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# On almost everywhere exponential convergence of the modified Jacobi–Perron algorithm: a corrected proof

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## 0. *Introduction*

The following theorem was published in [**2**].

THEOREM. *There exists a constant* $\delta > 0$ *such that for Lebesgue almost every* $(\alpha, \beta) \in X = [0, 1] \times [0, 1]$, *there exists* $n_0 = n_0(\alpha, \beta)$ *such that for any* $n \geq n_0$

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^{1+\delta}}, \quad \left|\beta - \frac{r_n}{q_n}\right| < \frac{1}{q_n^{1+\delta}},$$

*where the integers* $p_n, q_n, r_n$ *are provided by the modified Jacobi–Perron algorithm.*

There is a gap in the proof of this theorem, where Jensen's inequality is used, which has been pointed out by T. Fujita, one of the authors.

The gap is cleared by considering the metrics $\|\mathbf{x}\|_1 = |x| + |y|$ and $\|\mathbf{x}\|_\infty = \max(|x|, |y|)$ for $\mathbf{x} = (x, y) \in \mathbb{R}^2$ in addition to the Euclidean metric. The first part of the proof is the same as the original proof, but the latter part is improved and becomes simpler by using these new metrics. Therefore, we will present the improved proof here, referring to the original article for the unchanged parts.

A complete version is available (electronically) from the authors on request.

## 1. *The modified Jacobi–Perron algorithm*

This section remains unmodified.

## 2. *An application of the subadditive ergodic theorem*

This section remains unmodified until the end of the statement of the theorem. The corrected proof now follows.

To prove this result, it is sufficient to show

$$\int_X \frac{1}{2} \log \|D_2\|_2 \, d\mu < 0.$$

In fact, let us denote the infimum in the proposition above by $\gamma$. We have by formula (**) that

$$|q_n\alpha - p_n|^2 \quad \text{and} \quad |q_n\beta - r_n|^2$$

are bounded from above by

$$\|D_n f\|_2^2 \cdot \left\| \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} \right\|_2^2 \leq 2\|D_n\|_2^2.$$

By the proposition,

$$\lim_{n\to\infty} \frac{1}{n} \log \|D_n\|_2 = \gamma,$$

so that

$$\left|\alpha - \frac{p_n}{q_n}\right|, \; \left|\beta - \frac{r_n}{q_n}\right| \leq \frac{1}{q_n} e^{n\gamma'} = \frac{1}{q_n^{1+\delta'}}$$

for sufficiently large $n$, provided that

$$\delta' < -\frac{\gamma}{c},$$

with $c = -\int_X \log\theta \, d\mu > 0$ being the constant of the corollary. Therefore, it is sufficient to show that $\gamma < \frac{1}{2}\int_X \log \|D_2\|_2 \, d\mu < 0$.

## 3. *Calculus*

Before calculating the value of the integral, we note the following properties of the norms. Let us define the norms of $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$:

$$\|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2}, \quad \|\mathbf{x}\|_1 = \sum_{i=1,2} |x_i|, \quad \|\mathbf{x}\|_\infty = \max_{i=1,2} |x_i|.$$

For the corresponding norms of a $2 \times 2$ matrix $A$, we use the notation

$$\|A\|_2, \quad \|A\|_1, \quad \|A\|_\infty.$$

Then the following properties hold.

LEMMA. *Put* $A = (a_{ij})$, $1 \leq i, j \leq 2$. *Then:*

(i) $\|A\|_2 = \sqrt{\lambda}$ *where* $\lambda$ *is the maximum eigenvalue of* ${}^tAA$;

(ii) $\|A\|_1 = \max_{1\leq j\leq 2} \left( \sum_{i=1,2} |a_{ij}| \right)$;

(iii) $\|A\|_\infty = \max_{1\leq i\leq 2} \left( \sum_{j=1,2} |a_{ij}| \right)$;

(iv) $\|A\|_2^2 \leq \|A\|_1 \|A\|_\infty$.

