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A minor-monotone graph parameter based on oriented matroids

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Abstract

For an undirected graph $G = (V,E)$ let $\lambda'(G)$ be the largest d for which there exists an oriented matroid M on V of corank d such that for each nonzero vector (x^+,x^-) of M, x^+ is nonempty and induces a connected subgraph of G.

We show that $\lambda'(G)$ is monotone under taking minors and clique sums. Moreover, we show that $\lambda'(G) \leqslant 3$ if and only if G has no K_5- or V_8-minor; that is, if and only if G arises from planar graphs by taking clique sums and subgraphs.

1. Introduction

In [5] the following invariant $\lambda(G)$ for a graph $G = (V,E)$ was introduced: $\lambda(G)$ is equal to the largest dimension of any linear subspace X of $\mathbb{R}^V$ with the property that for any nonzero $x \in X$ the graph $\langle \mathrm{supp}_+(x) \rangle$ induced by $\mathrm{supp}_+(x)$ is nonempty and connected. (Here $\mathrm{supp}_+(x)$ denotes the *positive support* of x; that is, the set $\{v \in V \mid x(v) > 0\}$. Similarly, $\mathrm{supp}_-(x)$ denotes the *negative support* of x; that is, the set $\{v \in V \mid x(v) < 0\}$. Moreover, for any $U \subseteq V$, $\langle U \rangle$ denotes the subgraph of G *induced by* U; that is, the subgraph with vertex set U and edges all edges of G contained in U. In this paper, all graphs are assumed to be simple.)

This graph parameter can be easily seen to be monotone under taking minors. That is, if G is a minor of H, then $\lambda(G) \leqslant \lambda(H)$. So for each natural number d the class of graphs G with $\lambda(G) \leqslant d$ is closed under taking minors.

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In [5] it is also shown that $\lambda(G) = \max\{\lambda(G_1), \lambda(G_2)\}$ if G is a *clique sum* of G_1 and G_2 (that is, arises by identifying two cliques of equal size in G_1 and G_2). It was shown that

(i) $\lambda(G) \leqslant 1$ if and only if G is a forest;
(ii) $\lambda(G) \leqslant 2$ if and only if G is series-parallel;
(iii) $\lambda(G) \leqslant 3$ if and only if G arises by taking subgraphs and clique sums from planar graphs. (1)

The function $\lambda(G)$ was motivated by the graph invariant $\mu(G)$ introduced by Colin de Verdière [2] (cf. [3]), although we do not know a relation between the two numbers. (It might be that $\lambda(G) \leqslant \mu(G)$ holds for each graph G.)

In the discussion after presenting the results above at the *5ème Colloque International Graphes et Combinatoire* in Marseille Luminy (September 1995), the first author of the present paper raised the question of extending these results to oriented matroids. The present paper shows that indeed most results of [5] are maintained under such an extension.

We first give the definition of oriented matroid (see [1] for background). To this end it is convenient to introduce, for any ordered pair $x = (a, b)$, the notation $x^+ := a$ and $x^- := b$.

Let $M = (V, X)$ be an oriented matroid, where X is the set of 'vectors' of M. That is, X is a collection of ordered pairs $x = (x^+, x^-)$ of subsets of V satisfying

(i) for each $x \in X$, $x^+ \cap x^- = \emptyset$;
(ii) $\mathbf{0} := (\emptyset, \emptyset) \in X$;
(iii) if $x \in X$ then $-x := (x^-, x^+) \in X$;
(iv) if $x, y \in X$, then $x \cdot y := (x^+ \cup (y^+ \setminus x^-), x^- \cup (y^- \setminus x^+)) \in X$;
(v) if $x, y \in X$ and $u \in x^+ \cap y^-$, then there exists a $z \in X$ such that $u \notin z^+ \cup z^-$, $(x^+ \setminus y^-) \cup (y^+ \setminus x^-) \subseteq z^+ \subseteq x^+ \cup y^+$, and $(x^- \setminus y^+) \cup (y^- \setminus x^+) \subseteq z^- \subseteq x^- \cup y^-$. (2)

The elements of X are called the *vectors* of the oriented matroid ($\mathbf{0}$ is the *zero*). Any linear subspace Y of $\mathbb{R}^V$ gives an oriented matroid (V, X), by taking $X := \{(\mathrm{supp}_+(x), \mathrm{supp}_-(x)) \mid x \in Y\}$.

