Alexandroff and Scott Topologies for Generalized Metric Spaces

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ABSTRACT: Generalized metric spaces are a common generalization of preorders and ordinary metric spaces. Every generalized metric space can be isometrically embedded in a complete function space by means of a metric version of the categorical *Yoneda embedding*. This simple fact gives naturally rise to: 1. a topology for generalized metric spaces which for arbitrary preorders corresponds to the Alexandroff topology and for ordinary metric spaces reduces to the ε-ball topology; 2. a topology for algebraic generalized metric spaces generalizing both the Scott topology for algebraic complete partial orders and the ε-ball topology for metric spaces.

1. INTRODUCTION

Partial orders and metric spaces play a central role in the semantics of programming languages (see, e.g., [21] and [3]). Parts of their theory have been developed because of semantic necessity (see, e.g., [18] and [1]). Generalized metric spaces provide a common framework for the study of both preorders and ordinary metric spaces. A *generalized metric space* (gms for

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short) consists of a set X together with a distance function $X(-, -): X \times X \to [0, \infty]$ satisfying, for all x, y, and z in X,

1. $X(x, x) = 0$ and
2. $X(x, z) \leq X(x, y) + X(y, z)$.

Clearly every ordinary metric space is a gms. A preorder $\leq$ on X can be represented by the gms X with

$$X(x, y) = \begin{cases} 0 & \text{if } x \leq y \\ \infty & \text{if } x \not\leq y, \end{cases}$$

for x and y in X. Reflexivity and transitivity of $\leq$ imply 1. and 2., respectively. By a slight abuse of language, any gms stemming from a preorder in this way will itself be called a preorder.

In this paper we propose two topologies for gms's. The first one is a generalized Alexandroff topology. For preorders it coincides with the Alexandroff topology while for metric spaces it corresponds to the ε-ball topology. The second one is a generalized Scott topology. For algebraic complete partial orders it corresponds to the Scott topology, while for metric spaces it coincides with the ε-ball topology. Both topologies are defined in two ways: by specifying the open sets and by a closure operator. These two alternative definitions are shown to coincide.

Our definition of the generalized Alexandroff topology in terms of open sets is similar to the ones given by Smyth [15], [16] and Flagg and Kopperman [5]. A definition of a generalized Scott topology in terms of open sets similar to ours is briefly mentioned by Smyth in [15].

The definition of the generalized Alexandroff topology in terms of a closure operator already appears in [10], [11], [9]. New is the definition of the generalized Scott topology in terms of a closure operator. Both closure operators are defined by means of an adjunction between preorders. In defining these adjunctions we use the fact — first observed by Lawvere [10] — that, intuitively, one may identify elements x of a gms X with a description of the distances between any element y in X and x. Formally, this description is a function mapping every y in X to the distance $X(y, x)$. These functions from X to $[0, \infty]$ can be interpreted as fuzzy subsets of X. The value a function ϕ assigns to an element y in X is thought of as a measure for the extent to which y is an element of ϕ. This fact corresponds to a generalized metric version of the categorical Yoneda lemma [22]. The corresponding Yoneda embedding isometrically embeds a gms X into the gms of fuzzy subsets of X. By comparing the fuzzy subsets of X with the ordinary subsets of X we obtain an adjunction. This adjunction gives rise to the closure operator defining the generalized Alexandroff topology. Similarly, an algebraic gms X can be isometrically embedded into the gms of fuzzy subsets of its basis B. By comparing the fuzzy subsets of B with the subsets of X we obtain another adjunction inducing the closure operator defining the generalized Scott topology.

Like the ordinary Scott topology for complete partial orders, the generalized Scott topology encodes all information about order, convergence, and continuity (cf. [16]). The generalized Alexandroff topology only encodes the information about order, just like the ordinary Alexandroff topology for preorders (cf. [15], [5]).

The paper is organized as follows. Section 2 and 4 give some basic definitions and facts on gms's. The Yoneda lemma and the generalized Alexandroff topology are discussed in Section 3, while the generalized Scott topology is presented in Section 5. Finally, in Section 6 some related work is discussed.

2. GENERALIZED METRIC SPACES

In this section and Section 4 some basic facts and definitions on gms's are presented. This section is concluded with a table containing the preorder and ordinary metric notions corresponding to the notions introduced below.

An important example of a gms is the set of (extended) real numbers $[0, \infty]$ with the distance function defined, for r and s in $[0, \infty]$, by

$$[0, \infty](r, s) = \begin{cases} 0 & \text{if } r \geq s \\ s - r & \text{if } r < s. \end{cases}$$

This gms is a *quasimetric space* (qms for short): besides the axioms 1. and 2. of the introduction it also satisfies, for all x and y in X, if $X(x, y) = 0$ and $X(y, x) = 0$ then $x = y$. The gms $[0, \infty]$ has the following fundamental property. For all r, s, t in $[0, \infty]$,

$$r + s \geq t \quad \text{if and only if } r \geq [0, \infty](s, t). \tag{1}$$

The above equation expresses that the category with the elements in $[0, \infty]$ as objects and the relation $\geq$ defining the morphisms is a closed category with $+$ as tensor. Many properties of gms's derive from this categorical structure on $[0, \infty]$.

The gms *opposite* to a gms X, denoted by X^{op}, is the set X with the distance function defined, for x and y in X, by

$$X^{op}(x, y) = X(y, x).$$

Let X and Y be gms's. A function $f\colon X \to Y$ is *nonexpansive* if, for all x and y in X,

$$Y(f(x), f(y)) \leq X(x, y).$$

If the above inequality always is an equality then f is said to be *isometric*. The set of nonexpansive functions from X to Y, denoted by Y^X, together with the distance function defined, for f and g in Y^X, by

$$Y^X(f, g) = \sup_{x \in X} Y(f(x), g(x)),$$

is a gms.

As we have seen in the introduction, a preorder can be viewed as a gms. Conversely, a gms gives rise to a preorder. The *underlying preorder* of a gms X is defined, for x and y in X, by

$$x \leq_X y \quad \text{if and only if } X(x, y) = 0.$$

The preorder and metric notions corresponding to the ones introduced above are listed below.

gms	preorder	metric space
qms	partial order	metric space
opposite	opposite	identity
nonexpansive	monotone	nonexpansive
isometric	order equivalence	isometric
underlying preorder	preorder	identity relation

3. A GENERALIZED ALEXANDROFF TOPOLOGY

We present a generalized Alexandroff topology for gms's. The following lemma turns out to be of great importance for the definitions of topologies for gms's as we shall see below and in Section 5. It is the $[0, \infty]$-enriched version of the famous Yoneda lemma [22], [8] from category theory.