*Proof.* The proof of (i), (ii) and (iii) are well known (see [1]).

For any norm on $\mathbb{R}^2$, the following relation (submultiplicativity) is valid:

$$\|AB\| \leq \|A\|\|B\|, \quad \text{where } \|A\| := \sup_{\|v\|\leq 1} \|Av\|.$$

From (i), let $\|A\|_2^2 = \lambda$ and let $\mathbf{y}$ be an eigenvector of ${}^tAA$ w.r.t. $\lambda$. Then we have

$$\|A\|_2^2 = \lambda = |\lambda| = \frac{\|{}^tAA\mathbf{y}\|}{\|\mathbf{y}\|} \leq \|{}^tAA\| \leq \|{}^tA\|\|A\|.$$

In particular, putting $\|\cdot\| = \|\cdot\|_2$, from (ii) and (iii), we have

$$\|A\|_2^2 \leq \|A\|_1\|A\|_\infty.$$

□

Let us define the partitions $\{X_0,\ X_1\}$ of $X$ and $\{X_{n,0},\ X_{n,1} | n = 1, 2, \ldots\}$ of $X_0$ as follows:

$$X_0 := \{(\alpha,\ \beta) \mid \beta < \alpha\}, \quad X_1 := \{(\alpha,\ \beta) \mid \alpha < \beta\}$$

and

$$\begin{aligned} X_{n,0} &:= \left\{(\alpha,\ \beta) \in X_0 \mid \frac{1}{n+1} < \alpha < \frac{1}{n},\ \beta > 1 - n\alpha\right\} \\ X_{n,1} &:= \left\{(\alpha,\ \beta) \in X_0 \mid \frac{1}{n+1} < \alpha < \frac{1}{n},\ \beta < 1 - n\alpha\right\}. \end{aligned}$$

Let us calculate the $2 \times 2$ matrix $D_2(\alpha,\ \beta)$ explicitly:

$$D_2(\alpha,\ \beta) = \begin{cases} \begin{bmatrix} 0 & -\alpha \\ 1 & \beta \end{bmatrix} \begin{bmatrix} 0 & -\dfrac{\beta}{\alpha} \\ 1 & -\dfrac{1}{\alpha} + n \end{bmatrix} = \begin{bmatrix} -\alpha & 1 - \alpha n \\ -\beta & -n\beta \end{bmatrix} & \text{if } (\alpha,\ \beta) \in X_{n,0} \\ \begin{bmatrix} 0 & -\alpha \\ 1 & \beta \end{bmatrix} \begin{bmatrix} -\dfrac{\beta}{\alpha} & 1 \\ -\dfrac{1}{\alpha} + n & 0 \end{bmatrix} = \begin{bmatrix} 1 - \alpha n & 0 \\ -\beta n & 1 \end{bmatrix} & \text{if } (\alpha,\ \beta) \in X_{n,1}. \end{cases}$$

The other two cases are, when $(\alpha,\ \beta) \in X_1$, symmetric formulas in $\alpha$ and $\beta$. Therefore we see that

$$\|D_2\|_1 = \begin{cases} \max(\alpha + \beta, 1 - \alpha n + \beta n) & \text{if } (\alpha, \beta) \in X_{n,0} \\ \max(1 - \alpha n + \beta n, 1) & \text{if } (\alpha, \beta) \in X_{n,1}. \end{cases}$$

From the fact that $1 - (1 - n\alpha + n\beta) = n(\alpha - \beta) > 0$ if $(\alpha, \beta) \in X_{n,1}$, we know that

$$\|D_2\|_1 = 1 \quad \text{if } (\alpha, \beta) \in X_{n,1} \text{ for every } n; \tag{$***$}$$

also, for $(\alpha, \beta) \in X_{n,0}$,

$$\|D_2\|_1 \geq 1 \quad \text{if and only if } (\alpha, \beta) \in X_{1,0}. \tag{$****$}$$