For any oriented matroid $M = (V, X)$, the minimal nonempty subsets of $\{x^+ \cup x^- \mid x \in X\}$ form the circuit collection of a matroid, again denoted by M. Thus matroid terminology applies to oriented matroids. We give the concepts we need below, expressed in terms of the circuits of M.

The *rank* of a subset U of V is the size of a largest subset U' of U not containing a circuit of M. The *rank* $\mathrm{rank}(M)$ of M is the rank of V.

A *cobase* is a base of the dual matroid M^*; that is, it is an inclusionwise minimal subset intersecting each circuit of M. The *cospan* $\mathrm{cospan}(U)$ of a subset U of V is the set of elements $v \in V$ such that there is no circuit containing v and not intersecting U (so $U \subseteq \mathrm{cospan}(U)$). The *corank* $\mathrm{corank}(U)$ of a subset U of V is the size of a minimal subset U' of U such that $U \subseteq \mathrm{cospan}(U')$. A basic matroid theory formula is

$$\operatorname{corank}(U) = |U| + \operatorname{rank}(V \setminus U) - \operatorname{rank}(V). \tag{3}$$

The *corank* $\operatorname{corank}(M)$ of M is equal to $\operatorname{corank}(V)$, which is equal to $|V| - \operatorname{rank}(V)$.

Finally, we denote the deletion and contraction of U by $M \setminus U$ and M/U, respectively. In terms of oriented matroids, if $M = (V, X)$ is an oriented matroid and $U \subseteq V$, then $M \setminus U$ is the oriented matroid $(V \setminus U, X')$ with $X' := \{x \in X \mid (x^+ \cup x^-) \cap U = \emptyset\}$, and M/U is the oriented matroid $(V \setminus U, X'')$ with $X'' := \{(x^+ \setminus U, x^- \setminus U) \mid x \in X\}$.

We next describe our graph parameter based on oriented matroids. Let $G = (V, E)$ be an undirected graph. A *valid representation* for G is any oriented matroid $M = (V, X)$ with the property that for each nonzero $x \in X$, the subgraph $\langle x^+ \rangle$ of G induced by x^+ is nonempty and connected. Let $\lambda'(G)$ be the largest corank of any valid representation for G.

As each subspace of $\mathbb{R}^V$ gives an oriented matroid, with corank equal to the dimension of the subspace, we have for each graph G

$$\lambda(G) \leqslant \lambda'(G). \tag{4}$$

One of the consequences of this paper is that there are no graphs with $\lambda(G) \leqslant 3$ and $\lambda(G) < \lambda'(G)$. In fact, we do not know any graph G with strict inequality in (4).

2. λ' is minor-monotone

We now first show:

Theorem 1. *If G is a minor of H then $\lambda'(G) \leqslant \lambda'(H)$.*

Proof. Let $M = (V, X)$ be a valid representation of $G = (V, E)$ with $\operatorname{corank}(M) = \lambda'(G)$. If G arises from H by deleting an edge of G, then M is also a valid representation for H. So $\lambda'(H) \geqslant \operatorname{corank}(M) = \lambda'(G)$.

If G arises from H by contracting an edge $e = uv$ of H to vertex w of G, then replacing in any $x \in X$, x^+ by $(x^+ \setminus \{w\}) \cup \{u, v\}$ if $w \in x^+$, and similarly, x^- by $(x^- \setminus \{w\}) \cup \{u, v\}$ if $w \in x^-$, gives a valid representation M' for H, with $\operatorname{corank}(M') = \operatorname{corank}(M) = \lambda'(G)$. □

This theorem implies, by Robertson and Seymour's theorem [4], that for each fixed n there is a finite class $\mathscr{F}_n$ of graphs with the property that for any graph G: $\lambda'(G) \geqslant n$ if and only if G has no minor in $\mathscr{F}_n$.

