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Probability, Networks and Algorithms

Transient asymptotics of Lévy-driven queues

K.G. Dębicki, A. Es-Saghouani, M.R.H. Mandjes

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- We first identify conditions under which the probability of interest is dominated by the 'most demanding event', in the sense that it is asymptotically equivalent to $\mathbb{P}(Q > \max\{p, q\}B)$ for B large, where Q denotes the steady-state workload. These conditions essentially reduce to T_B being sublinear (i.e., $T_B/B \to 0$ as $B \to \infty$)

- A second condition is derived under which the probability of interest essentially 'decouples', in that it is asymptotically equivalent to $\mathbb{P}(Q > pB)\mathbb{P}(Q > qB)$ for B large. For various models considered in the literature this 'decoupling condition' reduces to requiring that T_B is superlinear (i.e., $T_B/B \to \infty$ as $B \to \infty$). Notable exceptions are two 'heavy-tailed' cases, viz. the situations in which the Lévy input process corresponds to an α-stable process, or to a compound Poisson process with regularly varying job sizes, in which the 'decoupling condition' reduces to $T_B/B^2 \to \infty$. For these input processes we also establish the asymptotics of the probability under consideration for T_B increasing superlinearly but subquadratically.

We pay special attention to the case $T_B = RB$ for some $R > 0$; for light-tailed input we derive intuitively appealing asymptotics, intensively relying on sample-path large deviations results. The regimes obtained have appealing interpretations in terms of most likely paths to overflow.

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Krzysztof Dębicki, Abdelghafour Es-Saghouani & Michel Mandjes*

December 9, 2008

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With $(Q_t)_t$ denoting the stationary workload process in a queue fed by a Lévy input process $(X_t)_t$, this paper focuses on the asymptotics of rare event probabilities of the type $\mathbb{P}(Q_0 > pB, Q_{T_B} > qB)$, for given positive numbers p, q, and a positive determinstic function T_B.

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We pay special attention to the case $T_B = RB$ for some $R > 0$; for light-tailed input we derive intuitively appealing asymptotics, intensively relying on sample-path large deviations results. The regimes obtained have appealing interpretations in terms of most likely paths to overflow.

1 Introduction

Lévy processes are widely used to model various real-life phenomena, for instance in finance and networking, see e.g. [14, 16]. In the literature special attention is paid to two intimately related subjects: fluctuation theory for Lévy processes (predominantly focusing on the analysis of the distribution of the maximal value attained by a Lévy process with negative drift), and to queues fed by Lévy input (studying the probabilistic properties of the workload).

Assuming that the Lévy process does not make negative jumps (i.e., the Lévy process is *spectrally-positive*), the Laplace transform of the steady-state workload Q has been known for over four decades, and is referred to as the (generalized) Pollaczek-Khinchine formula [20]; see also [5] for more background. In addition, the *asymptotics* of $\mathbb{P}(Q > B)$ (B large) have been identified, in various regimes. Asymptotically exact results for the light-tailed case (or: Cramér case) are presented in [4], cf. also [11], whereas the heavy-tailed case was covered by e.g. [2]; it is furthermore noted that there is also an intermediate case, cf. e.g. [13].

Substantially less attention has been paid to the analysis of transient characteristics of Lévy-driven queues. Again for the case of spectrally-positive Lévy input, in principle the full transient distribution is known, as we have an explicit expression for the double transform

$$F(q, \alpha) := \int_0^\infty e^{-qt} \mathbb{E}(e^{-\alpha Q_t} \mid Q_0 = x) \mathrm{d}t,$$

with Q_s denoting the workload at time $s > 0$, and $x \geq 0$; see e.g. [12]. In order to get a handle on the transient distribution, one may use inversion techniques. Note however that essentially two inversions then need to be performed: one to obtain $\mathbb{E}(e^{-\alpha Q_t} \mid Q_0 = x)$ from $F(q, \alpha)$, and one to obtain the transient distribution $\mathbb{P}(Q_t \leq \cdot \mid Q_0 = x)$ from $\mathbb{E}(e^{-\alpha Q_t} \mid Q_0 = x)$. We remark that [9] uses the double transform mentioned above to analyze the covariance function $r(t) := \mathbb{C}\mathrm{ov}(Q_0, Q_t)$; more specifically, it is proved that $r(\cdot)$ is positive, decreasing, and concave, and in addition its asymptotics (for large t) are determined.

In this paper we choose an alternative approach to analyze transient workload probabilities. We concentrate on probabilities of the type

$$\Pi_B := \mathbb{P}(Q_0 > pB, Q_{T_B} > qB),$$

which we analyze asymptotically, as $B \to \infty$. Here p and q are two positive constants, and T_B is a given positive function of B. It is stressed that we do not impose the assumption that the Lévy input process, say $(X_t)_t$, be spectrally-positive.

Interestingly, the shape of the function T_B essentially dictates the asymptotics of Π_B. More specifically, our paper makes the following contributions.

(i) Our first contribution is the identification of conditions under which

$$\Pi_B \sim \mathbb{P}(Q > \max\{p, q\}B), \tag{1}$$

or, in other words, the most demanding requirement determines the asymptotics (here '$\sim$' means that the ratio of the left-hand side and right-hand side converges to 1). These conditions essentially boil down to requiring that T_B is sublinear, that is $T_B/B \to 0$ as $B \to \infty$. The idea behind this

property is that the most demanding requirement essentially implies the other requirement with overwhelming probability, as $B \to \infty$.

(ii) A second contribution is the identification of a condition on T_B such that

$$\Pi_B \sim \mathbb{P}(Q > pB)\mathbb{P}(Q > qB). \tag{2}$$

If $\mathbb{P}(Q > B)$ decays (roughly) like e^{-B} (exponential decay), $\exp(-B^\alpha)$ with $0 < \alpha < 1$ (Weibullian decay), $B^{-\log B}$ (which could be called 'lognormal decay'), then this 'decoupling condition' reduces to $T_B/B \to \infty$. If $\mathbb{P}(Q > B)$ roughly looks like $B^{-\alpha}$ (polynomial decay), however, then the condition reads $T_B/B^2 \to \infty$; we have this type of decay in two relevant 'heavy-tailed' cases, viz. the situations in which the Lévy input process corresponds to an α-stable process, and to a compound Poisson process with regularly varying job sizes.

(iii) For the two 'heavy-tailed' scenarios mentioned above, we determine the asymptotics of Π_B for T_B increasing superlinearly but subquadratically; in this case the rare event under consideration is essentially due to a single big jump (whereas in the superquadratic case two big jumps are needed, leading to asymptotics (2)).

(iv) We pay special attention to the linear case, that is, $T_B = RB$ for some $R > 0$. For light-tailed input we derive intuitively appealing logarithmic asymptotics. If R is small (that is, fulfilling an explicit criterium in terms of p, q, and the characteristics of the Lévy process $(X_t)_t$, then we have asymptotics as in (1). If this condition does not apply, two cases are possible: for large R the most likely scenario is that the buffer drains, remains empty for a while, and starts building up relatively short before R (in this case the asymptotics look like the decoupled asymptotics (2)), and for moderate R the buffer remains (most likely) non-empty between 0 and R. These three regimes are in line with those identified in e.g. [8] for Gaussian input, [15] for exponential on-off input, as well as [19, Section 11.7] in the setting of an M/M/1 queue. The proofs of our 'trichotomy' rely intensively on large-deviations techniques, e.g., sample-path large deviations results [1].

The structure of the paper is as follows. In Section 2 we introduce the model, and present a number of preliminaries, such as a useful lemma taken from [8]. In Section 3 we address contributions (i) and (ii). Section 4 is devoted to the situation in which $\mathbb{P}(Q > B)$ decays polynomially, that is, contribution (iii). Finally, contribution (iv) is covered by Section 5. Section 6 contains a short summary, discussion, and directions for future research.

2   Notation and preliminaries

In this paper we consider a queue fed by a Lévy process $(X_t)_t$, emptied at a constant rate $C > 0$; recall that Lévy processes are stochastic processes with stationary independent increments [14]. Assume that $\mathbb{E}(X_1) = \varrho < C$, to ensure that the stationary workload exists.
More formally, the steady-state buffer-content process $\{Q_t : t \geq 0\}$ is given through

$$Q_t = \sup_{s \geq 0}(A(t - s, t) - Cs), \tag{3}$$

where $A(s,t) := X_t - X_s$ for $s \leq t$.

As mentioned in the introduction, this paper focuses on analyzing transient characteristics of the buffer-content process. We define

$$\Pi_B := \mathbb{P}\left(Q_0 > pB, Q_{T_B} > qB\right).$$

In this paper the primary focus lies on the asymptotics of Π_B as $B \to \infty$, for given $p, q > 0$ and some function T_B that tends to ∞ as $B \to \infty$.

