  to  $2^{-9}$  have all a regular behaviour and decrease for increasing  $N, K$ . For a fixed value of  $N$  at  $K = N$  the error stabilises for decreasing  $\varepsilon$ : the errors for  $\varepsilon$  from  $2^{-4}$  to  $2^{-9}$  are practically the same. In particular, for all  $\varepsilon, N = K = 256$ , the error is less than 0.5%. Also here, the numerical results illustrate the practical value of  $\varepsilon$ -convergent methods.

**6. Improved time-accuracy**

In this section we construct a discrete method based on defect correction, which also converges  $\varepsilon$ -uniformly to the solution of the boundary value problem, but with an order of accuracy (with respect to  $\tau$ ) higher than in the estimate (4.2).

The idea to improve time-accuracy is as follows. For the difference scheme (3.2), (4.1) the error in the approximation of the partial derivative  $\partial u(x, t)/\partial t$  is caused by the divided difference  $\delta_{\bar{t}}z(x, t)$  and is associated with the truncation error given by the relation

$$\frac{\partial}{\partial t} u(x, t) - \delta_{\bar{t}}u(x, t) = 2^{-1}\tau \frac{\partial^2}{\partial t^2} u(x, t) - 6^{-1}\tau^2 \frac{\partial^3}{\partial t^3} u(x, t - \vartheta), \tag{6.1}$$

where  $\vartheta \in [0, \tau]$ . Therefore it is natural to use for the approximation of the derivative  $\partial u(x, t)/\partial t$  the expression

$$\delta_{\bar{t}}u(x, t) + \tau\delta_{\bar{t}\bar{t}}u(x, t)/2,$$

where  $\delta_{\bar{t}\bar{t}}u(x, t) \equiv \delta_{\bar{t}\bar{t}}u(x, t - \tau)$ ,  $\delta_{\bar{t}\bar{t}}u(x, t)$  is the second central divided difference. We can realise this approximation by defect correction,

$$A_{(3.2)}z^c(x, t) = f(x, t) - 2^{-1}p(x, t)\tau^c \partial^2 u(x, t)/\partial t^2, \tag{6.2}$$

where  $x \in \bar{\omega}$ , where  $\bar{\omega}$  is as in (3.1); and  $t \in \omega_0^c$ , where  $\bar{\omega}_0^c$  is a time grid with step-size  $\tau^c$ , which – generally – can be different from the original time-grid  $\bar{\omega}_0$ ;  $z^c(x, t)$  is the “corrected” solution. Instead of  $(\partial^2/\partial t^2)u(x, t)$  we shall use  $\delta_{\bar{t}\bar{t}}z(x, t)$ , where  $z(x, t), (x, t) \in G_{h(4.1)}$ , is the solution of the difference scheme (3.2), (4.1). E.g. we can take for  $\bar{\omega}_0^c$  a uniform grid with step-size  $2\tau$ . We may hope that the new solution  $z^c(x, t)$  has an accuracy of  $O(\tau)^2$  with respect to the time variable. This is true, as will be shown in Section 6.3.

Moreover, in a similar way we can construct a difference approximation with a convergence order higher than two (with respect to the time variable) and  $O(N^{-1} \ln N)$  with respect to the space variable  $\varepsilon$ -uniformly (see Section (6.4)).

### 6.1 Auxiliary constructions

On the set  $\bar{G}$  we introduce the sequence of grids

$$\bar{G}_h^{(l)} = \bar{\omega} \times \bar{\omega}_0^{(l)}, \quad l = 0, 1, \dots, n, \quad (6.3)$$

where  $\bar{\omega}_0^{(l)}$  is a uniform grid with a step-size  $\tau^{(l)} = 2^{-l}\tau^{(0)}$ ,  $n \geq 0$  is an integer number,  $\tau^{(l)} = T/K^{(l)}$ , where  $K^{(l)} + 1$  is the number of nodes in the grid  $\bar{\omega}_0^{(l)}$ . We take  $\tau = \tau^{(n)}$ . Let a function  $z^{(l)}(x, t)$  be defined on each grid  $\bar{G}_h^{(l)}$ . We denote by  $\delta_{k\bar{x}}^{(l)} z^{(l)}(x, t)$  the backward difference of order  $k$ :

$$\begin{aligned} \delta_{0\bar{x}}^{(l)} z^{(l)}(x, t) &= z(x, t), \\ \delta_{k\bar{x}}^{(l)} z^{(l)}(x, t) &= (\delta_{k-1\bar{x}}^{(l)} z^{(l)}(x, t) - \delta_{k-1\bar{x}}^{(l)} z^{(l)}(x, t - \tau^{(l)})) / \tau^{(l)}, \quad (x, t) \in \bar{G}_h^{(l)}, \quad t \geq k\tau^{(l)}, \quad k \geq 1. \end{aligned}$$

On the grid  $\bar{G}_h^{(n)}$  we consider the finite difference scheme (3.2), writing

$$\left. \begin{aligned} A_{(6.4)}^{(n)} Z^{(n)}(x, t) &= f(x, t), \quad (x, t) \in G_h^{(n)}, \\ Z^{(n)}(x, t) &= \varphi(x, t), \quad (x, t) \in S_h^{(n)}. \end{aligned} \right\} \quad (6.4)$$

Here, for  $l = 0, 1, \dots, n$ ,

$$A_{(6.4)}^{(l)} z^{(l)}(x, t) \equiv \varepsilon^2 \delta_{\bar{x}}(a^h(x, t) \delta_{\bar{x}} z^{(l)}(x, t)) - c(x, t) z^{(l)}(x, t) - p(x, t) \delta_{\bar{t}}^{(l)} z^{(l)}(x, t).$$

For the solution of (6.4) Theorem 4.1 yields the estimate

$$|u(x, t) - Z^{(n)}(x, t)| \leq M[N^{-1} \ln N + \tau], \quad (x, t) \in \bar{G}_h^{*(n)}. \quad (6.5)$$

Here the adapted grid on level  $(l)$  is denoted by

$$\bar{G}_h^{*(l)} = \bar{\omega}^* \times \bar{\omega}_0^{(l)}, \quad l = 0, 1, \dots, n. \quad (6.6)$$

### 6.2 The difference scheme with second order accuracy in time

To start the defect correction we construct on the grid  $\bar{G}_h^{(n-1)}$  a difference scheme which converges to the solution of (2.1) with second order accuracy (with respect to  $\tau$ ). We take a difference scheme of the form

$$A_{(6.4)}^{(n-1)} Z^{(n,1)}(x, t) = C_1^{(n-1)}(x, t) \tau^{(n-1)} \delta_{2\bar{t}}^{(n)} Z^{(n,0)}(x, t) + f(x, t), \quad (x, t) \in G_h^{(n-1)}. \quad (6.7a)$$

Starting the iteration, we take as initial solution  $Z^{(n,0)}(x, t) = Z^{(n)}(x, t)$ , and find a corrected solution by the difference scheme

$$Z^{(n,1)}(x, t) = \varphi(x, t), \quad (x, t) \in S_h^{(n-1)}.$$

For simplicity, in this section we take a homogeneous initial condition:

$$\varphi(x, 0) = 0, \quad x \in \bar{D}. \quad (6.8)$$

The function  $C_1^{(n-1)}(x, t)$  is determined as follows. On the grid  $\bar{\omega}_0^{(n-1)}$ , with step-size  $\tau^{(n-1)}$ , for the divided difference operator (w.r.t. time) in (6.7) we have

$$\begin{aligned} -p(x, t) \delta_{\bar{t}}^{(n-1)} u(x, t) &= -p(x, t) \left( \frac{\partial}{\partial t} u(x, t) - \frac{\tau^{(n-1)}}{2} \frac{\partial^2}{\partial t^2} u(x, t) + \frac{(\tau^{(n-1)})^2}{6} \frac{\partial^3}{\partial t^3} u(x, t - \vartheta_1) \right), \\ t &\in \bar{\omega}_0^{(n-1)}, \quad \vartheta_1 \in [0, \tau^{(n-1)}]. \end{aligned}$$

For the second derivative in the right-hand side we have

$$\begin{aligned} C_1^{(n-1)}(x, t) \tau^{(n-1)} \delta_{2\bar{t}}^{(n)} u(x, t) &= C_1^{(n-1)}(x, t) \tau^{(n-1)} \left( \frac{\partial^2}{\partial t^2} u(x, t) - \tau^{(n)} \frac{\partial^3}{\partial t^3} u(x, t - \vartheta_2) \right), \\ t &\in \bar{\omega}_0^{(n-1)}, \quad \vartheta_2 \in [0, \tau^{(n)}]. \end{aligned}$$

Therefore, by choosing

$$C_1^{(n-1)}(x, t) = 2^{-1} p(x, t), \quad (6.7b)$$

we have

$$-p(x, t) \delta_{\bar{t}}^{(n-1)} u(x, t) - C_1^{(n-1)}(x, t) \tau^{(n-1)} \delta_{2\bar{t}}^{(n)} u(x, t) = -p(x, t) \frac{\partial}{\partial t} u(x, t) + O(\tau^2),$$

i.e., the divided difference operator w.r.t. the time in (6.7a) approximates  $p(x, t) (\partial/\partial t)$  with  $O(\tau^2)$ .

