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Fibrations with indeterminates: contextual and functional completeness for polymorphic lambda calculi

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Lambek used categories with indeterminates to capture explicit variables in simply typed λ-calculus. He observed that such categories with indeterminates can be described as Kleisli categories for suitable comonads. They account for 'functional completeness' for cartesian (closed) categories.
Here we refine this analysis, by distinguishing 'contextual' and 'functional' completeness, and extend it to polymorphic λ-calculi. Since the latter are described as certain fibrations, we are lead to consider indeterminates, not only for ordinary categories, but also for fibred categories. Following a 2-categorical generalisation of Lambek's approach, such fibrations with indeterminates are presented as 'simple slices' in suitable 2-categories of fibrations; more precisely, as Kleisli objects. It allows us to establish contextual and functional completeness results for some polymorphic calculi.

1. Introduction

The aim of this paper is to present the construction of polynomial fibrations, with the study of contextual and functional completeness of polymorphic λ-calculi as a main application.

We recall the original formulation of functional completenesss for combinatory algebras, as described in Barendregt (1984). Given a combinatory algebra $(\mathscr{A}, \cdot)$, a λ-term M with (at most) one free variable x induces a function $[\![M]\!][\_/x] : \mathscr{A} \to \mathscr{A}$, which maps an element a to $[\![M]\!][a/x]$. Such functions are called *algebraic*. An arbitrary function $f : \mathscr{A} \to \mathscr{A}$ is *representable* if there exists an element $\ulcorner f \urcorner \in \mathscr{A}$ such that for every $a \in \mathscr{A}$, $f(a) = \ulcorner f \urcorner \cdot a$. Combinatorial or functional completeness asserts that every algebraic function is representable. Note that representability asserts an equation for every element (or constant) a. One could reformulate such a statement as an equation involving an 'arbitrary' or generic constant, which we will call an *indeterminate*. Then, the equation involving a particular constant a is obtained by instantiation (or substitution) of the generic one. This will be made precise below.

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Lambek and Scott (Lambek and Scott 1986) use the term 'functional completeness' in a somewhat different way. Their formulation says: given a simply typed λ-calculus $\mathscr{L}$, consider the simply typed λ-calculus $\mathscr{L}(c)$ obtained from $\mathscr{L}$ by freely adjoining a new constant $c:\sigma$, that is, its typing relation $\vdash_c$ extends that of $\mathscr{L}$, $\vdash$, by $\vdash_c c:\sigma$; terms of $\mathscr{L}(c)$ are 'polynomials' in c; and one has that for each term

$$\Gamma \vdash_c M:\tau$$

there is a unique (up to $=$) term $\llcorner M \lrcorner$ in the original calculus $\mathscr{L}$,

$$\Gamma, x:\sigma \vdash \llcorner M \lrcorner:\tau$$

such that

$$\Gamma \vdash_c M = \llcorner M \lrcorner[c/x]:\tau.$$

We understand this as saying that the role of the new constant c in $\mathscr{L}(c)$ can be taken over by an additional variable in the contexts of $\mathscr{L}$. Here, this property will be called *contextual completeness* of $\mathscr{L}$. Thus contextual completeness refers to the possibility of performing 'context extension' $\Gamma \mapsto \Gamma, x:\sigma$. Note that contextual completeness shows that the traditional logical view of variables as 'unspecified constants' agrees with their treatment as 'projections' or de Bruijn indices.

We think of functional completeness – in line with the above combinatorial description – as something different: for each $\Gamma \vdash_c M:\tau$ in $\mathscr{L}(c)$ there is a unique (up to $=$) $\Gamma \vdash \ulcorner M \urcorner:\sigma \to \tau$ in $\mathscr{L}$ with $\Gamma \vdash_c M = \ulcorner M \urcorner c:\tau$. That is, the type $\sigma \to \tau$ internalises polynomials in an indeterminate of type σ, and the term $\ulcorner M \urcorner$ captures the functional behaviour of the polynomial term M.

Contextual and functional completeness are useful tools for reasoning in internal languages, see, for example, Sections 5 and 6 in Lambek and Scott (1986, Part II). For instance, functional completeness can be used to get an equational presentation of coproducts in a cartesian closed category, involving only constant terms in the equations, *cf.* Prop. 8.1 in *ibid.* A more fundamental application of contextual and functional completeness is in Lambek and Scott (1986, Part 0, Theorem 11.3), which establishes the equivalence of simply typed λ-calculus and cartesian closed categories.

In this paper we study contextual and functional completeness for polymorphic λ-calculi $\lambda\!\to$ and $\lambda\omega$ (or F_ω). Terms in these calculi are written as

$$\Gamma \mid \Theta \vdash M:\tau,$$

where Γ is a context of type variables $\alpha:A$ in a *kind* A, Θ is a context of term variables $x:\sigma$ in a *type* σ and '|' is a non-logical symbol used to separate these contexts. In the calculus $\lambda\!\to$ (as we use it here) one has finite products $1, \sigma \times \tau$ and exponential types $\sigma \to \tau$. In $\lambda\omega$ one additionally has finite products $1, A \times B$ and exponential kinds $A \to B$ and universal quantification $\forall \alpha:A.\,\sigma$ of types over kinds.

For these polymorphic calculi one can distinguish contextual and functional completeness *for types* and *for kinds*. For types one assumes a new constant $c:\sigma$, where σ is a closed type (*i.e.*, $\vdash \sigma:\mathsf{Type}$). Contextual completeness for types means that for each term $\Gamma \mid \Theta \vdash_c M:\tau$ in a polymorphic calculus $\mathscr{P}(c)$ extended with c, there is a unique (up to =) term $\Gamma \mid \Theta, x{:}\sigma \vdash \llcorner M \lrcorner : \tau$ with $\Gamma \mid \Theta \vdash_c M = \llcorner M \lrcorner [c/x] : \tau$. For functional completeness for types one requires for such an M a unique (up to =) $\Gamma \mid \Theta \vdash \ulcorner M \urcorner : \sigma \to \tau$ with $\Gamma \mid \Theta \vdash_c M = \ulcorner M \urcorner\, c : \tau$.

In order to describe contextual and functional completeness for kinds, one assumes a pair of constants: $\kappa{:}A$ in a kind A and $c{:}\sigma[\kappa/\alpha]$ in a type σ containing a variable $\alpha{:}A$. On the one hand, contextual completeness for kinds expresses that for each term $\Gamma \mid \Theta \vdash_{\kappa,c} M{:}\tau$ in a polymorphic calculus $\mathscr{P}(\kappa, c)$ extended with these constants (κ, c), there are unique (up to =) $\llcorner \tau \lrcorner$ and $\llcorner M \lrcorner$ in,

$$\Gamma, \alpha{:}A \vdash \llcorner \tau \lrcorner : \mathsf{Type} \qquad \text{and} \qquad \Gamma, \alpha{:}A \mid \Theta, x{:}\sigma \vdash \llcorner M \lrcorner : \llcorner \tau \lrcorner$$

such that

$$\Gamma \vdash_\kappa \tau = \llcorner \tau \lrcorner [\kappa/\alpha] : \mathsf{Type} \qquad \text{and} \qquad \Gamma \mid \Theta \vdash_{\kappa,c} M = \llcorner M \lrcorner [\kappa/\alpha][c/x] : \tau.$$

On the other hand, functional completeness for kinds means that for each such term $\Gamma \mid \Theta \vdash_{\kappa,c} M{:}\tau$ there are unique (up to =)

$$\Gamma \vdash \ulcorner \tau \urcorner : A \to \mathsf{Type} \qquad \text{and} \qquad \Gamma \mid \Theta \vdash \ulcorner M \urcorner : \forall \alpha{:}A.\,(\sigma \to \ulcorner \tau \urcorner \alpha)$$

with

$$\Gamma \vdash_\kappa \tau = \ulcorner \tau \urcorner\, \kappa : \mathsf{Type} \qquad \text{and} \qquad \Gamma \mid \Theta \vdash_{\kappa,c} M = \ulcorner M \urcorner\, \kappa\, c : \tau$$

Note that Θ above is such that for any $x{:}\tau \in \Theta$, τ does not involve occurrences of κ. This is the role of the parameter c in the above expression of functional completeness: to internalise the dependence of M on κ, we must also internalise the dependence on those term variables whose types involve κ.

The presence of exponentials (or $\to$-) types in $\lambda\!\to$, guarantees functional completeness for types in the calculus $\lambda\!\to$, while in $\lambda\omega$ one has all the structure to express a functional completeness result for kinds.

Here we are concerned with categorical versions of such contextual and functional completeness results. Section 2 below will describe the basic notions in terms of polynomial categories $\mathbb{B}[c{:}I]$ having a generic constant $c{:}1 \to I$. This section serves to motivate the 2-categorical treatment of objects with an indeterminate, as well as the subsequent version of Lambek's identification of such objects with Kleisli objects for certain comonads in Section 3. This section recalls the relevant concepts involved, notably that of Kleisli object for a comonad in a 2-category, and develops their relevant basic properties to give the 2-categorical version of Lambek's result. Subsequently, we apply this abstract treatment to the appropriate 2-categories of fibrations to deal with the above-mentioned contextual/functional completeness for types and for kinds. In Section 4 we describe 'polynomial' fibrations $p[c{:}s]$, for fibrations over a given base, and study contextual and functional completeness for types for $\lambda\!\to$-fibrations, the categorical counterpart of the calculus $\lambda\!\to$. In Section 5 we describe the corresponding version of 'polynomial' fibrations $p[\langle\kappa, c\rangle{:}X]$ for fibrations over arbitrary bases, and show contextual and functional completeness for kinds for $\lambda\omega$-fibrations. The construction of $p[\langle\kappa, c\rangle{:}X]$ is an instance of the construction of Kleisli objects in the 2-category **Fib**, a main technical contribution of the paper. The details of this construction are given in Appendix B.

Finally, in Section 6 we mention further applications of fibrations with indeterminates and contextual/functional completeness and consider some lines for further research.

2. Contextual and functional completeness for cartesian closed categories

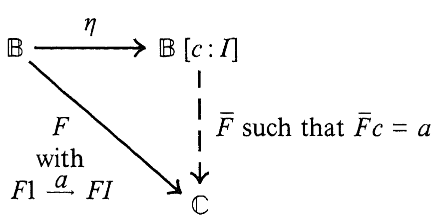
In this section we recall Lambek's functional completeness for cartesian closed categories. The technical tool required is the addition of an indeterminate to a category. After recalling this notion – both with an explicit syntactic description and a universal characterisation – we present a mild refinement of Lambek's analysis, distinguishing between *contextual* and *functional completeness*. A category with finite products enjoys the first kind of completeness, while a cartesian closed category is both contextually and functionally complete. This section contains essentially standard material in the way of motivation for the 2-categorical and fibrational versions to follow.

Given a category $\mathbb{B}$, with a terminal object 1, and an object $I \in \mathbb{B}$, we describe the polynomial category $\mathbb{B}[c:I]$ obtained by adding an extra morphism $c:1 \to I$ to $\mathbb{B}$ and characterise it in terms of a universal property. All this can be found in Lambek and Scott (1986).

Assume $\mathbb{B}$ is a category with structure, which is thought of as the interpretation of the calculus $\mathscr{L}$. $\mathbb{B}$ has thus a terminal object 1, which interprets the empty context. The construction of the category $\mathbb{B}[c:I]$ with an indeterminate global element c of an object $I \in \mathbb{B}$ (or, a generic constant of type I), which interprets the calculus $\mathscr{L}(c)$, proceeds as follows.

(i) Add an edge $c:1 \to I$ to the underlying graph $\mathscr{G}(\mathbb{B})$ of $\mathbb{B}$, obtaining a graph $\mathscr{G}(\mathbb{B})[c:I]$.
(ii) Form the free category with structure $\mathscr{F}(\mathscr{G}(\mathbb{B})[c:I])$ on this graph.
(iii) Make a suitable quotient of $\mathscr{F}(\mathscr{G}(\mathbb{B})[c:I])$ so that the inclusion $\mathscr{G}(\mathbb{B}) \hookrightarrow \mathscr{G}(\mathbb{B}[c:I])$ becomes a structure preserving functor $\eta:\mathbb{B} \to \mathbb{B}[c:I]$.

This category $\mathbb{B}[c:I]$, together with the functor $\eta:\mathbb{B} \to \mathbb{B}[c:I]$, is appropriately characterised by the following universal property. For each structure preserving functor $F:\mathbb{B} \to \mathbb{C}$ together with a morphism $(1 \cong)F1 \xrightarrow{a} FI$ in $\mathbb{C}$ there is an (up-to-isomorphism) unique structure preserving functor $\overline{F}:\mathbb{B}[x:I] \to \mathbb{C}$ with $\overline{F}\eta = F$ and $\overline{F}c = a$ in



The following definition captures contextual and functional completeness, as described in the previous section, categorically.

Definition 2.1. Let $\mathbb{B}$ be a category with a terminal object. We call $\mathbb{B}$

(i) *contextually complete* if for each $I \in \mathbb{B}$, the functor $\eta:\mathbb{B} \to \mathbb{B}[c:I]$ has a left adjoint;
(ii) *functionally complete* if each such η has a right adjoint.

Note that the above left and/or right adjoints are not required to preserve any specific structure.