For a gms X, let $\hat{X}$ denote the nonexpansive function space

$$\hat{X} = [0, \infty]^{X^{op}}.$$

An element ϕ in $\hat{X}$ can be interpreted as a fuzzy subset of X. The value that ϕ assigns to an element x in X is thought of as a measure for the extent to which x is an element of ϕ. The smaller this number, the more x should be viewed as an element of ϕ. Every gms can be mapped into the gms $\hat{X}$ of its fuzzy subsets by the *Yoneda embedding* $\mathsf{y}: X \to \hat{X}$ which is defined, for x in X, by

$$\mathsf{y}(x) = \lambda y \in X.\, X(y, x).$$

Note that the nonexpansiveness of $\mathsf{y}(x)$ is an immediate consequence of condition 2. of the introduction and (1).

LEMMA 3.1: (Yoneda) Let X be a gms. For all x in X and ϕ in $\hat{X}$,

$$\hat{X}(\mathsf{y}(x), \phi) = \phi(x).$$

Proof: Let x be in X and ϕ be in $\hat{X}$.

$$\begin{aligned}\phi(x) &= [0, \infty](X(x, x), \phi(x)) \\ &\leq \sup_{y \in X} [0, \infty](X(y, x), \phi(y)) \\ &= \hat{X}(\mathsf{y}(x), \phi)\end{aligned}$$

Because ϕ is nonexpansive, for all y in X,

$$[0, \infty](\phi(x), \phi(y)) \leq X^{op}(x, y) = X(y, x) = \mathsf{y}(x)(y).$$

According to (1), this is equivalent to

$$[0, \infty](\mathsf{y}(x)(y), \phi(y)) \leq \phi(x).$$

Consequently, $\hat{X}(\mathsf{y}(x), \phi) \leq \phi(x)$. ❑

The following corollary is immediate.

COROLLARY 3.2: Let X be a gms. The Yoneda embedding $\mathsf{y}\colon X \to \hat{X}$ is isometric.

The closure operator defining the generalized Alexandroff topology for a gms X is obtained by comparing the fuzzy subsets of X with the ordinary subsets of X. Given a fuzzy subset ϕ in $\hat{X}$, by taking only its real elements, i.e., the elements x in X for which $\phi(x) = 0$, we obtain its *extension*

$$\mathsf{e}_A(\phi) = \{x \in X \mid \phi(x) = 0\},$$

where the subscript A stands for Alexandroff. Note that

$$\begin{aligned}\mathsf{e}_A(\phi) &= \{x \in X \mid \phi(x) = 0\} \\ &= \{x \in X \mid \hat{X}(\mathsf{y}(x), \phi) = 0\} \text{ [Yoneda lemma 3.1]} \\ &= \{x \in X \mid \mathsf{y}(x) \leq_{\hat{X}} \phi\}.\end{aligned}$$

Any subset V in $\mathcal{P}(X)$ defines a fuzzy subset $\mathsf{k}_A(V)$ in $\hat{X}$ which is referred to as the *character* of the subset V. It is defined by

$$\mathsf{k}_A(V) = \lambda x \in X.\ \inf_{v \in V} X(x, v).$$

The closer an element x in X is to the subset V, the more x should be viewed as an element of the character of V. Note that

$$\mathsf{k}_A(V) = \lambda x \in X.\ \inf_{v \in V} \mathsf{y}(v)(x).$$

The functions $\mathsf{e}_A : \hat{X} \to \mathcal{P}(X)$ and $\mathsf{k}_A : \mathcal{P}(X) \to \hat{X}$ can be nicely related by considering $\hat{X}$ with the underlying preorder $\leq_{\hat{X}}$ and $\mathcal{P}(X)$ ordered by subset inclusion.

PROPOSITION 3.3: Let X be a gms. The functions $\mathsf{e}_A : (\hat{X}, \leq_{\hat{X}}) \to (\mathcal{P}(X), \subseteq)$ and $\mathsf{k}_A : (\mathcal{P}(X), \subseteq) \to (\hat{X}, \leq_{\hat{X}})$ are monotone. Moreover, k_A is left adjoint to e_A.

Proof: Monotonicity of e_A and k_A follows directly from their definitions. We will hence concentrate on the second part of the proposition by proving, for all V in $\mathcal{P}(X)$ and ϕ in $\hat{X}$, $V \subseteq \mathsf{e}_A(\mathsf{k}_A(V))$ and $\mathsf{k}_A(\mathsf{e}_A(\phi)) \leq_{\hat{X}} \phi$, which is equivalent to k_A being left adjoint to e_A (cf. Theorem 0.3.6 of [7]). Because, for all V in $\mathcal{P}(X)$ and v in V, $\mathsf{y}(v) \leq_{\hat{X}} \mathsf{k}_A(V)$, we have that

$$V \subseteq \{x \in X \mid \mathsf{y}(x) \leq_{\hat{X}} \mathsf{k}_A(V)\} = \mathsf{e}_A(\mathsf{k}_A(V)).$$

Furthermore, for ϕ in $\hat{X}$ and x in X,

$$\begin{aligned}
\mathsf{k}_A(\mathsf{e}_A(\phi))(x) &= \inf\{X(x, y) \mid y \in X \wedge \mathsf{y}(y) \leq_{\hat{X}} \phi\} \\
&= \inf\{\mathsf{y}(y)(x) \mid y \in X \wedge \forall z \in X\colon \mathsf{y}(y)(z) \geq \phi(z)\} \\
&\geq \inf\{\mathsf{y}(y)(x) \mid y \in X \wedge \mathsf{y}(y)(x) \geq \phi(x)\} \\
&\geq \phi(x).
\end{aligned}$$

Consequently, $\mathsf{k}_A(\mathsf{e}_A(\phi)) \leq_{\hat{X}} \phi$. (Note that the ordering underlying $[0, \infty]$ is the opposite of the usual ordering.) ❑

The above fundamental adjunction relates the character of subsets and the extension of fuzzy subsets and is often referred to as the *comprehension schema* [10], [9]. As with any adjoint pair between preorders, the composition $\mathsf{e}_A \circ \mathsf{k}_A$ is a closure operator on X (cf. Theorem 0.3.6 of [7]). It satisfies, for V in $\mathcal{P}(X)$,

$$\begin{aligned}
(\mathsf{e}_A \circ \mathsf{k}_A)(V) &= \{x \in X \mid \mathsf{k}_A(V)(x) = 0\} \\
&= \{x \in X \mid \hat{X}(\mathsf{y}(x), \mathsf{k}_A(V)) = 0\} \quad \text{[Yoneda lemma 3.1]} \\
&= \{x \in X \mid \forall y \in X\colon [0, \infty](\mathsf{y}(x)(y), \mathsf{k}_A(V)(y)) = 0\} \\
&= \{x \in X \mid \forall y \in X\colon \mathsf{y}(x)(y) \geq \mathsf{k}_A(V)(y)\} \\
&= \{x \in X \mid \forall y \in X\colon \forall \varepsilon > 0\colon X(y, x) < \varepsilon \Rightarrow \exists v \in V\colon X(y, v) < \varepsilon\}. \quad (2)
\end{aligned}$$

By using the above characterization (2) we can prove the following theorem.