We finish this section with two general lemmas that are used later in the paper. Directly from (3) it can be found that

$$\Pi_B = \mathbb{P}\left(\exists s \geq 0, t \geq 0 : A(-s,0) - Cs > pB, A(T_B - t, T_B) - Ct > qB\right). \tag{4}$$

The following lemma, featuring a reduction property proven in [8], formalizes the evident property that the start of the busy period in which T_B is contained (corresponding to time, say, $T_B - t^\star$), cannot take place before the start $-s^\star$ of the busy period in which 0 is contained, but also not in the interval $(-s^\star, 0]$. In order words: the only two options are that both busy periods start at the same epoch (then $t^\star = T_B + s^\star$), and that the busy period in which 0 is contained ends before T_B (then $t^\star \in [0, T_B)$). It means that in (4) we can restrict ourselves to a subset of $s, t \geq 0$.

Lemma 2.1. *Let*

$$\mathscr{E} := \{(s,t) : s \geq 0, t \in [0, T_B) \cup \{T_B + s\}\}.$$

Then

$$\Pi_B = \mathbb{P}\left(\exists(s,t) \in \mathscr{E} : A(-s,0) - Cs > pB, A(T_B - t, T_B) - Ct > qB\right).$$

We finally state a weak law of large numbers, which holds due to the fact that X_t is integrable.

Lemma 2.2. *For any $\delta > 0$,*

$$\lim_{t \to \infty} \mathbb{P}\left(\frac{X_t}{t} < \varrho - \delta\right) = \lim_{t \to \infty} \mathbb{P}\left(\frac{X_t}{t} > \varrho + \delta\right) = 0.$$

3 General results

In this section we prove two general results. The first says that (1) holds under the plausible condition that $T_B/B \to 0$; in the sequel we call this the *short time-scale regime.* The second identifies a condition under which the asymptotic decoupling (2) holds; notably, as mentioned in the introduction, this condition does not necessarily reduce to $T_B/B \to \infty$. We refer to the latter regime as the *long time-scale regime.*

3.1 Short time-scale regime

In this subsection we prove our result for the short time-scale regime; as before Q denotes the stationary workload. It consists of two cases: the case $p > q$ which holds under the condition $T_B/B \to 0$ as $B \to \infty$, and the case $q > p$ which holds under Assumption 3.1. We stress that later in this paper we will show that both in heavy-tailed scenarios and in light-tailed scenarios Assumption 3.1 is fulfilled as long as $T_B/B \to 0$ as $B \to \infty$.

4

Assumption 3.1. *One of the following two properties hold:*

(i) The sequence T_B is such that, for all $\eta > 0$,

$$\limsup_{B \to \infty} \frac{\mathbb{P}(\exists t \in (0, T_B) : X_t - Ct > \eta B)}{\mathbb{P}(Q > qB)} = 0.$$

(ii) The sequence T_B is such that, for all $\eta > 0$, $\mathbb{P}(Q > qB + \eta T_B) \sim \mathbb{P}(Q > qB)$ as $B \to \infty$.

Theorem 3.2. *Case $p > q$: If $T_B/B \to 0$ as $B \to \infty$, then*

$$\Pi_B \equiv \mathbb{P}(Q_0 > pB, Q_{T_B} > qB) \sim \mathbb{P}(Q > pB).$$

Case $q > p$: Under Assumption 3.1,

$$\Pi_B \equiv \mathbb{P}(Q_0 > pB, Q_{T_B} > qB) \sim \mathbb{P}(Q > qB).$$

Proof. First consider the case $p > q$. We are left to prove that

$$\liminf_{B \to \infty} \frac{\mathbb{P}(Q_0 > pB, Q_{T_B} > qB)}{\mathbb{P}(Q > pB)} \geq 1.$$

This is proven as follows. Fix $\varepsilon > 0$. Let B be sufficiently large such that $(p - q)B > (C - \varrho + \varepsilon)T_B$ (which is possible due to $T_B/B \to 0$ and $p > q$). Then

$$\mathbb{P}(Q_0 > pB, Q_{T_B} > qB) \geq \mathbb{P}(Q_0 > pB) \cdot \mathbb{P}(X_{T_B} > (\varrho - \varepsilon)T_B).$$

This is evidently true if T_B is bounded, and if it is not, then due Lemma 2.2 we have that for any $\delta > 0$ and for B large enough $\mathbb{P}(X_{T_B} > (\varrho - \varepsilon)T_B) > 1 - \delta$. The stated follows by letting $\delta \downarrow 0$.

Now focus on $q > p$, first under Assumption 3.1.(i). Now it suffices to prove that, as $B \to \infty$, we have that $\mathbb{P}(Q_0 < pB, Q_{T_B} > qB) = o(\mathbb{P}(Q > qB))$. Let $\mathscr{T}_B$ be the event that $Q_t > 0$ for all $t \in (0, T_B)$. First observe that, with $\eta := q - p > 0$,

$$\mathbb{P}(Q_0 < pB, Q_{T_B} > qB, \mathscr{T}_B) \leq \mathbb{P}(X_{T_B} > \eta B + CT_B) \leq \mathbb{P}(\exists t \in (0, T_B) : X_t - Ct > \eta B),$$

which is $o(\mathbb{P}(Q > qB))$ due to Assumption 3.1.(i). Also,

$$\mathbb{P}(Q_0 < pB, Q_{T_B} > qB, \mathscr{T}_B^c) \leq \mathbb{P}(Q_{T_B} > qB, \mathscr{T}_B^c) \leq \mathbb{P}(\exists t \in (0, T_B) : A(T_B - t, T_B) - Ct > qB),$$

which is also $o(\mathbb{P}(Q > qB))$, again by Assumption 3.1.(i).

Again consider the case $q > p$, but now under Assumption 3.1.(ii). It is clear that it suffices to show that $\liminf_{B \to \infty} \Pi_B/\mathbb{P}(Q > qB) \geq 1$. For each positive N,

$$\Pi_B \geq \mathbb{P}(Q_0 > qB + NT_B, Q_{T_B} > qB) \geq \mathbb{P}(Q > qB + NT_B) \cdot \mathbb{P}(X_{T_B} > (C - N)T_B).$$

Now observe that, by assumption, $\mathbb{P}(Q > qB + NT_B) \sim \mathbb{P}(Q > qB)$ as $B \to \infty$. Moreover, for each $\epsilon > 0$ there exists an N_0 such that, for each $N \geq N_0$, it holds that $\mathbb{P}(X_{T_B} > (C - N)T_B) \geq 1 - \epsilon$ for sufficiently large B. Thus, as $B \to \infty$,

$$\mathbb{P}(Q > qB + NT_B) \cdot \mathbb{P}(X_{T_B} > (C - N)T_B) \sim \mathbb{P}(Q > qB) \cdot \mathbb{P}(X_{T_B} > (C - N)T_B),$$

which is larger than $(1 - \epsilon)\mathbb{P}(Q > qB)$. The stated follows by letting $\epsilon \downarrow 0$. This completes the proof. $\square$

Remark 3.3. *The case $p = q$.* The case $p = q$ should be handled with care; it is readily checked from the proof of Thm. 3.2 that the argumentation for $q > p$ works for $q \geq p$ under Assumption 3.1.(ii), but not under Assumption 3.1.(i).

Let us now check how Assumption 3.1.(ii) relates to the condition $T_B/B \to 0$. In case that $\mathbb{P}(Q > B)$ decays (roughly) polynomially (i.e., $\mathbb{P}(Q > B) \sim KB^{-\zeta}$), then Assumption 3.1.(ii) indeed reduces to $T_B/B \to 0$ as $B \to \infty$. It is noted, however, that if $\mathbb{P}(Q > B)$ decays (roughly) exponentially, then Assumption 3.1.(ii) reads $T_B \to 0$.