The above definition gives a finer formulation of the structure required in $\mathbb{B}$ to interpret a simply typed theory: the terminal object 1 interprets the empty context and thus closed terms of type τ correspond to global elements $\mathbb{B}(1,\tau)$, identifying types with their interpretation in $\mathbb{B}$. Terms with a free variable $x{:}\sigma$ correspond to closed terms in $\mathbb{B}[x{:}\sigma]$. To interpret these terms in $\mathbb{B}$, we require contextual completeness. Note that the left adjoint $L \dashv \eta : \mathbb{B} \to \mathbb{B}[x{:}\sigma]$ determines a comonad on $\mathbb{B}$, which can be understood type-theoretically as performing 'context comprehension', as in Jacobs (1993a):

$$\Gamma \mapsto \Gamma, x{:}\sigma.$$

The counit of this comonad $\Gamma, x{:}\sigma \to \Gamma$ corresponds to *weakening* in the theory, and its comultiplication $\Gamma, x{:}\sigma \to \Gamma, x{:}\sigma, y{:}\sigma$ to *contraction*. This view of weakening and contraction as counit and comultiplication for a 'context comprehension comonad' can be generalised to the case of type dependency, via *comprehension categories*, *cf. ibid.*. This observation is applied in Jacobs and Streicher (1993) to give appropriate formulations of equality types in dependent type theory.

Note that contexts are inductively formed by context comprehension starting from the empty context. This is the reason to require finite products in $\mathbb{B}$: the above 'context comprehension' comonad is $\_ \times \sigma : \mathbb{B} \to \mathbb{B}$. A category with finite products is then contextually complete. Proposition 3.11 generalises this observation to cartesian objects in a 2-category.

To see that the above definition of contextual and functional completeness in categorical terms captures the type-theoretic version formulated in Section 1, we argue as follows. Via the equivalence between simply typed λ-calculi and cartesian closed categories, as in Lambek and Scott (1986, Part 0, Theorem 11.3), a calculus $\mathscr{L}$ has associated a ccc $\mathscr{C}(\mathscr{L})$ (its 'term model'), so that $\mathscr{L}$ can be recovered as the internal language of this category. $\mathscr{C}(\mathscr{L})$ has the types of $\mathscr{L}$ as objects, while a morphism $I \to J$ is an equivalence class of terms (modulo the equational theory of $\mathscr{L}$) $x{:}I \vdash M{:}J$, written $[M]$. The equivalence between calculi and ccc's holds also in the presence of indeterminates, *i.e.*, for an indeterminate $c{:}I$, $\mathscr{C}(\mathscr{L}(c)) \cong \mathscr{C}(\mathscr{L})[c{:}I]$, *cf.* Lambek and Scott (1986, Part 0, Proposition 11.2).

Proposition 2.2. A simply typed calculus $\mathscr{L}$ is contextually / functionally complete (as described in the introduction) iff its term model category $\mathscr{C}(\mathscr{L})$ is contextually / functionally complete (as in Definition 2.1).

Proof. Assuming $\mathscr{L}$ is contextually complete, we give a left adjoint to $\eta : \mathscr{C}(\mathscr{L}) \to \mathscr{C}(\mathscr{L}(c))$ (using the above identification of categories with an indeterminate $c{:}I$). We define this adjoint $\Sigma_I : \mathscr{C}(\mathscr{L}(c)) \to \mathscr{C}(\mathscr{L})$ as follows: on objects $\Sigma_I(J) = J \times I$. We have a morphism in $\mathscr{C}(\mathscr{L}(c))(J, J \times I)$, namely, $x{:}J \vdash_c \langle x, c\rangle{:}J \times I$. Given a term $x{:}J \vdash_c M{:}K$ (representing a morphism in $\mathscr{C}(\mathscr{L}(c))(J,K)$), we have by contextual completeness a (an equivalence class of) term $x{:}J, y{:}I \vdash \llcorner M \lrcorner{:}K$, unique such that

$$x{:}J \vdash_c \llcorner M \lrcorner[c/y] = M{:}K.$$

Notice that $[\llcorner M \lrcorner] \circ [\langle x, c\rangle] = [\llcorner M \lrcorner[c/y]]$. Thus we have established the universality of $J \times I$, as required. The argument for functional completeness is entirely analogous.

Conversely, if $\mathscr{C}(\mathscr{L})$ is contextually complete, the left adjoint $\Sigma_I \dashv \eta$ gives the desired correspondence between terms for $\mathscr{L}$ and $\mathscr{L}(c)$, as follows from the characterisation of $\mathscr{C}(\mathscr{L})[c{:}I]$ as the Kleisli category of the comonad $\_ \times I : \mathscr{C}(\mathscr{L}) \to \mathscr{C}(\mathscr{L})$ in Proposition 2.4 below. □

Similar statements relating the type-theoretic and categorical versions of contextual/ functional completeness for types and kinds (the latter as formulated in Sections 4 and 5 below) can be established, via the appropriate 'term model' of the polymorphic calculus involved. We omit the details, which are essentially the same as those given in the proof of the above proposition.

Categories with indeterminate elements are used in Jacobs (1993b) to describe datatypes with parameters. Contextual and functional completeness play an important role in this setting: the former guarantees the 'stability of terminal models' while the latter does the same with respect to initial models of datatypes. In particular, this stability applies to initial algebras for inductive datatypes and terminal coalgebras for coinductive ones. Such a stability property is required to define functions of several arguments by induction on one of them, for example, to get the usual schema of primitive recursion for a natural numbers object. We refer to *ibid.* for further details.

In order to describe contextual and functional completeness for categories with finite products, we introduce the following notation.

Notation 2.3. Let $\mathbb{B}$ be a category with binary products $\times$. Each object $I \in \mathbb{B}$ induces a comonad $\_ \times I : \mathbb{B} \to \mathbb{B}$ with counit and comultiplication at J given by $\pi : J \times I \to J$ and $\langle id, \pi' \rangle : J \times I \to (J \times I) \times I$, respectively.
The Kleisli category of this comonad is written as $\mathbb{B}/\!/I$ and called the *simple slice category* over I. Its objects are $X \in \mathbb{B}$ and its morphisms $X \to Y$ are morphisms $X \times I \to Y$ in $\mathbb{B}$. Such a morphism $f : X \times I \to Y$ can be seen as a family of maps $(f_i : X \to Y)$ parameterised by $i \in I$. For this Kleisli category, we write $F_I \dashv I^* : \mathbb{B} \to \mathbb{B}/\!/I$ for its associated resolution. The right adjoint I^* is given by $X \mapsto X$ and $f \mapsto f \circ \pi$, while the left adjoint is described by $X \mapsto X \times I$ and $f \mapsto \langle f, \pi' \rangle$.

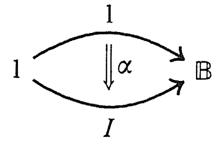
The following result is based on Lambek and Scott (1986, Part I, Proposition 7.1). It is an instance of the 2-categorical version of Propositions 3.11 and 3.12, so we omit the proof.

Proposition 2.4. For a category $\mathbb{B}$ with finite products, $\eta : \mathbb{B} \to \mathbb{B}[c{:}I]$ is $I^* : \mathbb{B} \to \mathbb{B}/\!/I$. Hence $\mathbb{B}$ is

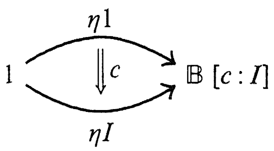
(i) contextually complete,
(ii) functionally complete if and only if it is cartesian closed.

In order to describe adjoining indeterminates to fibrations, we reformulate the universal property of $\eta : \mathbb{B} \to \mathbb{B}[c{:}I]$ more abstractly in 2-categorical terms.

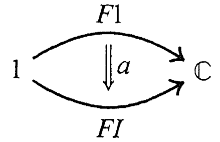
Recall that a global element $\alpha : 1 \to I$ in a category $\mathbb{B}$ is a 2-cell



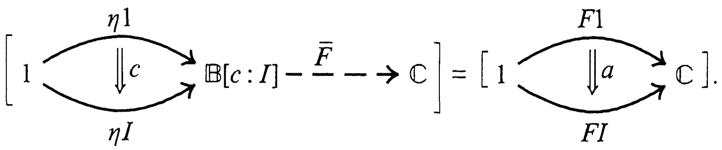
where **1** is the terminal category $\{*\}$ (one object, one arrow). Thus the above polynomial category comes equipped with a structure preserving 1-cell $\eta : \mathbb{B} \to \mathbb{B}[c:I]$ and a 2-cell



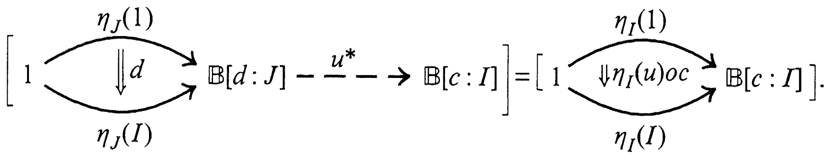
that is universal in the sense that for any structure preserving 1-cell $F : \mathbb{B} \to \mathbb{C}$ together with a 2-cell



there is a unique structure preserving 1-cell $\overline{F} : \mathbb{B}[c:I] \to \mathbb{C}$ with $\overline{F}\eta = F$ and



Remark 2.5. We notice that by the universal property of $\eta_I : \mathbb{B} \to \mathbb{B}[c:I]$, the assignment $I \mapsto \mathbb{B}[c:I]$ extends to a pseudo-functor $\mathbb{B}^{op} \to \mathbf{Cat}$. Explicitly, for a morphism $u : I \to J$ in $\mathbb{B}$, $u^* : \mathbb{B}[d:J] \to \mathbb{B}[c:I]$ arises in the following diagram



Proposition 2.4 implies that for a category $\mathbb{B}$ with finite products this functor is $I \mapsto \mathbb{B}/\!/I$. Applying the Grothendieck construction to this pseudo-functor $\mathbb{B}^{op} \to \mathbf{Cat}$, one obtains a fibration on $\mathbb{B}$. The resulting fibration is what we call the *simple fibration* and which we write as $\begin{smallmatrix} s(\mathbb{B}) \\ \downarrow \\ \mathbb{B} \end{smallmatrix}$.

Notice that contextual/functional completeness for $\mathbb{B}$ means that the reindexing functors $I^* : (B \cong)\mathbb{B}[*:1] \to \mathbb{B}[c:I]$ (induced by the unique morphisms $I : I \to 1$) have left/right adjoints. Using the simple fibration, we can formulate a 'stable' version of contex-

tual/functional completeness, which requires the existence of left/right adjoints for every reindexing functor arising from projections $\pi_{JI} : J\times I \to J$ (this amounts to requiring $\mathbb{B}[d{:}J]$ contextual/functional complete). In addition, we may require such left/right adjoints to satisfy the corresponding Beck–Chevalley condition, which means that every $\eta_J : \mathbb{B} \to \mathbb{B}[d{:}J]$ should preserve the adjunctions for contextual/functional completeness. In fact, any cartesian/cartesian closed category satisfies these stronger requirements of 'stable' contextual/functional completeness, as can be easily seen by inspection of the constructions involved.
This 'stable' version of contextual/functional completeness for $\mathbb{B}$, namely the requirement that the simple fibration have 'simple' coproducts/products as in the previous paragraph, is adopted by M. Hasegawa in Hasegawa (1994) as the semantic basis for his κ-/ ζ-calculus respectively. These calculi are introduced in *ibid.* in order to give a convenient syntax for categorical programming.

3. Comonads and Kleisli objects in a 2-category

We recall the concepts of comonad and its associated Kleisli object in a 2-category, following Street (1972). The reason for dealing with these concepts at a 2-categorical level is that we need to consider them not only in **Cat** (as in the previous section), but also in the 2-categories **Fib**($\mathbb{B}$) and **Fib**. For basic 2-categorical definitions see Kelly and Street (1974).

We denote an adjunction in a 2-category $\eta, \epsilon : f \dashv g : A \to B$, where $f : B \to A$ is left adjoint to $g : A \to B$. The 2-cells $\eta : 1 \Rightarrow gf$ and $\epsilon : fg \Rightarrow 1$ denote the unit and counit, respectively, and are often left implicit.

Definition 3.1. Given a 2-category $\mathscr{K}$, a *comonad* in it is a triple

$$\langle g : A \to A, \epsilon, \delta \rangle$$

where $\epsilon : g \Rightarrow 1_A$ and $\delta : g \Rightarrow g \circ g$ are called the *counit* and the *comultiplication*, respectively. The data must satisfy

$$g\epsilon \circ \delta = 1_g = \epsilon g \circ \delta \qquad \delta g \circ \delta = g\delta \circ \delta.$$

When $\mathscr{K} = \mathbf{Cat}$, we get the usual notion of a comonad. An adjunction $\eta, \epsilon : f \dashv u : A \to B$ in $\mathscr{K}$ generates a comonad $\langle fu : A \to A, \epsilon, f\eta u \rangle$. In this case $f \dashv u$ is a *resolution* for the comonad so generated (Lambek and Scott 1986, Part 0, Definition 6.4).

In order to define Kleisli objects for comonads in a 2-category $\mathscr{K}$, we organise comonads in $\mathscr{K}$ into a 2-category $\mathsf{Cmd}(\mathscr{K})$.