We note that for the complete graph K_n one has

Theorem 2. $\lambda'(K_n) = n - 1$.

Proof. Let $M = (V, X)$ be a valid representation for K_n. If $\operatorname{corank}(M) = n$, then $\operatorname{rank}(M) = 0$, and therefore $\{v\}$ is a circuit for each $v \in V$. So $\{v\}$ contains $x^+ \cup x^-$

for some nonzero $x \in X$. This contradicts the fact that both x^+ and x^- are non-empty.

On the other hand, the set X of all pairs (U, W) with $U = \emptyset = W$ or $U \neq \emptyset \neq W$ and $U \cap W = \emptyset$, gives a valid representation for K_n of corank $n - 1$. □

So Hadwiger's conjecture implies the conjecture that $\gamma(G) \leqslant \lambda'(G) + 1$ for each graph G, where $\gamma(G)$ is the chromatic number of G. (Hadwiger's conjecture states that $\gamma(G) \leqslant n$ if G does not have any K_{n+1}-minor.)

It is useful to note:

Theorem 3. *If graph G' arises from graph G by deleting one vertex, then $\lambda'(G) \leqslant \lambda'(G') + 1$.*

Proof. Let $M = (V, X)$ be a valid representation for $G = (V, E)$, of corank $\lambda'(G)$. Let G' arise from G by deleting vertex v. Then the matroid $M' := M \backslash \{v\}$ obtained from M by deleting v is a valid representation for G'. Moreover $\mathrm{rank}(M') \leqslant \mathrm{rank}(M)$, and hence $\mathrm{corank}(M') = |V| - 1 - \mathrm{rank}(M') \geqslant |V| - 1 - \mathrm{rank}(M) = \lambda'(G) - 1$. □

3. Clique sums

In this section we show that the function $\lambda'(G)$ does not increase by taking clique sums, and from this we directly derive characterizations of the classes of graphs G satisfying $\lambda'(G) \leqslant 1$ and $\lambda'(G) \leqslant 2$. We first prove a lemma on oriented matroids.

Lemma 1. *Let $M = (V, X)$ be an oriented matroid and let $x, y \in X$ with $\emptyset \neq x^+ \subseteq y^-$ and $x \neq -y$. Then there is a nonzero $z \in X$ such that $z^+ \subseteq y^+$ and $x^+ \not\subseteq z^-$.*

Proof. Choose a nonzero $z \in X$ such that (i) $x^+ \not\subseteq z^-$, (ii) $z^+ \subseteq x^+ \cup y^+$, (iii) $x^- \backslash y^+ \subseteq z^- \subseteq x^- \cup y^-$, and (iv) $|y^+ \cup z^+|$ as small as possible. Such a z exists, since $z = x$ satisfies (i)–(iii).

Assume $z^+ \not\subseteq y^+$, and choose $u \in z^+ \backslash y^+$. So $u \in x^+$, and hence $u \in y^-$. Therefore, applying (2)(v) to y, z, there is a $z' \in X$ such that $u \notin z'^+ \cup z'^-$, $z'^+ \subseteq y^+ \cup z^+$, $z'^- \subseteq y^- \cup z^-$, $(y^+ \backslash z^-) \cup (z^+ \backslash y^-) \subseteq z'^+$, and $(z^- \backslash y^+) \cup (y^- \backslash z^+) \subseteq z'^-$. Then $x^+ \not\subseteq z'^-$ (as $u \notin z^-$), $z'^+ \subseteq y^+ \cup z^+ \subseteq x^+ \cup y^+$, $x^- \backslash y^+ \subseteq z^- \backslash y^+ \subseteq z'^- \subseteq y^- \cup z^- \subseteq x^- \cup y^-$, and $y^+ \cup z'^+ \subset y^+ \cup z^+$ (as $u \notin y^+ \cup z'^+$). Since $|y^+ \cup z^+|$ is minimal it follows that $z' = \mathbf{0}$. Hence $y^+ \subseteq z^-$, $z^+ \subseteq y^-$, $z^- \subseteq y^+$, and $y^- \subseteq z^+$. So $z = -y$, and therefore $y^- \subseteq x^+$ and $y^+ \subseteq x^-$. Moreover, $x^- \backslash y^+ \subseteq y^+$, and hence $x^- \subseteq y^+$. So $x = -y$, contradicting our assumption. □