We now argue that Assumption 3.1.(ii) is, in the case $p = q$, 'minimal' if $\mathbb{P}(Q > B)$ decays exponentially, in the sense that $\liminf_{B \to \infty} T_B = M > 0$ leads to $\limsup_{B \to \infty} \Pi_B/\mathbb{P}(Q > pB) < 1$, as follows. Consider for instance the case that $(X_t)_t$ corresponds to (standard) Brownian motion. Decompose Π_B into $\Pi_B^{(1)} + \Pi_B^{(2)}$, where $\mathscr{T}_B$ is defined in the proof of Thm. 3.2 and

$$\Pi_B^{(1)} := \mathbb{P}\left(Q_0 > pB, Q_{T_B} > pB, \mathscr{T}_B\right), \ \ \Pi_B^{(2)} := \mathbb{P}\left(Q_0 > pB, Q_{T_B} > pB, \mathscr{T}_B^c\right).$$

First observe

$$\begin{aligned} \Pi_B^{(2)} &\leq \mathbb{P}(Q_0 > pB, \exists t \in [0, T_B] : A(t, T_B - t) > pB + Ct) \\ &= \mathbb{P}(Q_0 > pB) \cdot \mathbb{P}(\exists t \in [0, T_B] : A(t, T_B - t) > pB + Ct) \leq (\mathbb{P}(Q > pB))^2 = o(\mathbb{P}(Q > pB)). \end{aligned}$$

Regarding $\Pi_B^{(1)}$, first recall that $\mathbb{P}(Q > B) = e^{-2CB}$. We find

$$\begin{aligned} \Pi_B^{(1)} &\leq \mathbb{P}(Q_0 > pB, Q_0 + X_{T_B} > pB + CT_B) = \int_{pB}^{\infty} \mathbb{P}(X_{T_B} > pB + CT_B - x) \cdot 2Ce^{-2Cx}\mathrm{d}x \\ &= \int_0^{\infty} \mathbb{P}(X_{T_B} > CT_B - y) \cdot 2Ce^{-2C(y+pB)}\mathrm{d}y = \mathbb{P}(Q + X_{T_B} > CT_B) \cdot \mathbb{P}(Q > pB). \end{aligned}$$

Since $\liminf_{B \to \infty} T_B = M > 0$, we have that $\limsup_{B \to \infty} \mathbb{P}(Q + X_{T_B} > CT_B) < 1$, and as a consequence that $\limsup_{B \to \infty} \Pi_B^{(1)}/\mathbb{P}(Q > pB) < 1$, and therefore also $\limsup_{B \to \infty} \Pi_B/\mathbb{P}(Q > pB) < 1$. This shows that Assumption 3.1.(ii) is 'minimal' for the case $p = q$. $\Diamond$

3.2 Long time-scale regime

The main goal of this section is to prove our result for the long time-scale regime. A crucial role is played by the following assumption. Recall that Q denotes the stationary workload.

Assumption 3.4. *The sequence T_B is such that, for all $\eta > 0$,*

$$\limsup_{B \to \infty} \frac{\mathbb{P}(Q > \eta T_B)}{\mathbb{P}(Q > pB) \cdot \mathbb{P}(Q > qB)} = 0.$$

In the next sections we relate this assumption to the behavior of T_B as $B \to \infty$. It turns out that depending on the driving Lévy process being heavy-tailed or light-tailed, various regimes need to be distinguished.

Theorem 3.5. *Under Assumption 3.4, it holds that*

$$\Pi_B \equiv \mathbb{P}(Q_0 > pB, Q_{T_B} > qB) \sim \mathbb{P}(Q > pB) \cdot \mathbb{P}(Q > qB).$$

Proof. Let us start by establishing the lower bound. By definition,

$$\mathbb{P}(Q_0 > pB, Q_{T_B} > qB) = \mathbb{P}(\exists s \geq 0 : A(-s, 0) > pB + Cs, \exists t \geq 0 : A(T_B - t, T_B) > qB + Ct).$$

The probability in the right-hand side of the previous display majorizes

$$\mathbb{P}(\exists s \geq 0 : A(-s, 0) > pB + Cs, \exists t \in (0, T_B) : A(T_B - t, T_B) > qB + Ct)$$
$$= \mathbb{P}(\exists s \geq 0 : A(-s, 0) > pB + Cs) \cdot \mathbb{P}(\exists t \in (0, T_B) : A(T_B - t, T_B) > qB + Ct)$$
$$= \mathbb{P}(Q > pB) \cdot \mathbb{P}(\exists t \in (0, T_B) : A(-t, 0) > qB + Ct).$$

We observe that it is left to prove that

$$\frac{\mathbb{P}(\exists t > T_B : A(-t, 0) > qB + Ct)}{\mathbb{P}(Q > qB)} \to 0 \tag{5}$$

as $B \to \infty$. Let us consider the numerator of (5). It is trivial that

$$\mathbb{P}(\exists t > T_B : A(-t, 0) > qB + Ct)$$
$$= \mathbb{P}\left(\left(\sup_{t > T_B} A(-t, -T_B) - C(t - T_B)\right) + A(-T_B, 0) > qB + CT_B\right)$$
$$= \mathbb{P}\left(Q_{-T_B} + A(-T_B, 0) > qB + CT_B\right).$$

Now Assumption 3.4 implies that $\mathbb{P}(Q > \eta T_B)/\mathbb{P}(Q > qB) \to 0$ for all $\eta > 0$, and hence $CT_B > qB$ for B sufficiently large. It means that the previous display is majorized by

$$\mathbb{P}\left(Q_{-T_B} + A(-T_B, 0) > 2CT_B\right),$$

for B large enough. We now distinguish between Q_{-T_B} being either smaller or larger than $\frac{1}{2}CT_B$:

$$\mathbb{P}\left(Q_{-T_B} + A(-T_B, 0) > 2CT_B, Q_{-T_B} < \frac{1}{2}CT_B\right)$$
$$+ \quad \mathbb{P}\left(Q_{-T_B} + A(-T_B, 0) > 2CT_B, Q_{-T_B} > \frac{1}{2}CT_B\right), \tag{6}$$

Realize that Q_{-T_B} and $A(-T_B, 0)$ are independent, and that Q_{-T_B} is distributed as the steady-state workload Q. Expression (6) is therefore bounded from above by

$$\mathbb{P}\left(A(-T_B, 0) - CT_B > \frac{1}{2}CT_B\right) + \mathbb{P}\left(Q > \frac{1}{2}CT_B\right)$$
$$\leq \quad \mathbb{P}\left(\exists t \geq 0 : A(-t, 0) - Ct > \frac{1}{2}CT_B\right) + \mathbb{P}\left(Q > \frac{1}{2}CT_B\right)$$
$$= \quad 2\mathbb{P}\left(Q > \frac{1}{2}CT_B\right).$$

Now (5) follows from applying Assumption 3.4 once more.

We now proceed by establishing the upper bound. In view of Lemma 2.1, we can split the probability of interest on the basis of the queue having been empty in $(0, T_B)$ or not, thus obtaining

$$\mathbb{P}(Q_0 > pB, Q_{T_B} > qB, \mathscr{T}_B^c) + \mathbb{P}(Q_0 > pB, Q_{T_B} > qB, \mathscr{T}_B). \tag{7}$$

The first of the probabilities in (7) equals

$$\mathbb{P}(\exists s \geq 0 : A(-s, 0) > pB + Cs, \exists t \in (0, T_B) : A(T_B - t, T_B) > qB + Ct)$$
$$= \quad \mathbb{P}(\exists s \geq 0 : A(-s, 0) > pB + Cs) \cdot \mathbb{P}(\exists t \in (0, T_B) : A(T_B - t, T_B) > qB + Ct)$$
$$\leq \quad \mathbb{P}(\exists s \geq 0 : A(-s, 0) > pB + Cs) \cdot \mathbb{P}(\exists t \geq 0 : A(-t, 0) > qB + Ct) = \mathbb{P}(Q > pB) \cdot \mathbb{P}(Q > qB).$$

The second of the probabilities in (7) equals

$$\mathbb{P}(\exists s > 0 : A(-s, 0) > pB + Cs, A(-s, T_B) > qB + C(T_B + s))$$
$$\leq \quad \mathbb{P}(\exists s > 0 : A(-s, T_B) > qB + C(T_B + s)) = \mathbb{P}(\exists s > T_B : A(-s, 0) > qB + Cs).$$

Above we saw that $\mathbb{P}(\exists s > T_B : A(-s, 0) > qB + Cs) \leq 2\mathbb{P}(Q > CT_B/2)$ for B sufficiently large. The upper bound now immediately follows from applying Assumption 3.4. $\qquad\square$

4   Heavy-tailed Lévy input

In this section we focus on the situation that the tail distribution of Q decays essentially polynomially.

Assumption 4.1. *For an $\zeta, K > 0$,*

$$\mathbb{P}(Q > qB) \cdot B^\zeta \to K,$$

as $B \to \infty$.

Let us first check what Assumptions 3.1 and 3.4 look like in this situation.

- Consider Assumption 3.1.(ii). As has been noticed in Remark 3.3, this assumption is valid under $T_B/B \to 0$ as $B \to \infty$.

- Now consider Assumption 3.4. It is readily checked that under Assumption 4.1 this does *not* reduce to $T_B/B \to \infty$, but to $T_B/B^2 \to \infty$.

  We here mention that, interestingly, Assumption 3.4 *does* reduce to requiring that $T_B/B \to \infty$ for $B \to \infty$ in a number of specific situations in which the tail distribution of Q decays subexponentially (but faster than polynomially); this is for instance the case when $\mathbb{P}(Q > B)$ roughly decays as $\exp(-B^\alpha)$ for $\alpha \in (0, 1)$ (Weibullian decay), and also when it decays as $B^{-\log B}$ (which is a tail that resembles that of the lognormal distribution).

