



Centrum voor Wiskunde en Informatica

REPORTRAPPORT

MAS

Modelling, Analysis and Simulation



Modelling, Analysis and Simulation

Realization theory for linear hybrid systems, part II:
Reachability, observability and minimality

M. Petreczky, J.H. van Schuppen

REPORT MAS-R0803 FEBRUARY 2008

Centrum voor Wiskunde en Informatica (CWI) is the national research institute for Mathematics and Computer Science. It is sponsored by the Netherlands Organisation for Scientific Research (NWO). CWI is a founding member of ERCIM, the European Research Consortium for Informatics and Mathematics.

CWI's research has a theme-oriented structure and is grouped into four clusters. Listed below are the names of the clusters and in parentheses their acronyms.

Probability, Networks and Algorithms (PNA)

Software Engineering (SEN)

Modelling, Analysis and Simulation (MAS)

Information Systems (INS)

Copyright © 2008, Stichting Centrum voor Wiskunde en Informatica
P.O. Box 94079, 1090 GB Amsterdam (NL)
Kruislaan 413, 1098 SJ Amsterdam (NL)
Telephone +31 20 592 9333
Telefax +31 20 592 4199

ISSN 1386-3703

Realization theory for linear hybrid systems, part II: Reachability, observability and minimality

ABSTRACT

The paper is the second part of the series of papers started in [1]. The paper deals with observability, reachability and minimality of linear hybrid systems. Linear hybrid systems are continuous-time hybrid systems without guards, whose continuous dynamics is determined by time-invariant linear control systems. We will show that that if a set of input-output maps has a realization by a linear hybrid system, then it has a realization by a minimal linear hybrid system. We will present conditions for observability and span-reachability of linear hybrid systems and we will show that minimality is equivalent to observability and span-reachability. We will sketch algorithms for checking observability and span-reachability and for transforming a linear hybrid system to a minimal one.

2000 Mathematics Subject Classification: 93B15 93B20 93B25 93C99

Keywords and Phrases: hybrid systems; realization theory; minimal realization; observability; reachability

Realization Theory For Linear Hybrid Systems, Part II: Reachability, Observability and Minimality

Mihály Petreczky

Department of Mechanical Engineering, Eindhoven University of Technology

PO Box 513, 5600 MB Eindhoven, The Netherlands

`M.Petreczky@cwi.nl`

Jan H. van Schuppen

Centrum voor Wiskunde en Informatica (CWI)

P.O.Box 94079, 1090GB Amsterdam, The Netherlands

`J.H.van.Schuppen@cwi.nl`

Abstract

The paper is the second part of the series of papers started in [1]. The paper deals with observability, reachability and minimality of linear hybrid systems. Linear hybrid systems are continuous-time hybrid systems without guards, whose continuous dynamics is determined by time-invariant linear control systems. We will show that that if a set of input-output maps has a realization by a linear hybrid system, then it has a realization by a minimal linear hybrid system. We will present conditions for observability and span-reachability of linear hybrid systems and we will show that minimality is equivalent to observability and span-reachability. We will sketch algorithms for checking observability and span-reachability and for transforming a linear hybrid system to a minimal one.

This work was carried out during the first author's stay at Centrum voor Wiskunde en Informatica (CWI) in Amstredam, The Netherlands.

I. INTRODUCTION

The current paper is a continuation of [1] and its aim is to present results on observability, reachability and minimality of linear hybrid systems.

The problem of finding a minimal state-space realization of a certain class is a very fundamental one, and has important applications in systems identification and model reduction. Indeed, identifiability and minimality are closely related properties, hence characterizations of minimality may help to understand identifiability for a particular class of systems. In addition, minimization procedures may give rise to model reduction techniques, which, in turn, can facilitate analysis and control design for complex systems. Furthermore, it is widely accepted that understanding of observability and reachability is necessary for control and observer design.