THEOREM 3.4: Let X be a gms. The closure operator $\mathsf{e}_A \circ \mathsf{k}_A$ on X is topological.

Proof: It is an immediate consequence of (2) that $(\mathsf{e}_A \circ \mathsf{k}_A)(\varnothing) = \varnothing$. Because $\mathsf{e}_A \circ \mathsf{k}_A$ is a closure operator, for V, W in $\mathcal{P}(X)$,

$$(\mathsf{e}_A \circ \mathsf{k}_A)(V) \cup (\mathsf{e}_A \circ \mathsf{k}_A)(W) \subseteq (\mathsf{e}_A \circ \mathsf{k}_A)(V \cup W).$$

For the reverse inclusion, let x be in $(\mathsf{e}_A \circ \mathsf{k}_A)(V \cup W)$. Suppose x is not in $(\mathsf{e}_A \circ \mathsf{k}_A)(V)$. We will show that x is in $(\mathsf{e}_A \circ \mathsf{k}_A)(W)$. Let y_W be in X and $\varepsilon_W > 0$ with $X(y_W, x) < \varepsilon_W$. We should find a w in W with $X(y_W, w) < \varepsilon_W$. Because x is not in $(\mathsf{e}_A \circ \mathsf{k}_A)(V)$, there exist a y_V in X and an $\varepsilon_V > 0$ such that

$$X(y_V, x) < \varepsilon_V \wedge \forall v \in V: X(y_V, v) \geq \varepsilon_V. \tag{3}$$

Let $\varepsilon = \min\{\varepsilon_V - X(y_V, x), \varepsilon_W - X(y_W, x)\}$. Because x is in $(\mathsf{e}_A \circ \mathsf{k}_A)(V \cup W)$ and $X(x, x) < \varepsilon$, there exists a w in $V \cup W$ with $X(x, w) < \varepsilon$. The assumption that w is in V contradicts (3), because

$$X(y_V, w) \leq X(y_V, x) + X(x, w) < \varepsilon_V.$$

Thus w is in W. Furthermore,

$$X(y_W, w) \leq X(y_W, x) + X(x,w) < \varepsilon_W. \quad \square$$

The above theorem implies that the closure operator $\mathsf{e}_A \circ \mathsf{k}_A$ induces a topology on X. In Theorem 3.5 below, it is proved equivalent to the following generalized ε-ball topology. For x in X and $\varepsilon > 0$, we define the *generalized ε-ball* of x by

$$B_\varepsilon(x) = \{y \in X \mid X(x, y) < \varepsilon\}.$$

A subset V of a gms X is *generalized Alexandroff open* (gA-open for short) if, for all x in V,

$$\exists \varepsilon > 0: B_\varepsilon(x) \subseteq V.$$

For instance, for every x in X, the generalized ε-ball $B_\varepsilon(x)$ is gA-open. The set of all gA-open subsets of X is denoted by O_{gA}. One can easily verify that O_{gA} is a topology on X with $\{B_\varepsilon(x) \mid \varepsilon > 0 \wedge x \in X\}$ as basis.

For ordinary metric spaces, the above introduced generalized ε-balls are as usual, while for preorders they are upper-closed sets: if X is a preorder then

$$\begin{aligned} B_\varepsilon(x) &= \{y \in X \mid X(x, y) < \varepsilon\} \\ &= \{y \in X \mid X(x, y) = 0\} \\ &= \{y \in X \mid x \leq_X y\}. \end{aligned} \tag{4}$$

Consequently, the generalized Alexandroff topology restricted to metric spaces is the ε-ball topology, while restricted to preorders it is the ordinary Alexandroff topology.

For V in $\mathcal{P}(X)$, we write $cl_A(V)$ for the closure of V in the generalized Alexandroff topology.

THEOREM 3.5: Let X be a gms. For all V in $\mathcal{P}(X)$, $cl_A(V) = (\mathsf{e}_A \circ \mathsf{k}_A)(V)$.

Proof: For every topology O on X, the induced topological closure operator cl on X satisfies, for all V in $\mathcal{P}(X)$, $cl(V) = V \cup V^d$, where V^d is the derived set, the set of all accumulation points, of V. It follows from this fact and characterization (2) of $\mathsf{e}_A \circ \mathsf{k}_A$ that it is sufficient to prove

$$V \cup V^d = \{x \in X \mid \forall y \in X\colon \forall \varepsilon > 0\colon X(y, x) < \varepsilon \Rightarrow \exists v \in V\colon X(y, v) < \varepsilon\}. \quad (5)$$

From the definition of accumulation point and the fact that the set of all generalized ε-balls is a basis for the generalized Alexandroff topology, it follows that, for every x in X,

$$\begin{aligned} x \in V^d &\Leftrightarrow \forall W \in O_{gA}\colon x \in W \Rightarrow W \cap (V\backslash\{x\}) \neq \varnothing \\ &\Leftrightarrow \forall y \in X\colon \forall \varepsilon > 0\colon x \in B_\varepsilon(y) \Rightarrow B_\varepsilon(y) \cap (V\backslash\{x\}) \neq \varnothing \\ &\Leftrightarrow \forall y \in X\colon \forall \varepsilon > 0\colon X(y, x) < \varepsilon \Rightarrow \exists v \in (V\backslash\{x\})\colon X(y, v) < \varepsilon. \end{aligned} \quad (4)$$

Therefore, (5) holds. ❑

Every topology O on X induces a preorder on X called the *specialization preorder*: for all x and y in X, $x \leq_O y$ if and only if, for all V in O, if x is in V then y is in V. The specialization preorder on a gms X induced by its generalized Alexandroff topology coincides with the preorder underlying X.