The above observations indicate that, for $\mathbb{P}(Q > B)$ behaving as $B^{-\zeta}$, the situations that are left to investigate are those in which T_B is between linear and quadratic. In this section we analyze this case.
As a first observation, we notice that Lemma 2.1 entails that we can decompose Π_B into

$$\mathbb{P}\left(\begin{array}{l} \exists s \geq 0, \exists t \in [0, T_B] : \ A(-s, 0) - Cs > pB, \ A(T_B - t, T_B) - Ct > qB \\ \qquad \vee \ \exists s \geq 0 : \ A(-s, 0) - Cs > pB, \ A(-s, T_B) - C(s + T_B) > qB \end{array} \right)$$
$$= \quad \mathbb{P}(E_1) + \mathbb{P}(E_2) - \mathbb{P}(E_1 \cap E_2),$$

where

$$E_1 := \{\exists s \geq 0, \exists t \in [0, T_B] : \ A(-s, 0) - Cs > pB, \ A(T_B - t, T_B) - Ct > qB\},$$

$$E_2 := \{\exists s \geq 0 : \ A(-s,0) - Cs > pB, \ A(-s, T_B) - C(s + T_B) > qB\}.$$

The following two lemmas are useful in our proofs.

Lemma 4.2. *The following three statements hold:*

(i) for any $B > 0$,

$$\mathbb{P}(E_1) = \mathbb{P}(Q > pB) \cdot \mathbb{P}\left(\sup_{t \in [0, T_B]} (X_t - Ct) > qB\right);$$

(ii) for $\varepsilon \in (0,1)$, if $T_B \to \infty$ as $B \to \infty$, then $\mathbb{P}(E_2) \sim p_1(B) + p_2(B)$, where

$$p_1(B) \ := \ \mathbb{P}(Q > pB) \cdot \mathbb{P}\left(X_{T_B} > CT_B + (q - p)B\right)$$
$$p_2(B) \ := \ \mathbb{P}\left(Q > qB + (C - \varrho)T_B + \varepsilon(T_B + B)\right) \cdot \mathbb{P}\left(X_{T_B} \in [\varrho T_B - \varepsilon(T_B + B), CT_B + (q - p)B]\right);$$

(iii) if $T_B = RB^2$ for some $R > 0$, then under Assumption 4.1 we have $\mathbb{P}(E_1 \cap E_2) = o(\mathbb{P}(E_1))$ as $B \to \infty$.

Proof. Claim (i) follows directly from the independence of the increments of $(X_t)_t$. Now concentrate on Claim (ii). To make the notation a bit lighter, we write T instead of T_B throughout this proof. Observe that

$$\mathbb{P}(E_2) = \mathbb{P}\left(Q_0 > \max\{pB, qB + CT - X_T\}\right)$$
$$= \ \mathbb{P}\left(Q_0 > \max\{pB, qB + CT - X_T\}, \ X_T > CT + (q - p)B\right)$$
$$+ \mathbb{P}\left(Q_0 > \max\{pB, qB + CT - X_T\}, \ X_T \leq CT + (q - p)B\right)$$
$$= \ \mathbb{P}(Q_0 > pB)\mathbb{P}(X_T > CT + (q - p)B) + \mathbb{P}(E_{21}),$$

where $E_{21} := \{Q_0 > qB + CT - X_T, \ X_T \leq CT + (q - p)B\}$. We first consider $\mathbb{P}(E_{21})$. Let $\varepsilon \in (0,1)$. Then

$$\mathbb{P}(E_{21}) = \mathbb{P}(E_{211}) + \mathbb{P}(E_{212}), \tag{8}$$

with

$$E_{211} \ := \ \{Q_0 > qB + CT - X_T, \ X_T \in [\varrho T - \varepsilon(T + B), CT + (q - p)B]\}$$
$$E_{212} \ := \ \{Q_0 > qB + CT - X_T, \ X_T < \varrho T - \varepsilon(T + B)\}.$$

Observe that

$$\mathbb{P}(E_{212}) \ \leq \ \mathbb{P}\left(Q_0 > qB + (C - \varrho)T + \varepsilon(T + B), \ X_T < \varrho T - \varepsilon(T + B)\right)$$
$$= \ \mathbb{P}\left(Q > qB + (C - \varrho)T + \varepsilon(T + B)\right)\mathbb{P}\left(X_T < \varrho T - \varepsilon(T + B)\right)$$
$$= \ o\left(\mathbb{P}\left(Q > qB + (C - \varrho)T\right)\right) \tag{9}$$

as $B \to \infty$, because

$$\mathbb{P}\left(X_T < \varrho T - \varepsilon(T + B)\right) = \mathbb{P}\left(\frac{X_T}{T} < \varrho - \frac{\varepsilon(T + B)}{T}\right) \leq \mathbb{P}\left(\frac{X_T}{T} < \varrho - \varepsilon\right) \to 0$$

due to Lemma 2.2. We also have that $\mathbb{P}(E_{211})$ equals

$$\mathbb{P}\left(Q_0 > qB + CT + \max\{\varepsilon(T + B) - \varrho T, -X_T\}, \ X_T \in [\varrho T - \varepsilon(T + B), CT + (q - p)B]\right)$$

$$+ \mathbb{P}\left(qB + CT - X_T < Q_0 \le qB + (C - \varrho)T + \varepsilon(T + B), \ X_T \in [\varrho T - \varepsilon(T + B), CT + (q - p)B]\right)$$

$$= \mathbb{P}\left(Q_0 > qB + CT + \max\{\varepsilon(T + B) - \varrho T, -X_T\}, \ X_T \in [\varrho T - \varepsilon(T + B), CT + (q - p)B]\right)$$

$$= \mathbb{P}\left(Q_0 > qB + (C - \varrho)T + \varepsilon(T + B), \ X_T \in [\varrho T - \varepsilon(T + B), CT + (q - p)B]\right)$$

$$= \mathbb{P}\left(Q_0 > qB + (C - \varrho)T + \varepsilon(T + B)\right)\mathbb{P}\left(X_T \in [\varrho T - \varepsilon(T + B), CT + (q - p)B]\right). \tag{10}$$

Upon combining (8) with (9) and (10), we have established Claim (ii).

Finally consider Claim (iii). Let $\delta \in (0, \frac{1}{2})$ and $\varepsilon > 0$. We have

$$\mathbb{P}\left(E_1 \cap E_2\right) = \mathbb{P}\left(E_1 \cap E_2, \ X_T \ge (\varrho + \varepsilon)T + T^{1-\delta}\right) + \mathbb{P}\left(E_1 \cap E_2, \ X_T \le (\varrho + \varepsilon)T + T^{1-\delta}\right)$$

$$\le \mathbb{P}\left(Q_0 > pB, \ X_T \ge (\varrho + \varepsilon)T + T^{1-\delta}\right) +$$

$$\mathbb{P}\left(Q_0 > qB + (C - \varrho - \varepsilon)T - T^{1-\delta}, \ \sup_{t \in [0,T]}(X_T - X_{T-t} - Ct) > qB\right)$$

$$= \mathbb{P}\left(Q > pB\right)\mathbb{P}\left(X_T \ge (\varrho + \varepsilon)T + T^{1-\delta}\right) +$$

$$\mathbb{P}\left(Q > qB + (C - \varrho - \varepsilon)T - T^{1-\delta}\right)\mathbb{P}\left(\sup_{t \in [0,T]}(X_t - Ct) > qB\right). \tag{11}$$

Since $T = RB^2$ and $\delta \in (0, \frac{1}{2})$, for some constant $\bar{K} > 0$,

$$\mathbb{P}\left(X_T \ge (\varrho + \varepsilon)T + T^{1-\delta}\right) \le \mathbb{P}\left(\sup_{t \ge 0}(X_t - (\varrho + \varepsilon)t) \ge T^{1-\delta}\right) \sim \bar{K}(T^{1-\delta})^{-\zeta} = o(B^{-\zeta});$$

use Assumption 4.1. We thus conclude that

$$\mathbb{P}\left(Q > pB\right)\mathbb{P}\left(X_T \ge (\varrho + \varepsilon)T + T^{1-\delta}\right) = o(\mathbb{P}(E_1)).$$

We also have

$$\mathbb{P}\left(Q > qB + (C - \varrho - \varepsilon)T - T^{1-\delta}\right) \sim \mathbb{P}\left(Q > (C - \varrho - \varepsilon)T\right) = O(B^{-2\zeta}) = O(\mathbb{P}(E_1));$$

$$\mathbb{P}\left(\sup_{t \in [0,T]}(X_t - Ct) > qB\right) \le \mathbb{P}\left(\sup_{t \ge 0}(X_t - Ct) > qB\right) = \mathbb{P}(Q > qB) \to 0$$

as $B \to \infty$, which in view of (11) implies that $\mathbb{P}\left(E_1 \cap E_2\right) = o(\mathbb{P}(E_1))$. This completes the proof of (iii). $\square$

Lemma 4.3. *Under Assumption 4.1, for each $R > 0$, as $B \to \infty$,*

$$\mathbb{P}\left(\sup_{t \in [0,RB^2]}(X_t - Ct) > B\right) \sim \mathbb{P}(Q > B).$$