For the norm $\|\cdot\|_\infty$ we have

$$\|D_2\|_\infty = \begin{cases} \max(\alpha + 1 - \alpha n, \beta + \beta n) & \text{if } (\alpha, \beta) \in X_{n,0} \\ \max(1 - \alpha n, 1 + \beta n) & \text{if } (\alpha, \beta) \in X_{n,1}. \end{cases}$$

Therefore,

$$\|D_2\|_\infty \geq 1 \quad \text{if and only if } (\alpha, \beta) \in X_{n,1} \text{ or } (\alpha, \beta) \in X_{n,0} \text{ and } (n+1)\beta \geq 1. \qquad (*****)$$

For the norm $\|\cdot\|_2$, the trace and determinant of ${}^tD_2D_2$ on $X_0$ are given by

$$t = \text{trace of } {}^tD_2D_2 = \begin{cases} \alpha^2 + \beta^2 + (1-\alpha n)^2 + n^2\beta^2 & \text{if } (\alpha, \ \beta) \in X_{n,0} \\ 1 + (1-\alpha n)^2 + n^2\beta^2 & \text{if } (\alpha, \ \beta) \in X_{n,1} \end{cases}$$

and

$$d = \text{determinant of } D_2^T D_2 = \begin{cases} \beta^2 & \text{if } (\alpha, \ \beta) \in X_{n,0} \\ (1-\alpha n)^2 & \text{if } (\alpha, \ \beta) \in X_{n,1}. \end{cases}$$

Therefore, the norm $\|D_2\|_2^2$ is given as the maximum solution of $\lambda^2 - t\lambda + d = 0$:

$$\|D_2\|_2^2 = \frac{t + \sqrt{t^2 - 4d}}{2}.$$

Now, let us begin calculating the integral. By using property (iv), the value is estimated as follows:

$$\begin{aligned} \int_{X_0} \log \|D_2\|_2^2 \rho(\alpha, \beta)\, d\alpha\, d\beta &= \sum_{n=1}^{\infty} \int_{X_{n,0}} \log \|D_2\|_2^2 \rho(\alpha, \beta)\, d\alpha\, d\beta \\ &\leq \int_{X_{1,0}} \log \|D_2\|_2^2 \rho(\alpha, \beta)\, d\alpha\, d\beta \qquad (\mathrm{I}_1) \\ &\quad + \int_{X_{1,1}} \log \|D_2\|_2^2 \rho(\alpha, \beta)\, d\alpha\, d\beta \qquad (\mathrm{I}_2) \\ &\quad + \sum_{n=2}^{\infty} \int_{X_{n,0}} \log \|D_2\|_1 \rho(\alpha, \beta)\, d\alpha\, d\beta \qquad (\mathrm{I}_3) \\ &\quad + \sum_{n=2}^{\infty} \int_{X_{n,1}} \log \|D_2\|_1 \rho(\alpha, \beta)\, d\alpha\, d\beta \qquad (\mathrm{I}_4) \\ &\quad + \sum_{n=2}^{\infty} \int_{X_{n,0}} \log \|D_2\|_\infty \rho(\alpha, \beta)\, d\alpha\, d\beta \qquad (\mathrm{I}_5) \\ &\quad + \sum_{n=2}^{\infty} \int_{X_{n,1}} \log \|D_2\|_\infty \rho(\alpha, \beta)\, d\alpha\, d\beta. \qquad (\mathrm{I}_6) \end{aligned}$$

Each of the six values given above is now estimated.