Similarly to Part I, we will investigate linear hybrid system realizations of a *family of input-output maps*. By formulating our results for families of input-output maps rather than for a single input-output map we obtain more general, and hence potentially more widely applicable results. In addition, considering families of input-output maps should facilitate an easier connection to the framework of *behaviours* [2]. Furthermore, results on minimal realizations of families of input-output maps should help understanding the relationship between minimality and bisimulation (see the discussion later on).

Recall from [1] that a linear hybrid system is a hybrid system without guards whose continuous dynamics is determined by time-invariant continuous-time linear systems and whose discrete dynamics is determined by a finite Moore-automaton. For more on hybrid systems see [3] and the references therein. Recall that in Part I [1] we already presented sufficient and necessary conditions on existence of a realization by linear hybrid systems. The current paper presents a solution to the following problems.

- 1) **Observability, reachability and dimension** Find a suitable notion of observability, reachability and dimension for linear hybrid systems.
- 2) **Minimality** Consider a linear hybrid system H , and assume that H is a realization input-output maps Φ . Does there exist a minimal linear hybrid system realization of Φ , and if yes, is it unique? In addition, find a necessary and sufficient conditions for H being a minimal realization of Φ and present a procedure for transforming H to a minimal linear hybrid system which realizes Φ .

We will propose a notion of observability and span-reachability for linear hybrid system which will enable us to develop realization theory. We will show that observability and span-reachability can be characterized via linear algebraic conditions, which can be checked numerically. If applied to linear systems, the proposed notions of observability and span-reachability yield the classical definitions. We will define a notion of dimension for linear hybrid systems, which yields the classical definition if applied to linear systems. It will be shown that a linear hybrid system is minimal if and only if it is observable and span-reachable. We will show that if a family of input-output maps has a realization by a linear hybrid system, then it has a realization by minimal linear hybrid system and all minimal linear hybrid systems realizing the same family of input-output maps are isomorphic. In addition, we will show that any linear hybrid system can be transformed to a span-reachable and observable, and hence minimal, linear hybrid system which realizes the same input-output behaviour. In addition, this transformation can be done by an algorithm.

As it was already pointed out in Part I of the current series of papers [1], to the best of our knowledge, the only results on realization theory of hybrid systems are in [4], [5], [6], [7], [8], [9]. Except [10], [8], none of the papers cited above deal with linear hybrid systems. In [10], [8] some of the results of the current paper were stated, but most of the proofs were omitted. The results of the current paper were included into the first author's PhD thesis [11].

There is a link between the notion of minimal realization and the notion of biggest bisimulation. The latter was investigated in several papers, see [12], [13], [14], [15], [16]. In particular, bisimulation theory was developed for linear switching systems in [12], [17]. The main difference between linear switching systems and linear hybrid systems is that in the former the discrete events are viewed as disturbances, while in the latter discrete events viewed as inputs. In addition, in the case of linear switching systems the automaton is nondeterministic, while for linear hybrid systems the automaton is deterministic. For deterministic systems, the biggest bisimulation relation essentially coincides with the indistinguishability relation. More generally, the concept of bisimulation can be viewed as an extension of the classical notions of system morphism and observability to non-deterministic systems. In particular, if only span-reachable systems are considered, then a minimal linear hybrid system, in the sense defined in this paper, roughly corresponds to a linear hybrid system where the biggest bisimulation is the identity relation. Although the existence of a strong relationship between minimality and bisimulation is clear,

much more work needs to be done to explore the details of this relationship.

Theory of rational formal power series [18], [19], and classical automata theory [20], [21] are the main mathematical tools used in the paper. As it was noted in Part I, formal power series were already used for realization theory of nonlinear systems, see [22], [23], [24] and the references therein.

Part I, [1] is a prerequisite for the current paper. In particular, we will use the same notation and terminology as described in Part I. The proofs of the main result rely heavily on the results presented in Part I [1]. The outline of the paper is the following. Section II recalls from Part I the notions related to linear hybrid systems and presents the definitions of span-reachability, observability, dimension and hybrid system morphism. Section III presents the main theorems of the paper formally. Section V contains the proof of Theorem 3 which characterizes minimal linear hybrid systems. Section VI discusses briefly the computational aspects of minimality. Note that a more detailed exposition is planned in the form of a separate paper. The appendix presents some of the technical proofs.