For the solution of the discrete equations (6.7), (6.6) we find the estimate

$$|u(x, t) - Z^{(n,1)}(x, t)| \leq M[N^{-1} \ln N + \tau^2], \quad (x, t) \in \bar{G}_h^{*(n-1)}. \tag{6.9}$$

**Theorem 6.1:** *Let condition (6.8) hold and let for equation (2.1)  $a \in H^{(a+2n+1)}(\bar{G})$ ,  $c, p, f \in H^{(a+2n)}(\bar{G})$ ,  $\varphi \in H^{(a+2n)}(\bar{G})$ ,  $\alpha > 4$ ,  $n \geq 0$ , and let the condition (8.3) be satisfied for  $n \geq 1$ . Then for the solution of difference scheme (6.7), (6.6), the estimate (6.9) holds.*

The proof of this theorem is found in the appendix in Section (8.2).

**Remark:** If the condition (6.8) is not satisfied, we can use the standard technique to homogenise the initial condition, i.e., instead of the function  $u(x, t)$  we consider the function  $u^{(1)}(x, t) = u(x, t) - \varphi(x, 0)$ ,  $(x, t) \in \bar{G}$ , which is the solution of the problem

$$L_{(2.1)} u^{(1)}(x, t) = f^{(1)}(x, t), \quad (x, t) \in G, \quad u^{(1)}(x, t) = \varphi^{(1)}(x, t), \quad (x, t) \in S,$$

where  $f^{(1)}(x, t) = f(x, t) - \varepsilon^2 \frac{\partial}{\partial x} \left( a(x, t) \frac{\partial}{\partial x} \varphi(x, 0) \right)$ ,  $\varphi^{(1)}(x, t) = \varphi(x, t) - \varphi(x, 0)$ .

### 6.3 The difference scheme of third order in time

In an analogous way we construct a difference scheme with third order accuracy in  $\tau$ . On the grid  $\bar{G}_h^{(n-2)}$  we consider the difference scheme

$$\left. \begin{aligned} A_{(6.4)}^{(n-2)} Z^{(n,2)}(x, t) &= \sum_{k=1}^2 C_k^{(n-2)}(x, t) (\tau^{(n-2)})^k \delta_{k+1\bar{t}}^{(n-2+k)} Z^{(n,2-k)}(x, t) + f(x, t), \quad (x, t) \in G_h^{(n-2)}, \\ Z^{(n,2)}(x, t) &= \varphi(x, t), \quad (x, t) \in S_h^{(n-2)}. \end{aligned} \right\} \tag{6.10 a}$$

Again we suppose the homogeneous initial condition

$$\varphi(x, 0) = 0, \quad f(x, 0) = 0, \quad x \in \bar{D}. \tag{6.11}$$

The functions  $C_1^{(n-2)}(x, t)$ ,  $C_2^{(n-2)}(x, t)$  are determined as follows. On the grid  $\bar{\omega}_0^{(n-2)}$  we have the following representation for the divided differences for the solution  $u(x, t)$ :

$$\begin{aligned} -p(x, t) \delta_{\bar{t}}^{(n-2)} u(x, t) &= -p(x, t) \left( \frac{\partial}{\partial t} u(x, t) - \frac{\tau^{(n-2)}}{2} \frac{\partial^2}{\partial t^2} u(x, t) + \frac{(\tau^{(n-2)})^2}{6} \frac{\partial^3}{\partial t^3} u(x, t) \right) + O(\tau^3), \\ C_1^{(n-2)}(x, t) \tau^{(n-2)} \delta_{2\bar{t}}^{(n-1)} u(x, t) &= C_1^{(n-2)}(x, t) \tau^{(n-2)} \left( \frac{\partial^2}{\partial t^2} u(x, t) - \tau^{(n-1)} \frac{\partial^3}{\partial t^3} u(x, t) \right) + O(\tau^3), \\ C_2^{(n-2)}(x, t) (\tau^{(n-2)})^2 \delta_{3\bar{t}}^{(n)} u(x, t) &= C_2^{(n-2)}(x, t) (\tau^{(n-2)})^2 \frac{\partial^3}{\partial t^3} u(x, t) + O(\tau^3), \end{aligned}$$

If we choose

$$C_1^{(n-2)}(x, t) = p(x, t)/2, \quad C_2^{(n-2)}(x, t) = p(x, t)/12, \tag{6.10 b}$$

we find

$$\begin{aligned} -p(x, t) \delta_{\bar{t}}^{(n-2)} u(x, t) - C_1^{(n-2)}(x, t) \tau^{(n-2)} \delta_{2\bar{t}}^{(n-1)} u(x, t) - C_2^{(n-2)}(x, t) (\tau^{(n-2)})^2 \delta_{3\bar{t}}^{(n)} u(x, t) \\ = -p(x, t) \frac{\partial}{\partial t} u(x, t) + O(\tau^3). \end{aligned}$$

Thus, the divided difference operator with respect to the time approximates  $p(x, t) \frac{\partial}{\partial t}$  with an error  $O(\tau^3)$ .

For the solution of (6.10), (6.6), where  $Z^{(n,0)}(x, t)$  and  $Z^{(n,1)}(x, t)$  are the solutions of the schemes (6.4), (6.6) and (6.7), (6.6) respectively, we find the estimate

$$|u(x, t) - Z^{(n,2)}(x, t)| \leq M[N^{-1} \ln N + \tau^3], \quad (x, t) \in \bar{G}_h^{*(n-2)}. \tag{6.12}$$

Thus we have the following theorem.

**Theorem 6.2:** *Let the conditions (6.11) hold and let for equation (2.1)  $a \in H^{(a+2n+1)}(\bar{G})$ ,  $c, p, f \in H^{(a+2n)}(\bar{G})$ ,  $\varphi \in H^{(a+2n)}(\bar{G})$ ,  $\alpha > 4$ ,  $n \geq 0$ , and let the condition (8.3) be satisfied with  $n = 2$ . Then for the solution of problem*

(6.10), (6.6), where  $Z^{(n,1)}(x, t)$  and  $Z^{(n,0)}(x, t)$  are the solutions of problems (6.7), (6.6) and (6.4), (6.6), respectively, the estimate (6.12) holds.

The proof of the estimate (6.12) is given in Section (8.3).

Remark 1: If the condition (6.11) is not satisfied, instead of the function  $u(x, t)$  we may consider the function

$$u^{(2)}(x, t) = u(x, t) - \varphi(x, 0) - p^{-1}(x, 0) \left[ \varepsilon^2 \frac{\partial}{\partial x} \left( a(x, t) \frac{\partial}{\partial x} \varphi(x, 0) \right) - c(x, 0) \varphi(x, 0) - f(x, 0) \right] t, \quad (x, t) \in \bar{G},$$

and then we can solve the boundary value problem for the function  $u^{(2)}(x, t)$ .

Remark 2: In a similar way we can construct difference schemes with an order of accuracy

$$O(N^{-1} \ln N + \tau^{n+1}).$$

#### 6.4 Modified difference schemes of the second and third order accuracy in $\tau$

Here we discuss the construction of alternative difference schemes with higher order accuracy with respect to the time variable. In this variant we consider all intermediate difference equations on the same time-grid  $\bar{\omega}_{0(3.1)}$  (with step-size  $\tau$ ).

When constructing difference schemes of the second order accuracy in  $\tau$  in (6.2), instead of  $\delta_{\bar{\tau}} z(x, t)$  we use  $\delta_{2\bar{\tau}} z(x, t)$ , which (as in Section 6.1) is the second divided difference of the solution to the discrete problem (3.2), (4.1). For the boundary value problem (2.1) we now get for the difference equations for  $t = \tau$  and  $t \geq 2\tau$ :

$$\left. \begin{aligned} A_{(3.2)} z^{(2)}(x, t) &= f(x, t) + \frac{p(x, t)}{2} \tau \frac{\partial^2}{\partial t^2} u(x, 0), & t = \tau, \\ A_{(3.2)} z^{(2)}(x, t) &= f(x, t) + \frac{p(x, t)}{2} \tau \delta_{2\bar{\tau}} z^{(1)}(x, t), & t \geq 2\tau, (x, t) \in G_h, \\ z^{(2)}(x, t) &= \varphi(x, t), & (x, t) \in S_h. \end{aligned} \right\} \quad (6.13)$$

Here  $z^{(1)}(x, t)$  is the solution of the discrete problem (3.2), (4.1), and the derivative  $\partial^2 u(x, 0)/\partial t^2$  is obtained from the equation (2.1a). We shall call  $z^{(2)}(x, t)$  the solution of the difference scheme (6.13), (3.2), (4.1).