$\mathsf{Cmd}(\mathscr{K})$

objects comonads $\langle g : A \to A, \epsilon, \delta \rangle$ in $\mathscr{K}$

morphisms $(h,\theta) : \langle g{:}A \to A, \epsilon, \delta\rangle \to \langle g'{:}B \to B, \epsilon', \delta'\rangle$ consists of a morphism $h : A \to B$ and a 2-cell $\theta : g'h \Rightarrow hg$, displayed as

$$\begin{array}{ccc} A & \xrightarrow{\ h\ } & B \\ {\scriptstyle g}\Big\downarrow & \overset{\theta}{\Leftarrow} & \Big\downarrow{\scriptstyle g'} \\ A & \xrightarrow[\ h\ ]{} & A \end{array}$$

subject to the following two conditions:

$$h\epsilon \circ \theta = \epsilon' h \qquad h\delta \circ \theta = \theta g \circ g'\theta \circ \delta h$$

2-cells $\gamma : (h,\theta) \Rightarrow (h',\theta') : \langle g{:}A \to A, \epsilon, \delta\rangle \to \langle g'{:}B \to B, \epsilon', \delta'\rangle$ is a 2-cell $\gamma : h \Rightarrow h'$ compatible with the θ's, that is,

$$\gamma g \circ \theta = \theta' \circ g'\gamma.$$

Identities and composition are inherited from $\mathscr{K}$.

There is a (full and faithful) 2-functor $\Delta : \mathscr{K} \to \mathsf{Cmd}(\mathscr{K})$, which sends an object A to the identity comonad on it $\langle 1_A, 1, 1\rangle$.

Definition 3.2. A 2-category $\mathscr{K}$ *admits Kleisli objects for comonads* if $\Delta : \mathscr{K} \to \mathsf{Cmd}(\mathscr{K})$ has a left 2-adjoint,

$$\mathsf{Kl} \dashv \Delta : \mathsf{Cmd}(\mathscr{K}) \to \mathscr{K}.$$

In elementary terms, we have for a comonad $\langle g{:}A \to A, \epsilon, \delta\rangle$ in $\mathscr{K}$:

(i) An *oplax cocone* for $\langle g{:}A \to A, \epsilon, \delta\rangle$ is a morphism $(l,\sigma) : \langle g{:}A \to A, \epsilon, \delta\rangle \to \Delta(B)$. It amounts to a morphism $l{:}A \to B$ and a 2-cell $\sigma : l \Rightarrow lg$, satisfying

$$l\epsilon \circ \sigma = 1_l \qquad \sigma g \circ \sigma = l\delta \circ \sigma$$

B is called the *vertex* of the cocone.

(ii) A *Kleisli object* for $\langle g{:}A \to A, \epsilon, \delta\rangle$ is an initial oplax cocone $(u{:}A \to A_g, \lambda)$. It is characterised by the following isomorphism of categories

$$\mathscr{K}(A_g, B) \cong \mathsf{Cmd}(\mathscr{K})(\langle g{:}A \to A, \epsilon, \delta\rangle, \Delta(B)).$$

We refer to the object A_g itself as the Kleisli object.

Remarks 3.3.

(i) The above isomorphism means that, given an oplax cocone $(l{:}A \to B, \sigma)$, there is a unique morphism $\overline{(l,\sigma)}{:}A_g \to B$ such that $\overline{(l,\sigma)} \circ u = l$ and $\overline{(l,\sigma)}\lambda = \sigma$. The 2-dimensional aspect means that given a 2-cell $\gamma : (l,\sigma) \Rightarrow (l',\sigma')$, there is a unique 2-cell $\bar{\gamma} : \overline{(l,\sigma)} \Rightarrow \overline{(l',\sigma')}$ such that $\bar{\gamma} u = \gamma$.

(ii) Any resolution $f \dashv u{:}A \to B$ for the comonad g induces an oplax cocone $(u, \eta u)$. As a partial converse, the oplax colimit (u, λ) is such that u has a left adjoint f and the adjunction $f \dashv u$ generates the comonad. See Street (1972) for details.

Recall that in **Cat**, the Kleisli category $\mathbb{A}_G$ for a comonad G on $\mathbb{A}$ has the same objects as $\mathbb{A}$ and has hom-sets $\mathbb{A}_G(X,Y) = \mathbb{A}(GX,Y)$. Identities are given by instances of ϵ. For $f:GX \to Y$ and $g:GY \to Z$, their composite is $g \circ Gf \circ \delta_X$. There is an adjunction, written $\eta,\epsilon : F_G \dashv U_G : \mathbb{A} \to \mathbb{A}_G$ that generates G. The induced oplax cocone $(U_G, \eta U_G)$ is an oplax colimit: given an oplax cocone $(L:\mathbb{A} \to \mathbb{B}, \sigma)$, $\overline{(L,\sigma)}:\mathbb{A}_G \to \mathbb{B}$ is given by $\overline{(L,\sigma)}(f:GX \to Y) = Lf \circ \sigma_X : LX \to LY$, and a morphism $\gamma : (L,\sigma) \Rightarrow (L',\sigma')$, induces $\overline{\gamma}_X = \gamma_X : \overline{(L,\sigma)}X \to \overline{(L',\sigma')}X$.

In dealing with categories and fibrations that model certain calculi, we are interested in some kind of structure, for example, cartesian closure. Quite often, such structure is specified in terms of adjunctions, for example, right adjoints in the case of a ccc. When adding indeterminates to such objects, given by Kleisli objects for suitable comonads on them, we are interested in preserving such structure, *i.e.*, the Kleisli object should have it and the morphism from the original object into it should be structure preserving.

By 2-functoriality, $\mathsf{Kl} : \mathsf{Cmd}(\mathscr{K}) \to \mathscr{K}$ takes adjunctions in $\mathsf{Cmd}(\mathscr{K})$ to adjunctions in $\mathscr{K}$ between the Kleisli objects for the comonads involved. This fact leads to the following adjoint lifting theorem. It yields a uniform way of transferring structure, given by adjoints, to Kleisli objects.

Theorem 3.4. Given comonads $\langle g:A \to A, \epsilon, \delta\rangle$ and $\langle g':B \to B, \epsilon', \delta'\rangle$ and a morphism between them

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle g}\downarrow & \overset{\theta}{\Leftarrow} & \downarrow{\scriptstyle g'} \\ A & \xrightarrow[h]{} & A \end{array}$$

(i) If θ is an isomorphism and h has right adjoint, $\underline{\eta},\underline{\epsilon} : h \dashv r : B \to A$, this adjunction induces an adjunction

$$\overline{\eta},\overline{\epsilon} : \overline{h} \dashv \overline{r} : A_g \to B_{g'}$$

such that

$$\begin{array}{ccccc} A_g & \xrightarrow{\overline{h}} & B_{g'} & \xrightarrow{\overline{r}} & A_g \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle u} \\ A & \xrightarrow[h]{} & B & \xrightarrow[r]{} & A \end{array}$$

and $u\eta = \overline{\eta}u$ *i.e.* $(u : A \to A_g, u' : B \to B_{g'})$ is a (strict) map of adjunctions in the sense of Mac Lane (1971, Section IV.7).

(ii) If h has a left adjoint $\underline{\eta},\underline{\epsilon} : l \dashv h : A \to B$ such that the adjoint mate of θ, $\overline{\epsilon}gl \circ l\theta l \circ lg'\overline{\eta}$, is an isomorphism, this adjunction induces an adjunction

$$\overline{l} \dashv \overline{h} : B_{g'} \to A_g$$

such that $(u : A \to A_g, u' : B \to B_{g'})$ is a map of adjunctions.

Proof. The argument for both cases is entirely similar, so we only prove (i). The adjoint mate of θ^{-1}, $\phi = \underline{\eta} gr \circ r\theta^{-1} r \circ rg' \underline{\epsilon} : gr \Rightarrow rg'$, makes

$$(r, \phi) : \langle g' : B \to B, \epsilon', \delta' \rangle \to \langle g : A \to A, \epsilon, \delta \rangle$$

a morphism in $\mathsf{Cmd}(\mathscr{K})$ right adjoint to (h, θ) in $\mathsf{Cmd}(\mathscr{K})$. Hence, the 2-functor Kl applied to this adjunction gives the desired adjunction

$$\bar{h} \dashv \bar{r} : A_g \to B_{g'}.$$

The fact that (u, u') become a map of adjunctions follows from the fact the oplax cocone of a Kleisli object is the (instance of the) unit of the 2-adjunction $\Delta \dashv \mathsf{Kl}$, which is a 2-natural transformation. □

An alternative formulation of the above theorem can be given as a 'lifting' of adjunctions. We spell out the right-adjoint case in the following corollary.

Corollary 3.5. For comonads $(g : A \to A, \epsilon, \delta)$ and $(g' : B \to B, \epsilon', \delta')$ in $\mathscr{K}$, let $\eta, \rho : f \dashv u : A \to A_g$ and $\eta', \rho' : f' \dashv u' : B \to B_{g'}$ be their associated Kleisli resolutions. Consider a commutative diagram

$$\begin{array}{ccc} A_g & \xrightarrow{\bar{h}} & B_{g'} \\ {\scriptstyle u}\uparrow & & \uparrow{\scriptstyle u'} \\ A & \xrightarrow[h]{} & A \end{array}$$

such that the 2-cell $\tau = \rho' h \circ \bar{h}\eta : f'\bar{h} \Rightarrow hf$ is an isomorphism. Then, if h has a right adjoint $h \dashv r : B \to A$, $\bar{h}$ has a right adjoint $\bar{r}$ and (u, u') is a map of adjunctions from $h \dashv r$ to $\bar{h} \dashv \bar{r}$

Proof. It follows from Theorem 3.4.(i) by the equivalence of the following two statements

(i) τ is an isomorphism
(ii) There is an isomorphism (2-cell) $\theta : g'h \Rightarrow hg$ such that (h, θ) is a morphism of comonads from g to g' (with $\bar{h} = \mathsf{Kl}(h, \theta) : A_g \to B_{g'}$)

To show these statements equivalent, let (u, λ) and (u', λ') be the Kleisli objects for $(g : A \to A, \epsilon, \delta)$ and $(g' : B \to B, \epsilon', \delta')$, respectively.

(i)$\Longrightarrow$(ii) $(h, \tau u)$ is a moprhism of comonads since $(h, \bar{h})$ with 1 and τ form a pseudo-map of adjunctions, as in Kelly and Street (1974).

(ii)$\Longrightarrow$(i) Since $g = fu$ and $g' = f'u'$, $\theta : f'\bar{h}u \Rightarrow hfu$ is a 2-cell between $(f'\bar{h}u, f'\bar{h}\lambda)$ and $(hfu, hf\lambda)$, which induces $\tau : f'\bar{h} \Rightarrow hf$. Hence τ is an isomorphism, since θ is.

□

It is worth spelling out the lifted right adjoint $\bar{r} : \mathbb{B}_{g'} \to \mathbb{A}_g$ in **Cat**, the case used most often. Given $\eta, \rho : h \dashv r : \mathbb{B} \to \mathbb{A}$ we have

$$\begin{aligned} \bar{r}X &= rX \\ \bar{r}(f : G'X \to Y) &= rf \circ \phi_X^{-1} : grX \to rY, \end{aligned}$$

where $\phi = rg'\rho \circ r\theta^{-1} r \circ \eta gr : gr \Rightarrow rg'$ is the adjoint mate of $\theta^{-1} : hg \Rightarrow g'h$, as given in the proof of the above theorem.

From the lifting of adjoints presented in the above corollary, we can easily infer that these lifted adjoints would be preserved by morphisms induced by oplax cocones, provided the latter preserve the original adjunctions. We make this precise in the following corollary, for the right-adjoint version we use in the sequel.

Corollary 3.6. Consider a morphism of comonads

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle g}\downarrow & \overset{\theta}{\Leftarrow} & \downarrow{\scriptstyle g'} \\ A & \xrightarrow[h]{} & A \end{array}$$

with θ an isomorphism, where h has right adjoint, $\eta, \zeta : h \dashv r : B \rightarrow A$. Let there be given oplax cocones $(l : A \rightarrow C, \gamma : l \Rightarrow lg)$ and $(k : B \rightarrow D, \rho : k \Rightarrow kg')$ for g and g', respectively, with an adjunction $\underline{\eta}, \underline{\zeta} : h' \dashv r' : D \rightarrow C$, and an isomorphism χ, as shown below,

$$\begin{array}{ccc} A & \xrightarrow{l} & C \\ {\scriptstyle h}\downarrow & \overset{\chi}{\Rightarrow} & \downarrow{\scriptstyle h'} \\ B & \xrightarrow[k]{} & D \end{array}$$

satisfying $\chi g \circ h'\gamma = k\theta \circ \rho h \circ \chi$. Then, we have an induced isomorphism.

$$\begin{array}{ccc} A_g & \xrightarrow{\bar{l}} & C \\ {\scriptstyle \bar{h}}\downarrow & \overset{\bar{\chi}}{\Rightarrow} & \downarrow{\scriptstyle h'} \\ B_{g'} & \xrightarrow[\bar{k}]{} & D \end{array}$$

Furthermore, if the adjoint mate of χ, $r'k\zeta \circ r'\chi r \circ \underline{\eta} hr$ is an isomorphism, so is the adjoint mate of $\bar{\chi}$, $r'\overline{k\zeta} \circ r'\overline{\chi r} \circ \underline{\eta}\overline{hr}$

Proof. The compatibility condition

$$\chi g \circ h'\gamma = k\theta \circ \rho h \circ \chi$$

means that χ is a 2-cell from the morphism $(f'h, f'\gamma) : g \rightarrow D$ to $(kh, k\theta \circ \rho h) : g \rightarrow D$ in $\mathsf{Cmd}(\mathscr{K})$, which thereby induces $\bar{\chi}$ by universality of A_g. The remaining statement is verified by simple diagram chasing (or rather, pasting). □

3.1. *Objects with an indeterminate*

Given a category $\mathbb{B}$ with a terminal object 1, and any object I of $\mathbb{B}$, we recalled in Section 2 the universal property of $\mathbb{B}[x:I]$, the category with an indeterminate element of 'type' I. We also mentioned that, when $\mathbb{B}$ has finite products, $\mathbb{B}[x:I]$ could be presented as a Kleisli category. We now give the 2-categorical version of this result.