Proof. It clearly suffices to establish the lower bound. We have

$$\mathbb{P}\left(\sup_{t \in [0,RB^2]}(X_t - Ct) > B\right) \ge \mathbb{P}(Q > B) - \mathbb{P}\left(\sup_{t > RB^2}(X_t - Ct) > B\right). \tag{12}$$

10

Also, with Q, as before, denoting a random variable that is distributed as the stationary workload, and which is independent of X_{RB^2},

$$\mathbb{P}\left(\sup_{t>RB^2}(X_t - Ct) > B\right) = \mathbb{P}\left(X_{RB^2} + \sup_{t>RB^2}(X_t - X_{RB^2} - C(t - RB^2)) > B + CRB^2\right)$$

$$= \mathbb{P}(X_{RB^2} + Q > B + CRB^2) \sim \mathbb{P}(X_{RB^2} + Q > CRB^2).$$

Now realize that $\mathbb{P}(Q > CRB^2)$ is asymptotically $(B \to \infty)$ proportional to $B^{-2\zeta}$ by Assumption 4.1, and

$$\mathbb{P}(X_{RB^2} > CRB^2) = \mathbb{P}(X_{RB^2} - (\varrho + \varepsilon)RB^2 > (C - \varrho - \varepsilon)RB^2)$$

$$\leq \mathbb{P}\left(\sup_{t>0}(X_t - (\varrho + \varepsilon)t) > (C - \varrho - \varepsilon)RB^2\right) \sim \check{K}B^{-2\zeta},$$

for some positive constant $\check{K}$ (again by Assumption 4.1), so that $\mathbb{P}(X_{RB^2} + Q > CRB^2)$ is roughly proportional to $B^{-2\zeta}$ as well, as follows from basic properties of regularly varying distributions. The stated is now a direct consequence of (12) and the fact that $P(Q > B)$ is asymptotically proportional to $B^{-\zeta}$. $\square$

We now present two propositions that, for the case that T_B is at least linear but slower than quadratic, express the asymptotics of Π_B in terms of the asymptotics of $\mathbb{P}(Q > B)$, viz. Prop. 4.4 for the case $q \geq p$ and Prop. 4.6 for the case $p > q$. Corollaries 4.5 and 4.7 summarize the findings so far.

Proposition 4.4. *Let $q \geq p$.*

(i) *If $\liminf_{B \to \infty} T_B/B \geq R$ for some $R > 0$ and $T_B/B^2 \to 0$ as $B \to \infty$, then*

$$\Pi_B \sim \mathbb{P}(Q > qB + (C - \varrho)T_B); \tag{13}$$

(ii) *If $T_B = RB^2$ for some $R > 0$, then*

$$\Pi_B \sim \mathbb{P}(Q > pB)\mathbb{P}(Q > qB) + \mathbb{P}(Q > (C - \varrho)T_B). \tag{14}$$

Proof. To prove Claim (i), it suffices to show $\mathbb{P}(E_1) = o(\mathbb{P}(E_2))$. From Lemma 4.2.(i) it immediately follows that $\mathbb{P}(E_1) \leq \mathbb{P}(Q > pB)\mathbb{P}(Q > qB)$. As, for $q \geq p$, $p_1(B) = o(p_2(B))$, we also have, by letting $\varepsilon \downarrow 0$ in Lemma 4.2.(ii), $\mathbb{P}(E_2) \sim \mathbb{P}(Q > qB + (C - \varrho)T_B)$. It also holds that

$$\mathbb{P}(Q > pB)\mathbb{P}(Q > qB) = o(\mathbb{P}(Q > qB + (C - \varrho)T_B))$$

as $B \to \infty$. This completes the proof of Claim (i).
Now consider Claim (ii). If $T_B = RB^2$, then, following Lemmas 4.2 and 4.3,

$$\mathbb{P}(E_1) = \mathbb{P}(Q > pB)\mathbb{P}\left(\sup_{t \in [0, T_B]}(X(t) - Ct) > qB\right) \sim \mathbb{P}(Q > pB)\mathbb{P}(Q > qB)$$

and

$$\mathbb{P}(E_2) \sim \mathbb{P}(Q > pB + (C - \varrho)T_B) \sim \mathbb{P}(Q > (C - \varrho)T_B),$$

as $B \to \infty$. Since $\mathbb{P}(E_1) = O(\mathbb{P}(E_2))$, it now suffices to recall that due to Lemma 4.2.(iii) it holds that $\mathbb{P}(E_1 \cap E_2) = o(\mathbb{P}(E_1))$. We thus establish Claim (ii). $\square$

11

The following corollary is an immediate consequence of Thms. 3.2, 3.5, and 4.4, Remark 3.3 and Lemma 4.3.

Corollary 4.5. *Let $q \geq p$.*

(i) *If $T_B/B \to 0$ as $B \to \infty$, then $\Pi_B \sim K q^{-\varsigma} B^{-\varsigma}$;*

(ii) *If $T = RB$ for some $R > 0$, then $\Pi_B \sim K(q + CR)^{-\varsigma} B^{-\varsigma}$;*

(iii) *If $T_B/B \to \infty$ and $T_B/B^2 \to 0$ as $B \to \infty$, then $\Pi_B \sim K(CT_B)^{-\varsigma}$;*

(iv) *If $T_B = RB^2$ for some $R > 0$, then $\Pi_B \sim \left(K^2(pq)^{-\varsigma} + (CR)^{-\varsigma}\right) B^{-2\varsigma}$;*

(v) *If $T_B/B^2 \to \infty$ as $B \to \infty$, then $\Pi_B \sim K^2(pq)^{-\varsigma} B^{-2\varsigma}$.*

We now switch to the case $q < p$.

Proposition 4.6. *Let $q < p$.*

(i) *If $T_B = RB$ with $R \leq (p - q)/B$, then $\Pi_B \sim \mathbb{P}(Q > pB)$;*

(ii) *If $\liminf_{B \to \infty} T_B/B > (p - q)/C$ and $T_B/B^2 \to 0$ as $B \to \infty$, then $\Pi_B \sim \mathbb{P}(Q > qB + (C - \varrho)T_B)$;*

(iii) *If $T = RB^2$ as $B \to \infty$ for some $R > 0$, then $\Pi_B \sim \mathbb{P}(Q > pB)\mathbb{P}(Q > qB) + \mathbb{P}(Q > (C - \varrho)T_B)$.*

Proof. We only consider Claim (i); the other claims can be proven as the corresponding statements in Prop. 4.4. Notice that

$$\mathbb{P}(Q > pB) \cdot \mathbb{P}(X_{T_B} > \varrho T_B + (C - \varrho + (q - p)/R)T_B) \sim \mathbb{P}(Q > pB),$$

due to the weak law of large numbers (Lemma 2.2); the probability $p_2(B)$ corresponds to two rare events (use $C - \varrho + (q - p)/R < 0$), such that $p_2(B) = o(p_1(B))$. As a consequence, Lemma 4.2.(ii) entails that $\mathbb{P}(E_2) \sim \mathbb{P}(Q > pB)$. Combining this with Lemma 4.2.(i), we conclude that $\mathbb{P}(E_1) = o(\mathbb{P}(E_2))$. This implies $\Pi_B \sim \mathbb{P}(Q > pB)$, which completes the proof of (i). $\square$

Corollary 4.7. *Let $q < p$.*

(i) *If $T_B/B \to 0$ as $B \to \infty$, or $T = RB$ with $R \leq (p - q)/C$, then $\Pi_B \sim K p^{-\varsigma} B^{-\varsigma}$;*

(ii) *If $T_B = RB$ for $R > (p - q)/C$, then $\Pi_B \sim K(q + CR)^{-\varsigma} B^{1-\varsigma}$;*

(iii) *If $T_B/B \to \infty$ and $T_B/B^2 \to 0$ as $B \to \infty$, then $\Pi_B \sim K(CT_B)^{-\varsigma}$;*

(iv) *If $T_B = RB^2$ as $B \to \infty$ for some $R > 0$, then $\Pi_B \sim \left(K^2(pq)^{-\varsigma} + (CR)^{-\varsigma}\right) B^{-2\varsigma}$;*

(v) *If $T_B/B^2 \to \infty$ as $B \to \infty$, then $\Pi_B \sim K^2(pq)^{-\varsigma} B^{-2\varsigma}$.*

In the remainder of this section we consider two special cases: (A) α-stable input, and (B) compound Poisson input with polynomially decaying job size distribution.

(A) *α-stable input.* Let X_t be an α-stable Lévy process [18] with $\alpha \in (1,2)$ and $\beta \in (-1,1]$. We use the notation

$$\mathbb{B}(\alpha,\beta) := \frac{\Gamma(1+\alpha)}{\pi} \sqrt{1 + \beta^2 \tan^2(\pi\alpha/2)} \, \sin\left(\frac{\pi\alpha}{2} + \arctan\left(\beta\tan\left(\pi\alpha/2\right)\right)\right).$$

Then, due to [17], Assumption 4.1 is valid with

$$K = \frac{\mathbb{B}(\alpha,\beta)}{C\alpha(\alpha-1)},$$

and $\zeta = \alpha - 1$. Hence the theory developed earlier in this section can be applied.