Under the homogeneous initial condition (6.8), for the solution of problem (6.13), (3.2), (4.1) the following estimate (similar to (6.9)) holds:

$$|u(x, t) - z^{(2)}(x, t)| \leq M[N^{-1} \ln N + \tau^2], \quad (x, t) \in \bar{G}_h. \quad (6.14)$$

Theorem 6.3: Let the condition of the theorem 6.1 be fulfilled. Then for the solution of the difference scheme (6.13), (3.2), (4.1) the estimate (6.14) holds.

The proof of this theorem is similar to the proof of Theorem 6.1.

A difference scheme of third order accuracy in  $\tau$  can, similarly, be constructed as

$$\left. \begin{aligned} A_{(3.2)} z^{(3)}(x, t) &= f(x, t) + p(x, t) \left( C_{11} \tau \frac{\partial^2}{\partial t^2} u(x, 0) + C_{12} \tau^2 \frac{\partial^3}{\partial t^3} u(x, 0) \right), & t = \tau, \\ L_{(3.2)} z^{(3)}(x, t) &= f(x, t) + p(x, t) \left( C_{21} \tau \frac{\partial^2}{\partial t^2} u(x, 0) + C_{22} \tau^2 \frac{\partial^3}{\partial t^3} u(x, 0) \right), & t = 2\tau, \\ L_{(3.2)} z^{(3)}(x, t) &= f(x, t) + p(x, t) (C_{31} \tau \delta_{2\bar{\tau}} z^{(2)}(x, t) + C_{32} \tau^2 \delta_{3\bar{\tau}} z^{(1)}(x, t)), & t \geq 3\tau, (x, t) \in G_h, \\ z^{(3)}(x, t) &= \varphi(x, t), & (x, t) \in S_h. \end{aligned} \right\} \quad (6.15 a)$$

Here  $z^{(1)}(x, t)$  and  $z^{(2)}(x, t)$  are solutions of the problems (6.13), (3.2), (4.1) and (3.2), (4.1), respectively, the derivatives  $(\partial^2/\partial t^2) u(x, 0)$ ,  $(\partial^3/\partial t^3) u(x, 0)$  are obtained from equation (2.1a), the coefficients  $C_{ij}$  are determined below. They are chosen such that the following conditions are satisfied:

$$\begin{aligned} \frac{\partial}{\partial t} u(x, t) &= \delta_{\bar{\tau}} u(x, t) + C_{11} \tau \frac{\partial^2}{\partial t^2} u(x, t - \tau) + C_{12} \tau^2 \frac{\partial^3}{\partial t^3} u(x, t - \tau) + O(\tau^3), \\ \frac{\partial}{\partial t} u(x, t) &= \delta_{\bar{\tau}} u(x, t) + C_{21} \tau \frac{\partial^2}{\partial t^2} u(x, t - 2\tau) + C_{22} \tau^2 \frac{\partial^3}{\partial t^3} u(x, t - 2\tau) + O(\tau^3), \\ \frac{\partial}{\partial t} u(x, t) &= \delta_{\bar{\tau}} u(x, t) + C_{31} \tau \delta_{2\bar{\tau}} u(x, t) + C_{32} \tau^2 \delta_{3\bar{\tau}} u(x, t) + O(\tau^3). \end{aligned}$$

It follows that

$$C_{11} = C_{21} = C_{31} = 1/2, \quad C_{12} = C_{32} = 1/3, \quad C_{22} = 5/6. \quad (6.15 b)$$



We shall call  $z^{(3)}(x, t)$  the solution of the difference scheme (6.15), (6.13), (3.2), (4.1). Under condition (6.11) for the solution of the difference scheme (6.15), (6.13), (3.2), (4.1) the following estimate holds:

$$|u(x, t) - z^{(3)}(x, t)| \leq M[N^{-1} \ln N + \tau^3], \quad (x, t) \in \bar{G}_h. \quad (6.16)$$

**Theorem 6.4:** *Let the condition of Theorem 6.2 be satisfied, then for the solution of the scheme (6.15), (6.13), (3.2), (4.1) the estimate (6.16) is valid.*

The proof of Theorem 6.4 is similar to the proof of Theorem 6.2.

## 7. Numerical results for the time-accurate schemes

The solution of problem (5.3) in the quarter plane is given by the formula

$$V_{(7.1)}(x, t) = \operatorname{erfc} \left( \frac{x}{2\varepsilon \sqrt{t}} \right) \left( \frac{x^8}{1680\varepsilon^8} + \frac{x^6}{30\varepsilon^6} t + \frac{x^4}{2\varepsilon^4} t^2 + \frac{2x^2}{\varepsilon^2} t^3 + t^4 \right) - \frac{1}{\sqrt{\pi}} \exp \left( \frac{-x^2}{4\varepsilon^2 t} \right) \left( \frac{x^7}{840\varepsilon^7} t^{1/2} + \frac{9x^5}{140\varepsilon^5} t^{3/2} + \frac{37x^3}{42\varepsilon^3} t^{5/2} + \frac{93x}{35\varepsilon} t^{7/2} \right). \quad (7.1)$$

We consider the model problem

$$L_{(5.1)}u(x, t) = 0, \quad (x, t) \in G, \quad u(x, t) = V_{(7.1)}(x, t), \quad (x, t) \in S, \quad (7.2)$$

then the function  $V_{(7.1)}(x, t)$ , is the solution of problem (7.2).

According to the Theorems 6.1 and 6.2, the difference schemes (6.7), (6.6) and (6.10), (6.6) converge respectively with orders 2 and 3 with respect to  $\tau$ . To demonstrate this effect numerically, we consider the difference schemes (6.4), (6.6); (6.7), (6.6); and (6.10), (6.6) for problem (7.2). We denote the solutions of these schemes for  $n=2$ , i.e.  $Z^{(2,0)}(x, t)$ ,  $(x, t) \in \bar{G}_h^{*(2)}$ ;  $Z^{(2,1)}(x, t)$ ,  $(x, t) \in \bar{G}_h^{*(1)}$ ;  $Z^{(2,2)}(x, t)$ ,  $(x, t) \in \bar{G}_h^{*(0)}$ , by  $Z_1(x, t)$ ,  $Z_2(x, t)$  and  $Z_3(x, t)$ , respectively. The grid  $\bar{G}_h^{*(3-k)}$ ,  $k=1, 2, 3$ , is determined by numbers  $N$ ,  $K_k = K^{(3-k)}$ ,  $k$ , and by the parameter  $\varepsilon$ . So  $\bar{G}_{h,k}^* = \bar{G}_{h(6.6)}^{*(3-k)}(N, K_k = K^{(3-k)}, \varepsilon)$ . Note that  $\bar{G}_{h(6.6)}^{*(n)}(N, K^{(n)}, \varepsilon) = \bar{G}_{h(4.1)}^*(N, K, \varepsilon)$  for  $K^{(n)} = K$ , and the scheme (6.4), (6.6) is scheme (3.2), (4.1) with  $\bar{G}_{h(6.6)}^*(N, K, \varepsilon) = \bar{G}_{h(4.1)}^*(N, K, \varepsilon)$ . We solve the problem (5.1), using the schemes (6.4), (6.6); (6.7), (6.6); and (6.10), (6.6) for various numbers  $N, K$ , values of the parameter  $\varepsilon$  and  $m = m_{(4.1)} = 2$ .

Table 3. Table errors  $E(N, K_1, \varepsilon)$  for scheme (6.4), (7.3)