First, we must reformulate the 'category with an indeterminate' concept in a 2-category. Since we are interested in cartesian objects, we give a formulation of 'cartesian objects with an indeterminate'.

Definition 3.7. Let $\mathscr{K}$ be a 2-category with finite products. Let B be a cartesian object of $\mathscr{K}$ and let $i : 1 \to B$ be a global element. The *cartesian object with an i-indeterminate* $B[x:i]$ is a cartesian object together with a morphism $\eta : B \to B[x:i]$ that preserves finite products and a 2-cell $x : \eta 1 \Rightarrow \eta i : 1 \to B[x:i]$ with the following universal property: given a cartesian object C, a finite product preserving morphism $f : B \to C$ and a 2-cell $\alpha : f1 \Rightarrow fi$, there is a unique finite product preserving morphism $\overline{(f,\alpha)} : B[x:i] \to C$ such that $\overline{(f,\alpha)}\eta = f$ and $\overline{(f,\alpha)}x = \alpha$. Further, given any other such pair (f',α') and a 2-cell $\gamma : f \Rightarrow f'$, there is a unique 2-cell $\overline{\gamma} : \overline{(f,\alpha)} \Rightarrow \overline{(f',\alpha')}$ such that $\overline{\gamma}\eta = \gamma$.

Now, we want to show that if $\mathscr{K}$ admits Kleisli objects for comonads, *i.e.*, if the appropriate oplax colimits exist, the Kleisli object $B_{\_\otimes i}$ for the comonad $\_ \otimes i$ in the following definition.

Definition 3.8. Let A be a cartesian object in $\mathscr{K}$, with $\eta' \dashv \epsilon' : \delta_A \to \otimes$. A global element $i : 1 \to A$ induces a comonad $\langle g_i : A \to A, \epsilon, \delta \rangle$, where

(i) $g_i = \otimes \langle 1_A, i!_A \rangle$

(ii) $\epsilon = p_{1_A, i!_A} (= \pi\epsilon' \langle 1_A, i!_A \rangle) : g_i \Rightarrow 1_A$

(iii) $\delta = \langle\!\langle 1_{g_i}, q\langle 1_A, i!_A \rangle \rangle\!\rangle g_i 1_A (= \otimes \langle 1_\otimes, \pi'\epsilon' \rangle \langle 1_A, i!_A \rangle \circ \eta' \otimes \langle 1_A, i!_A \rangle) : g_i \Rightarrow g_i^2$.

The verification of the comonad laws proceeds by 2-categorical pasting. The above definition generalises the fact that an object I of a category $\mathbb{A}$ with finite products induces a comonad $\_ \times I$ on it. Notice that to give a counit and a comultiplication making $\_ \times I$ a comonad is equivalent to giving a comonoid structure on I. The latter corresponds, type-theoretically, to proofs of weakening and contraction at 'type' I.

We must show, among other facts, that $B_{\_\otimes i}$ is cartesian. In **Cat**, this follows from the fact that the Kleisli category $\mathbb{B}_G$, for a comonad G on $\mathbb{B}$ with finite products, has finite products. Consider objects X and Y in $\mathbb{B}_G$, then

$$\mathbb{B}_G(Z, X \times Y) \cong \mathbb{B}(GZ, X \times Y) \cong \mathbb{B}(GZ, X) \times \mathbb{B}(GZ, Y) \cong \mathbb{B}_G(Z, X) \times \mathbb{B}_G(Z, Y),$$

so products in $\mathbb{B}_G$ are obtained from those in $\mathbb{B}$. To generalise this to $\mathscr{K}$, we must assume the following property.

Definition 3.9. Let $\mathscr{K}$ be a category with finite products that admits Kleisli objects for comonads. A comonad $\langle g : A \to A, \epsilon, \delta \rangle$ induces a comonad $\langle g \times g : A \times A \to A \times A, \epsilon \times \epsilon, \delta \times \delta \rangle$. Let $(u : A \to A_g, \lambda)$ be the Kleisli object of g. $\mathscr{K}$ satisfies PCK (*products commute with Kleisli objects*) if the oplax cocone $(U \times U : A \times A \to A_g \times A_g, \lambda \times \lambda)$ is an oplax colimit.

Remark 3.10. The above definition means that the Kleisli object of the product comonad

$g \times g$ is given by the product of those for g. In **Cat**, we have

$$\begin{aligned}(\mathbb{A} \times \mathbb{A})_{G\times G}((X,X'),(Y,Y')) &\cong \mathbb{A} \times \mathbb{A}((GX,GX'),(Y,Y')) \\ &\cong \mathbb{A}(GX,Y) \times \mathbb{A}(GX',Y') \\ &\cong \mathbb{A}_G(X,Y) \times \mathbb{A}_G(X',Y').\end{aligned}$$

Now, we can prove the 2-categorical version of Lambek's identification of $B[x{:}i]$ and $B_{\_\otimes i}$, a Kleisli object for the comonad in the following definition.

Proposition 3.11. Let $\mathscr{K}$ be a 2-category with finite products satisfying PCK. Let B be a cartesian object, with a global element $i : 1 \to B$. $B_{\_\otimes i}$ has the universal property of $B[x{:}I]$

Proof. (i) First, we must show that $B_{\_\otimes i}$ is a cartesian object and $u_i{:}B \to B_{\_\otimes i}$ preserves finite products. This follows from Theorem 3.4.((i)). We show the argument for binary products. We have a morphism of comonads

$$\begin{array}{ccc} B & \xrightarrow{\ \Delta\ } & B\times B \\ {\scriptstyle -\otimes i}\downarrow & & \downarrow{\scriptstyle -\otimes i\times -\otimes i} \\ B & \xrightarrow[\ \Delta\ ]{} & B\times B \end{array}$$

The coherence conditions follow from the 2-naturality of Δ. Since Δ has a right adjoint $\otimes{:}B \times B \to B$, it induces an adjunction

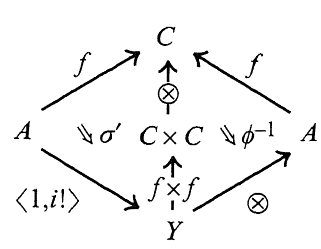
$$\overline{\Delta} \dashv \overline{\otimes} : (B \times B)_{\_\otimes i \times \_\otimes i} \to B_{\_\otimes i}$$

between the corresponding Kleisli objects. By condition PCK, $(B \times B)_{\_\otimes i\times\_\otimes i} \cong, B_{\_\otimes i} \times B_{\_\otimes i}$, and hence the above adjunction yields binary products for $B_{\_\otimes i}$, as condition PCK implies that we may take $\overline{\Delta} = \Delta'{:}B_{\_\otimes i} \to B_{\_\otimes i} \times B_{\_\otimes i}$, the diagonal morphism on $B_{\_\otimes i}$. Furthermore, this adjunction is preserved by $u_i{:}B \to B_{\_\otimes i}$ (on the nose).

(ii) Given a cartesian object C, with product $\hat{\otimes}$ and a finite product preserving morphism $f : B \to C$, there is one-to-one correspondence between 2-cells $\alpha : f1 \Rightarrow fi$ and 2-cells $\sigma : f \Rightarrow f\_\otimes i$, making (f,σ) an oplax cocone for the comonad $\_\otimes i$. The correspondence is set up by the assignment

$$\sigma \mapsto fq_{1,i} \circ \sigma 1$$

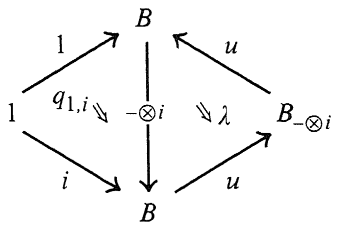
where $1{:}1 \to B$ is the terminal object, and $q_{1,i} :\otimes \langle 1, i\rangle \Rightarrow i$ is the second projection (which is an isomorphism), *cf.* Appendix A.2. In the other direction, given $\alpha : f1 \Rightarrow fi$, we get σ as the composite



where $(\phi^{-1} =)\phi_f^{-1}$, the inverse of ϕ_f, exists because f preserves finite products, and

$\sigma' = \langle\langle 1_f, \alpha !_A \circ f\tau \rangle\rangle$, with $\tau : 1_B \Rightarrow 1 \circ !_A$ is the unique such into the terminal object 1, *cf.* Appendix A.2.

(iii) We can now verify the universal property of B_{gi} as $B[x:I]$. We have a finite product preserving morphism $u : B \rightarrow B_{\_\otimes i}$ by (3.1), with a 2-cell $x : u1 \Rightarrow ui$ given by the composite



Given another cartesian object C, with product $\hat{\otimes}$, a finite product preserving morphism $f : B \rightarrow C$ and a 2-cell $\alpha : f1 \Rightarrow fi$, we have an oplax cocone on $\_\otimes i$ (f, σ), where σ is obtained from α by the correspondence in (3.1) above. Hence, by universality there exists a unique $\overline{(f,\sigma)} : B_{\_\otimes i} \rightarrow C$, such that $\overline{(f,\sigma)}\lambda = \sigma$. This implies $\overline{(f,\sigma)}x = \alpha$ as follows:

$$\begin{aligned}
\overline{(f,\sigma)}x &= f q_{1,i} \circ \sigma 1 \\
&= f q_{1,i} \circ \phi_f^{-1} \langle 1, i \rangle \circ \langle\langle 1_{f1}, \alpha \rangle\rangle \\
&= q_{f1,fi} \circ \langle\langle 1_{f1}, \alpha \rangle\rangle, \text{ because } \phi_f^{-1} \text{ is coherent with respect to projections.} \\
&= \alpha.
\end{aligned}$$

Finally, we must show $\overline{(f,\sigma)}$ preserves finite products. This follows from Corollary 3.6, taking the corresponding comparison 2-cell χ in that corollary to be the identity, which clearly satisfies the hypothesis of the corollary. □

Finally, we mention a relationship between Kleisli objects for monads and comonads, related by an adjunction. It allows us to give a characterisation of functional completeness whenever a (cartesian) object with an indeterminate can be presented as in Proposition 3.11 above.

First, let us recall that the notion of monad is dual to that of comonad. Formally, a monad in a 2-category $\mathcal{K}$ is a comonad in $\mathcal{K}^{co}$, the 2-category obtained from $\mathcal{K}$ by reversing the direction of the 2-cells. Then, from Definition 3.2, we get the 2-category $\mathsf{Mnd}(\mathcal{K}) = \mathsf{Cmd}(\mathcal{K}^{co})^{co}$ and $\mathcal{K}$ *admits Kleisli objects for monads* if the 2-functor $\Delta : \mathcal{K} \rightarrow \mathsf{Mnd}(\mathcal{K})$ has a left 2-adjoint.

Proposition 3.12. Given a comonad $\langle g : A \rightarrow A, \epsilon, \delta \rangle$, consider its associated Kleisli resolution $f \dashv u : A \rightarrow A_g$. The following are equivalent:

(i) g has a right adjoint $g \dashv t : A \rightarrow A$

(ii) u has a right adjoint $u \dashv r : A_g \rightarrow A$.

Under either of the above equivalent hypotheses, t $(= ru)$ is the morphism part of a monad and the corresponding Kleisli object A_t is isomorphic to A_g.

Proof. The equivalence is easily established, in view of the fact that if t is part of a monad, it induces a right adjoint r via its Kleisli resolution. The monad structure on t is induced as follows. Let η' and ϵ' be the unit and counit of $g \dashv t$. The unit of the monad is $\overline{\eta} = t\epsilon o\eta'$ and the multiplication is $\overline{\mu} = t(\epsilon' o g \epsilon' t o \delta t^2) o \eta' t^2$ (these 2-cells are the adjoint mates of ϵ and δ, resepctively). With a similar argument we establish the following isomorphism

$$\mathscr{K}(A_g, X) \cong \mathsf{Cmd}(\mathscr{K})(\langle g : A \to A, \epsilon, \delta\rangle, \Delta X) \cong \mathsf{Mnd}(\mathscr{K})(\langle t : A \to A, \overline{\eta}, \overline{\mu}\rangle, \Delta X) \cong \mathscr{K}(A_t, X)$$

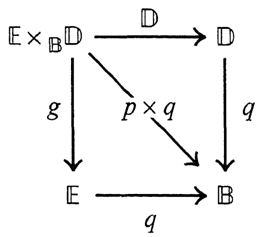
2-natural in X, whence the correspoonding Kleisli objects are isomrphic, that is, $A_g \cong A_t$. $\square$

4. Indeterminates for fibrations over a given base

Fibrations can be organised in two different 2-categories: **Fib**($\mathbb{B}$) with fibrations over a fixed base category $\mathbb{B}$ and **Fib** with fibrations over arbitrary bases. In general, 2-categorical constructions in **Fib**($\mathbb{B}$) are different from those in **Fib**, although they can usually be related via change-of-base: see Hermida (1993) for a study of these matters. In this section we concentrate on indeterminates in **Fib**($\mathbb{B}$); in the next section we study these in **Fib**. It turns out that contextual and functional completeness *for types* (see Section 1) is described in **Fib**($\mathbb{B}$), whereas contextual and functional completeness *for kinds* is dealt with in **Fib**. Constructions in these categories generalise the ordinary ones since **Fib(1)** $\simeq$ **Cat**.