II. PROBLEM FORMULATION

A. Linear Hybrid System

Recall from Part I [1] the definition of linear hybrid systems. That is, a linear hybrid system is a system of the form

$$H : \begin{cases} \frac{d}{dt}x(t) = A_{q(t)}x(t) + B_{q(t)}u(t) \\ y(t) = C_{q(t)}x(t) \\ q(t+) = \delta(q(t), \gamma(t)), \quad x(t+) = M_{q(t+), \gamma(t), q(t)}x(t-) \\ o(t) = \lambda(q(t)) \end{cases} \quad (1)$$

Here $q(t) \in Q$ is the *discrete state* at time t , $x(t) \in \mathbb{R}^{n_{q(t)}} = \mathcal{X}_{q(t)}$ is the *continuous state* at time t , $y(t) \in \mathbb{R}^p$ is the *continuous output* at time t , and $o(t) \in O$ is the *discrete output* at time t . The behaviour of the system at time t is influenced by the *continuous input* $u(t) \in \mathbb{R}^m$, and the *discrete input* $\gamma(t) \in \Gamma$. Further, Q is the finite set of *discrete states* of H , $\mathcal{X}_q = \mathbb{R}^{n_q}$, $n_q > 0$ is the *continuous state-space associated with the discrete state* $q \in Q$, O is the finite set of *discrete outputs*, Γ is the finite set of *discrete inputs (events)*, \mathbb{R}^m is the set of *continuous input values*,

and \mathbb{R}^p is the set of *continuous output values*. The *state-space of H* is the set of all pairs (q, x) where $q \in Q$ is a discrete state and $x \in \mathcal{X}_q$ is a continuous state. For each discrete state $q \in Q$, the matrices $A_q \in \mathbb{R}^{n_q \times n_q}$, $B_q \in \mathbb{R}^{n_q \times m}$ and $C_q \in \mathbb{R}^{p \times n_q}$ define a continuous-time linear system (A_q, B_q, C_q) on $\mathcal{X}_q = \mathbb{R}^{n_q}$. The map $\delta : Q \times \Gamma \rightarrow Q$ is called the *discrete state-transition map*, and the map $\lambda : Q \rightarrow O$ is called the *discrete readout map*. For each discrete state $q \in Q$ and discrete input $\gamma \in \Gamma$, the matrix $M_{\delta(q,\gamma),\gamma,q} \in \mathbb{R}^{n_{\delta(q,\gamma)} \times n_q}$ is referred to as a *reset map*. Recall from Part I [1] the following notation.

Notation 1 (Linear hybrid systems): A linear hybrid system of the form (1) is denoted by

$$H = (\mathcal{A}, \mathbb{R}^m, \mathbb{R}^p, (\mathcal{X}_q, A_q, B_q, C_q)_{q \in Q}, \{M_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\})$$

where $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ is the Moore-automaton formed by the discrete-state transition and discrete readout map of the system H . The automaton \mathcal{A} is denoted by \mathcal{A}_H , and the *state space of H* will be denoted by $\mathcal{H}_H = \bigcup_{q \in Q} \{q\} \times \mathcal{X}_q$.