The lemma is used to prove

Theorem 4. *Let $M = (V, X)$ be a valid representation for $G = (V, E)$ and let $y, z \in X$. If $y \neq -z$ then $\langle y^+ \cup z^+ \rangle$ is connected.*

Proof. Suppose $y \neq -z$ and $\langle y^+ \cup z^+ \rangle$ is disconnected. So y and z are nonzero and $y^+ \cap z^+ = \emptyset$. Consider $z \cdot y = (z^+ \cup (y^+ \setminus z^-), z^- \cup (y^- \setminus z^+))$. Since $\langle z^+ \cup (y^+ \setminus z^-) \rangle$ is connected, $y^+ \setminus z^- = \emptyset$, that is, $y^+ \subseteq z^-$. This implies by Lemma 1 that there is a nonzero $w \in X$ such that $w^+ \subseteq z^+$ and $y^+ \not\subseteq w^-$. Consider $w \cdot y = (w^+ \cup (y^+ \setminus w^-), w^- \cup (y^- \setminus w^+))$. Then w^+ is a nonempty subset of z^+ and $y^+ \setminus w^-$ is a nonempty subset of y^+, contradicting the fact that $\langle w^+ \cup (y^+ \setminus w^-) \rangle$ is connected. □

This theorem does not apply if $y = -z$. This case can be described as follows.

Theorem 5. *Let $M = (V, X)$ be a valid representation for $G = (V, E)$. Then for all $y \in X$ with $\langle y^+ \cup y^- \rangle$ not connected, there exist* corank(M) *pairwise openly vertex-disjoint paths connecting y^+ and y^-, except if* corank$(M) = 1$ *and y^+ and y^- are contained in different components of G.*

Proof. Suppose not. Then by Menger's theorem there exists a subset U of V such that y^+ and y^- are contained in different components of $G - U$ and such that $|U| < \text{corank}(M)$. By Theorem 4, $y^+ \cup y^-$ is the unique circuit of M contained in $V \setminus U$. (Indeed, by Theorem 4, for any nonzero $x \in X$ and $x^+ \cup x^- \subseteq V \setminus U$ one has $x \in \{y, -y\}$.) Therefore $\text{rank}(V \setminus U) = |V \setminus U| - 1$.

If $U = \emptyset$ then $\text{rank}(M) = |V| - 1$, and hence $\text{corank}(M) = 1$. If $U \neq \emptyset$, we can choose some $u \in U$. Let $x \in X$ be such that $x^+ \cup x^- \subseteq (V \setminus U) \cup \{u\}$. If $u \notin x^+$ then $x \in \{y, -y\}$ (by Theorem 4). Similarly, if $u \notin x^-$ then again $x \in \{y, -y\}$. Concluding, $y^+ \cup y^-$ is the unique circuit contained in $(V \setminus U) \cup \{u\}$ and hence $\text{rank}((V \setminus U) \cup \{u\}) = |V \setminus U|$. Hence $\text{rank}(M) \geqslant \text{rank}((V \setminus U) \cup \{u\}) = |V \setminus U|$. This contradicts the fact that $\text{rank}(M) = |V| - \text{corank}(M) = |V| - d < |V \setminus U|$. □

We use Theorems 4 and 5 to investigate the behaviour of $\lambda'(G)$ upon taking a 'clique sum', which is defined as follows. Let $G = (V, E)$ be a graph and let V_1 and V_2 be subsets of V such that $V = V_1 \cup V_2$, $K := V_1 \cap V_2$ is a clique in G and such that there is no edge connecting $V_1 \setminus K$ and $V_2 \setminus K$. Then G is called a *clique sum* of $G_1 := \langle V_1 \rangle$ and $G_2 := \langle V_2 \rangle$.