PROPOSITION 3.6: Let X be a gms. For all x and y in X, $x \leq_{O_{gA}} y$ if and only if $x \leq_X y$.

Proof: For any gA-open set V, if x is in V and $X(x, y) = 0$ then y is in V. From this observation the implication from right to left is clear. For the converse, suppose $x \leq_{O_{gA}} y$. Then, for every $\varepsilon > 0$, x is in $B_\varepsilon(x)$ implies y is in $B_\varepsilon(x)$, because generalized ε-balls are gA-open sets. Hence $X(x, y) < \varepsilon$. Since ε was arbitrary, $X(x, y) = 0$, that is $x \leq_X y$. ❑

The above proposition tells us that the underlying preorder of a gms can be reconstructed from its generalized Alexandroff topology.

Note that the specialization preorder $\leq_{O_{gA}}$ is a partial order — this is equivalent to the generalized Alexandroff topology being T_0 — if and only if X is a qms.

4. CAUCHY SEQUENCES, LIMITS, AND COMPLETENESS

Some further basic facts and definitions on gms's are presented. Like Section 2, this section is concluded with a table containing the preorder and ordinary metric notions corresponding to the notions introduced below.

A sequence $(x_n)_n$ in a gms X is *forward-Cauchy* if

$$\forall \varepsilon > 0 : \exists N : \forall n \geq m \geq N : X(x_m, x_n) \leq \varepsilon.$$

Since our distance functions need not be symmetric, the following variation exists. A sequence $(x_n)_n$ is *backward-Cauchy* if

$$\forall \varepsilon > 0 : \exists N : \forall n \geq m \geq N : X(x_n, x_m) \leq \varepsilon.$$

The *forward-limit* of a forward-Cauchy sequence $(r_n)_n$ in $[0, \infty]$ is given by

$$\underrightarrow{\lim}_n r_n = \sup_n \inf_{k \geq n} r_k.$$

Dually, the *backward-limit* of a backward-Cauchy sequence $(r_n)_n$ in $[0, \infty]$ is defined by

$$\underleftarrow{\lim}_n r_n = \inf_n \sup_{k \geq n} r_k.$$

Forward-limits and backward-limits in $[0, \infty]$ are related as follows. For all forward-Cauchy sequences $(r_n)_n$ in $[0, \infty]$ and r in $[0, \infty]$,

$$[0, \infty](\underrightarrow{\lim}_n r_n, r) = \underleftarrow{\lim}_n [0, \infty](r_n, r). \tag{6}$$

Forward-limits in an arbitrary gms can now be defined in terms of backward-limits in $[0, \infty]$. An element x is a *forward-limit* of a forward-Cauchy sequence $(x_n)_n$ in a gms X, denoted by $x \in \underrightarrow{\lim}_n x_n$, if, for all y in X,

$$X(x, y) = \underleftarrow{\lim}_n X(x_n, y).$$

Note that if $(x_n)_n$ is a forward-Cauchy sequence in X, then, for all y in X, $(X(x_n, y))_n$ is a backward-Cauchy sequence in $[0, \infty]$ because of (1). Our earlier definition of the forward-limit of forward-Cauchy sequences in $[0, \infty]$ is consistent with this definition for arbitrary gms's because of (6). Similarly one can define backward-limits in an arbitrary gms. Since these will not play a role in this paper, their definition is omitted. For simplicity, we shall use *Cauchy* instead of forward-Cauchy and *limit* instead of forward-limit. Note that Cauchy sequences may have more than one limit. Let $(x_n)_n$ be a Cauchy sequence in a gms X and x be in X, with $x \in \underrightarrow{\lim}_n x_n$. For all y in X,

$$y \in \underrightarrow{\lim}_n x_n \text{ if and only if } X(x, y) = 0 \text{ and } X(y, x) = 0. \tag{7}$$

Consequently, limits are unique in qms's. In that case we sometimes write $x = \underrightarrow{\lim}_n x_n$.

A gms X is *complete* if every Cauchy sequence in X has a limit. For example, $[0, \infty]$ is complete. Let X and Y be gms's. If Y is complete then Y^X is also complete (cf. [14, Theorem 6.5]). Consequently, for every gms X, the function space $\hat{X}$ is complete.[1] Limits of the complete function space Y^X are taken pointwise. For all Cauchy sequences $(f_n)_n$ in Y^X and f in Y^X,

$$f \in \underrightarrow{\lim}_n f_n \quad \text{if and only if, for all } x \text{ in } X,\ f(x) \in \underrightarrow{\lim}_n (f_n(x)). \tag{8}$$

Let X and Y be gms's. A nonexpansive function $f: X \to Y$ is *continuous* if it preserves limits: for all Cauchy sequences $(x_n)_n$ in X and x in X, with $x \in \underrightarrow{\lim}_n x_n$, $f(x) \in \underrightarrow{\lim}_n f(x_n)$.

The preordered notion finite can be generalized as follows. An element x in a gms X is *finite* if the function $\lambda y \in X \,.\, X(x, y)$ from X to $[0, \infty]$ is continuous. In order to conclude that x is finite in X, it suffices to prove that, for all Cauchy sequences $(y_n)_n$ in X and y in X, with $y \in \underrightarrow{\lim}_n y_n$, $\underrightarrow{\lim}_n X(x, y_n) \le X(x, y)$. For example, for all x in X, one can show that

$$\mathsf{y}(x) \text{ is finite in } \hat{X} \tag{9}$$

(cf. [2, Lemma 4.3]).

A subset B of finite elements of a gms X is a *basis* for X if every element in X is a limit of a Cauchy sequence in B. A gms is *algebraic* if there exists a basis. Such a basis is generally not unique.

Below we give the table with the corresponding preorder and metric notions.

gms	preorder	metric space
forward-Cauchy	eventually increasing	Cauchy
backward-Cauchy	eventually decreasing	Cauchy
forward-limit	eventually minimal upperbound	limit
backward-limit	eventually maximal lowerbound	limit
complete	ω-complete	complete
continuous	ω-continuous	continuous
finite	finite	arbitrary
basis	basis	dense subset
algebraic	algebraic	arbitrary

5. A GENERALIZED SCOTT TOPOLOGY

In the Scott topology of a complete partial order X least upperbounds of increasing sequences in X are topological limits. Also, in the ε-ball topology of an ordinary metric space, X limits of Cauchy sequences in X are topolog-

[1]As a consequence, the Yoneda embedding y isometrically embeds a gms X into the complete gms $\hat{X}$. One can define the completion of X as the smallest complete subspace of $\hat{X}$ containing the y-image of X. For details we refer the reader to [2].

ical limits. However, a similar result does not hold for our generalized Alexandroff topology. For example, for complete partial orders the generalized Alexandroff topology coincides with the ordinary Alexandroff topology, for which this result does not hold in general. The Scott topology is the coarsest topology refining the Alexandroff topology with this property (cf.[7], [12], [17]). Also for gms's a suitable refinement of the generalized Alexandroff topology exists.