Below we will briefly recall the dynamics of linear hybrid systems, which follows the classical definition [3]. Denote the set of timed sequences of discrete inputs by $(\Gamma \times T)^*$, i.e. a typical element of $(\Gamma \times T)^*$ is a finite sequence of the form $w = (\gamma_1, t_1)(\gamma_2, t_2) \cdots (\gamma_k, t_k)$ where $k \geq 0$, $\gamma_1, \dots, \gamma_k \in \Gamma$, $t_1, \dots, t_k \in T$. The interpretation of the sequence w is the following. The event γ_i took place *after* the event γ_{i-1} and t_{i-1} is the *elapsed time between the arrival of γ_{i-1} and the arrival of γ_i* . That is, t_i is the difference of the arrival times of γ_i and γ_{i-1} . Consequently, $t_i \geq 0$ but we allow $t_i = 0$, that is, we allow γ_i to arrive instantly after γ_{i-1} . If $i = 1$, then t_1 is simply the time when the event γ_1 arrived. The inputs of the linear hybrid system H are piecewise-continuous input functions $u \in PC(T, \mathbb{R}^m)$ and timed sequences of discrete inputs (events) $w = (\gamma_1, t_1) \cdots (\gamma_k, t_k) \in (\Gamma \times T)^*$. Recall from Part I that for an arbitrary state $h_0 = (q_0, x_0)$ of H the *continuous state $x_H(h_0, u, w, t_{k+1}) \in \mathcal{X}_{q_k}$ reached from h_0 with inputs u and w at time $\sum_{j=1}^k t_j + t_{k+1}$ is of the form*

$$\begin{aligned} x_H(h_0, u, w, t_{k+1}) = & e^{A_{q_k} t_{k+1}} M_{q_k, \gamma_k, q_{k-1}} e^{A_{q_{k-1}} t_k} \cdots M_{q_1, \gamma_1, q_0} e^{A_{q_0} t_1} x_0 + \\ & + \sum_{i=0}^k e^{A_{q_k} t_{k+1}} M_{q_k, \gamma_k, q_{k-1}} e^{A_{q_{k-1}} t_k} \cdots M_{q_{i+1}, \gamma_i, q_i} \times \\ & \times \int_0^{t_{i+1}} e^{A_{q_i} (t_{i+1} - s)} B_{q_i} u(s + \sum_{j=1}^i t_j) ds \end{aligned} \quad (2)$$

Define the *output* $v_H(h_0, u, w, t_{k+1})$ induced by h_0 under inputs u, w at time $\sum_{j=1}^{k+1} t_j$ as

$$v_H(h_0, u, w, t_{k+1}) = (\lambda(q_0, w), C_{q_k} x_H(h_0, u, w, t_{k+1}))$$

Recall from Part I the *input-output map of the system H induced by the state $h_0 \in \mathcal{H}_H$ of H* as the function

$$\begin{aligned} v_H(h, \cdot) : PC(T, \mathbb{R}^m) \times (\Gamma \times T)^* \times T \ni (u, w, t) \mapsto \\ v_H(h, u, w, t) \in O \times \mathbb{R}^p \end{aligned} \quad (3)$$

Let \mathcal{H}_0 be a subset of the state space \mathcal{H}_H and for each discrete state $q \in Q$ define the set $Reach_q(H, \mathcal{H}_0)$ as the linear span of all the continuous states which belong to \mathcal{X}_q and are reachable from some initial state in \mathcal{H}_0 , i.e.

$$\begin{aligned} Reach_q(H, \mathcal{H}_0) = \text{Span}\{x_H(h_0, u, w, t) \in \mathcal{X}_q \mid \\ h_0 = (q_0, x_0) \in \mathcal{H}_0, u \in PC(T, \mathbb{R}^m), t \in T, \text{ and} \\ t_1, \dots, t_k \in T, w = (\gamma_1, t_1) \cdots (\gamma_k, t_k), k \geq 0, \delta(q_0, \gamma_1 \gamma_2 \cdots \gamma_k) = q\} \end{aligned} \quad (4)$$

The linear hybrid system H is called *span-reachable* from \mathcal{H}_0 if,

- 1) The automaton \mathcal{A}_H is reachable from $\Pi_Q(\mathcal{H}_0) = \{q \in Q \mid \exists x \in \mathcal{X}_q : (q, x) \in \mathcal{H}_0\}$, and
- 2) For each discrete state $q \in Q$, $Reach_q(H, \mathcal{H}_0) = \mathcal{X}_q$.