(B) *Compound Poisson input with polynomially decaying job sizes.* Consider a Poissonian arrival stream (with rate λ) of i.i.d. jobs. Let the distribution of the jobs obey $\mathbb{P}(J^r > x) \sim \kappa x^{-\zeta}$, for positive ζ, κ, where J^r denotes the *residual* job length:

$$\mathbb{P}(J^r > x) = \frac{1}{\mathbb{E}J} \int_x^\infty \mathbb{P}(J > y) \mathrm{d}y.$$

Note that $\varrho = \lambda \cdot \mathbb{E}J$. Then [6, 7]

$$\mathbb{P}(Q > x) \sim \frac{\varrho}{C - \varrho} \kappa x^{-\zeta}.$$

Conclude that again Assumption 4.1 (and hence the theory of this section) applies, with an obvious value for K.

5 Light-tailed input

In this section we derive the logarithmic asymptotics of Π_B as $B \to \infty$, for the case of light-tailed input. We impose the following assumption.

Assumption 5.1. *With*

$$\beta^\star := \inf\{\beta : \mathbb{E}\left(e^{-\beta X_1}\right) < \infty\},$$

assume that $\beta^\star < 0$. Let $\varphi(\vartheta) := \log \mathbb{E}\exp(-\vartheta X_1)$, and assume that there exists $\vartheta^\star \in (\beta^\star, 0)$, such that $\varphi(\vartheta^\star) + C\vartheta^\star = 0$.

We first recall in Prop. 5.2 a result that is a special case of [11, Thm. 4], which states that the tail probabilities of the steady-state workload decay essentially exponentially. Bearing in mind Assumption 3.4, this means that Thm. 3.5 holds when $T_B/B \to \infty$. In Lemma 5.5 we will show that Assumption 3.1 applies if $T_B/B \to 0$ as $B \to \infty$, and hence the case $T_B/B \to 0$ is covered by Thm. 3.2.

The above means that the only case left to analyze is the linear case, and therefore the rest of this section concentrates on $T_B = RB$ for some $R > 0$. It turns out that three intuitively appealing regimes can be distinguished (small R, moderate R, large R); at the end of this section we provide more insight in these regimes.

In the following proposition, we let Q denote the stationary workload of a Lévy-driven queue, i.e., let Q be distributed as $\sup_{t>0}(X_t - Ct)$.

Proposition 5.2. *Under Assumption 5.1 it holds that*

$$\lim_{B \to \infty} \frac{1}{B} \log \mathbb{P}(Q > B) = \vartheta^\star. \tag{15}$$

Remark 5.3. We give here an alternative proof of the upper bound associated with the above result, as it provides interesting additional insight, and the proof technique will be used again in the proof of Lemma 5.5. Importantly, we obtain the uniform upper bound $\mathbb{P}(Q > B) \le e^{\vartheta^\star B}$.

Under the assumption $\varrho < C$, evidently the queueing system is stable under the measure $\mathbb{P}$. We will now perform a change of measure, with which we associate $\mathbb{Q}$, under which overflow occurs with probability 1, by application of an exponential twist $\vartheta^\star$. Under the light-tailed assumption, we have that the Laplace exponent $\varphi(\vartheta)$ of X_t is well defined and characterized through, with $\sigma^2 > 0$ and a measure $\Pi_\varphi(\cdot)$ such that $\int_{(0,\infty)} \min\{1, x^2\} \Pi_\varphi(\mathrm{d}x) < \infty$,

$$\varphi(\vartheta) = -\vartheta \varrho + \frac{1}{2} \vartheta^2 \sigma^2 + \int_{(0,\infty)} (e^{-\vartheta x} - 1 + \vartheta x 1_{(0,1)}) \Pi_\varphi(\mathrm{d}x).$$

It is now a matter of straightforward calculations to show that $\bar{\varphi}(\vartheta) := \varphi(\vartheta + \vartheta^\star) - \varphi(\vartheta^\star)$ is a Laplace exponent as well. Under $\mathbb{Q}$, the Lévy process has Laplace exponent $\bar{\varphi}(\vartheta)$; from the convexity of $\varphi(\cdot)$ it is concluded that (in self-evident notation) $\mathbb{E}_\mathbb{Q} X_1 = -\bar{\varphi}'(\vartheta^\star) > \varrho$, so that the system under the new measure is indeed unstable. (One can check that under $\mathbb{Q}$ the drift has increased to $\varrho - \vartheta^\star \sigma^2$, the Brownian term remains unchanged, whereas the measure $\Pi_{\bar{\varphi}}(\mathrm{d}x)$ is given through the exponentially twisted version $e^{-\vartheta^\star x} \Pi_\varphi(\mathrm{d}x)$).

Suppose one would compute $\mathbb{P}(\sup_{t>0} X_t - Ct > B)$ by simulating under $\mathbb{Q}$. There is the fundamental equality, with I denoting the indicator function of the event $\{\sup_{t>0} X_t - Ct > B\}$

$$\mathbb{P}\left(\sup_{t>0} X_t - Ct > B\right) = \mathbb{E}_\mathbb{Q}(LI),$$

cf. [3, Thm. XIII.3.2], where L denotes the likelihood ratio (to be understood as a Radon-Nikodým derivative) of the value of the Lévy process under $\mathbb{P}$ with respect to $\mathbb{Q}$; it is a standard result that at time t this likelihood ratio equals $e^{\vartheta^\star X_t} \exp(\varphi(\vartheta^\star) t)$. Let τ_B be defined as the first epoch at which X_t exceeds $B + Ct$ (which is a stopping time); as $I = 1$ with $\mathbb{Q}$-probability 1, we thus obtain

$$\mathbb{P}\left(\sup_{t>0} X_t - Ct > B\right) = \mathbb{E}_\mathbb{Q} e^{\vartheta^\star X_{\tau_B}} e^{\varphi(\vartheta^\star) \tau_B} = \mathbb{E}_\mathbb{Q} e^{\vartheta^\star X_{\tau_B}} e^{-C \vartheta^\star \tau_B}.$$

As by definition $X_{\tau_B} \ge B + C\tau_B$, we thus find that $\mathbb{P}(Q > B) \le e^{\vartheta^\star B}$. $\diamond$

In the next lemma we relate the decay rate $\vartheta^\star$ to the large deviations rate function, defined through $I(r) := \sup_{\vartheta \ge 0} (\vartheta r - \varphi(-\vartheta))$, and an associated variational problem.

Lemma 5.4. *It holds that*

$$-\vartheta^\star = \inf_{r > C} \frac{I(r)}{r - C}. \tag{16}$$

Proof. Let the minimizer in the right-hand side of (16) be $r^\star$, satisfying $(r - C) I'(r) = I(r)$. Define in addition $\vartheta(r) := \arg \sup_{\vartheta \ge 0} (\vartheta r - \varphi(-\vartheta))$, so that $I(r) = \vartheta(r) r - \varphi(-\vartheta(r))$. Noticing that $\vartheta(r)$ satisfies $r + \varphi'(-\vartheta) = 0$, we find that

$$I'(r) = \vartheta'(r) r + \vartheta(r) + \vartheta'(r) \varphi(-\vartheta(r)) = \vartheta(r).$$

14

¿From the facts that $\vartheta^\star$ solves $\varphi(\vartheta^\star) + C\vartheta^\star = 0$ and

$$\vartheta(r^\star)r^\star - \varphi(-\vartheta(r^\star)) = I(r^\star) = (r^\star - C)I'(r^\star) = (r^\star - C)\vartheta(r^\star),$$

we conclude that $-\vartheta(r^\star) = \vartheta^\star$, we proves the claim. $\qquad\square$

As indicated in the beginning of this section, like in the heavy-tailed case, in this light-tailed case we again have that Assumption 3.1 is valid if $T_B/B \to 0$ as $B \to \infty$. This is proven in the following lemma. We recall that it entails that the only case left to analyze is the linear case, that is, $T_B = RB$ for some $R > 0$.