$N$		8	16	32	64	128	256	512
$\varepsilon$	$K_1$							
1	16	1.79 (-2)	1.75 (-2)	1.74 (-2)	1.74 (-2)	1.74 (-2)	1.74 (-2)	1.74 (-2)
	32	9.49 (-3)	9.02 (-3)	8.90 (-3)	8.87 (-3)	8.87 (-3)	8.86 (-3)	8.86 (-3)
	64	5.17 (-3)	4.65 (-3)	4.52 (-3)	4.49 (-3)	4.48 (-3)	4.48 (-3)	4.48 (-3)
	128	2.97 (-3)	2.43 (-3)	2.30 (-3)	2.26 (-3)	2.25 (-3)	2.25 (-3)	2.25 (-3)
	256	1.86 (-3)	1.31 (-3)	1.17 (-3)	1.14 (-3)	1.13 (-3)	1.13 (-3)	1.13 (-3)
512	1.30 (-3)	7.51 (-4)	6.12 (-4)	5.77 (-4)	5.68 (-4)	5.66 (-4)	5.65 (-4)	
$2^{-2}$	16	3.66 (-2)	2.68 (-2)	2.42 (-2)	2.37 (-2)	2.35 (-2)	2.34 (-2)	2.34 (-2)
	32	2.58 (-2)	1.55 (-2)	1.27 (-2)	1.21 (-2)	1.19 (-2)	1.18 (-2)	1.18 (-2)
	64	2.03 (-2)	9.70 (-3)	6.87 (-3)	6.19 (-3)	6.00 (-3)	5.96 (-3)	5.95 (-3)
	128	1.76 (-2)	6.81 (-3)	3.94 (-3)	3.22 (-3)	3.04 (-3)	2.99 (-3)	2.98 (-3)
	256	1.62 (-2)	5.35 (-3)	2.46 (-3)	1.74 (-3)	1.55 (-3)	1.51 (-3)	1.50 (-3)
512	1.55 (-2)	4.62 (-3)	1.73 (-3)	9.94 (-3)	8.08 (-4)	7.62 (-4)	7.50 (-4)	
$2^{-4}$	16	5.66 (-2)	4.67 (-2)	3.26 (-2)	2.68 (-2)	2.42 (-2)	2.37 (-2)	2.35 (-2)
	32	4.87 (-2)	3.65 (-2)	2.19 (-2)	1.55 (-2)	1.27 (-2)	1.21 (-2)	1.19 (-2)
	64	4.48 (-2)	3.14 (-3)	1.65 (-3)	9.70 (-3)	6.87 (-3)	6.19 (-3)	6.00 (-3)
	128	4.28 (-2)	2.88 (-2)	1.38 (-2)	6.81 (-3)	3.94 (-3)	3.22 (-3)	3.04 (-3)
	256	4.19 (-2)	2.75 (-2)	1.24 (-2)	5.35 (-3)	2.46 (-3)	1.74 (-3)	1.55 (-3)
512	4.14 (-2)	2.68 (-2)	1.17 (-2)	4.62 (-3)	1.73 (-3)	9.94 (-4)	8.08 (-4)	
$2^{-4}$	16	5.66 (-2)	4.67 (-2)	3.26 (-2)	2.72 (-2)	2.47 (-2)	2.38 (-2)	2.35 (-2)
	32	4.87 (-2)	3.65 (-2)	2.19 (-2)	1.58 (-2)	1.32 (-2)	1.23 (-2)	1.20 (-2)
$2^{-8}$	64	4.48 (-2)	3.14 (-3)	1.65 (-3)	1.00 (-2)	7.33 (-3)	6.40 (-3)	6.09 (-3)
	128	4.28 (-2)	2.88 (-2)	1.38 (-2)	7.13 (-3)	4.40 (-3)	3.44 (-3)	3.13 (-3)
	256	4.19 (-2)	2.75 (-2)	1.24 (-2)	5.68 (-3)	2.92 (-3)	1.96 (-3)	1.64 (-3)
	512	4.14 (-2)	2.68 (-2)	1.17 (-2)	4.95 (-3)	2.19 (-3)	1.22 (-3)	8.97 (-4)

$E(N, K_1, \varepsilon)$  is defined by (5.5), where  $z(x, t) = Z_1(x, t)$ ,  $u^*(x, t) = V_{(7.1)}(x, t)$ ,  $\bar{G}_h = \bar{G}_{h,1}(N, K_1, \varepsilon)$ .

Table 4. Table of errors  $E(N, K_2, \varepsilon)$  for scheme (6.7), (7.7)

$N$		8	16	32	64	128	256	512
$\varepsilon$	$K_2$							
1	8	2.14 (-3)	1.60 (-3)	1.47 (-3)	1.43 (-3)	1.42 (-3)	1.42 (-3)	1.42 (-3)
	16	1.11 (-3)	5.55 (-4)	4.15 (-4)	3.80 (-4)	3.71 (-4)	3.69 (-4)	3.36 (-4)
	32	8.37 (-4)	2.81 (-4)	1.41 (-4)	1.05 (-4)	9.66 (-5)	9.44 (-5)	9.38 (-5)
	64	7.68 (-4)	2.12 (-4)	7.07 (-5)	3.54 (-5)	2.66 (-5)	2.44 (-5)	2.38 (-5)
	128	7.51 (-4)	1.94 (-4)	5.30 (-5)	1.77 (-5)	8.88 (-6)	6.67 (-6)	6.07 (-6)
	256	7.46 (-4)	1.89 (-4)	4.85 (-5)	1.32 (-5)	4.38 (-6)	2.17 (-6)	1.62 (-6)
$2^{-2}$	8	1.57 (-2)	4.86 (-3)	1.99 (-3)	1.31 (-3)	1.14 (-3)	1.09 (-3)	1.08 (-3)
	16	1.50 (-2)	4.14 (-3)	1.24 (-3)	5.09 (-4)	3.32 (-4)	2.88 (-4)	2.78 (-4)
	32	1.48 (-2)	3.95 (-3)	1.05 (-3)	3.11 (-4)	1.28 (-4)	8.35 (-5)	7.26 (-5)
	64	1.48 (-2)	3.91 (-3)	1.00 (-3)	2.63 (-4)	7.78 (-5)	3.20 (-5)	2.09 (-5)
	128	1.48 (-2)	3.90 (-3)	9.90 (-4)	2.52 (-4)	6.60 (-5)	1.95 (-5)	7.99 (-6)
	256	1.48 (-2)	3.89 (-3)	9.87 (-4)	2.50 (-4)	6.31 (-5)	1.65 (-5)	4.86 (-6)
$2^{-4}$	8	4.09 (-2)	2.67 (-2)	1.20 (-2)	4.86 (-3)	1.99 (-3)	1.31 (-3)	1.14 (-3)
	16	4.09 (-2)	2.63 (-2)	1.13 (-2)	4.14 (-3)	1.24 (-3)	5.09 (-4)	3.32 (-4)
	32	4.09 (-2)	2.62 (-2)	1.11 (-2)	3.95 (-3)	1.05 (-3)	3.11 (-4)	1.28 (-4)
	64	4.09 (-2)	2.62 (-2)	1.10 (-2)	3.91 (-3)	1.00 (-3)	2.63 (-4)	7.78 (-5)
	128	4.09 (-2)	2.62 (-2)	1.10 (-2)	3.90 (-3)	9.90 (-4)	2.52 (-4)	6.68 (-5)
	256	4.09 (-2)	2.62 (-2)	1.10 (-2)	3.89 (-3)	9.87 (-4)	2.50 (-4)	6.31 (-5)
$2^{-6}$	8	4.09 (-2)	2.67 (-2)	1.20 (-2)	5.16 (-3)	2.46 (-3)	1.52 (-3)	1.22 (-3)
	16	4.09 (-2)	2.63 (-2)	1.13 (-2)	4.45 (-3)	1.69 (-3)	7.31 (-4)	4.16 (-4)
$2^{-8}$	32	4.09 (-2)	2.62 (-2)	1.11 (-2)	4.72 (-3)	1.50 (-3)	5.38 (-4)	2.14 (-4)
	64	4.09 (-2)	2.62 (-2)	1.10 (-2)	4.23 (-3)	1.46 (-3)	4.92 (-4)	1.66 (-4)
	128	4.09 (-2)	2.62 (-2)	1.10 (-2)	4.22 (-3)	1.45 (-4)	4.80 (-4)	1.55 (-4)
	256	4.09 (-2)	2.62 (-2)	1.10 (-2)	4.22 (-3)	1.45 (-3)	4.77 (-4)	1.52 (-4)

$E(N, K_2, \varepsilon)$  is defined by (5.5), where  $z(x, t) = Z_2(x, t)$ ,  $u^*(x, t) = V_{(7.1)}(x, t)$ ,  $\bar{G}_h = \bar{G}_{h,2}(N, K_2, \varepsilon)$ .

Notice, that the solution of boundary value problem (7.2) has one boundary layer at the left hand side. For its solution we use the locally condensed grid

$$\bar{G}_h^{(*)} = \bar{\omega}^{(*)} \times \bar{\omega}_0, \quad (7.3)$$

where  $\bar{\omega}^{(*)} = \bar{\omega}^{(*)}(\sigma)$  is a special mesh, condensed in the neighbourhood of left end interval  $[0, d]$ ;  $\sigma$  is the parameter depending on  $\varepsilon$  and  $N$ . The mesh  $\bar{\omega}^{(*)}(\sigma)$  is constructed as follows. The interval  $[0, d]$  is divided into two parts  $[0, \sigma]$ ,  $[\sigma, d]$ ,  $0 < \sigma \leq d/2$ . We divide each interval  $[0, \sigma]$ ,  $[\sigma, d]$  into  $N/2$  equal parts.