Theorem 6. *If G is a clique sum of G_1 and G_2 then $\lambda'(G) = \max\{\lambda'(G_1), \lambda'(G_2)\}$, except if $G = \bar{K}_2$.*

Proof. Since G_1 and G_2 are subgraphs of G, we have $\lambda'(G) \geqslant \max\{\lambda'(G_1), \lambda'(G_2)\}$. So it suffices to show that $\lambda'(G) = \lambda'(G_i)$ for $i = 1$ or 2. Assume that $\lambda'(G) > \max\{\lambda'(G_1), \lambda'(G_2)\}$. Let $d := \lambda'(G)$, $G = (V, E)$, $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, $K := V_1 \cap V_2$, and $t := |K|$. We may assume that we have chosen this counterexample so that t is as small as possible.

Then $\langle V_1 \setminus K \rangle$ has a component L such that each vertex in K is adjacent to at least one vertex in L. Otherwise G would be a repeated clique sum of subgraphs of G_1 and G_2 with common clique sum smaller than t. In that case $\lambda'(G) = \max\{\lambda'(G_1), \lambda'(G_2)\}$

would follow by the minimality of t. Concluding, G_1 has a K_{t+1}-minor, and therefore $\lambda'(G_1) \geqslant t$. Hence $\lambda'(G) > t$.

Let $M = (V, X)$ be a valid representation for G with corank$(M) = d$. There exists a nonzero $y \in X$ such that $y^+ \cup y^- \subseteq V \setminus K$ (otherwise rank$(M) \geqslant |V \setminus K|$, contradicting the fact that $d > t$).

By Theorem 5 both y^+ and y^- are contained in the same component of $G - K$. Hence we may assume that $y^+ \cup y^- \subseteq V_1 \setminus K$. Hence by Theorem 4 we have that there is no nonzero $x \in X$ with $x^+ \subseteq V_1 \setminus K$. So $M/(V_1 \setminus K)$ has corank equal to corank(M). Moreover, for each nonzero $x \in X$, $x^+ \cap V_2$ induces a nonempty connected subgraph of G_2. Hence $\lambda'(G_2) \geqslant$ corank$(M) = \lambda'(G)$, contradicting our assumption that $\lambda'(G_2) < d$. □

This theorem directly implies characterizations of those graphs G satisfying $\lambda'(G) \leqslant 1$ and $\lambda'(G) \leqslant 2$.

Corollary 6a. *For any graph G, $\lambda'(G) \leqslant 1$ if and only if G does not have a K_3-minor; that is, if and only if G is a forest.*

Proof. If $\lambda'(G) \leqslant 1$ then G has no K_3-minor, as $\lambda'(K_3) = 2$. Conversely, if G is a forest, then G arises by taking clique sums and subgraphs from the graph K_2. As $\lambda'(K_2) = 1$, Theorem 6 gives the corollary. □

Corollary 6b. *For any graph G, $\lambda'(G) \leqslant 2$ if and only if G does not have a K_4-minor; that is, if and only if G is a series–parallel graph.*

Proof. If $\lambda'(G) \leqslant 2$ then G has no K_4-minor, as $\lambda'(K_4) = 3$. Conversely, if G is a series–parallel graph, then G arises by taking clique sums and subgraphs from the graph K_3. As $\lambda'(K_3) = 2$, Theorem 6 gives the corollary. □

4. Graphs satisfying $\lambda'(G) \leqslant 3$

We characterize in this section the graphs G satisfying $\lambda'(G) \leqslant 3$. The main step consists in proving that $\lambda'(G) \leqslant 3$ if G is planar.

Theorem 7. *If G is planar then $\lambda'(G) \leqslant 3$.*

Proof. Suppose $G = (V, E)$ is a planar graph with $\lambda'(G) \geqslant 4$ and $|V|$ minimal. We assume that we have an embedding of G in the sphere. For each face f of G let V_f be the set of vertices incident with f. Note that G is 4-connected, since otherwise it would be a subgraph of clique sums of smaller planar graphs, and hence we would have $\lambda'(G) \leqslant 3$ by Theorem 6.