*The estimation of* $\mathrm{I}_1$ *and* $\mathrm{I}_2$. For each $\epsilon = 0$ or $1$ we have

$$\begin{aligned} &\int_{X_{1,\epsilon}} \log \|D_2\|_2^2 \rho(\alpha, \beta)\, d\alpha\, d\beta \\ &\leq \int_{X_{1,\epsilon}} (\|D_2\|_2^2 - 1) \rho(\alpha, \beta)\, d\alpha\, d\beta \\ &= \frac{1}{2} \int_{X_{1,\epsilon}} t(\alpha, \beta) \rho(\alpha, \beta)\, d\alpha\, d\beta + \frac{1}{2} \int_{X_{1,\epsilon}} \sqrt{t^2 - 4d}\, \rho(\alpha, \beta)\, d\alpha\, d\beta \\ &\quad - \int_{X_{1,\epsilon}} \rho(\alpha, \beta)\, d\alpha\, d\beta \end{aligned}$$

By using Maple, the values of the integrations can be calculated:

$$\int_{X_{1,0}} (\alpha^2+\beta^2+(1-\alpha)^2+\beta^2)\rho(\alpha,\beta)\,d\alpha\,d\beta = 0.168378576\ldots$$

$$\int_{X_{1,0}} \sqrt{(2\alpha^2-2\alpha+1+2\beta^2)^2-4\beta^2}\,\rho(\alpha,\beta)\,d\alpha\,d\beta = 0.1036941092\ldots$$

$$\int_{X_{1,0}} \rho(\alpha,\beta)\,d\alpha\,d\beta = 0.1338263206\ldots$$

$$\int_{X_{1,1}} (1+(1-\alpha)^2+\beta^2)\rho(\alpha,\beta)\,d\alpha\,d\beta = 0.117559369\ldots$$

$$\int_{X_{1,1}} \sqrt{(1+(1-\alpha)^2+\beta^2)^2-4(1-\alpha)^2}\,\rho(\alpha,\beta)\,d\alpha\,d\beta = 0.09320701616\ldots$$

$$\int_{X_{1,1}} \rho(\alpha,\beta)\,d\alpha\,d\beta = 0.1008454268\ldots.$$

Therefore, the values $\mathrm{I}_1$ and $\mathrm{I}_2$ can be estimated as

$$\mathrm{I}_1 \le 0.00221002185\ldots$$
$$\mathrm{I}_2 \le 0.00453776578\ldots.$$

*The estimation of* $\mathrm{I}_3$. We know from $(****)$ that $\log\|D_2\|_1 \le 0$ for all $n \ge 2$. For each $n$, let us divide the integration into two parts:

$$\int_{X_{n,0}} \log\|D_2\|_1\rho(\alpha,\beta)\,d\alpha\,d\beta$$
$$= \int_{X_{n,0,1}} \log\|D_2\|_1\rho(\alpha,\beta)\,d\alpha\,d\beta \qquad (\mathrm{I}_{3,1,n})$$
$$+\int_{X_{n,0,2}} \log\|D_2\|_1\rho(\alpha,\beta)\,d\alpha\,d\beta, \qquad (\mathrm{I}_{3,2,n})$$

where $X_{n,0,1} = X_{n,0}\cap\{(\alpha,\beta)\mid \alpha+\beta\ge 1-n\alpha+n\beta\}$ and $X_{n,0,2} = X_{n,0}\cap\{(\alpha,\beta)\mid \alpha+\beta\le 1-n\alpha+n\beta\}$.

Using the change of variables

$$\alpha = \frac{1}{n+y},\quad \beta = \frac{x}{n+y},$$

$$\mathrm{I}_{3,1,n} = \int_{X_0\cap\{(x,y)\mid y\le(1-n)x+1\}} \log\left(\frac{1}{n+y}+\frac{x}{n+y}\right) g(x,y)\,dx\,dy$$

$$\mathrm{I}_{3,2,n} = \int_{X_0\cap\{(x,y)\mid y\ge(n-1)x+1\}} \log\left(1-\frac{n}{n+y}+\frac{nx}{n+y}\right) g(x,y)\,dx\,dy$$
$$= \int_0^{1/n} dy \int_{(1-y)/(n-1)}^{1} \log\left(\frac{y+nx}{n+y}\right) g(x,y)\,dx \qquad (\mathrm{I}_{3,3,n})$$
$$+\int_{1/n} dy \int_y^1 \log\left(\frac{y+nx}{n+y}\right) g(x,y)\,dx, \qquad (\mathrm{I}_{3,4,n})$$