Lemma 5.5. *Under Assumption 5.1, Assumption 3.1.(i) applies if $T_B/B \to 0$ as $B \to \infty$.*

Proof. Let $\mathbb{Q}(\vartheta)$ be the Lévy process obtained after exponentially twisting the original Lévy process with twist $\vartheta < 0$, as in Remark 5.3. In a similar fashion, it follows that

$$\mathbb{P}(\exists t \in (0, T_B) : X_t - Ct > qB) \leq \mathbb{E}_{\mathbb{Q}(\vartheta)} e^{\vartheta(B + C\tau_B)} e^{\varphi(\vartheta)\tau_B},$$

where τ_B is the minimum of T_B and the first epoch at which $X_t - Ct$ exceeds B (which is a stopping time). It then follows that for all $\vartheta < 0$, bearing in mind that $\tau_B \leq T_B = o(B)$,

$$\limsup_{B \to \infty} \frac{1}{B} \log \mathbb{P}(\exists t \in (0, T_B) : X_t - Ct > qB) \leq \limsup_{B \to \infty} \left(\vartheta + \vartheta C \frac{\tau_B}{B} + \varphi(\vartheta) \frac{\tau_B}{B} \right) = \vartheta.$$

This entails that $\mathbb{P}(\exists t \in (0, T_B) : X_t - Ct > qB)$ decays superexponentially:

$$\limsup_{B \to \infty} \frac{1}{B} \log \mathbb{P}(\exists t \in (0, T_B) : X_t - Ct > qB) \leq \inf_{\vartheta < 0} \vartheta = -\infty.$$

Combining this with Prop. 5.2, the stated follows. $\qquad\square$

¿From now on we just consider the case that $T_B = RB$. The next proposition shows that for small R the decay rate of interest equals the decay rate of the 'most binding event', cf. Thm. 3.2. We denote

$$\bar{R} := \max \left\{ \frac{p - q}{C - \varrho}, \frac{q - p}{r^\star - C} \right\}.$$

Proposition 5.6. *If $R < \bar{R}$, then*

$$\lim_{B \to \infty} \frac{1}{B} \log \Pi_B = \max\{p, q\}\vartheta^\star.$$

Proof. First suppose $p > q > 0$. The upper bound follows immediately from Prop. 5.2:

$$\limsup_{B \to \infty} \frac{1}{B} \log \Pi_B \leq \limsup_{B \to \infty} \frac{1}{B} \log \mathbb{P}(Q_0 > pB) = p\vartheta^\star.$$

Now consider the lower bound, which we establish by applying the lower bound of a sample-path large-deviations result. We here rely on de Acosta [1, Thm. 5.1], which can be applied to obtain

$$\liminf_{B \to \infty} \frac{1}{B} \log \Pi_B \geq -\mathbb{I}(f), \tag{17}$$

for any $f \in \mathscr{A}$, where

$$\mathbb{I}(f) := \int_{-\infty}^{\infty} I(f'(\tau)) \mathrm{d}\tau.$$

and the set of paths $\mathscr{A}$ is given by

$$\mathscr{A} := \{f : \exists (\sigma, \tau) \in \mathcal{E} : -f(-\sigma) \ge C\sigma + p, f(R) - f(R - \tau) \ge C\tau + q\}.$$

Now consider the continuous path $f^\star$ through the origin that has slope $r^\star$ between $-p/(r^\star - C)$ and 0 and slope ϱ elsewhere; clearly

$$\mathbb{I}(f) := \int_{-p/(r^\star - C)}^{0} I(f'(\tau)) \mathrm{d}\tau = \frac{p}{r^\star - C} \cdot I(r^\star) = -p\vartheta^\star.$$

Claim 1 now follows from the observation that $f^\star \in \mathscr{A}$, as

$$-f\left(-\frac{p}{r^\star - C}\right) = \frac{pr^\star}{r^\star - C} = \frac{pC}{r^\star - C} + p,$$

and, by virtue of $R < (p - q)/(C - \varrho)$,

$$f(R) - f\left(-\frac{p}{r^\star - C}\right) = \varrho R + \frac{pr^\star}{r^\star - C} > C\left(R + \frac{p}{r^\star - C}\right) + q.$$

Claim (2) can be proven along the same lines. The upper bound is identical, and in de lower bound we again use Theorem 5.1 of [1], but now with a path $f^\star$ that has slope $r^\star$ between $R - q/(r^\star - C)$ and R and ϱ elsewhere. The stated follows after checking that this path is in $\mathscr{A}$ if $R < (q - p)/(r^\star - C)$. $\qquad \square$

In the sequel we use the following lemma extensively, see [10, Lemma 1.2.15].

Lemma 5.7. *For any finite integer M,*

$$\lim_{n \to \infty} \frac{1}{n} \log \left(\sum_{i=1}^{M} \alpha_{i,n} \right) = \max_{i=1,\dots,M} \left(\lim_{n \to \infty} \frac{1}{n} \log \alpha_{i,n} \right).$$

Proposition 5.8. *If $R > \bar{R}$, then*

$$\lim_{B \to \infty} \frac{1}{B} \log \Pi_B = p\vartheta^\star + \max\{q\vartheta^\star, -\psi(R)\}, \quad \text{where } \psi(R) := R \cdot I\left(C + \frac{q - p}{R}\right).$$

Proof. First we establish the upper bound, which consists of five steps.

STEP I. The probability of interest Π_B can be decomposed as $\Pi_B^{(1)} + \Pi_B^{(2)}$, with

$$\begin{aligned}
\Pi_B^{(1)} &:= \mathbb{P}(Q_0 > pB, Q_{RB} > qB, \forall t \in (0, RB) : Q_t > 0), \\
\Pi_B^{(2)} &:= \mathbb{P}(Q_0 > pB, Q_{RB} > qB, \exists t \in (0, RB) : Q_t = 0).
\end{aligned}$$

STEP II. We first observe that we can bound $\Pi_B^{(2)}$ as follows:

$$\begin{aligned}
\Pi_B^{(2)} &= \mathbb{P}(Q_0 > pB, \exists t \in (0, RB) : A(RB - t, RB) - Ct \ge qB) \\
&= \mathbb{P}(Q_0 > pB)\mathbb{P}(\exists t \in (0, RB) : A(RB - t, RB) - Ct \ge qB) \\
&\le \mathbb{P}(Q_0 > pB)\mathbb{P}(\exists t \ge 0 : A(RB - t, RB) - Ct \ge qB) = \mathbb{P}(Q > pB)\mathbb{P}(Q > qB),
\end{aligned}$$

16

and hence

$$\lim_{B\to\infty} \frac{1}{B} \log \Pi_B^{(2)} \leq \lim_{B\to\infty} \frac{1}{B} \log \mathbb{P}(Q > pB) + \lim_{B\to\infty} \frac{1}{B} \log \mathbb{P}(Q > qB) = (p+q)\vartheta^\star. \tag{18}$$

STEP III. Now let us focus on $\Pi_B^{(1)}$; in this scenario the busy period in which R is contained starts at the same epoch as the busy period in which 0 is contained. Hence

$$\Pi_B^{(1)} = \mathbb{P}(\exists s \geq 0 : A(-s,0) - Cs > pB, \ A(-s,RB) - C(RB+s) > qB).$$

Let $\varepsilon > 0$ be picked arbitrary; let M be some natural number, whose value we specify later. Then $\Pi_B^{(1)}$ is majorized by

$$\sum_{k=0}^{M-1} \mathbb{P}\left(\exists s \geq 0 : \begin{array}{l} A(-s,0) - Cs \in ((p+k\varepsilon)B, (p+(k+1)\varepsilon)B]; \\ A(-s,RB) - C(RB+s) > qB \end{array}\right)$$
$$+ \mathbb{P}(\exists s \geq 0 : A(-s,0) - Cs > (p+M\varepsilon)B). \tag{19}$$

Now the k-th term in the summation of the previous display is bounded from above by

$$\mathbb{P}(\exists s \geq 0 : A(-s,0) - Cs > (p+k\varepsilon)B) \times \mathbb{P}(A(0,RB) - CRB > (q - (p+(k+1)\varepsilon))B),$$

which we call $\zeta_B^{(k)}$. Due to Prop. 5.2 and Cramér's theorem,

$$\lim_{B\to\infty} \frac{1}{B} \log \zeta_B^{(k)} = (p+k\varepsilon)\vartheta^\star - R \cdot I\left(C + \frac{q-p-(k+1)\varepsilon}{R}\right).$$

We have now found that (19) is not larger than

$$\sum_{k=0}^{M-1} \zeta_B^{(k)} + \mathbb{P}(Q > (p+M\varepsilon)B),$$

and therefore, due to Lemma 5.7,

$$\lim_{B\to\infty} \frac{1}{B} \log \Pi_B^{(1)} \leq \max\left\{ \max_{k=0,\dots,M-1} \left\{ (p+k\varepsilon)\vartheta^\star - R \cdot I\left(C + \frac{q-p-(k+1)\varepsilon}{R}\right)\right\}, (p+M\varepsilon)\vartheta^\star\right\}.$$

STEP IV. We now study how $g_k := (p+k\varepsilon)\vartheta^\star - R \cdot I(\Delta_k/R)$ behaves when varying k, with $\Delta_k := CR + q - p - (k+1)\varepsilon$. Because of the convexity of $I(\cdot)$, we see that g_k is concave in k. This means that proving $g_1 \leq g_0$ also yields that $\max_{k=0,\dots,M-1} g_k = g_0$. To this end, first observe that, owing to the convexity of $I(\cdot)$ and using that $\Delta_1 < \Delta_0$,

$$g_0 - g_1 = -\varepsilon\vartheta^\star + R\left(I\left(\frac{\Delta_1}{R}\right) - I\left(\frac{\Delta_0}{R}\right)\right) \geq -\varepsilon\vartheta^\star + (\Delta_1 - \Delta_0)I'\left(\frac{\Delta_1}{R}\right) = -\varepsilon\left(I'\left(\frac{\Delta_1}{R}\right) + \vartheta^\star\right).$$

Now recall that $\vartheta^\star = -I'(r^\star)$, and that $I'(\cdot)$ is increasing. It follows that $g_1 \leq g_0$ if $\Delta_1 < r^\star R$, which is true under $R > (q-p)/(r^\star - C)$ and ε sufficiently small. We conclude, noting that we can take M arbitrarily large,

$$\lim_{B\to\infty} \frac{1}{B} \log \Pi_B^{(1)} \leq g_0 = p\vartheta^\star - R \cdot I\left(C + \frac{q-p-\varepsilon}{R}\right). \tag{20}$$

STEP V. By letting $\varepsilon \downarrow 0$ in (20), applying the upper bound on the decay rates of both $\Pi_B^{(1)}$ and $\Pi_B^{(2)}$, and Lemma 5.7 once more, we have

$$\lim_{B\to\infty} \frac{1}{B} \log \Pi_B \leq p\vartheta^\star + \max\left\{q\vartheta^\star, -R \cdot I\left(C + \frac{q-p}{R}\right)\right\}.$$

This completes the upper bound.