We take that  $\sigma = \min[d/2, m\varepsilon \ln N]$ , where  $m$  is an arbitrary positive number. Then for the solution of the discrete problem (3.2), (7.3) have the estimate:

$$|u(x, t) - z(x, t)| \leq M(N^{-1} \ln N + \tau), \quad (x, t) \in \bar{G}_h^{(*)}. \quad (7.4)$$

For the solution of the discrete problem (6.4), (7.3) the following estimate holds:

$$|u(x, t) - Z^{(n)}(x, t)| \leq M(N^{-1} \ln N + \tau), \quad (x, t) \in \bar{G}_h^{(*)^{(n)}}, \quad (7.5)$$

where

$$\bar{G}_h^{(*)^{(l)}} = \bar{\omega}^{(*)} \times \bar{\omega}_0^{(l)}, \quad l = 0, 1, \dots, n. \quad (7.6)$$

For the solution of the discrete problem (6.7), (7.3), where  $Z^{(n,0)}(x, t)$  is the solution of problem (6.4), (7.3), the following estimate holds:

$$|u(x, t) - Z^{(n,1)}(x, t)| \leq M(N^{-1} \ln N + \tau^2), \quad (x, t) \in \bar{G}_h^{(*)^{(n-1)}}. \quad (7.7)$$

For the solution of the discrete problem (6.10), (7.3), where  $Z^{(n,0)}(x, t)$  and  $Z^{(n,1)}(x, t)$  are the solutions of problems (6.7), (7.3) and (6.4), (7.3), respectively, the following estimate holds:

$$|u(x, t) - Z^{(n,2)}(x, t)| \leq M(N^{-1} \ln N + \tau^3), \quad (x, t) \in \bar{G}_h^{(*)^{(n-2)}}, \quad (7.8)$$

which is similar to the estimate (6.12).

Taking into account the easier notation we summarise (7.5), (7.7), and (7.8) as

$$|u(x, t) - Z_1(x, t)| \leq M(N^{-1} \ln N + \tau), \quad (x, t) \in \bar{G}_{h,1}, \quad (7.9)$$

$$|u(x, t) - Z_2(x, t)| \leq M(N^{-1} \ln N + \tau^2), \quad (x, t) \in \bar{G}_{h,2}, \quad (7.10)$$

$$|u(x, t) - Z_3(x, t)| \leq M(N^{-1} \ln N + \tau^3), \quad (x, t) \in \bar{G}_{h,3}. \quad (7.11)$$

The results for a set of numerical experiments are given in the Tables 3–5. From these tables we can see that the order of  $\varepsilon$ -uniform convergence with respect to  $\tau$  is better for scheme (6.7), (7.6) than that for scheme (6.4), (7.6), and the order for scheme (6.10), (7.6) is better than for scheme (6.7), (7.6). We see that the order of convergence with respect to  $\tau$  increases for the functions  $Z_k(x, t)$  for increasing  $k$ .

Because the space error is smaller for  $\varepsilon = 1$ , this case shows more clearly the effect of the defect correction in time. The function  $E(N, N_0, \varepsilon)$  for  $N = 512$  for  $Z_1$  decreases by a factor 2 when  $N_0$  is doubled, for  $Z_2$  it decreases by a factor 4, and for  $Z_3$  by a factor of 8. When the parameter  $\varepsilon$  decreases, a similar behaviour is observed, however for much larger values of the number  $N$ . To see this effect more clearly it would be necessary to extend Table 5 to values of  $N$  much larger than 512.

Table 5. Table of errors  $E(N, K_3, \varepsilon)$  for scheme (6.10), (7.7)

$N$		8	16	32	64	128	256	512
$\varepsilon$	$K_3$							
1	4	7.11 (-4)	1.54 (-4)	7.55 (-5)	8.05 (-5)	8.18 (-5)	8.21 (-5)	8.22 (-5)
	8	7.42 (-4)	1.84 (-4)	4.34 (-5)	1.17 (-5)	1.22 (-5)	1.23 (-5)	1.24 (-5)
	16	7.45 (-4)	1.88 (-4)	4.67 (-5)	1.13 (-5)	2.49 (-6)	1.68 (-6)	1.70 (-6)
	32	7.45 (-4)	1.88 (-4)	4.70 (-5)	1.17 (-5)	2.89 (-6)	6.38 (-7)	2.19 (-7)
	64	7.45 (-4)	1.88 (-4)	4.70 (-5)	1.17 (-5)	2.89 (-6)	6.82 (-7)	1.73 (-7)
	128	7.45 (-4)	1.88 (-4)	4.71 (-5)	1.17 (-5)	2.89 (-6)	6.87 (-7)	1.77 (-7)
$2^{-2}$	4	1.45 (-2)	3.59 (-3)	6.92 (-4)	3.15 (-4)	4.15 (-4)	4.45 (-4)	4.52 (-4)
	8	1.47 (-2)	3.85 (-3)	9.48 (-4)	2.09 (-4)	2.98 (-5)	4.97 (-5)	5.67 (-5)
	16	1.48 (-2)	3.89 (-3)	9.81 (-4)	2.43 (-4)	5.71 (-5)	1.08 (-5)	5.47 (-5)
	32	1.48 (-2)	3.89 (-3)	9.86 (-4)	2.48 (-4)	6.16 (-5)	1.49 (-5)	3.26 (-6)
	64	1.48 (-2)	3.89 (-3)	9.86 (-4)	2.49 (-4)	6.21 (-5)	1.55 (-5)	3.83 (-6)
	128	1.48 (-2)	3.89 (-3)	9.86 (-4)	2.49 (-4)	6.22 (-5)	1.56 (-5)	3.91 (-6)
$2^{-4}$	4	4.05 (-2)	2.58 (-2)	1.08 (-2)	3.60 (-3)	6.92 (-4)	3.13 (-4)	4.14 (-4)
	8	4.08 (-2)	2.61 (-2)	1.10 (-2)	3.85 (-3)	9.48 (-4)	2.09 (-4)	2.98 (-5)
	16	4.09 (-2)	2.62 (-2)	1.10 (-2)	3.89 (-3)	9.81 (-4)	2.43 (-4)	5.71 (-5)
	32	4.09 (-2)	2.62 (-2)	1.10 (-2)	3.89 (-3)	9.86 (-4)	2.48 (-4)	6.16 (-5)
	64	4.09 (-2)	2.62 (-2)	1.10 (-2)	3.89 (-3)	9.86 (-4)	2.49 (-4)	6.21 (-5)
	128	4.09 (-2)	2.62 (-2)	1.10 (-2)	3.89 (-3)	9.86 (-4)	2.49 (-4)	6.22 (-5)
$2^{-6}$	4	4.05 (-2)	2.58 (-2)	1.08 (-2)	3.91 (-3)	1.15 (-3)	2.22 (-4)	3.63 (-4)
	8	4.08 (-2)	2.61 (-2)	1.10 (-2)	4.18 (-3)	1.40 (-3)	4.37 (-4)	1.13 (-4)
$2^{-8}$	16	4.09 (-2)	2.62 (-2)	1.10 (-2)	4.21 (-3)	1.44 (-3)	4.71 (-4)	1.46 (-4)
	32	4.09 (-2)	2.62 (-2)	1.10 (-2)	4.21 (-3)	1.45 (-3)	4.76 (-4)	1.51 (-4)
	64	4.09 (-2)	2.62 (-2)	1.10 (-2)	4.22 (-3)	1.45 (-3)	4.76 (-4)	1.51 (-4)
	128	4.09 (-2)	2.62 (-2)	1.10 (-2)	4.22 (-3)	1.45 (-3)	4.76 (-4)	1.51 (-4)

The function  $E(N, K_3, \varepsilon)$  is defined by (5.5), where  $z(x, t) = Z_3(x, t)$ ,  $u^*(x, t) = V_{(7.1)}(x, t)$ ,  $\bar{G}_h = \bar{G}_{h,3}(N, K_3, \varepsilon)$ .

## Conclusion

For the parabolic boundary value problem (2.1), where a small parameter  $\varepsilon$  multiplies the highest derivative, we have analysed different approaches for the construction of discrete methods. We look for methods for which the accuracy of the discrete solution does not depend on the value of the small parameter, but only on the number of points in the discretisation ( $\varepsilon$ -uniform methods extended to non-uniform meshes). With a special, adapted, non-uniform mesh and a simple classical difference scheme, we are able to construct an  $\varepsilon$ -uniform approximation.

To illustrate the practical importance of our study, we show by a numerical example for a model problem that, on a uniform grid the classical difference scheme is not  $\varepsilon$ -uniformly convergent. In our example, the error is about 3% for any meshwidth in space and time that is smaller than  $1/128$ , and  $\varepsilon \leq 2^{-7}$ . The same example shows that we obtain an  $\varepsilon$ -uniformly convergent solution if we use the adapted mesh. Now the error is not larger than 0.5% for 127 mesh-points for the time and the space discretisation and for all  $\varepsilon \leq 1$ .

By defect correction we are able to increase the accuracy of the approximate solution essentially, i.e. from 1st to 2nd and 3rd order with respect to the time.

The total error of approximation consists of two parts. One part is the error due to the approximation of the space-variable and the second is the error due to the time-variable. Note that the defect correction process improves only the accuracy with respect to the time-variable and does not change the approximation with respect to the space-variable. Therefore, if we apply the defect correction technique, the principal part of the total error is the error due to the approximation of the space-variable.