Two states $h_1 \neq h_2 \in \mathcal{H}_H$ of the linear hybrid system H are *indistinguishable* if $v_H(h_1, \cdot) = v_H(h_2, \cdot)$, that is the input-output map induced by the state h_1 is the same as the input-output map induced by the state h_2 . H is called *observable* if it has no pair of distinct indistinguishable states. In other words, H is observable if for any two states h_1, h_2 , the equality $v_H(h_1, \cdot) = v_H(h_2, \cdot)$ implies $h_1 = h_2$.

Recall from Part I [1] that the input-output maps of interest are maps of the form $f : PC(T, \mathbb{R}^m) \times (\Gamma \times T)^* \times T \rightarrow O \times \mathbb{R}^p$ and the class of all such maps is denoted by $F(PC(T, \mathbb{R}^m) \times (\Gamma \times T)^* \times T, O \times \mathbb{R}^p)$. Recall from Part I [1] the definition of a linear hybrid realization. That is, let H be a linear hybrid system of the form (1) and let Φ be a subset of the set of input-output maps. Let $\mu : \Phi \rightarrow \mathcal{H}_H$ be any map. Then the pair (H, μ) is called a *realization*. The map μ just specifies a way to associate an initial state to each element of Φ . The set Φ is said to be *realized* by a hybrid realization (H, μ) , if $\mu : \Phi \rightarrow \mathcal{H}_H$, and for each

$$v_H(\mu(f), \cdot) = f$$

That is, for each input $u \in PC(T, \mathbb{R}^m)$, for each timed sequence of discrete inputs $w \in (\Gamma \times T)^*$ and for each time $t \in T$,

$$v_H(\mu(f), u, w, t) = f(u, w, t)$$

We will say that H *realizes* Φ if there exists a map $\mu : \Phi \rightarrow \mathcal{H}_H$ such that (H, μ) realizes Φ . With slight abuse of terminology, sometimes we will call both H and (H, μ) a *realization* of Φ . We say that a realization (H, μ) is *observable* if H is observable and we say that (H, μ) is *span-reachable* if H is span-reachable from the range of μ , i.e. if H is span-reachable from $\text{Im}\mu = \{\mu(f) \mid f \in \Phi\}$. Recall that denoted by μ_D the Q -valued component of μ , and by μ_C the continuous valued component of μ , that is, for each $f \in \Phi$, $\mu(f) = (\mu_D(f), \mu_C(f))$.

For a linear hybrid system H the *dimension* $\dim H$ of H is defined as a pair of natural numbers; the first component of $\dim H$ is the cardinality of the discrete state-space, the second component is the sum of dimensions of the continuous state-spaces, that is $\dim H = (\text{card}(Q), \sum_{q \in Q} \dim \mathcal{X}_q) \in \mathbb{N} \times \mathbb{N}$. For each two pairs of natural numbers $(m, n), (p, q) \in \mathbb{N} \times \mathbb{N}$ define the partial order relation as $(m, n) \leq (p, q)$, if $m \leq p$ and $n \leq q$. That is, the pair (m, n) is *smaller than or equal to* the pair (p, q) , if m is not greater than p and n is not greater than q . The reason for defining the dimension of a linear hybrid system as above is that there is a trade-off between the number of discrete states and dimensionality of each continuous state-space component. Notice that the above ordering of dimensions of linear hybrid systems is a *partial order*. That is, there can be two linear hybrid systems, dimensions of which are impossible to compare. For example, it may happen that one has two hybrid system realization of the same input/output map such that one of the realization has more discrete states but the dimensionality of the continuous state-spaces is small, while the other realization has fewer discrete states, but the dimensionality of each continuous state space is bigger.

A realization H of Φ is called a *minimal realization* of Φ , if for any linear hybrid system realization H' of Φ : $\dim H \leq \dim H'$, i.e. the following two conditions hold,

- 1) for any realization H' of Φ , the dimension of H' is comparable with the dimension of H ,
and
- 2) the dimension of H' is not smaller than the dimension of H .

Since not all hybrid realizations of Φ have comparable dimensions, it is not at all clear that one can choose a hybrid system realization of Φ whose dimension is minimal. *Hence, the existence*

of a minimal linear hybrid realization does not follow trivially, it has to be proven.