We write **Fib**($\mathbb{B}$) for the 2-category of fibrations with $\mathbb{B}$ as base category; morphisms $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix} \xrightarrow{F} \begin{smallmatrix}\mathbb{D}\\ \downarrow q\\ \mathbb{B}\end{smallmatrix}$ are ($\mathbb{B}$-)*fibred* functors $F: \mathbb{E} \to \mathbb{D}$, which preserve cartesian morphisms and satisfy $q \circ F = p$. A 2-cell $\alpha: F \Rightarrow G$ is a *vertical* natural transformation $F \Rightarrow G$ (that is, with $q(\alpha) = id$). In line with the previous 2-categorical considerations about indeterminates, we investigate first what global elements s and 2-cells $1 \Rightarrow s$ are.

The 2-category **Fib**($\mathbb{B}$) has finite products: the identity functor on $\mathbb{B}$ is a terminal object and the binary product of $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ and $\begin{smallmatrix}\mathbb{D}\\ \downarrow q\\ \mathbb{B}\end{smallmatrix}$ is given by their pullback A global element



of a fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ in **Fib**($\mathbb{B}$) is thus a fibred functor $s: \mathbb{B} \to \mathbb{E}$ that is a section of p (that is, satisfies $p \circ s = id$). Such a global element corresponds to a family of objects $\{s(I) \in \mathbb{E}_I\}_{I \in \mathbb{B}}$ stable under reindexing. The latter means that for each $u: I \to J$ in $\mathbb{B}$ one has $u^*(s(J)) \cong s(I)$. When $\mathbb{B}$ has a terminal object 1, a global element s is determined by the object $s(1)$ in the fibre over 1. Given such an object X in the fibre over 1, we write s_X for the corresponding global element $\mathbb{B} \to \mathbb{E}$, given by $I \mapsto !_I^*(X)$.

In **Fib**($\mathbb{B}$), a cartesian object is a fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ with fibred finite products, *i.e.*, (i) there is a global element $1: \mathbb{B} \to \mathbb{E}$ such that $1(I)$ is terminal in the fibre $\mathbb{E}_I$, and (ii) there is a fibred product functor $\times: \mathbb{E} \times_{\mathbb{B}} \mathbb{E} \to \mathbb{E}$ (more precisely, $\times: p \times p \to p$ in **Fib**($\mathbb{B}$)) such that for each pair X, Y in the same fibre, $X \times Y$ is the binary product in that fibre. These fibrewise finite products are stable under reindexing. A morphism in **Fib($\mathbb{B}$)** preserves fibred finite products if it preserves them fibrewise.

Assume $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ is a fibration with a terminal object 1 and a global element s. A 2-cell $\alpha : 1 \Rightarrow s$ then consists of a family of vertical morphisms $\{\alpha_I : 1(I) \to s(I) \text{ in } \mathbb{E}_I\}_{I \in \mathbb{B}}$ stable under reindexing: for each $u : I \to J$ in $\mathbb{B}$ one has that α_I is unique in making the following diagram commute.

$$\begin{array}{ccc} 1(I) & \xrightarrow{1(u)} & 1(J) \\ {\scriptstyle \alpha_I}\downarrow & & \downarrow{\scriptstyle \alpha_J} \\ s(I) & \xrightarrow[s(u)]{} & s(J) \end{array}$$

In this diagram both horizontal morphisms are cartesian, and hence the square is a pullback. When the base catgory $\mathbb{B}$ has a terminal object 1, such a natural transformation $\alpha : 1 \to s$ is determined by the component α_1 at 1.

Remark 4.1. The above characterisation of global elements and 2-cells amounts to the equivalence $\mathbf{Fib}(\mathbb{B})(1_{\mathbb{B}}, p) \simeq \mathbb{E}_1$ when $\mathbb{B}$ has a terminal object. This equivalence is an instance of what Bénabou calls the 'fibred Yoneda lemma': see Jacobs (1991, Lemma 1.1.9).

For a given global element $s : 1_{\mathbb{B}} \to p$, we write $p[c{:}s]$ for the fibration with an *indeterminate* 2-cell $c : 1 \Rightarrow s$ equipped with a $\mathbb{B}$-fibred functor $\eta : p \to p[c{:}s]$ that preserves fibred finite products and a 2-cell $c : \eta 1 \Rightarrow \eta s$, universal among such, as specified in Definition 3.7. We can now formulate contextual and functional completeness for types as follows.

Definition 4.2. Let $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ be a fibration, with fibred finite products and $\mathbb{B}$ with a terminal object. $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ is

(i) *contextually complete for types* if for every global element $s_X : 1_{\mathbb{B}} \to p$, $\eta : p \to p[c{:}s_X]$ has a $\mathbb{B}$-fibred left adjoint, and

(ii) *functionally complete for types* if every such η has a $\mathbb{B}$-fibred right adjoint.

By Propositions 3.11 and B.1, every fibration with fibred finite products is contextually complete for types. Hence we can describe the fibration $p[c{:}s]$ concretely, as the 'simple slice' fibration $p/\!/s$ obtained as the Kleisli fibration of the comonad $\_ \times s : p \to p$ – $X \mapsto X \times s(pX)$ – in the following definition, *cf.* Proposition B.1 .

Definition 4.3. Let $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ be a fibration with finite products and s a global element. Define the category $\mathbb{E}/\!/s$ with,

objects	$X \in \mathbb{E}$
morphisms	$X \to Y$ in $\mathbb{E}/\!/s$ are morphisms $X \times s(pX) \to Y$ in $\mathbb{E}$.

The identity on $X \in \mathbb{E}$ is given by the (vertical) projection $\pi : X \times s(pX) \to X$, and the composition of $f : X \times s(pX) \to Y$ and $g : Y \times s(pY) \to Z$ is $g \circ (f \times s(pf)) \circ \langle id, \pi' \rangle$.
The fibration $p/\!/s : \mathbb{E}/\!/s \to \mathbb{B}$ has the same action as p on objects and morphisms, and $s^* : \mathbb{E} \to \mathbb{E}/\!/s$ for the functor given by $X \mapsto X$ and $f \mapsto f \circ \pi$

Notice that the above description of $p/\!/s$ agrees with the expected type-theoretic interpretation: 'terms' in $p[c{:}s]$ have an extra variable of 'type' s_1.

We may now consider contextual and functional completeness for types for the calculus $\lambda\!\!\rightarrow$. Recall (for example, from Jacobs (1991)) that a $\lambda\!\!\rightarrow$-fibration is a fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ having

(i) finite products in the base category $\mathbb{B}$;
(ii) a fibrewise CCC-structure (that is, every fibre is cartesian closed and each reindexing functor preserves the CCC-structure);
(iii) a generic object $T \in \mathbb{E}$, which is an object such that for each $X \in \mathbb{E}$ there is a cartesian morphism $X \to T$.

Such a $\lambda\!\!\rightarrow$-fibration gives a categorical model of a polymorphic calculus of types and terms (with type variables) in which one can form finite products and exponential types.

A morphism of $\lambda\!\!\rightarrow$-fibrations (over $\mathbb{B}$) is a fibred functor that preserves all of the above structure (i)–(iii).

In order to show contextual and functional completeness for $\lambda\!\!\rightarrow$-fibrations, we show that the description of $p[c{:}s]$ as the 'simple slice' fibration $p/\!/s$ is adequate for the $\lambda\!\!\rightarrow$-fibrations, *i.e.*, that $p/\!/s$ is a $\lambda\!\!\rightarrow$-fibration when p is, and its universal property holds with respect to other $\lambda\!\!\rightarrow$-fibrations. Theorem 3.4 accounts for the structure given by right adjoints. Generic objects are handled by the following lemma.

Lemma 4.4. Let $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ have a generic object T and let $\langle G : p \to p, \epsilon, \delta\rangle$ be a $\mathbb{B}$-fibred comonad. Then, the Kleisli fibration $p_G : \mathbb{E}_G \to \mathbb{B}$ has a generic object and $U_G : p \to p_G$ (the right adjoint of the resolution) preserves generic objects.

Proof. It is routine to verify that T is a generic object for p_G: for $X \in |\mathbb{E}|$, let $\chi_X : X \to T$ be a p-cartesian morphism in $\mathbb{E}$. Then $\chi_X \circ \epsilon_X : X \to T$ is p_G-cartesian in $\mathbb{E}_G$. Preservation by U_G is immediate.

□

Proposition 4.5. Each $\lambda\!\!\rightarrow$-fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ is both contextually and functionally complete over $\mathbb{B}$.

Proof. (i) We must show $p/\!/s$ is a $\lambda\!\!\rightarrow$-fibration. By Proposition 3.11, it has fibred finite products. Fibred exponentials are obtained by Theorem 3.4.((i)). To see this, consider an object X in the fibre $\mathbb{E}_I$. We have the following morphism of comonad

$$\begin{array}{ccc}
\mathbb{E}_I & \xrightarrow{\;-\times X\;} & \mathbb{E}_I \\
{\scriptstyle -\times s(I)}\Big\downarrow & \overset{\theta}{\Leftarrow} & \Big\downarrow{\scriptstyle -\times s(I)} \\
\mathbb{E}_I & \xrightarrow[\;-\times X\;]{} & \mathbb{E}_I
\end{array}$$

where θ is the canonical comparison isomorphism induced by $X \times s(I) \cong s(I) \times X$. Then we have an induced exponential adjunction on the Kleisli category $(\mathbb{E}_I)_{-\times s(I)}$, the fibres of the simple slice fibration $p/\!/s$, which is easily seen to be stable under reindexing.

Lemma 4.4 accounts for generic objects. Hence $p/\!/s$ is a $\lambda\!\rightarrow$-fibration. Preservation of such structure by the unique mediating morphism $\overline{F}: p/\!/s \to q$ induced by a morphism $F: p \to q$ of $\lambda\!\rightarrow$-fibrations and a 2-cell $\alpha : F1 \Rightarrow Fs$ is immediate, by Corollary 3.6 and the construction of a generic object in Lemma 4.4.

(ii) Contextual completeness is guaranteed by the above verification that $p[c:s]$ can be presented as the Kleisli fibration $p/\!/s$. Functional completeness follows from Proposition 3.12, since the comonad $\_ \times s: p \to p$ has a fibred right adjoint $s \Rightarrow \_: p \to p$ with action $X \mapsto s(pX) \Rightarrow X$. □

Remark 4.6. The above proof of contextual and functional completeness for types for $\lambda\!\rightarrow$-fibrations, based on the identification of $p[c:s]$ with the Kleisli fibration $p/\!/s$ can be easily extended to show a similar completeness property for $\lambda 2$- and $\lambda\omega$-fibrations, *i.e.*, models for second and higher-order lambda calculi. The stronger completeness properties of $\lambda 2$-fibrations are not dealt with by the techniques in the present paper. $\lambda\omega$-fibrations instead enjoy additional completeness properties, which we deal with in the following section, from which their contextual and functional completeness for types is obtained as an instance.

5. Indeterminates for fibrations over arbitrary bases

Let **Fib** denote the 2-category of fibrations over arbitrary base categories. A morphism (1-cell) $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix} \xrightarrow{(K,L)} \begin{smallmatrix}\mathbb{D}\\ \downarrow q\\ \mathbb{A}\end{smallmatrix}$ in **Fib** consists of two functors $K: \mathbb{B} \to \mathbb{A}$ and $L: \mathbb{E} \to \mathbb{D}$ such that $q \circ L = K \circ p$ and L maps cartesian morphisms in $\mathbb{E}$ to cartesian morphisms in $\mathbb{D}$. A 2-cell $(K, L) \Rightarrow (K', L')$ between two such 1-cells consists of two natural transformations $\sigma: K \Rightarrow K'$ and $\tau: L \Rightarrow L'$ such that τ is above σ, that is, $q(\tau_X) = \sigma_{pX}$.