A key step towards the definition of the generalized Scott topology for algebraic gms's is the following restriction of the Yoneda embedding.

LEMMA 5.1: Let X be a gms. If B is a basis for X, then the function $\mathsf{y}_B: X \to \hat{B}$ defined, for x in X, by

$$\mathsf{y}_B(x) = \lambda b \in B.\, X(b, x)$$

is isometric and continuous.

Proof: According to Corollary 3.2, y is isometric. Consequently, y_B is nonexpansive. According to (8), to prove that y_B is continuous it suffices to show that, for all Cauchy sequences $(x_n)_n$ in X and x in X, with $x \in \underrightarrow{\lim}_n x_n$, and b in B,

$$\mathsf{y}_B(x)(b) \in \underrightarrow{\lim}_n \mathsf{y}_B(x_n)(b).$$

This follows immediately from the fact that b is finite in X. The function y_B is isometric, because, for x and y in X, since B is a basis there exist Cauchy sequences $(b_m)_m$ and $(c_n)_n$ in B such that $x \in \underrightarrow{\lim}_m b_m$ and $y \in \underrightarrow{\lim}_n c_n$, and

$$\begin{aligned}
\hat{B}(\mathsf{y}_B(x), \mathsf{y}_B(y)) &= \underleftarrow{\lim}_m B(\mathsf{y}_B(b_m), \mathsf{y}_B(y)) && [x \in \underrightarrow{\lim}_m b_m,\ \mathsf{y}_B \text{ is continuous}]\\
&= \underleftarrow{\lim}_m \underrightarrow{\lim}_n \hat{B}(\mathsf{y}_B(b_m), \mathsf{y}_B(c_n)) && [y \in \underrightarrow{\lim}_n c_n,\ \mathsf{y}_B \text{ is continuous,}\\
& && \ \mathsf{y}_B(b_m) \text{ is finite in } \hat{B} \text{ according to (9)}]\\
&= \underleftarrow{\lim}_m \underrightarrow{\lim}_n B(b_m, c_n) && [\text{Corollary 3.2}]\\
&= \underleftarrow{\lim}_m \underrightarrow{\lim}_n X(b_m, c_n)\\
&= \underleftarrow{\lim}_m X(b_m, y) && [y \in \underrightarrow{\lim}_n c_n,\ b_m \text{ is finite in } X]\\
&= X(x, y). && [x \in \underrightarrow{\lim}_m b_m] \quad \square
\end{aligned}$$

The converse of the above lemma holds as well (cf. [2, Theorem 5.6]).

The closure operator defining the generalized Scott topology for an algebraic gms X with basis B is obtained by comparing the fuzzy subsets of B, rather than the fuzzy subsets of X as we have done in Section 3, with the ordinary subsets of X. The *extension* function $\mathsf{e}_S: \hat{B} \to \mathcal{P}(X)$ is defined, for ϕ in $\hat{B}$, by

$$\mathsf{e}_S(\phi) = \{x \in X \mid \mathsf{y}_B(x) \leq_{\hat{B}} \phi\}$$

and the *character* function $\mathsf{k}_S: \mathcal{P}(X) \to \hat{B}$ is defined, for V in $\mathcal{P}(X)$, by

$$\mathsf{k}_S(V) = \lambda b \in B.\ \inf_{v \in V} \mathsf{y}_B(v)(b).$$

As in Proposition 3.3, the functions e_S: $(\hat{B}, \leq_{\hat{B}}) \rightarrow (\mathcal{P}, (X), \subseteq)$ and k_S: $(\mathcal{P}(X), \subseteq) \rightarrow (\hat{B}, \leq_{\hat{B}})$ are monotone and k_S is left adjoint to e_S. Thus, $\mathsf{e}_S \circ \mathsf{k}_S$ is a closure operator on X. Since a basis is generally not unique, one might think that the definition of the closure operator $\mathsf{e}_S \circ \mathsf{k}_S$ depends on the choice of the basis. That this is not the case is a consequence of Theorem 5.6 below. In a way similar to (2), this closure operator can be characterized, for V in $\mathcal{P}(X)$, by

$$(\mathsf{e}_S \circ \mathsf{k}_S)(V) = \{x \in X \mid \forall b \in B: \forall \varepsilon > 0: X(b, x) < \varepsilon \Rightarrow \exists v \in V: X(b, v) < \varepsilon\}. \quad (10)$$

Also this closure operator is topological.

THEOREM 5.2: Let X be an algebraic gms. The closure operator $\mathsf{e}_S \circ \mathsf{k}_S$ on X is topological.

Proof: This theorem is proved along the same lines as Theorem 3.4, but one needs the following additional observation. If B is a basis for X then, for any b_V and b_W in B, $\varepsilon_V, \varepsilon_W > 0$, and x in X, such that $X(b_V, x) < \varepsilon_V$ and $X(b_W, x) < \varepsilon_W$, there exists a b in B such that $X(b_V, b) < \varepsilon_V$, $X(b_W, b) < \varepsilon_W$, and $X(b, x) < \varepsilon$, where $\varepsilon = \min\{\varepsilon_V - X(b_V, b),\ \varepsilon_W - X(b_W, b)\}$. This fact can be proved as follows. Because X is an algebraic gms with B as basis, there exists a Cauchy sequence $(b_n)_n$ in B with $x \in \underrightarrow{\lim}_n\, b_n$. Because

$$\begin{aligned} \varepsilon_V &> X(b_V, x) \\ &= \underrightarrow{\lim}_n X(b_V, b_n) \quad [x \in \underrightarrow{\lim}_n b_n,\ b_V \text{ is finite in } X], \end{aligned}$$

there exists an N_V such that, for all $n \geq N_V$, $X(b_V, b_n) < \varepsilon_V$. Similarly, there exists an N_W such that, for all $n \geq N_W$, $X(b_W, b_n) < \varepsilon_W$. Since

$$\begin{aligned} 0 &= X(x, x) \\ &= \underleftarrow{\lim}_n X(b_n, x) \quad [x \in \underrightarrow{\lim}_n b_n], \end{aligned}$$

there exists an N such that, for all $n \geq N$, $X(b_n, x) < \varepsilon$. The element $b_{\max\{N_V, N_W, N\}}$ in B is the one we were looking for. ❑

Thus, the operator $\mathsf{e}_S \circ \mathsf{k}_S$ induces a topology on X. In the case that X is a preorder with basis B, for every V in $\mathcal{P}(X)$,

$$(\mathsf{e}_S \circ \mathsf{k}_S)(V) = \{x \in X \mid \forall b \in B: b \leq_X x \Rightarrow \exists v \in V: b \leq_X v\},$$

which we recognize as the closure operator induced by the ordinary Scott topology.