where

$$g(x,y)=\frac{2n+2y+x+1}{(n+y)(n+y+1)(n+y+x)(n+y+x+1)}.$$

Using Maple, we have

$$\begin{aligned}\sum_{n=2}^{10} \mathrm{I}_{3,1,n} &= -0.01334879149\ldots\\ \sum_{n=2}^{10} \mathrm{I}_{3,3,n} &= -0.006379319311\ldots\\ \sum_{n=2}^{10} \mathrm{I}_{3,4,n} &= -0.007224728418\ldots\end{aligned}$$

and so

$$\sum_{n=2}^{10} \mathrm{I}_{3,2,n} = -0.013604047729\ldots.$$

Therefore we have

$$\mathrm{I}_3 \le -0.026952739219\ldots.$$

*The estimation of* $\mathrm{I}_4$. From the fact that $\|D_2\|_1 = 1$ if $(\alpha,\beta) \in X_{n,1}$ for every $n$, we know that $\log \|D_2\|_1 = 0$ if $(\alpha,\beta) \in X_{n,1}$. Therefore, we have $\mathrm{I}_4 = 0$.

*The estimation of* $\mathrm{I}_5$. From $(*****)$, we know that for $(\alpha,\beta) \in X_{n,0}$, $\|D_2\|_\infty \ge 1$ if and only if $(n+1)\beta \ge 1$. Using the change of variables as above, we have

$$\begin{aligned}\mathrm{I}_{5,n} &= \int_{X_{n,0}} \log \|D_2\|_\infty \rho(\alpha,\beta)\,d\alpha\,d\beta\\ &\le \int_{X_{n,0}\cap\{(\alpha,\beta)|(n+1)\beta\ge\alpha+1-\alpha n\}} \log \|D_2\|_\infty \rho(\alpha,\beta)\,d\alpha\,d\beta\\ &= \int_{X_0\cap\{(x,y)|y\le(n+1)x-1\}} \log\left(\frac{(n+1)x}{n+y}\right) g(x,y)\,dx\,dy.\end{aligned}$$

Notice that if $3x-1 \le y \le (n+1)x-1$ and $n \ge 2$, then

$$\log\frac{(n+1)x}{n+y} \le 0.$$

Thus,

$$\begin{aligned}\mathrm{I}_{5,n} &\le \int_{X_0\cap\{(x,y)|y\le 3x-1\}} \log\left(\frac{(n+1)x}{n+y}\right) g(x,y)\,dx\,dy\\ &= \int_{X_0\cap\{(x,y)|y\le 3x-1\}} \log\left(\frac{(n+1)x}{n+y}\right)\\ &\quad\times\left\{\frac{1}{(n+y)(n+y+x)} - \frac{1}{(n+y+1)(n+1+y+x)}\right\} dx\,dy.\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{I}_5 &= \sum_{n=2}^{\infty} \mathrm{I}_{5,n} \\
&\le \sum_{n=2}^{\infty} \int_{X_0 \cap \{(x,y) | y \le 3x-1\}} \log\left(\frac{(n+1)x}{n+y}\right) \\
&\quad \times \left\{\frac{1}{(n+y)(n+y+x)} - \frac{1}{(n+y+1)(n+1+y+x)}\right\} dx\,dy \\
&= \sum_{n=3}^{\infty} \int \left\{\log\frac{(n+1)x}{n+y} - \log\frac{nx}{n-1+y}\right\} \frac{1}{(n+y)(n+y+x)}\,dx\,dy \\
&\quad + \int_{X_0 \cap \{(x,y) | y \le 3x-1\}} \log\frac{3x}{2+y} \frac{1}{(2+y)(2+y+x)}\,dx\,dy.
\end{aligned}$$