Fibrations are stable under pullback: given a functor $F: \mathbb{A} \to \mathbb{B}$ and a fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$, we obtain a fibration $\begin{smallmatrix}F^*(\mathbb{E})\\ \downarrow F^*(p)\\ \mathbb{A}\end{smallmatrix}$ by *change-of-base along* F, as shown in the following pullback square:

$$\begin{array}{ccc} F^*(\mathbb{E}) & \xrightarrow{p^*(F)} & \mathbb{E} \\ {\scriptstyle F^*(p)}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{A} & \xrightarrow[F]{} & \mathbb{B} \end{array}$$

The 2-category **Fib** has finite products: the fibration $\begin{smallmatrix}1\\ \downarrow\\ 1\end{smallmatrix}$ (also written **1**) is a terminal object and, given fibrations $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ and $\begin{smallmatrix}\mathbb{D}\\ \downarrow q\\ \mathbb{A}\end{smallmatrix}$, their product is $\begin{smallmatrix}\mathbb{E}\times\mathbb{D}\\ \downarrow p\times q\\ \mathbb{B}\times\mathbb{A}\end{smallmatrix}$. Consequently, a cartesian object in **Fib** is a fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ that has *finite products*, that is

— the unique map $p \to \mathbf{1}$ has a right adjoint in **Fib**. This means that both $\mathbb{E}$ and $\mathbb{B}$ have a terminal object and that p preserves it.

— the diagonal $\Delta: p \to p \times p$ has a right adjoint in **Fib**. This means that both $\mathbb{E}$ and

$\mathbb{B}$ have binary products and that p preserves them. We write $X \tilde{\times} Y$ for the binary product of X and Y in $\mathbb{E}$ over $pX \times pY$ in $\mathbb{B}$.

Equivalently (see, for example, Jacobs (1991) and Hermida (1993)), $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ has finite products in **Fib** if the base category $\mathbb{B}$ has finite products and p has *fibred* finite products.

A morphism $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix} \xrightarrow{(K,L)} \begin{smallmatrix}\mathbb{D}\\ \downarrow q\\ \mathbb{A}\end{smallmatrix}$ of fibrations between p and q with finite products preserves finite products if both $K : \mathbb{B} \to \mathbb{A}$ and $L : \mathbb{E} \to \mathbb{D}$ preserve finite products (the latter either globally or fibrewise).

A global element of $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ in **Fib** is given by a morphism of fibrations

$$\begin{smallmatrix}1\\ \downarrow\\ 1\end{smallmatrix} \xrightarrow{(K,L)} \begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$$

that is, by an object $X = L(*) \in \mathbb{E}$ above $I = K(*) = pX$. We write $X : \mathbf{1} \to p$ for such a global element, where it is understood that X is an object of the total category $\mathbb{E}$ of p. In particular, when p has finite products, the terminal object $1 \in \mathbb{E}$ is such a global element $1 : \mathbf{1} \to p$.

Notice that a 2-cell $1 \Rightarrow X$ between $1, X : \mathbf{1} \to p$ consists of a morphism $f : 1 \to X$ in the total category of p. It can be identified with a map $u : 1 \to pX$ in the base category together with a map $f : 1 \to u^*(X)$ in the fibre over 1. Therefore, such a 2-cell will often be written as a pair $\langle u, f\rangle : 1 \Rightarrow X$. Type-theoretically, a global element corresponds to a type X with a free variable of kind pX.

Before proceeding with the study of contextual and functional completeness for kinds in this setting, we should make a technical warning. If we want to work with the 2-category **Fib** as defined, we should take the appropriate bicategorical versions of the notions considered in Section 3, *i.e.*, an object with an indeterminate element and Kleisli object. For instance, a biKleisli object for a comonad $\langle g : A \to A, \epsilon, \delta\rangle$ is an object A_g that induces an *equivalence* (rather than an isomorphism) of categories

$$\mathcal{K}(A_g, B) \simeq \mathsf{Cmd}(\mathcal{K})(\langle g, \epsilon, \delta\rangle, \Delta B)$$

pseudo-natural in B. This means that, in the universal property of the Kleisli object A_g, the relevant diagrams commute only up to a (coherent) isomorphism, rather than up-to-equality.

Although this is the natural categorical approach to follow, and indeed all the statements and constructions we present in this section apply to this setting, we think the explicit treatment of the (coherent) isomorphisms involved will make the presentation heavier than necessary. We thus assume we work in the more restricted setting of split fibrations and splitting-preserving morphisms, for which the (strict) 2-categorical versions we have considered so far continue to apply. We emphasise that this is only a choice of presentation; all the general results in Section 3 hold (suitably translated) in their bicategorical versions, and apply to **Fib**. The same considerations apply to the presentation of Kleisli objects in **Fib** in Appendix B.

As before, we can now specify polynomial fibrations $p[\langle \kappa, c\rangle : X]$ in **Fib**, instantiating Definition 3.7. This is used to express contextual and functional completeness for kinds, as follows.

Definition 5.1. Let $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ be a fibration with finite products. p is

(i) *contextually complete for kinds* if, for every $X \in |\mathbb{E}|$, $\eta : p \to p[\langle\kappa, c\rangle : X]$ has a left adjoint in **Fib**, and

(ii) *functionally complete for kinds* if every such η has a right adjoint in **Fib**.

This categorical expression of completeness for kinds reflects the type-theoretic version for polymorphic λ-calculi in Section 1.

By Proposition 3.11 and Theorem B.6, a fibration with finite products is contextually complete, as the fibration $p[\langle\kappa, c\rangle : X]$ can be presented as a Kleisli fibration. We give a concrete description of the resulting 'simple slice' fibration $p/\!/X$, for X a global element of p in the following definition.

Definition 5.2. Let $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ be a fibration with finite products in **Fib**. For a global element $X \in \mathbb{E}$, consider the category $\mathbb{E}/\!/(X)$ with

objects (I, Y) with $pY = I \times pX$

morphisms $(I, Y) \to (J, Z)$ are $u : I \times pX \to J$ in $\mathbb{B}$ and $f : Y \times \pi'^*(X) \to Z$ in $\mathbb{E}$ above $\langle u, \pi'\rangle : I \times pX \to J \times pX$

The fibration $p/\!/X : \mathbb{E}/\!/(X) \to \mathbb{B}/\!/pX$ sends (I, Y) to I and (u, f) to u. Furthermore, we write X^* for the morphism $p \to p/\!/X$ given by $Y \mapsto (pY, \pi^*(Y))$ from $\mathbb{E}$ to $\mathbb{E}/\!/(X)$, and whose base part is $pX^* : \mathbb{B} \to \mathbb{B}/\!/pX$.

Note that a morphism in $(u, f) : \eta(Y) \to (J, Z)$ in $\mathbb{E}/\!/(X)$ corresponds to a term

$$\alpha : pY \mid x : \pi^*_{pY,pX}(Y) \vdash_{\kappa,c} f : Z(u, c)$$

with $\kappa : pX$ and $c : X(\kappa)$, where the 'context for type variables' $\pi^*_{pY,pX}(Y)$ does not depend on κ. This reflects precisely the restriction on contexts required in the formulation of functional completeness for kinds in Section 1.

Remark 5.3. There are two special cases of the simple slice fibration $p/\!/X$ associated with a fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ in **Fib**, which deserve separate attention.

(1) If the object X is in the fibre above the terminal object, it corresponds to a global element s_X of p in **Fib**($\mathbb{B}$), as described in the previous section. Then, the fibration $p/\!/X$ (as described in Definition 5.2) is isomorphic to $p/\!/s_X$ (as in Definition 4.2).

(2) If the object X is a terminal object in a fibre, say over $I = pX$, then the fibration $p/\!/X$ can be understood type theoretically as the fibration arising by adjoining only a constant $\kappa : I$ in a kind (but no constant in a type).

Thus we can actually 'decompose' the polynomial fibration $p/\!/X$ in two: we may first consider the case in (2) with an indeterminate $\kappa : I = pX$ in a *kind*, and then add an indeterminate $c : X(\kappa)$ at the level of types to the fibration so obtained. This two-stage construction is reflected in the way Kleisli objects in **Fib** are obtained in Theorem B.6 in Appendix B.

We now intend to show contextual and functional completeness for the categorical version of the higher-order polymorphic λ-calculus, $\lambda\omega$.

We recall from Jacobs (1991) that a $\lambda\omega$-**fibration** is a $\lambda\!\to$-fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ (as described in Section 4), which has the following additional properties.

(i) The base category $\mathbb{B}$ is cartesian closed.

(ii) The fibration p has 'simple products': that is, each reindexing functor π^* induced by a cartesian projection $\pi: I \times J \to I$ in $\mathbb{B}$ has a right adjoint $\prod_{(I,J)}: \mathbb{E}_{I\times J} \to \mathbb{E}_I$, and these right adjoints satisfy the Beck–Chevalley condition.

A morphism of $\lambda\omega$-fibrations is a morphism that preserves the above structure. As for fibrations, we may consider such morphisms over a fixed base category or between fibrations over different bases.

In order to show contextual and functional completeness of $\lambda\omega$-fibrations, we must show that the presentation of $p[\langle\kappa, c\rangle : X]$ as the Kleisli fibration $p/\!/X$ is adequate for $\lambda\omega$-fibrations. Once again, Theorem 3.4.(i) takes account of the structure given by right adjoints. The following Lemma shows that change-of-base along a left adjoint functor preserves the property of having a generic object. It shows that such a property is inherited by Kleisli fibrations for comonads in **Fib**, according to the construction in Theorem B.6.

Lemma 5.4. Given $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$, with a generic object G (over Ω) and an adjunction $\eta, \epsilon : F \dashv U : \mathbb{B} \to \mathbb{A}$, let $(\overline{U}, U): p \to F^*(p)$ be the right adjoint to $(p^*(F), F)$ (in **Fib**) induced by change-of-base, $\overline{U}X = (\epsilon_{pX})^*(X)$. Then, $F^*(p)$ has a generic object, preserved by $(\overline{U}, U)$.

Proof. The fact that $(\overline{U}, U)$ is right adjoint to $(p^*(F), F)$ is in Hermida (1993, Lemma 3.2.1). We must simply verify that $\overline{U}G$ $(= \langle\epsilon_\Omega^*(G), G\Omega\rangle)$ over $G\Omega$ is a generic object for $F^*(p)$. Let X be an object of $F^*(\mathbb{E})$. We obtain a cartesian morphism $\chi_X : X \to \overline{U}G$ as the adjoint transpose of a cartesian morphism $\chi_{p^*(F)X} : p^*(F)X \to G$. □

Proposition 5.5. Each $\lambda\omega$-fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ is both contextually and functionally complete for kinds.

Proof. Let $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ be a $\lambda\omega$-fibration and let X be a global element with $pX = I$.

(i) First we show that the presentation of $p[\langle\kappa, c\rangle : X]$ as the Kleisli fibration $p/\!/X$ is adequate for $\lambda\omega$-fibrations. To show that it has fibred exponentials and simple products, we apply Theorem 3.4.(i). We illustrate the case of simple products, which is the most involved.

Let J be an object of the base category. The natural transformation $\pi_{\_,J} : \_ \times J \Rightarrow id$ induces a fibred functor $\Delta_J : p \to (\_ \times J)^*(p)$, $X \mapsto \pi_{pX,J}^*(X)$. Then, p has simple products at J if and only if Δ_J has a fibred right adjoint, and it has simple products when it has them at every object of the base $\mathbb{B}$. To show that $p/\!/X$ has simple products at J, observe that we have the following pseudo-morphism of comonads in **Fib**

$$\begin{array}{ccc} p & \xrightarrow{\;\Delta_J\;} & (-\times J)^*(p) \\ {\scriptstyle -\times s(I)}\big\downarrow & \overset{\theta}{\Leftarrow} & \big\downarrow{\scriptstyle (-\times J)^*(-\overline{\times}\, X)} \\ p & \xrightarrow[\;\Delta_J\;]{} & (-\times J)^*(p) \end{array}$$

where θ is the isomorphism induced by a cleavage of p. Of course, when p is split,

θ is the identity. Hence, a right adjoint to Δ_J induces a right adjoint for $\overline{\Delta_J}: p/\!/X \to (\_ \times J)^*(p_{(\_\times J)^*(\_\tilde{\times} X)})$ between the corresponding Kleisli fibrations. But

$$(\_ \times J)^*(p_{(\_\times J)^*(\_\tilde{\times} X)}) \cong (\_ \times J)^*(p/\!/X),$$

as can be seen from inspection of the construction in Theorem B.6 (assuming p split, otherwise we obtain an equivalence). Thus the induced adjunction shows that $p/\!/X$ has simple products at J, preserved by the morphism $X^*: p \to p/\!/X$, as required. Lemmas 5.4 and 4.4 show that $p/\!/X$ has a generic object, preserved by X^*, according to the construction of $p/\!/X$ in Theorem B.6.

(ii) Contextual completeness follows at once from the above verification that $p/\!/X$ is the appropriate $\lambda\omega$-fibration with an indeterminate. To show functional completeness, we need to define a right adjoint $X \Rightarrow (\_): \mathbb{E}/\!/(X) \to \mathbb{E}$ to X^* above the right adjoint $pX \Rightarrow (\_): \mathbb{B}/\!/pX \to \mathbb{B}$ to $(pX)^*$. It is given by

$$(I, Y) \mapsto \prod\nolimits_{(pX \Rightarrow I, pX)} (\pi'^*(X) \Rightarrow \langle ev, \pi' \rangle^*(Y)).$$

□

The notation $X \Rightarrow (\_)$ used in the above proof is justified because the total category $\mathbb{E}$ of a $\lambda\omega$-fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ is cartesian closed: the exponential $X \Rightarrow Y$ in $\mathbb{E}$ of $X, Y \in \mathbb{E}$, with $pX = I$ and $pY = J$, is given by

$$X \Rightarrow Y \;=\; \prod\nolimits_{(I \Rightarrow J, I)} (\pi'^*(X) \Rightarrow ev^*(Y)),$$

where $\Rightarrow$ on the right-hand side is the exponential in the fibre over $(I \Rightarrow J) \times I$ in $\mathbb{B}$. This is the "logical predicate implication": in informal notation, writing X and Y as predicates (or families of propositions) $X = \{X_i\}_{i \in I}$ and $Y = \{Y_j\}_{j \in J}$, we have

$$(X \Rightarrow Y)_{f \in I \Rightarrow J} \;=\; \forall i \in I.\,(X_i \Rightarrow Y_{f(i)}).$$

See Hermida (1993) for a more abstract explanation of this situation, in terms of adjunctions in **Fib**.