Next, an alternative definition of this topology is given by specifying the open sets (this time starting from a gms X). In Theorem 5.6 below, it will be shown that the two definitions coincide whenever X is algebraic.

A subset V of a gms X is *generalized Scott open* (gS-open for short) if, for all Cauchy sequences $(x_n)_n$ in X and x in V, with $x \in \underrightarrow{\lim}_n x_n$,

$$\exists \varepsilon > 0 : \exists N : \forall n \geq N : B_\varepsilon(x_n) \subseteq V.$$

The following proposition gives an example of gS-open sets.

PROPOSITION 5.3: Let X be a gms. An element b in X is finite if and only if, for all $\varepsilon > 0$, the set $B_\varepsilon(b)$ is gS-open.

Proof: Let b be finite in X and $\varepsilon > 0$. We have to show that the generalized ε-ball $B_\varepsilon(b)$ is gS-open. Let $(x_n)_n$ be a Cauchy sequence in X and x be in $B_\varepsilon(b)$, with $x \in \underrightarrow{\lim}_n x_n$. It suffices to prove that

$$\exists \delta > 0 : \exists N : \forall n \geq N : X(b, x_n) < \varepsilon - \delta. \tag{11}$$

Because x is in $B_\varepsilon(b)$, we have that $\exists \delta > 0 : X(b, x) < \varepsilon - \delta$. Since

$$\begin{aligned} \varepsilon - \delta &> X(b, x) \\ &= \underrightarrow{\lim}_n X(b, x_n) \quad [x \in \underrightarrow{\lim}_n x_n,\ b \text{ is finite in } X] \end{aligned}$$

and the sequence $(X(b, x_n))_n$ is Cauchy, we can conclude (11).

Conversely, assume that, for all $\varepsilon > 0$, the set $B_\varepsilon(b)$ is gS-open. Let $(x_n)_n$ be a Cauchy sequence in X and x be in X, with $x \in \underrightarrow{\lim}_n x_n$. Then

$$\forall \varepsilon > 0 : x \in B_{X(b,x)+\varepsilon}(b).$$

Because the set $B_{X(b,x)+\varepsilon}(b)$ is gS-open,

$$\forall \varepsilon > 0 : \exists \delta > 0 : \exists N : \forall n \geq N : B_\delta(x_n) \subseteq B_{X(b,x)+\varepsilon}(b).$$

Hence, $\underrightarrow{\lim}_n X(b, x_n) \leq X(b, x)$. ❑

The set of all gS-open subsets of X is denoted by O_{gS}. This collection forms indeed a topology.

PROPOSITION 5.4: Let X be a gms. O_{gS} is a topology on X. If X is algebraic with basis B, then the set $\{B_\varepsilon(b) \mid \varepsilon > 0 \wedge b \in B\}$ forms a basis for O_{gS}.

Proof: One can easily verify that O_{gS} is closed under finite intersections and arbitrary unions. We will only prove that, for an algebraic gms X with basis B, every gS-open set V in $\mathcal{P}(X)$ is the union of generalized ε-balls of finite elements. Let x be in V. Since X is algebraic, there is a Cauchy sequence $(b_n)_n$ in B with $x \in \underrightarrow{\lim}_n b_n$. Because V is gS-open and $x \in \underrightarrow{\lim}_n b_n$,

$$\exists \varepsilon_x > 0 : \exists N_x : \forall n \geq N_x : B_{\varepsilon_x}(b_n) \subseteq V \wedge x \in B_{\varepsilon_x}(b_n).$$

Therefore, $V \subseteq \bigcup_{x \in V} B_{\varepsilon_x}(b_{N_x})$. Since the other inclusion trivially holds we have that the collection of all generalized ε-balls of finite elements forms a basis for O_{gS}. ❑

Note that every gS-open set is gA-open, because every element x in a gms X is a limit of the constant Cauchy sequence $(x)_n$. Therefore, the generalized Scott topology refines the generalized Alexandroff topology.

Any ordinary metric space X is an algebraic gms in which all elements are finite. Therefore, by the previous proposition, the basic open sets of the generalized Scott topology are all the generalized ε-balls. Hence, for ordinary metric spaces the generalized Scott topology coincides with the standard ε-ball topology.

For a preorder,[2] a subset is gS-open precisely when it is Scott open as is shown in the following proposition. Consequently, the generalized Scott topology restricted to preorders is the ordinary Scott topology.

PROPOSITION 5.5: Let X be a preorder. A set V in $\mathcal{P}(X)$ is gS-open if and only if:

1. for all x, y in X, if x is in V and $x \leq_X y$ then y is in V, and
2. for all sequences $(x_n)_n$ in X satisfying, for all n, $x_n \leq_X x_{n+1}$, and x in V, with $x \in \underrightarrow{\lim}_n x_n$,

$$\exists N : x_N \in V.$$

Proof: Assume the set V in $\mathcal{P}(X)$ is gS-open. Let x, y be in X with x in V and $x \leq_X y$. Because V is gS-open, and hence gA-open,

$$\exists \varepsilon > 0 : B_\varepsilon(x) \subseteq V.$$

Since $x \leq_X y$, we have that y is in $B_\varepsilon(x)$ and consequently y is in V. Let $(x_n)_n$ be a sequence in X satisfying, for all n, $x_n \leq_X x_{n+1}$, and x be in V, with $x \in \underrightarrow{\lim}_n x_n$. Clearly, the sequence $(x_n)_n$ is Cauchy. Because V is gS-open,

$$\exists \varepsilon > 0 : \exists N : \forall n \geq N : B_\varepsilon(x_n) \subseteq V.$$

Hence, x_N is in V.