Acknowledgement: This research was supported in part by the Dutch Research Organisation NWO, through grant IB 07-30-012 and by the Russian Foundation for Basic Research under grand N 95-01-00039.

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Received February 24, 1995, revised June 15, 1995, accepted October 6, 1995

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## Appendix

## 8.1 Estimates of the solution and its derivatives

As for Dirichlet problems for elliptic and parabolic equations [11, 13, 14], the estimate for the solution of problem (2.1) and its derivatives is derived.

We denote  $H^{(\alpha)}(\bar{G}) = H^{\alpha, \alpha/2}(\bar{G})$ , where  $\alpha$  is an arbitrary positive number [7]. We suppose that the functions  $f(x, t)$ , and  $\varphi(x, t)$  satisfy conditions at the corner points, so that the solution of the boundary value problem is smooth for every fixed value of the parameter  $\varepsilon$ . As mentioned in the introduction, such smoothness is indispensable for higher order accurate approximation.

For simplicity, we assume that on  $S_0 \cap \bar{S}_1$  the following conditions hold:

$$\frac{\partial^k}{\partial x^k} \varphi(x, t) = \frac{\partial^{k_0}}{\partial t^{k_0}} \varphi(x, t) = 0, \quad k + 2k_0 \leq [\alpha] + 2n, \quad \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} f(x, t) = 0, \quad k + 2k_0 \leq [\alpha] + 2n - 2, \quad (8.1)$$

where  $[\alpha]$  is the integer part of a number  $\alpha$ ,  $\alpha > 0$ ,  $n \geq 0$  is an integer number. We also suppose that  $[\alpha] + 2n \geq 2$ . The compatibility condition (8.1) provides suitable smoothness of the solution on  $\bar{G}$ , for a fixed value of the parameter  $\varepsilon$ , which is used in the following analysis.

Using interior a-priori estimates and estimates up to the boundary for the regular function  $\tilde{u}(\xi, t)$  [7], where  $\tilde{u}(\xi, t) = u(x(\xi), t)$ ,  $\xi = x/\varepsilon$ , we find for  $(x, t) \in \bar{G}$  the estimate

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} u(x, t) \right| \leq M\varepsilon^{-k}, \quad k + 2k_0 \leq 2n + 4, \quad n \geq 0. \quad (8.2)$$

This estimate holds, for example, for

$$u \in H^{(2n+4+\nu)}(\bar{G}), \quad \nu > 0, \quad (8.3)$$

where  $\nu$  is some small number.

For example, (8.3) is guaranteed for the solution of (2.1) if  $a \in H^{(\alpha+2n-1)}(\bar{G})$ ,  $c, p, f \in H^{(\alpha+2n-2)}(\bar{G})$ ,  $\varphi \in H^{(\alpha+2n)}(\bar{G})$ ,  $\alpha > 4$ ,  $n \geq 0$ , and condition (8.1) is fulfilled.

In fact, we need a more accurate estimate than (8.2). Therefore, we represent the solution of the boundary value problem (2.1) in the form of a sum,

$$u(x, t) = U(x, t) + W(x, t), \quad (x, t) \in \bar{G}, \quad (8.4)$$

where  $U(x, t)$  represents the regular part, and  $W(x, t)$  the singular part, i.e. the parabolic boundary layer. The function  $U(x, t)$  is the smooth solution of equation (2.1a) satisfying condition (2.1b) for  $t = 0$ . For example, under suitable assumptions for the data of the problem, we can consider the solution of the Dirichlet boundary value problem for equation (2.1a) smoothly extended to the set  $\bar{G}^*$ . ( $\bar{G}^*$  is a sufficiently large neighbourhood of  $\bar{G}$ .) On the set  $\bar{G}$  the coefficients and the initial value of the extended problem are the same as for the problem (2.1). Then the function  $U(x, t)$  is the restriction (on the set  $\bar{G}$ ) of the solution to the extended problem, and

$U \in H^{(2n+4+\nu)}(\bar{G})$ ,  $\nu > 0$ . The function  $W(x, t)$  is the solution of a boundary value problem for the parabolic equation

$$L_{(2.1)}W(x, t) = 0, \quad (x, t) \in G, \quad W(x, t) = u(x, t) - U(x, t), \quad (x, t) \in S. \tag{8.5}$$

If (8.3) is true then  $W \in H^{(2n+4+\nu)}(\bar{G})$ . Now, for the functions  $U(x, t)$  we derive the estimates

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} U(x, t) \right| \leq M, \tag{8.6}$$

and

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} W(x, t) \right| \leq M\epsilon^{-k} \exp(-m_{(8.7)}\epsilon^{-1}r(x, \gamma)), \quad (x, t) \in \bar{G}, \quad k + 2k_0 \leq 2n + 4, \tag{8.7}$$

where  $r(x, \gamma)$  is the distance between the point  $x \in \bar{D}$  and the set  $\gamma = \bar{D} \setminus D$ ,  $m_{(8.7)}$  is a sufficiently small, positive number. The estimates (8.6) and (8.7) hold, for example, when

$$U, W \in H^{(2n+4+\nu)}(\bar{G}), \quad \nu > 0. \tag{8.8}$$

The inclusions (8.8) are guaranteed if  $a \in H^{(\alpha+2n+1)}(\bar{G})$ ,  $c, p, f \in H^{(\alpha+2n)}(\bar{G})$ ,  $\varphi \in H^{(\alpha+2n)}(\bar{G})$ ,  $\alpha > 4$ ,  $n \geq 0$ , and condition (8.3) is fulfilled. We summarise these results in the following theorem.

**Theorem 8.1:** *Let for equation (2.1)  $a \in H^{(\alpha+2n+1)}(\bar{G})$ ,  $c, p, f \in H^{(\alpha+2n)}(\bar{G})$ ,  $\varphi \in H^{(\alpha+2n)}(\bar{G})$ ,  $\alpha > 4$ ,  $n \geq 0$ , and let the condition (8.3) be fulfilled. Then, for the solution,  $u(x, t)$ , of problem (2.1), and for its components in representation (8.4), the inclusion  $u, U, W \in H^{(4+2n)}(\bar{G})$  is fulfilled and the estimates (8.2), (8.6), (8.7) hold.*

The proof of the theorem is similar to the proof in [14], where the equation

$$\epsilon^2 a(x, t) \frac{\partial^2}{\partial x^2} u(x, t) - c(x, t) u(x, t) - p(x, t) \frac{\partial}{\partial t} u(x, t) = f(x, t)$$

was considered.

### 8.2 The proof of Theorem 6.1

Let us show that the function  $\delta_{\bar{t}z}(x, t)$ , where  $z(x, t) = z_{(6.4)}(x, t)$  is the solution of difference problem (6.4), approximates the function  $\delta_{\bar{t}u}(x, t)$   $\epsilon$ -uniformly. For simplicity we assume  $a(x, t)$  to be constant on  $\bar{G}$ .

The function  $\delta_{\bar{t}z}(x, t)$  is the solution of the difference problem

$$A_{(8.9)}\delta_{\bar{t}z}(x, t) = f_{(8.9)}(x, t), \quad (x, t) \in G_h^{[1]}; \quad \delta_{\bar{t}z}(x, t) = \varphi_{(8.9)}(x, t), \quad (x, t) \in S_h^{[1]}. \tag{8.9 a); (8.9 b)}$$

Here

$$\bar{G}_h^{[k]} = \bar{G}_h \cap \{t \geq k\tau\}, \quad G_h^{[k]} = G_h \cap \{t > k\tau\}, \quad S_h^{[k]} = \bar{G}_h^{[k]} \setminus G_h^{[k]}, \quad k \geq 1,$$

$$A_{(8.9)}\delta_{\bar{t}z}(x, t) \equiv \{\epsilon^2 a \delta_{\bar{t}\bar{t}} - \check{c}(x, t) - p_{\bar{t}}(x, t) - \check{p}(x, t) \delta_{\bar{t}}\} \delta_{\bar{t}z}(x, t),$$

$$f_{(8.9)}(x, t) = f_{\bar{t}}(x, t) + c_{\bar{t}}(x, t) z(x, t),$$

$$\varphi_{(8.9)}(x, t) = \varphi_{\bar{t}}(x, t), \quad x = 0, d, \quad (x, t) \in S_h^{[1]},$$

$$\varphi_{(8.9)}(x, t) = \varphi_{(8.9)}^0(x) \equiv \tau^{-1}[z(x, \tau) - \varphi(x, 0)], \quad t = \tau, \quad (x, t) \in S_h^{[1]},$$

$$\check{v}(x, t) = v(x, t - \tau).$$

The function  $\delta_{\bar{t}u}(x, t) \equiv [u(x, t) - u(x, t - \tau)]/\tau$ ,  $(x, t) \in \bar{G}$ ,  $t \geq \tau$ , is the solution of the differential problem

$$L_{(8.10)}\delta_{\bar{t}u}(x, t) = f_{(8.10)}(x, t), \quad (x, t) \in G_h^{[1]}; \quad \delta_{\bar{t}u}(x, t) = \varphi_{(8.10)}(x, t), \quad (x, t) \in S^{[1]}. \tag{8.10 a); (8.10 b)}$$