The lower bound follows again from sample-path large-deviations arguments [1].

- Let us first consider the case that

$$q\vartheta^\star > -R \cdot I\left(C + \frac{q-p}{R}\right). \tag{21}$$

Condition (21) implies that $R \geq q/(r^\star - C)$, as can be seen as follows. Supposing $R < q/(r^\star - C)$, and recalling that we have $R > (q-p)/(r^\star - C)$, it would follow that

$$R \cdot I\left(C + \frac{q-p}{R}\right) < \frac{q}{r^\star - C}I(r) = -q\vartheta^\star,$$

which is a contradiction; note that we also used that $C + (q-p)/R > \varrho$.

Using that we know that (21) implies $R \geq q/(r^\star - C)$, it can be seen that the path $f^\star$ through the origin that has slope $r^\star$ between $-p/(r^\star - C)$ and 0, and also between $R - q/(r^\star - C) > 0$ and R, and slope ϱ elsewhere, is indeed feasible (that is, lies in $\mathscr{A}$). It is also readily verified that $\mathbb{I}(f^\star) = -(p+q)\vartheta^\star$, as required.

- Now suppose that (21) does not hold. Define $f^\star$ as the path through the origin with slope $r^\star$ between $-p/(r^\star - C)$ and 0, slope $C + (q-p)/R$ between 0 and R, and slope ϱ elsewhere. It is easily seen that this path is feasible and, by applying the definition of $\mathbb{I}(\cdot)$,

$$\mathbb{I}(f^\star) = -p\vartheta^\star + R \cdot I\left(C + \frac{q-p}{R}\right),$$

as desired.

This concludes the proof of the lower bound. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 5.9. *For all $R > \bar{R}$, we have that $\psi(R)$ is increasing. In addition we have that $\psi(\bar{R}) \leq -q\vartheta^\star$.*

Proof. Observe, recalling that $I'(r) = \vartheta(r)$, that

$$\psi'(R) = -\frac{q-p}{R} \cdot \vartheta\left(C + \frac{q-p}{R}\right) + I\left(C + \frac{q-p}{R}\right).$$

First consider the case $p > q$, such that $\bar{R} = (p-q)/(C-\varrho)$. It then holds that $C + (q-p)/\bar{R} = \varrho$, so that

$$\psi'(\bar{R}) = -\frac{q-p}{R} \cdot I'(\varrho) + I(\varrho) = 0,$$

due to $I(\varrho) = I'(\varrho) = 0$. We are done if we can prove that $\psi'(R)$ increases for $R \geq \bar{R}$. To this end, we compute $\psi''(R)$; it is easily verified that $I'(r) = \vartheta(r)$ entails that

$$\psi''(R) = \frac{(q-p)^2}{R^3}I''\left(C + \frac{q-p}{R}\right),$$

which is indeed non-negative because of the convexity of $I(\cdot)$.

We now consider the case $q \geq p$, i.e., $\bar{R} = (q - p)/(r^\star - C)$. It then holds that $C + (q - p)/\bar{R} = r^\star$, so that

$$\psi'(\bar{R}) = (c - r^\star) \cdot I'(r^\star) + I(r^\star) = 0,$$

see the proof of Lemma 5.4. Again, we are done if we can prove that $\psi'(R)$ increases for $R \geq \bar{R}$, which follows in the same fashion as above.

We finally consider $\psi(\bar{R})$. In case $p > q$, this equals 0, which is evidently below $-q\vartheta^\star$. In case $q \geq p$, we have

$$\psi(\bar{R}) = \frac{q - p}{r^\star - C} I(r^\star) = -(q - p)\vartheta^\star \leq -q\vartheta^\star.$$

This completes the proof. □

The following claim is an immediate consequence of the previous lemma.

Corollary 5.10. *There is a unique solution (larger than $\bar{R}$) to $\psi(R) = -q\vartheta^\star$, say $\check{R}$. For all $R \in (\bar{R}, \check{R})$ we have $\psi(\bar{R}) \leq -q\vartheta^\star$, for all $> \check{R}$ we have $\psi(\bar{R}) > -q\vartheta^\star$.*

Application of Props. 5.6, 5.8 and this corollary immediately lead to the following theorem.

Theorem 5.11.   *(i) For $R \leq \bar{R}$ we have*

$$\lim_{B \to \infty} \frac{1}{B} \log \Pi_B = \max\{p, q\}\vartheta^\star.$$

(ii) For $R \in (\bar{R}, \check{R})$ we have

$$\lim_{B \to \infty} \frac{1}{B} \log \Pi_B = p\vartheta^\star - \psi(R).$$

(iii) For $R \geq \check{R}$ we have

$$\lim_{B \to \infty} \frac{1}{B} \log \Pi_B = (p + q)\vartheta^\star.$$

Summarizing, we have identified the decay rate of $\Pi(B)$, and found three regimes for R. This could be dealt with explicitly, in that we presented closed-form expressions for the decay rate, as well as for the critical values of R that separate three regimes, which could be anticipated in view of earlier work, see e.g. [8, 15] and [19, Section 11.7]. The three regimes have an appealing intuitive explanation.

- For small values of R, the 'tightest' of the events $\{Q_0 > pB\}$ and $\{Q_{RB} > qB\}$ will essentially imply the other, thus leading to the decay rate $\max\{p, q\}\vartheta^\star$.

- Then there is an intermediate range of values of R for which both $\{Q_0 > pB\}$ and $\{Q_{RB} > qB\}$ are tight, but that the time epochs 0 and RB lie in the same busy period with overwhelming probability. The decay rate $p\vartheta^\star$ represents the requirement that pB has to be exceeded at time 0, and then $CRB + (q - p)B$ traffic has to be generated in the next RB time units, leading to the contribution $-\psi(R)$.

19

- Finally, for large R still both events are tight, but now they occur in different busy periods with overwhelming probability, so that the joint probability effectively decouples (thus leading to the decay rate $(p+q)\vartheta^\star$).

Theorem 5.11 has made this heuristic rigorous. We finish this section with an example.

Example 5.12. Consider the Brownian case, that is, $\varphi(\vartheta) = -\varrho\vartheta + \frac{1}{2}\vartheta^2$. It is easy to derive that $I(a) = \frac{1}{2}(a-\varrho)^2$ and $\vartheta^\star = -2(C-\varrho)$. The solution $\check{R}$ (larger than $\bar{R}$) of $q\vartheta^\star = -\psi(R)$ is

$$\check{R} = \frac{(\sqrt{p}+\sqrt{q})^2}{(C-\varrho)},$$

in line with Prop. 5.1 of [8]. $\diamond$

6 Discussion and concluding remarks

In this paper we analyzed the asymptotics of Π_B for B large. We showed that for T_B increasing sublinearly, the asymptotics reduce to those of the most demanding event, cf. (1). We also identified a criterion under which the event become asymptotically independent ('decoupling'), cf. (2). The latter criterion reduces to $T_B/B \to \infty$ in many situations, a notable exception being the case that $\mathbb{P}(Q > B)$ decays polynomially (in which case the condition is $T_B/B^2 \to \infty$).

While this paper gives a fairly complete picture of all possible regimes, a number of special cases are still open. For instance when $\mathbb{P}(Q > B)$ looks like $\exp(-B^\alpha)$ for some $\alpha \in (0,1)$, or like $B^{-\log B}$, the above mentioned criterion for decoupling is $T_B/B \to \infty$, but remains unclear what happens when $T_B = RB$ for some $R > 0$. It is expected that delicate analysis is needed to obtain the asymptotics in these situations. Another topic for future research concerns the exact asymptotics for the light-tailed case and $T_B = RB$.

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