For the converse, assume 1. and 2. Let $(x_n)_n$ be a Cauchy sequence in X and x be in V, with $x \in \underrightarrow{\lim}_n x_n$. Because X is a preorder, there exists an N such that, for all n, $x_{N+n} \leq_X x_{N+n+1}$. One can easily verify that $x \in \underrightarrow{\lim}_n x_{N+n}$. According to 2.,

$$\exists M : x_{N+M} \in V.$$

From 1. we can conclude that, for all $m \geq M$, x_{N+m} is in V. Again using 1. and (4) we can deduce that, for all $m \geq M$, $B_1(x_{N+m}) \subseteq V$. ❑

[2]Although the Scott topology is usually only defined for complete partial orders, the construction also produces a topology for preorders.

To reconcile the Scott topology for preorders with the ε-ball topology for metric spaces, one could define topologies on gms's which are finer than our generalized Scott topology. For example, call a subset V of a gms X *naive generalized Scott* open (ngS-open for short) if V is generalized Alexandroff open and, for all Cauchy sequences $(x_n)_n$ in X and x in V, with $x \in \underrightarrow{\lim}_n x_n$,

$$\exists N : \forall n \geq N : x_n \in V.$$

Evidently, every gS-open set is ngS-open. Next we show that the naive generalized Scott topology is strictly finer than our generalized Scott topology and that the second part of Proposition 5.4 does not hold for the naive generalized Scott topology. Consider the set

$$X = \{x_1, x_2, ...,\} \cup \{x^1, x^2, ...\} \cup \{x\},$$

with the distance function defined by the following diagram.

$$\begin{tikzcd}
x^1 & x^2 & & \\
x_1 \arrow[u, "\frac{1}{2}"] \arrow[r, "\frac{1}{2}"] & x_2 \arrow[u, "\frac{1}{4}"] \arrow[r, "\frac{1}{4}"] & \ldots & x
\end{tikzcd}$$

If there is no path from y to z then $X(y, z) = 1$. Otherwise, $X(y, z)$ is the maximum of the labels of the path from y to z. For example, $X(x_2, x) = \frac{1}{4}$ and $X(x, x_2) = 1$. One can easily verify that, for all n, both x_n and x^n are finite in X and that $x \in \underrightarrow{\lim}_n x_n$. Consequently, X is an algebraic gms with basis $X \backslash \{x\}$. Consider now the set

$$V = \{x_1, x_2, ...\} \cup \{x\}.$$

Obviously, the set V is ngS-open. However, it is not gS-open as can be proved as follows. By Proposition 5.4, the set

$$\{B_\varepsilon(b) \mid \varepsilon > 0 \wedge b \in X \backslash \{x\}\}$$

forms a basis for the generalized Scott topology of X. Towards a contradiction, assume that V is gS-open. Since x is in V and the above set is a basis, there exists an $\varepsilon > 0$ and a b in $X \backslash \{x\}$ such that x is in $B_\varepsilon(b) \subseteq V$. Because b is in $B_\varepsilon(b) \subseteq V$, we have that $b = x_n$ for some n. Hence $X(x_n, x) < \varepsilon$. By definition, $X(x_n, x) = 2^{-n}$, thus $2^{-n} < \varepsilon$. Since $X(x_n, x^n) = 2^{-n}$, we have that x^n is in V, a contradiction. The above not only proves that V is not gS-open, but it also shows that the set $\{B_\varepsilon(b) \mid \varepsilon > 0 \wedge b \in B\}$ cannot be a basis for the naive generalized Scott topology.

As already announced, we show that the definitions of the generalized Scott topology defined in terms of the closure operator $\mathsf{e}_S \circ \mathsf{k}_S$ and the one defined in terms of open sets coincide. For V in $\mathcal{P}(X)$, we write $cl_S(V)$ for the closure of V in the generalized Scott topology defined in terms of open sets.

THEOREM 5.6: Let X be an algebraic gms. For all V in $\mathcal{P}(X)$, $cl_S(V) = (\mathsf{e}_S \circ \mathsf{k}_S)(V)$.

Proof: This theorem can be proved along the same lines as Theorem 3.5. It follows from characterization (10) of $\mathsf{e}_S \circ \mathsf{k}_S$ and the fact that the generalized ε-balls of finite elements form a basis for the generalized Scott topology. ❑

Since the definition of the closure operator cl_S does not use the basis, the above theorem implies that the choice of the basis is irrelevant for the definition of the closure operator $\mathsf{e}_S \circ \mathsf{k}_S$.

A subset V of a gms X is *generalized Scott closed* (gS-closed for short) if its complement $X \backslash V$ is gS-open. This is equivalent to the following condition. For all Cauchy sequences $(x_n)_n$ in X and x in X, with $x \in \underrightarrow{\lim}_n x_n$,

$$(\forall \varepsilon > 0 : \forall N : \exists n \geq N : \exists y \in V : X(x_n, y) < \varepsilon) \Rightarrow x \in V.$$

Note that if V is a gS-closed set and $(x_n)_n$ is a Cauchy sequence in V, then all its limits should belong to V. Consequently, if V is a subset of X and $(x_n)_n$ is a Cauchy sequence in V, then all its limits belong to $cl_S(V)$. This implies that if B is basis for X then $X \subseteq cl_S(B)$. The converse inclusion is obvious. Hence, the basis B of an algebraic gms X is dense in X.

The following lemma, due to Flagg and Sünderhauf, gives an example of gS-closed sets.

LEMMA 5.7: Let X be a gms. For all x in X and $\delta \geq 0$, the set $\overline{B}^{op}_\delta(x) = \{y \in X \mid X(y, x) \leq \delta\}$ is gS-closed.

Proof: Let $(z_n)_n$ be a Cauchy sequence in X and z be in X, with $z \in \underrightarrow{\lim}_n z_n$, and

$$\forall \varepsilon > 0 : \forall N : \exists n \geq N : \exists y \in \overline{B}^{op}_\delta(x) : X(z_n, y) < \varepsilon.$$

Then

$$\forall \varepsilon > 0 : \forall N : \exists n \geq N : X(z_n, x) < \varepsilon + \delta.$$

Because the sequence $(z_n)_n$ is Cauchy,

$$\forall \varepsilon > 0 : \exists N : \forall n \geq N : X(z_n, x) < \varepsilon + \delta.$$

Consequently, $\underleftarrow{\lim}_n X(z_n, x) \leq \delta$, and hence $X(z, x) \leq \delta$. ❑

Like the generalized Alexandroff topology, the generalized Scott topology provides us all information about the underlying preorder.

PROPOSITION 5.8: Let X be a gms. For all x and y in X, $x \leq_{O_{gS}} y$ if and only if $x \leq_X y$.