Here

$$\bar{G}^{[k]} = \bar{G} \cap \{t \geq k\tau\}, \quad G^{[k]} = G \cap \{t > k\tau\}, \quad S^{[k]} = \bar{G}^{[k]} \setminus G^{[k]}, \quad k \geq 1,$$

$$L_{(8.10)}\delta_{\bar{t}u}(x, t) \equiv \left\{ \epsilon^2 a \frac{\partial^2}{\partial x^2} - \check{c}(x, t) - p_{\bar{t}}(x, t) - \check{p}(x, t) \frac{\partial}{\partial t} \right\} \delta_{\bar{t}u}(x, t),$$

$$f_{(8.10)}(x, t) = f_{\bar{t}}(x, t) + c_{\bar{t}}(x, t) u(x, t) + p_{\bar{t}}(x, t) \left( \frac{\partial}{\partial t} u(x, t) - \delta_{\bar{t}u}(x, t) \right),$$

$$\varphi_{(8.10)}(x, t) = \varphi_{\bar{t}}(x, t), \quad x = 0, d, \quad (x, t) \in S^{[1]},$$

$$\varphi_{(8.10)}(x, t) = \varphi_{(8.10)}^0(x) \equiv \tau^{-1}[u(x, \tau) - \varphi(x, 0)], \quad t = \tau, \quad (x, t) \in S^{[1]}.$$

Let us estimate

$$\varphi_{(8.10)}^0(x) - \varphi_{(8.9)}^0(x) = \tau^{-1}\omega(x, \tau),$$

where

$$\omega(x, t) = u(x, t) - z(x, t), \quad (x, t) \in \bar{G}_h.$$

The function  $\omega(x, t)$  is the solution of the problem

$$A_{(6.4)}\omega(x, t) = (A_{(6.4)} - L_{(2.1)})u(x, t), \quad (x, t) \in G_h, \quad \omega(x, t) = 0, \quad (x, t) \in S_h.$$

The a-priori estimates of Theorem 8.1 lead to

$$|(A_{(6.4)} - L_{(2.1)})u(x, t)| \leq M[N^{-1} \ln N + \tau], \quad (x, t) \in G_h.$$

Using the maximum principle we estimate  $\omega(x, t)$ ,

$$|\omega(x, t)| \leq M[N^{-1} \ln N + \tau]t, \quad (x, t) \in \bar{G}_h.$$

Then we have

$$|\delta_{\bar{t}}u(x, \tau) - \delta_{\bar{t}}z(x, \tau)| = |\varphi_{(8.10)}^0(x) - \varphi_{(8.9)}^0(x)| \leq M[N^{-1} \ln N + \tau], \quad (x, t) \in S_h^{[1]}, \quad t = \tau,$$

i.e., the function  $\delta_{\bar{t}}z(x, \tau)$  approximates  $\delta_{\bar{t}}u(x, \tau)$   $\varepsilon$ -uniformly.

It is easy to see that difference problem (8.9) approximates the solution of the differential equation for the divided difference (8.10). Then we derive the estimate

$$|\delta_{\bar{t}}u(x, \tau) - \delta_{\bar{t}}z(x, t)| \leq M[N^{-1} \ln N + \tau], \quad (x, t) \in \bar{G}_h^{[1]}.$$

Let us show that under condition (6.8) the function  $\delta_{2\bar{t}}z(x, t)$  approximates the function  $\delta_{2\bar{t}}u(x, t)$   $\varepsilon$ -uniformly on the set  $\bar{G}_h^{[2]}$ . So, the functions  $\delta_{2\bar{t}}z(x, t)$  and  $\delta_{2\bar{t}}u(x, t)$  are solutions of equations

$$A_{(8.11)}\delta_{2\bar{t}}z(x, t) = f_{(8.11)}(x, t), \quad (x, t) \in G_h^{[2]}, \quad (8.11a)$$

$$L_{(8.12)}\delta_{2\bar{t}}u(x, t) = f_{(8.12)}(x, t), \quad (x, t) \in G_h^{[2]}. \quad (8.12a)$$

The equations are found by applying the operator  $\delta_{\bar{t}}$  to the equations (8.9.a), (8.10.a). At the left and the right boundary the following conditions are satisfied:

$$\delta_{2\bar{t}}z(x, t) = \varphi_{(8.11)}(x, t), \quad (x, t) \in S_h^{[2]}, \quad (8.11b)$$

$$\delta_{2\bar{t}}u(x, t) = \varphi_{(8.12)}(x, t), \quad (x, t) \in S_h^{[2]}, \quad (8.12b)$$

where

$$\left. \begin{aligned} \varphi_{(8.11)}(x, t) &= \varphi_{2\bar{t}}(x, t); & x = 0, d; & (x, t) \in S_h^{[2]}, \\ \varphi_{(8.11)}(x, t) &= \varphi_{(8.11)}^0(x) \equiv \delta_{2\bar{t}}z_{(6.4)}(x, t), & t = 2\tau, & (x, t) \in S_h^{[2]}, \end{aligned} \right\} \quad (8.11c)$$

$$\left. \begin{aligned} \varphi_{(8.12)}(x, t) &= \varphi_{2\bar{t}}(x, t); & x = 0, d; & (x, t) \in S^{[2]}, \\ \varphi_{(8.12)}(x, t) &= \varphi_{(8.12)}^0(x) \equiv \delta_{2\bar{t}}u(x, t), & t = 2\tau, & (x, t) \in S^{[2]}. \end{aligned} \right\} \quad (8.12c)$$

First we estimate

$$\varphi_{(8.12)}^0(x) - \varphi_{(8.11)}^0(x) = \delta_{2\bar{t}}u(x, t) - \delta_{2\bar{t}}z(x, t), \quad t = 2\tau.$$

For this purpose we write the function  $u(x, t)$  in a Taylor expansion for  $t$ ,

$$u(x, t) = a^{(1)}(x)t + a^{(2)}(x)t^2 + v_2(x, t) \equiv u^{[2]}(x, t) + v_2(x, t), \quad (x, t) \in \bar{G}, \quad (8.13)$$

where the coefficients  $a^{(1)}(x)$ ,  $a^{(2)}(x)$  are to be determined. Inserting  $u(x, t)$  in form (8.13) into equation (2.1a) we come to the systems

$$\begin{aligned} -p(x, 0)a^{(1)}(x) &= f(x, 0), \\ -2p(x, 0)a^{(2)}(x) + \varepsilon^2 a \frac{\partial^2}{\partial x^2} a^{(1)}(x) - \left( c(x, 0) - \frac{\partial}{\partial t} p(x, 0) \right) a^{(1)}(x) &= \frac{\partial}{\partial t} f(x, 0), \end{aligned}$$

from which the functions  $a^{(1)}(x)$ ,  $a^{(2)}(x)$  are found successively. The function  $v_2(x, t)$  is the solution of the boundary value problem

$$\left. \begin{aligned} L_{(2.1)}v_2(x, t) &= f_{(8.14)}(x, t) \equiv f(x, t) - L_{(2.1)}u^{[2]}(x, t), & (x, t) \in G, \\ v_2(x, t) &= \varphi_{(8.14)}(x, t) \equiv \varphi(x, t) - u^{[2]}(x, t), & (x, t) \in S. \end{aligned} \right\} \quad (8.14)$$

Estimating  $f_{(8.14)}(x, t)$ ,  $\varphi_{(8.14)}(x, t)$ , and using the maximum principle we derive the estimate

$$|v_2(x, t)| \leq Mt^3, \quad (x, t) \in \bar{G}. \quad (8.15)$$

Further we have to construct the function  $z(x, t)$  in the form

$$z(x, t) = (b_0^{(1)}(x) + b_1^{(1)}(x)\tau)t + b_0^{(2)}(x)t^2 + v_2^h(x, t) \equiv z^{[2]}(x, t) + v_2^h(x, t), \quad (x, t) \in \bar{G}_h,$$

i.e. as an expansion in powers of  $\tau$  and  $t$ . Inserting  $z(x, t)$  into equation (6.4), we arrive at the equations

$$\begin{aligned} -p(x, 0)b_0^{(1)}(x) &= f(x, 0), \\ -2p(x, 0)b_0^{(2)}(x) + \varepsilon^2 a \frac{\partial^2}{\partial x^2} b_0^{(1)}(x) - \left( c(x, 0) + \frac{\partial}{\partial t} p(x, 0) \right) b_0^{(1)}(x) &= \frac{\partial}{\partial t} f(x, 0), \\ b_0^{(2)}(x) + b_1^{(1)}(x) &= 0. \end{aligned}$$