Remarks 5.6.

(i) The construction in Definition 5.2 is adequate for *first-order-logic*-fibrations, that is, fibrations with the appropriate structure to interpret first-order predicate logic. These are fibrations over a cartesian (closed) category, with a fibred-ccc structure and simple products and coproducts. The latter two model universal and existential quantifiers, respectively. With the same argument as in the proof of Proposition 5.5 we can show contextual and functional completeness for these kinds of fibrations. See Hermida and Jacobs (1994) for an application of these facts in the context of induction principles for data types.

(ii) After the present paper was completed, it came to our notice that the above construction of fibrations with indeterminates in the context of first-order logic occurs in Makkai (1993), and even uses the same notation $p/\!/X$! The construction there, however, is not exhibited as a Kleisli construction, and its universal property is not spelled out, given the rather different nature of its applications.

6. Conclusions and further work

We have given concrete descriptions of fibrations with indeterminate elements, both at the base and fibre levels. The constructions given are instances of a 2-categorical generalisation of Lambek's presentation of cartesian (closed) categories with indeterminates as Kleisli categories for suitable comonads.

Of course, the Kleisli construction does not cope with all kinds of structure. For instance, it is well known that for a category with finite limits $\mathbb{C}$ and an object I in it, the category with finite limits with an indeterminate at the object I is the slice category $\mathbb{C}/I$. And this construction applies to a good many categories with structure, for example, locally cartesian closed categories and toposes.

Future work should consider the construction of fibrations with an inderminate for comprehension categories (Jacobs 1991) in order to study contextual and functional completeness for dependent types. We should also examine the connections between functional completeness and systems of combinators for polymorphic calculi.

It is worth mentioning that fibrations with an indeterminate (over arbitrary bases) occur implicitly in Burstall and McKinna (1992) to obtain so-called second-order deliverables as a means of structuring program development in type theory. We thank M. Takeyama for pointing this out.

Another application of $\lambda{\rightarrow}$-/$\lambda\omega$-fibrations with indeterminates is to give semantics for ML-style modules, following the approach in Fourman and Phoa (1992). We illustrate this with a simple example. Consider a $\lambda{\rightarrow}$-fibration p, with basic types those of SML. The following signature

```
signature Order =
   sig
      type t;
      val le: t * t → bool
   end;
```

is interpreted as the fibration $p[\langle t, \mathrm{le}\rangle : T \times T \Rightarrow !_\Omega^*(B)]$, with indeterminates $t : 1 \rightarrow \Omega$ (corresponding to the type `t` in the signature) and $\mathrm{le} : 1 \rightarrow t^*(T \times T \Rightarrow !_\Omega^*(B)) = t \times t \Rightarrow B$ (corresponding to the function `le`), where B is the object in the fibre over 1 corresponding to the (closed) type `bool`. We can then associate to a signature Σ a $\lambda{\rightarrow}$-fibration p_Σ. A structure S matching the above signature corresponds to a $\lambda{\rightarrow}$-fibration q with specified morphisms for `t` and `le`; by universality, it corresponds to a morphism of $\lambda{\rightarrow}$-fibrations $S : p[\langle t, \mathrm{le}\rangle : T \times T \Rightarrow !_\Omega^*(B)] \rightarrow q$; more generally, a structure for Σ is a morphism $S_\Sigma : p_\Sigma \rightarrow q$. An SML-functor `functor F(structure S:`$\Sigma_1$`):`$\Sigma_2$ corresponds to a morphism of $\lambda{\rightarrow}$-fibrations $F : p_{\Sigma_2} \rightarrow p_{\Sigma_1}$, induced by a 'transformation' of structures from Σ_1 to Σ_2. The application of such a functor `F` to a structure matching Σ_1 corresponds to the composite $S_{\Sigma_1} \circ F : p_{\Sigma_2} \rightarrow q$.

As we pointed out in Remark 5.6.(i), the construction of fibrations with indeterminates (in **Fib**) as Kleisli fibrations is adequate to cope with the structure required to interpret first-order predicate logic over a simply typed λ-calculus. Just as contextual and functional completeness guarantees 'stability' of terminal and initial algebras for endofunctors, respectively, the same result at the level of fibrations gurantees the 'stability' of the associated coinduction/induction principles. See Hermida and Jacobs (1994) for details.

Appendix A. Cartesian objects

A.1. *Products in a 2-category*

In order to state that a category has binary products and terminal object in terms of adjunctions, we use the fact that the 2-category **Cat** itself has finite products. Their definition in an arbitrary 2-category (Kelly 1989) is as follows.

Definition A.1.

(i) A 2-category $\mathscr{K}$ has a *terminal object* if there is an object 1 such that for every object A, there is an isomorphism

$$\mathscr{K}(A,1) \cong \{*\}, \text{ the one-object one-morphism category}$$

2-natural in A.

(ii) $\mathscr{K}$ has *binary products* if for any two objects A and B, there is an object $A \times B$ such that, for any object C there is an isomorphism

$$\mathscr{K}(C, A \times B) \cong \mathscr{K}(C,A) \times \mathscr{K}(C,B)$$

2-natural in C.

We say that $\mathscr{K}$ has finite products if it has binary ones and a terminal object. The above isomorphisms mean that the underlying category $\mathscr{K}_0$ has finite products as an ordinary category, and that they have a 2-dimensional universal property. Specifically, 1 is such that for every object A, there is a unique 1-cell $!_A : A \to 1$ and a unique 2-cell $\alpha : !_A \Rightarrow !_A$, whence $\alpha = 1_{!_A}$. Similarly, for objects A and B, the projections $A \stackrel{\pi}{\leftarrow} A \times B \stackrel{\pi'}{\rightarrow} B$ are such that for any span $A \stackrel{f}{\leftarrow} C \stackrel{g}{\rightarrow} B$ there is a unique $\langle f,g \rangle : C \to A \times B$ with $\pi \circ \langle f,g \rangle = f$ and $\pi' \circ \langle f,g \rangle = g$. And for any two 2-cells $\alpha : f \Rightarrow f' : C \to A$ and $\beta : g \Rightarrow g' : C \to B$ there is a unique 2-cell $\langle \alpha, \beta \rangle : \langle f,g \rangle \Rightarrow \langle f',g' \rangle$ with $\pi \langle \alpha, \beta \rangle = \alpha$ and $\pi' \langle \alpha, \beta \rangle = \beta$.

The non-elementary definition of products in $\mathscr{K}$ is given in terms of 2-adjoints: $\mathscr{K}$ has a terminal object if $!_{\mathscr{K}} : \mathscr{K} \to \{*\}$ has a right 2-adjoint; it has binary products if the diagonal 2-functor $\Delta : \mathscr{K} \to \mathscr{K} \times \mathscr{K}$ has a right 2-adjoint.

A.2. *Cartesian objects in a 2-category with products*

Rephrasing the definition of a category with finite products in **Cat** in terms of adjoints, we get the following definition of cartesian objects in a 2-category with finite products (Carboni *et al.* 1990).

Definition A.2. Let $\mathscr{K}$ be a 2-category with finite products. An object A is *cartesian* if both

(i) the unique morphism $!_A : A \to 1$ has a right adjoint $1 : 1 \to A$, and
(ii) the diagonal morphism $\delta_A : A \to A \times A$ has a right adjoint $\otimes : A \times A \to A$.

Note that the counit of $!_A \dashv 1$ must be the identity. If $\tau : 1_A \Rightarrow 1 \circ !_A$ is the unit, the adjunction laws reduce to $\tau 1 = 1_1$.

A cartesian object in **Cat** is a category with assigned finite products. A cartesian object in **Fib**($\mathbb{B}$) is a fibration with assigned fibred finite products, while a cartesian object in **Fib** is a fibration $\begin{array}{c}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{array}$ such that both $\mathbb{E}$ and $\mathbb{B}$ have assigned finite products and p preserves them strictly.

We now spell out how the usual operations of pairing and projection in a category with finite products, as in Lambek and Scott (1986, Part I), are obtained in this abstract setting.

In **Cat**, the projections associated to the binary product functor $\times : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ are natural transformations $\pi_{\_\_} : \times \dot{\to} \pi : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ and $\pi'_{\_\_} : \times \dot{\to} \pi'$. Then, for objects X, Y of $\mathbb{A}$, $\pi_{X,Y} : X \times Y \to X$ is the first projection. The projections are the components of the counit $\epsilon' : \delta_{\mathbb{A}} \times \dot{\to} 1_{\mathbb{A}\times\mathbb{A}}$.

Let A with $\eta', \epsilon' : \delta_A \dashv \otimes : A \times A \to A$ be a cartesian object in $\mathscr{K}$. The associated projections are $p = \pi\epsilon' : \otimes \Rightarrow \pi$ and $q = \pi'\epsilon' : \otimes \Rightarrow \pi'$.

As for pairing, recall that objects of a category $\mathbb{A}$ correspond to functors $1 \to \mathbb{A}$, where 1 is the terminal category, and morphisms of $\mathbb{A}$ correspond to natural transformations between the respective functors. Given morphisms $f : Z \Rightarrow I : 1 \to \mathbb{A}$ and $g : Z \Rightarrow J : 1 \to \mathbb{A}$, their pairing is $\langle f, g\rangle = (f \times g) \circ \delta_Z$. The diagonal morphism δ_Z is the component at Z of the unit $\eta' : 1_{\mathbb{A}} \dot{\to} \times \delta_{\mathbb{A}}$.

Generalising to $\mathscr{K}$, given 'objects' $f, g : B \to A$ of A, their product is $\otimes \langle f, g\rangle : B \to A$. For 'morphisms' $\alpha : f \Rightarrow g : B \to A$ and $\beta : f \Rightarrow h : B \to A$, their pairing is

$$\langle\!\langle \alpha, \beta \rangle\!\rangle = \otimes \langle \alpha, \beta\rangle \circ \eta' f : f \Rightarrow \otimes \langle g, h\rangle.$$

Let $p_{g,h} = p\langle g, h\rangle : \otimes \langle g, h\rangle \Rightarrow g$ and $q_{g,h} = q\langle g, h\rangle : \otimes \langle g, h\rangle \Rightarrow h$. Then,

$$p_{g,h} \circ \langle\!\langle \alpha, \beta \rangle\!\rangle = \alpha \qquad q_{g,h} \circ \langle\!\langle \alpha, \beta \rangle\!\rangle = \beta \qquad \langle\!\langle p_{g,h}, q_{g,h} \rangle\!\rangle = 1_{\otimes\langle g,h\rangle}.$$

Given cartesian objects A and B, with products $\otimes$ and $\hat{\otimes}$, respectively, a morphism $f : A \to B$ induces a 2-cell $\phi_f = \langle\!\langle fp, fq \rangle\!\rangle : f \otimes \Rightarrow \otimes' (f \times f)$ (the pairing is that of B). Then, f *preserves finite products* if ϕ_f is an isomorphism. This agrees in **Cat** with the usual definition.

B. Kleisli objects for fibred comonads

B.1. *Kleisli fibrations over a fixed base*

In this section we construct a Kleisli object for a comonad in **Fib**($\mathbb{B}$). The construction is based on that of Kleilsi categories for comonads in **Cat** given in 3.

Proposition B.1. **Fib**($\mathbb{B}$) admits Kleisli objects for comonads and satisfies PCK.

Proof. Given a fibred comonad G: $\begin{array}{c}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{array} \to \begin{array}{c}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{array}$, with counit $\epsilon : G \Rightarrow 1$ and comultiplication $\delta : G \Rightarrow G^2$, define a fibred oplax cocone $(U_G : p \to p_G, \lambda)$ as follows:

(i) $(U_G : \mathbb{E} \to \mathbb{E}_G, \lambda)$ is the Kleisli object associated with the (ordinary) comonad $G : \mathbb{E} \to \mathbb{E}$ in **Cat**.

(ii) $p_G: \mathbb{E}_G \to \mathbb{B}$ is the functor induced by the (oplax) cocone $(p: \mathbb{E} \to \mathbb{B}, 1)$, that is,

$$p_G(f: GX \to Y) = pf: pX \to pY.$$

Note that the 2-cell $\lambda : U \Rightarrow UG$ is vertical. p_G is a fibration: given $X \in \mathbb{E}_G$ and $u: I \to pX$, let $\bar{u}: u^*(X) \to X$ be a p-cartesian morphism; $\bar{u} \circ \epsilon_X : G(u^*(X)) \to X$ is a p_G-cartesian morphism over u. This construction of p_G-cartesian morphisms also shows that $U_G: p \to p_G$ is a fibred functor.
As for the universal property of $(U_G: p \to p_G, \lambda)$, it follows from that of $(U_G: \mathbb{E} \to \mathbb{E}_G, \lambda)$, as one easily verifies that the induced mediating functor to another fibred oplax cocone is fibred, and similarly for the mediating 2-cells. Finally, condition PCK follows from that in **Cat**. □

Note that the above construction of Kleisli objects in **Fib**($\mathbb{B}$) (which is a sub-2-category of **Cat**/$\mathbb{B}$), based on those of **Cat**, is analogous to the construction of colimits in a slice category $\mathbb{C}/I$ out of those of $\mathbb{C}$ Mac Lane (1971, Section IV.6).