Let $M = (V, X)$ be a valid representation for G with $\text{corank}(M) \geqslant 4$. Then $\text{corank}(\{u\}) = 1$ for each $u \in V$; that is, u is contained in at least one circuit of M. Otherwise, we can delete u from G and M.

We may assume that, for each edge uv, $\text{corank}(\{u, v\}) = 2$; that is, there is a circuit containing u but not v. Otherwise, either for each $x \in X$ one has $u \in x^+ \Leftrightarrow v \in x^-$, in which case we can delete the edge $\{u, v\}$ from G, or for each $x \in X$ one has $u \in x^+ \Leftrightarrow v \in x^+$, in which case we can contract the edge $\{u, v\}$ in G and identify elements u and v in M.

Note that this implies that if f and f' are adjacent faces (that is, have an edge in common) and $\text{corank}(V_f) = 2 = \text{corank}(V_{f'})$, then $\text{cospan}(V_f) = \text{cospan}(V_{f'})$.

Fixing V we choose E maximal under the condition that $\text{corank}(\{u, v\}) = 2$ for each edge $\{u, v\}$. Then $\text{corank}(V_f) \in \{2, 3\}$ for each face f. Indeed, $\text{corank}(V_f) \geqslant 2$, as each edge e has $\text{corank}(e) \geqslant 2$. Moreover, if $\text{corank}(V_f) \geqslant 4$, V_f contains at least two nonadjacent vertices u, v with $\text{corank}(\{u, v\}) = 2$. This contradicts the maximality of E.

For $x \in X$ let $\mathscr{F}_x$ be the set of faces f for which $V_f \cap x^+ \neq \emptyset$ and $V_f \cap x^- \neq \emptyset$. Then:

Let f and f' be two faces with $\text{corank}(V_f \cup V_{f'}) \geqslant 4$.
Then there is an $x \in X$ with $f, f' \in \mathscr{F}_x$. (5)

As $\text{corank}(V_f) \geqslant 2$, $\text{corank}(V_{f'}) \geqslant 2$, and $\text{corank}(V_f \cup V_{f'}) \geqslant 4$, there exist $u, v \in V_f$, $u', v' \in V_{f'}$ with $\text{corank}(\{u, v, u', v'\}) = 4$. Therefore, we can find $x \in X$ such that $u, u' \in x^+$ and $v, v' \in x^-$. So $f, f' \in \mathscr{F}_x$, proving (5).

For $x \in X$ let $W_x := \bigcup \{V_f \mid f \in \mathscr{F}_x\}$. We show:

$$\text{corank}(W_x) \leqslant 3 \text{ for all } x \in X. \tag{6}$$

Note that (6) implies an immediate contradiction with (5), as $\text{corank}(V) \geqslant 4$.

We show that (6) holds. It suffices to show the result for $x \in X$ such that $x^+ \cup x^- = V$. (Indeed, if there exists $u \notin x^+ \cup x^-$, let $y \in X$ with $u \in y^+$ and set $z := x \cdot y$. Then, $z^+ \supseteq x^+ \cup \{u\}$, $z^- \supseteq x^-$ and $W_z \supseteq W_x$. Hence validity of the result for z will imply validity for x.)

Let $x \in X$ with $x^+ \cup x^- = V$ be given. Observe that if f and f' are faces with $\text{corank}(V_f) = \text{corank}(V_{f'}) = 2$ and having a common edge, e say, then $\text{cospan}(V_f) = \text{cospan}(V_{f'})$, as it is equal to $\text{cospan}(e)$. Similarly, $\text{cospan}(V_f) \subseteq \text{cospan}(V_{f'})$ if $\text{corank}(V_f) = 2$, $\text{corank}(V_{f'}) = 3$ and f, f' share a common edge.

As both $\langle x^+ \rangle$ and $\langle x^- \rangle$ are connected, the cut $\delta(x^+)$ corresponds in the dual graph of G to a circuit C which traverses exactly two edges in each face $f \in \mathscr{F}_x$.