So, we have

$$z_0^{[2]}(x, t) = u^{[2]}(x, t) + b_1^{(1)}(x) \tau t, \quad (x, t) \in \bar{G}_h. \quad (8.16)$$

The function  $v_2^h(x, t)$  is the solution of the discrete boundary value problem

$$\begin{cases} \mathcal{A}_{(6.4)} v_2^h(x, t) = f_{(8.17)}(x, t) \equiv f(x, t) - \mathcal{A}_{(6.4)} z^{[2]}(x, t), & (x, t) \in G_h, \\ v_2^h(x, t) = \varphi_{(8.17)}(x, t) \equiv \varphi(x, t) - z^{[2]}(x, t), & (x, t) \in S_h. \end{cases} \quad (8.17)$$

Taking into account estimates of the functions  $f_{(8.17)}(x, t)$ ,  $\varphi_{(8.17)}(x, t)$ , we derive the estimate

$$|v_2^h(x, t)| \leq M[N^{-1} \ln N + t] t^2, \quad (x, t) \in \bar{G}_h. \quad (8.18)$$

From relations (8.15), (8.16), (8.18) the inequality

$$|\varphi_{(8.12)}^0(x) - \varphi_{(8.11)}^0(x)| = |\delta_{2\bar{t}} u(x, t) - \delta_{2\bar{t}} z(x, t)| \leq M[N^{-1} \ln N + \tau], \quad (x, t) \in \bar{G}_h, \quad t = 2\tau, \quad (8.19)$$

follows.

We continue to estimating  $\delta_{2\bar{t}} u(x, t) - \delta_{2\bar{t}} z(x, t)$  for  $t > 2\tau$ , noting that the functions  $\delta_{2\bar{t}} u(x, t)$  and  $\delta_{2\bar{t}} z(x, t)$  are solutions of differential and difference equations, obtained from the equations (2.1) and (6.4), respectively, by applying the difference operator  $\delta_{2\bar{t}}$ . Moreover, the difference equation for  $\delta_{2\bar{t}} z(x, t)$  approximates the differential equation for  $\delta_{2\bar{t}} u(x, t)$   $\varepsilon$ -uniformly. On the boundary  $S_h$ , for  $x = 0$  or  $x = 1$ , we have  $\delta_{2\bar{t}} u(x, t) = \delta_{2\bar{t}} z(x, t)$ . Taking into account estimate (8.19) we find

$$|\delta_{2\bar{t}} u(x, t) - \delta_{2\bar{t}} z(x, t)| \leq M[N^{-1} \ln N + \tau], \quad (x, t) \in \bar{G}_h, \quad t \geq 2\tau. \quad (8.20)$$

Using estimate (8.20) we derive the estimate

$$|\mathcal{A}_{(6.4)}^{(n-1)}(u(x, t) - Z^{(n,1)}(x, t))| \leq M[N^{-1} \ln N + \tau^2], \quad (x, t) \in G_h^{(n-1)}.$$

This combines to

$$|u(x, t) - Z^{(n,1)}(x, t)| \leq M[N^{-1} \ln N + \tau^2], \quad (x, t) \in \bar{G}_h^{(n-1)}.$$

This completes the proof.

**Remark 1:** For the functions  $Z^{(n,k)}(x, t)$ ,  $(x, t) \in \bar{G}_h^{(n-k)}$ ,  $k \leq 0$ , the following estimate holds,

$$|\delta_{2\bar{t}} u(x, t) - \delta_{2\bar{t}} Z^{(n,k)}(x, t)| \leq M[N^{-1} \ln N + \tau], \quad (x, t) \in \bar{G}_h^{(n-k)}, \quad t \leq 2\tau^{(n-k)}. \quad (8.21)$$

**Remark 2:** Let the condition (6.11) hold. Then, similar to the derivation of the estimate (8.20), we can find

$$|\delta_{3\bar{t}} u(x, t) - \delta_{3\bar{t}} Z^{(n,0)}(x, t)| \leq M[N^{-1} \ln N + \tau], \quad (x, t) \in \bar{G}_h^{(n)}, \quad t \geq 3\tau^{(n)}. \quad (8.22)$$

We only show the few essential differences to the proof given above for (8.20). To estimate the difference between  $\delta_{3\bar{t}} u(x, t)$  and  $\delta_{3\bar{t}} z(x, t)$  for  $t = 3\tau$  we represent the function  $u(x, t)$  (with condition (6.11)) in the form

$$u(x, t) = a^{(2)}(x) t^2 + a^{(3)}(x) t^3 + v_3(x, t) \equiv u^{[3]}(x, t) + v_3(x, t), \quad (x, t) \in \bar{G},$$

and the function  $z(x, t)$  in the form

$$z(x, t) = u^{[3]}(x, t) + (b_1^{(1)}(x) \tau + b_2^{(1)}(x) \tau^2) t + b_1^{(2)}(x) \tau t^2 + v_3^h(x, t) \equiv z^{[3]}(x, t) + v_3^h(x, t), \quad (x, t) \in \bar{G}_h.$$

The coefficients of these expansions are found using the equations (2.1) and (6.4), respectively. For the coefficients we have the system

$$\begin{aligned} -p(x, 0) a^{(2)}(x) &= \frac{\partial}{\partial t} f(x, 0), \\ -3p(x, 0) a^{(3)}(x) + \varepsilon^2 a \frac{\partial^2}{\partial x^2} a^{(2)}(x) - \left( c(x, 0) + 2 \frac{\partial}{\partial t} p(x, 0) \right) a^{(2)}(x) &= 2^{-1} \frac{\partial}{\partial t} f(x, 0), \\ -b_1^{(1)}(x) + a^{(2)}(x) &= 0, \\ -2p(x, 0) b_1^{(2)}(x) + \frac{\partial}{\partial t} p(x, 0) a^{(2)}(x) + 3p(x, 0) a^{(3)}(x) + \left( -\frac{\partial}{\partial t} p(x, 0) - c(x, 0) \right) b_1^{(1)}(x) + \varepsilon^2 a \frac{\partial^2}{\partial x^2} b_1^{(1)}(x) &= 0, \\ -b_2^{(1)}(x) - a^{(3)}(x) + b_1^{(2)}(x) &= 0. \end{aligned}$$

The unknown functions  $a^{(2)}$ ,  $a^{(3)}$ ,  $b_1^{(1)}$ ,  $b_1^{(2)}$ ,  $b_2^{(1)}$  can be found successively. For the functions  $v_3(x, t)$  and  $v_3^h(x, t)$  the following estimates are derived:

$$|v_3(x, t)| \leq M t^4, \quad (x, t) \in \bar{G}, \quad |v_3^h(x, t)| \leq M[N^{-1} \ln N + t] t^3, \quad (x, t) \in \bar{G}_h.$$

For these inequalities and the expression for  $z^{[3]}(x, t)$  it follows that (8.22) holds  $\varepsilon$ -uniformly for  $t = 3\tau^{(n)}$ .

The remainder of the proof of the estimate (8.22) repeats with small variations the proof of the estimate (8.20).

### 8.3 The proof of Theorem 6.2

Notice that, if for the functions  $Z^{(n,0)}(x, t)$ ,  $Z^{(n,1)}(x, t)$  the following relations hold:

$$\left. \begin{aligned} |\delta_{3\bar{t}}^{(n)} u(x, t) - \delta_{3\bar{t}} Z^{(n,1)}(x, t)| &\leq M[N^{-1} \ln N + \tau], & (x, t) \in G_h^{(n)}, & t \geq 3\tau^{(n)}, \\ |\delta_{2\bar{t}}^{(n-1)} u(x, t) - \delta_{2\bar{t}} Z^{(n,1)}(x, t)| &\leq M[N^{-1} \ln N + \tau], & (x, t) \in G_h^{(n-1)}, & t \geq 2\tau^{(n-1)}, \end{aligned} \right\} \quad (8.23)$$

then for the difference  $u(x, t) - Z^{(n,2)}(x, t) \equiv \omega^{(n,2)}(x, t)$  the following inequalities hold:

$$|\mathcal{A}_{(6.4)}^{(n-2)} \omega^{(n,2)}(x, t)| \leq M[N^{-1} \ln N + \tau^3], \quad (x, t) \in G_h^{(n-2)},$$

$$\omega^{(n,2)}(x, t) = 0, \quad (x, t) \in S_h^{(n-2)}.$$

Hence we have

$$|u(x, t) - Z^{(n,2)}(x, t)| \leq M[N^{-1} \ln N + \tau^3], \quad (x, t) \in \tilde{G}_h^{(n-2)}.$$

Thus, for the proof of the theorem it is sufficient to show inequalities (8.23). These inequalities (8.23) follow from (8.21), (8.22).