Proof: For any gS-open set V, if x is in V and $X(x, y) = 0$, then y is in V. From this observation, the implication from right to left is clear. For the converse, suppose $X(x, y) \neq 0$. Then x is in $X \backslash B_0^{op}(y)$ but y is not in $X \backslash B_0^{op}(y)$. Since, by Lemma 5.7, the set $X \backslash B_0^{op}(y)$ is gS-open it follows that $x \not\leq_{O_{gS}} y$. ❑

An element x is a *topological limit* of a sequence $(x_n)_n$ in a topology O, denoted by $x \in \lim_{O,n} x_n$, if, for all V in O with x in V,

$$\exists N : \forall n \geq N : x_n \in V.$$

The generalized Scott topology also encodes all information about convergence.

PROPOSITION 5.9: Let X be a gms. For all Cauchy sequences $(x_n)_n$ in X and x in X, with $x \in \underrightarrow{\lim}_n x_n$, and y in X, $y \in \lim_{O_{gS},n} x_n$ if and only if $y \leq_{O_{gS}} x$.

Proof: Clearly $x \in \lim_{O_{gS},n} x_n$, and hence $y \leq_{O_{gS}} x$ implies $y \in \lim_{O_{gS},n} x_n$. For the converse, let $y \in \lim_{O_{gS},n} x_n$. Assume $y \not\leq_{O_{gS}} x$. According to Proposition 5.8, there is a $\delta > 0$ such that $X(y, x) \not\leq \delta$. Hence, y is in $X \backslash \overline{B}_\delta^{op}(x)$, which is a gS-open set by Lemma 5.7. Since $y \in \lim_{O_{gS},n} x_n$,

$$\exists N : \forall n \geq N : x_n \in X \backslash \overline{B}_\delta^{op}(x).$$

But

$$\begin{aligned} 0 &= X(x, x) \\ &= \underleftarrow{\lim}_n X(x_n, x) \quad [x \in \underrightarrow{\lim}_n x_n] \end{aligned}$$

so

$$\exists M : \forall m \geq M : X(x_m, x) \leq \delta.$$

This gives a contradiction. Therefore, $y \leq_{O_{gS}} x$. ❑

From the above proposition we can conclude that in a gms every Cauchy sequence topologically converges to its metric limits. However, not every topologically convergent sequence is Cauchy. For example, provide the set $X = \{1, 2, \ldots, \omega\}$ with the distance function

$$X(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1/n & \text{if } x = \omega \text{ and } y = n \\ 1 & \text{otherwise.} \end{cases}$$

Then X is an algebraic complete qms with X itself as basis, since there are no nontrivial Cauchy sequences. The sequence $(n)_n$ topologically converges to ω but is not Cauchy.

Continuity is also encoded by the generalized Scott topology.

PROPOSITION 5.10: Let X and Y be complete gms's. A nonexpansive function $f: X \to Y$ is metrically continuous if and only if it is topologically continuous.

Proof: Let $f: X \to Y$ be a nonexpansive and metrically continuous function and let V in $\mathcal{P}(Y)$ be gS-open. We need to prove that $f^{-1}(V)$ in $\mathcal{P}(X)$ is gS-open in order to conclude that f is topologically continuous. Indeed, for any Cauchy sequence $(x_n)_n$ in X and x in X, with $x \in \underrightarrow{\lim}_n x_n$, we have

$$
\begin{aligned}
x \in f^{-1}(V) &\Rightarrow f(x) \in V \\
&\Rightarrow \exists \varepsilon > 0 : \exists N : \forall n \geq N : B_\varepsilon(f(x_n)) \subseteq V \\
&\quad [f \text{ is metrically continuous, } V \text{ is gS-open}] \\
&\Rightarrow \exists \varepsilon > 0 : \exists N : \forall n \geq N : B_\varepsilon(x_n) \subseteq f^{-1}(V) \\
&\quad [f \text{ is nonexpansive}]
\end{aligned}
$$

For the converse, assume $f: X \to Y$ is topologically continuous and let $(x_n)_n$ be a Cauchy sequence in X and x be in X, with $x \in \underrightarrow{\lim}_n x_n$. Let $y \in \underrightarrow{\lim}_n f(x_n)$. According to (7) it suffices to prove that $Y(y, f(x)) = 0$ and $Y(f(x), y) = 0$. We have that

$$
\begin{aligned}
Y(y, f(x)) &= \underleftarrow{\lim}_n Y(f(x_n), f(x)) \; [y \in \underrightarrow{\lim}_n f(x_n)] \\
&\leq \underleftarrow{\lim}_n X(x_n, x) \; [f \text{ is nonexpansive}] \\
&= X(x, x) \; [x \in \underrightarrow{\lim}_n x_n] \\
&= 0.
\end{aligned}
$$

According to Proposition 5.9, $x \in \lim_{O_{gS},n} x_n$. Because f is continuous, $f(x) \in \lim_{O_{gS},n} f(x_n)$. Using Proposition 5.9 again, we can conclude that $f(x) \leq_{O_{gS}} y$. By Proposition 5.8, $Y(f(x), y) = 0$. ❑

6. RELATED WORK

In this paper we have presented two topologies for gms's. The main contribution of our paper is the reconciliation of the enriched categorical approach of Lawvere [10], [11] (cf.[9], [19], [20]) and the topological approach of Smyth [15], [16] (cf. [5]). The present paper continues the work of Rutten [14] and is part of [2]. In the latter paper, besides the topologies presented in this paper, completion and powerdomains for generalized ultrametric spaces are also defined by means of the Yoneda embedding.

The basic definitions and facts on ordered spaces, metric spaces and topology, and gms's are taken from [7], [13], and [4], [17], and [16], [19], [14], respectively.

Smyth [16], and Flagg and Kopperman [5] have represented algebraic complete partial orders by another gms than the one given in the introduction. The distance function they use is in general not two-valued. In that case, the generalized Alexandroff topology reconciles the Scott topology for algebraic complete partial orders and the ε-ball topology for metric spaces. This approach is simpler than ours, since much of the standard theorems for ordinary metric spaces can be adapted. The price to be paid for this simplicity is that most of their results only hold for a restricted class of spaces: they have to be spectral.

Wagner [20] has also presented a generalized Scott topology. Although for complete partial orders his topology corresponds to the Scott topology, for metric spaces it does not coincide with the ε-ball topology.

Recently, Flagg and Sünderhauf [6] have proved that our generalized Scott topology of an algebraic complete qms arises as the sobrification of its basis taken with the generalized Alexandroff topology.

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