Suppose, to obtain a contradiction, that $\text{corank}(W_x) \geqslant 4$. Then there exist faces $f, f' \in \mathscr{F}_x$ with $\text{corank}(V_f) = \text{corank}(V_{f'}) = 3$ and such that $\text{cospan}(V_f) \neq \text{cospan}(V_{f'})$. They correspond to two nodes on C. Denote by $f_1, \ldots, f_t$ the faces between f and f' when traveling from f to f' along C (in a given direction). Then we may assume that $\text{corank}(V_{f_i}) = 2$ for all $i = 1, \ldots, t$. For $i = 0, 1, \ldots, t$, let $u_i v_i$ be the edge common to the faces f_i and f_{i+1}, setting $f_0 := f$ and $f_{t+1} := f'$. So each $u_i v_i$ belongs to $\delta(x^+)$ (as G is 4-connected). We may assume that $u_i \in x^+$ and $v_i \in x^-$ for each i.

Now choose $w \in V_f \setminus \mathrm{cospan}(V_{f'})$ and $w' \in V_{f'} \setminus \mathrm{cospan}(V_f)$. Then the set $\{u_0, v_0, w, w'\}$ has corank 4. Hence, there exists $y \in X$ such that $w, w' \in y^+$ and $u, v \notin y^+ \cup y^-$. Hence, the set $y^+ \cup y^-$ contains none of the vertices on the faces $f_1, \dots, f_t$ (since $V_{f_i} \subseteq \mathrm{cospan}(\{u_0, v_0\})$ for all $i = 1, \dots, t$). In particular, $u_i, v_i \notin y^+ \cup y^-$ for $i = 1, \dots, t$. By connectivity of $\langle y^+ \rangle$ there exists a path P from w to w' which is entirely contained in y^+.

Consider the region $R := \bigcup_{i=0}^{t+1} f_i$ (where faces are assumed to be topologically closed). As P joins two nodes on the boundary of R, $R \cup P$ partitions the rest of the sphere into two regions R_1 and R_2. We choose indices such that R_1 has the vertices $u_0, \dots, u_t$ on its boundary, while R_2 has the vertices $v_0, \dots, v_t$ on its boundary.

By the connectivity of $\langle y^- \rangle$, y^- is contained either in $\bar{R}_1$ or in $\bar{R}_2$. Suppose first that y^- is contained in $\bar{R}_1$. Consider the element $z := y \cdot x$ of X. Then, $z^- \supseteq \{v_0, \dots, v_t\} \cup y^-$, while $u_0, \dots, u_t \in z^+$. Then there is no path joining v_0 and y^- which is entirely contained in z^-, contradicting the connectivity of $\langle z^- \rangle$.

Suppose next that y^- is contained in $\bar{R}_2$. Set $z := y \cdot (-x)$. Then we arrive similarly at a contradiction. □

We can now characterize the graphs G satisfying $\lambda'(G) \leqslant 3$. It follows from Theorems 6 and 7 that $\lambda'(G) \leqslant 3$ if G can be obtained from planar graphs by taking clique sums and subgraphs. On the other hand, it follows from a result by Wagner [6] that the graphs that can be obtained from planar graphs by taking clique sums and subgraphs are precisely the graphs with no K_5- or V_8-minor. (V_8 is the graph with vertices $v_1, \dots, v_8$, where v_i and v_j are adjacent if and only if $|i-j| \in \{1, 4, 7\}$.) It is shown in [5] that $\lambda(V_8) = 4$. Hence $\lambda'(V_8) \geqslant 4$. As deleting any vertex of V_8 gives a planar graph, Theorem 3 implies that $\lambda'(V_8) = 4$. Moreover, by Theorem 2 $\lambda'(K_5) = 4$. Therefore,

Theorem 8. *A graph G satisfies $\lambda'(G) \leqslant 3$ if and only if G has no K_5- or V_8-minor; that is, if and only if G can be obtained from planar graphs by taking clique sums and subgraphs.*

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