The construction in Proposition B.1 shows that $p/\!/s$ is the Kleisli fibration corresponding to the comonad $\_ \times s: p \to p$, for s a global element of p.

B.2. *Kleisli fibrations over arbitrary bases*

We show that the 2-category **Fib** has Kleisli objects for comonads. The construction of such objects uses the construction given in Proposition B.1.

We should hasten to remark that we get Kleisli objects in the sense of Definition 3.2 only if we restrict our attention to split fibrations and splitting preserving functors. **Fib** admits Kleisli objects in a suitable bicategorical sense, where the diagrams involved commute only up to coherent isomorphism. The constructions given here apply in either case, but we concentrate on the split case for simplicity.

Some auxiliary technical lemmas are stated without a detailed proof. Full details are in Hermida (1993, Section 5).

The following lemma allows us to reduce the problem of constructing Kleisli objects for comonads in **Fib** to that of constructing them over a fixed base.

Lemma B.2. A comonad $(\tilde{G}, G)$: $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix} \to \begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ and a resolution $F \dashv R: \mathbb{B} \to \mathbb{A}$ induce a comonad $\overline{G}: F^*(p) \to F^*(p)$ $(F^*(p): \mathbb{A} \times_{\mathbb{B}} \mathbb{E} \to \mathbb{B})$ in **Fib**($\mathbb{A}$).

Proof. Recall that objects and morphisms in $\mathbb{A} \times_{\mathbb{B}} \mathbb{E}$ are denoted by pairs

$$(I, X) \stackrel{(u,f)}{\to} (J, Y), \text{ with } FI = pX, Fu = pf, FJ = pY.$$

Let η and ϵ be the unit and counit of $F \dashv R$, and let $(\tilde{\epsilon}, \epsilon')$ and $(\tilde{\delta}, \delta)$ be the counit and comultiplication, respectively, of $(\tilde{G}, G)$.

(i) The fibred functor $\overline{G}: F^*(p) \to F^*(p)$ acts as follows:

$$\begin{array}{ccc} (I,X) & & (I,F\eta_I^*(\tilde{G}X)) \\ {\scriptstyle (u,f)}\downarrow & \mapsto & \downarrow{\scriptstyle (u,f')} \\ (J,Y) & & (J,F\eta_J^*(\tilde{G}Y)) \end{array}$$

where f' is uniquely determined in the following diagram

$$\begin{array}{ccccccc} F\eta_I^*(\tilde{G}X) & \xrightarrow{\overline{F\eta_I}} & \tilde{G}X & & FI & \xrightarrow{F\eta_I} & GpX \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle \tilde{G}f} & \overset{p}{\mapsto} & \downarrow{\scriptstyle Fu} & & \downarrow{\scriptstyle Gf} \\ F\eta_J^*(\tilde{G}Y) & \xrightarrow[\overline{F\eta_J}]{} & \tilde{G}Y & & FJ & \xrightarrow[F\eta_J]{} & GpY \end{array}$$

(ii) The counit $\overline{\epsilon} : \overline{G} \Rightarrow 1$ has components

$$\overline{\epsilon}_{(I,X)} = (id, \phi) : (I, F\eta_I^*(\tilde{G}X)) \to (I, X),$$

where $\phi = F\eta_I^*(\tilde{G}X) \overset{\overline{F\eta_I}}{\to} \tilde{G}X \overset{\tilde{\epsilon}_X}{\to} X$

(iii) The comultiplication $\overline{\delta} : \overline{G} \Rightarrow \overline{G}^2$ has components

$$\overline{\delta}_{(I,X)} = (id, \psi) : (I, F\eta_I^*(\tilde{G}X)) \to (I, F\eta_I^*(\tilde{G}F\eta_I^*(\tilde{G}X)))$$

where ψ is uniquely determined in the following diagram

$$\begin{array}{ccccc} F\eta_I^*(\tilde{G}X) & \xrightarrow{\overline{F\eta_I}} & \tilde{G}X & & \\ {\scriptstyle \psi}\downarrow & & & \searrow{\scriptstyle \tilde{\delta}_X} & \\ F\eta_I^*(\tilde{G}F\eta_I^*(\tilde{G}X)) & \xrightarrow[\overline{F\eta_I}]{} & \tilde{G}F\eta_I^*(\tilde{G}X) & \xrightarrow[\tilde{G}\overline{F\eta_I}]{} & \tilde{G}^2X \end{array}$$

$$\Big\downarrow p$$

$$\begin{array}{ccccc} FI & \xrightarrow{F\eta_I} & GFI & & \\ {\scriptstyle id}\downarrow & & & \searrow{\scriptstyle \delta_{FI}} & \\ FI & \xrightarrow[F\eta_I]{} & GFI & \xrightarrow[GF\eta_I]{} & G^2FI \end{array}$$

□

The following two lemmas are only used to prove that the construction in Theorem B.6 below has the required universal property. The proofs are rather technical and therefore omitted.

Lemma B.3. Given

(i) a comonad $\langle(\tilde{G}, G) : \begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix} \to \begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}, (\tilde{\epsilon}, \epsilon), (\tilde{\delta}, \delta)\rangle$ for $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$,

(ii) a fibred oplax cocone $((\tilde{L}, L) : \begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix} \to \begin{smallmatrix}\mathbb{D}\\ \downarrow q\\ \mathbb{C}\end{smallmatrix}, (\tilde{\sigma}, \sigma))$,

(iii) an oplax cocone $(K : \mathbb{B}\to\mathbb{A}, v : K \Rightarrow KG)$ for G, and

(iv) a functor $J : \mathbb{A}\to\mathbb{C}$ such that $JK = L$ and $Jv = \sigma$.

there is a unique oplax cocone $(L' : \mathbb{E}\to J^*(\mathbb{D}), \sigma^\dagger : L'\Rightarrow L'\tilde{G})$ such that $((L', K), (v, \sigma))$ is a fibred oplax cocone for $(\tilde{G}, G)$, $q^*(J)L' = L$ and $q^*(J)\sigma^\dagger = \tilde{\sigma}$, where $q^*(J) : \mathbb{A} \times_{\mathbb{C}} \mathbb{D} \to \mathbb{D}$ is the pullback projection.

Proof. L' and $\sigma^\dagger$ are uniquely determined by the (2-dimensional) universal property of the pullback $\mathbb{A} \times_{\mathbb{C}} \mathbb{D}$

□

Lemma B.4. Consider a comonad $(\tilde{G}, G)$: $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix} \to \begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$. Let $(U : \mathbb{B} \to \mathbb{B}_G, \lambda)$ be the Kleisli object for the base comonad G. A fibred oplax cocone for $(\tilde{G}, G)$ over (U, λ), $((\tilde{U}, U)$: $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix} \to \begin{smallmatrix}\mathbb{D}\\ \downarrow q\\ \mathbb{C}\end{smallmatrix})$, induces an oplax cocone $(U' : F^*(p) \to q, \lambda')$ (in $\mathbf{Fib}(\mathbb{B}_G)$) for the comonad $\overline{G}$ associated to the Kleisli resolution $F_G \dashv U_G : \mathbb{B} \to \mathbb{B}_G$ (*cf.* Lemma B.2).

Remark B.5. The proof of the above lemma is based on the fact that an adjunction $F \dashv U : \mathbb{B} \to \mathbb{A}$ induces a (bi)adjunction $F^* \dashv U^* : \mathbf{Fib}(\mathbb{B}) \to \mathbf{Fib}(\mathbb{A})$.

Theorem B.6. Fib admits Kleisli objects for comonads and satisfies PCK.

Proof. Given a comonad $(\tilde{G}, G)$: $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix} \to \begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$, let $(U : \mathbb{B} \to \mathbb{B}_G, \lambda)$ be the Kleisli object for G and $F \dashv U$ its associated resolution. By Lemma B.2, this resolution induces a comonad $\overline{G} : F^*(p) \to F^*(p)$ in $\mathbf{Fib}(\mathbb{B}_G)$, which admits a Kleisli fibration $F^*(p)_{\overline{G}} : (\mathbb{B}_G \times_{\mathbb{B}} \mathbb{E})_{\overline{G}} \to \mathbb{B}_G$, by Proposition B.1. This is the fibration corresponding to the Kleisli object for $(\tilde{G}, G)$. The corresponding oplax cocone is $((U_1U_2, U) : p \to F^*(p)_{\overline{G}}, (\lambda', \lambda))$, and is given as follows:

(i) $U_1 : F^*(p) \to F^*(p)_{\overline{G}}$ is the fibred functor corresponding to the Kleisli object (U_1, λ_1) for $\overline{G}$ in $\mathbf{Fib}(\mathbb{B}_G)$. The associated resolution is $F_1 \dashv U_1$.

(ii) $(U_2, U) : p \to F^*(p)$ is the right adjoint (in $\mathbf{Fib}$) to $(p^*(F), F) : F^*(p) \to p$ (change-of-base square), where $p^*(F) : \mathbb{B}_G \times_{\mathbb{B}} \mathbb{E} \to \mathbb{E}$ is the pullback projection. The existence of such a right adjoint is proved in Hermida (1993, Lemma 3.2.1). The action of U_2 on objects is given by

$$X \in \mathbb{E}_I \mapsto \epsilon_I^*(X)$$

where $\epsilon : FU \Rightarrow 1$ is the counit of $F \dashv U$, which in this case is that of the comonad G.

(iii) λ' is obtained from the resolution $p^*(F)F_1 \dashv U_1U_2$ for $\tilde{G}$ given by the adjunctions $F_1 \dashv U_1$ and $p^*(F) \dashv U_2$. That one gets a resolution in this way follows from Hermida (1993, Proposition 5.4.9).

The universal property of the oplax cocone $((U_1U_2, U): p \to F^*(p)_{\overline{G}}, (\lambda', \lambda))$ follows from Lemmas B.3 and B.4: given another oplax cocone $((\tilde{L}, L): \begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix} \to \begin{smallmatrix}\mathbb{D}\\ \downarrow q\\ \mathbb{C}\end{smallmatrix}, (\tilde{\sigma}, \sigma))$, we get a functor $J: \mathbb{B}_G \to \mathbb{C}$ by universality of (U, λ). Applying Lemma B.3 we get a fibred oplax cocone $((L', U): p \to J^*(q), (\sigma', \lambda))$. This cocone then yields, by Lemma B.4, an oplax cocone for $\overline{G}: F^*(p) \to F^*(p)$, which yields the desired mediating fibred functor between $F^*(p)_{\overline{G}}$ and q (composing with $(q^*(J), J): J^*(q) \to q$).
Finally, condition PCK follows from the above construction, given that it holds both in **Cat** and **Fib**($\mathbb{B}_G$). □

Finally, we indicate how the above construction of Kleisli objects in **Fib**, applied to the comonad $(\_ \times X, \_ \times pX): \begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix} \to \begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ (p with finite products, $X \in \mathbb{E}$) yields the fibration $p/\!/X: \mathbb{E}/\!/(X) \to \mathbb{B}/\!/pX$ of Definition 5.2.

To calculate $F^*(p)_{\overline{G}}$, we consider first $F^*(p): \mathbb{B}/\!/pX \times_{\mathbb{B}} \mathbb{E} \to \mathbb{B}/\!/pX$, where $F: \mathbb{B}/\!/pX \to \mathbb{B}$ is the left adjoint of the Kleisli resolution for $\_ \times pX: \mathbb{B} \to \mathbb{B}$.

$\mathbb{B}/\!/pX \times_{\mathbb{B}} \mathbb{E}$	**objects**	(I, Y) with $pY = I \times pX\ (= FI)$
	morphisms	$(u, f): (I, Y) \to (J, Z)$ is a pair of morphisms $u: I \times pX \to J$ in $\mathbb{B}$ and $f: Y \to Z$ in $\mathbb{E}$ above $(Fu =)\langle u, \pi' \rangle: I \times pX \to J \times pX$

We obtain $F^*(p)_{\overline{G}}$ by changing the morphisms in the total category: now a morphism $(u, f): (I, Y) \to (J, Z)$ has $f: Y' \to Z$ above $(Fu =)\langle u, \pi' \rangle: I \times pX \to J \times pX$, where $\overline{G}(I, Y) = (I, Y')$. The comonad $\overline{G}: F^*(p) \to F^*(p)$, from Lemma B.2, acts on objects as follows:

$$\overline{G}(I, Y) = (I, F\eta_I^*(Y \times X) = (I, \langle id, \pi \rangle^*(\pi^*(Y) \times \pi'^*(X))) \cong (I, Y \times \pi'^*(X))$$

Thus, the Kleisli object $F^*(p)_{\overline{G}}$ agrees with the fibration $p/\!/X$ of Definition 5.2.

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