Here, we see that

$$\log\frac{(n+1)x}{n+y} - \log\frac{nx}{(n-1)+y} = \log\left(1 - \frac{1-y}{n(n+y)}\right) \le 0.$$

Thus, we obtain

$$\begin{aligned}
\mathrm{I}_5 &\le \int_{X_0 \cap \{(x,y) | y \le 3x-1\}} \log\left(\frac{3x}{2+y}\right) \frac{1}{(2+x)(2+y+x)}\,dx\,dy \\
&= \int_0^{1/2} dy \int_{(1+y)/3}^{1} \log\left(\frac{3x}{2+y}\right) \frac{1}{(2+x)(2+y+x)}\,dx \\
&\quad + \int_{1/2}^{1} dy \int_y^1 \log\left(\frac{3x}{2+y}\right) \frac{1}{(2+x)(2+y+x)}\,dx \\
&= -0.004356831580\ldots + (-0.001098840366\ldots) \\
&= -0.005455671946\ldots.
\end{aligned}$$

*The estimation of* $\mathrm{I}_6$. We know from the definition of $\|D_2\|_\infty$ that $\|D_2\|_\infty = 1 + \beta n$ if $(\alpha, \beta) \in X_{n,1}$. Thus,

$$\begin{aligned}
\mathrm{I}_{6,n} &= \int_{X_{n,1}} \log \|D_2\|_\infty \rho(\alpha, \beta)\,d\alpha\,d\beta \\
&= \int_{X_{n,1}} \log(1+\beta n) \frac{(2+\alpha+\beta)}{(1+\alpha)(1+\beta)(1+\alpha+\beta)}\,d\alpha\,d\beta \\
&= \int_{X_1} \log\left(1 + \frac{nx}{n+y}\right) \frac{(2n+2y+x+1)}{(n+y)(n+y+1)(n+y+x)(n+y+x+1)}\,dx\,dy \\
&\le \int_{X_1} \frac{nx}{n+y} \frac{(2n+2y+x+1)}{(n+y)(n+y+1)(n+y+x)(n+y+x+1)}\,dx\,dy \\
&\le \int_{X_1} x \frac{(2n+2y+x+1)}{(n+y)(n+y+1)(n+y+x)(n+y+x+1)}\,dx\,dy.
\end{aligned}$$

Therefore, we obtain that

$$\mathrm{I}_6 = \sum_{n=2}^{\infty} \mathrm{I}_{6,n} = \sum_{n=2}^{\infty} \int_{X_1} x \frac{(2n+2y+x+1)}{(n+y)(n+y+1)(n+y+x)(n+y+x+1)}\,dx\,dy$$

$$
\begin{aligned}
&= \sum_{n=2}^{\infty} \int_{X_1} x \left( \frac{1}{(n+y)(n+y+x)} - \frac{1}{(n+1+y)(n+1+y+x)} \right) dx\, dy \\
&= \int_{X_1} \sum_{n=2}^{\infty} x \left\{ \frac{1}{(n+y)(n+y+x)} - \frac{1}{(n+1+y)(n+1+y+x)} \right\} dx\, dy \\
&= \int_{X_1} x \frac{1}{(2+y)(2+y+x)} dx\, dy \\
&= 0.019707476\ldots.
\end{aligned}
$$

Summing up the estimated values, we have

$$
\begin{aligned}
\int_{X_0} \log \|D_2\|_2^2 \rho(\alpha, \beta)\, d\alpha\, d\beta &\leq \mathrm{I}_1 + \mathrm{I}_2 + \cdots + \mathrm{I}_6 \\
&= -0.00595324\ldots \\
&< 0.
\end{aligned}
$$

## REFERENCES


[1] F. R. Gantmacher. *The Theory of Matrices* (2 vols). Chelsea, New York, 1964.

[2] S. Ito, M. Keane and M. Ohtsuki. Almost everywhere exponential convergence of the modified Jacobi–Perron algorithm. *Ergod. Th. & Dynam. Sys.* **13** (1993), 319–334.