Below we will define the notion of linear hybrid system morphisms. Linear hybrid morphisms play a role, similar to the notion of algebraic similarity for linear systems. Let H' be a linear hybrid system of the form

$$H' = (\mathcal{A}', \mathbb{R}^m, \mathbb{R}^p, (\mathcal{X}'_q, A'_q, B'_q, C'_q)_{q \in Q'}, \{M'_{\delta'(q,\gamma),\gamma,q} \mid q \in Q', \gamma \in \Gamma\})$$

and assume that \mathcal{A}' is of the form $(Q', \Gamma, O, \delta', \lambda')$. Let H be a linear hybrid system of the form (1). Let Φ be a set of input-output maps and let $\mu : \Phi \rightarrow \mathcal{H}_H$ and $\mu' : \Phi \rightarrow \mathcal{H}_{H'}$ be two maps (recall that \mathcal{H}_H and $\mathcal{H}_{H'}$ denote the state-space of H and H' respectively). A map $T : \mathcal{H}_H \rightarrow \mathcal{H}_{H'}$ is called a *linear hybrid morphism* from (H, μ) to (H', μ') , if the following conditions hold.

- 1) There exists a map $T_D : Q \rightarrow Q'$ and for each discrete state $q \in Q$ of H there exist a linear map $T_{C,q} : \mathcal{X}_q \rightarrow \mathcal{X}'_{T_D(q)}$ such that
 - a) The map T_D forms an automaton morphism $T_D : (\mathcal{A}, \mu_D) \rightarrow (\mathcal{A}', \mu'_D)$,
 - b) for any state $(q, x) \in \mathcal{H}_H$ of H , $T((q, x)) = (T_D(q), T_{C,q}(x)) \in \mathcal{H}_{H'}$ holds.
- 2) For each discrete state $q \in Q$ of H ,
 - a) The matrices A_q, B_q, C_q , the map $T_{C,q}$ and the matrices $A'_{T_D(q)}, B'_{T_D(q)}, C'_{T_D(q)}$ commute, i.e.
$$T_{C,q}A_q = A'_{T_D(q)}T_{C,q}, \quad T_{C,q}B_q = B'_{T_D(q)}, \quad \text{and}$$

$$C_q = C'_{T_D(q)}T_{C,q}$$
 - b) For each input event $\gamma \in \Gamma$, the reset maps $M_{\delta(q,\gamma),\gamma,q}$, $M'_{\delta'(T_D(q),\gamma),\gamma,T_D(q)}$ and the maps $T_{C,q}$, $T_{C,\delta(q,\gamma)}$ commute, i.e. $T_{C,\delta(q,\gamma)}M_{\delta(q,\gamma),\gamma,q} = M'_{\delta'(T_D(q),\gamma),\gamma,T_D(q)}T_{C,q}$
- 3) For each input-output map f from Φ , $T(\mu(f)) = \mu'(f)$.

The fact that T is a linear hybrid system morphism from (H, μ) to (H', μ') will be denoted by $T : (H, \mu) \rightarrow (H', \mu')$. It is easy to see that there is a one-to-one correspondence between linear hybrid morphism $T : (H, \mu) \rightarrow (H', \mu')$ and pairs maps (T_D, T_C) such that $T_D : (\mathcal{A}, \mu_D) \rightarrow (\mathcal{A}', \mu'_D)$ is an automaton morphism, and $T_C : \bigoplus_{q \in Q} \mathcal{X}_q \rightarrow \bigoplus_{q \in Q'} \mathcal{X}'_q$ is a linear map defined between the direct sums of the continuous state spaces, such that for each $q \in Q$ and for all $x \in \mathcal{X}_q$, $T_C x = T_{C,q}x \in \mathcal{X}'_{T_D(q)}$ and the maps $T_D, T_{C,q}$ satisfy the condition 1), 2), and 3) described above. Using the identification of T with a pair (T_D, T_C) as described above, the linear hybrid system morphism T is said to be injective, surjective or *isomorphism* if both T_D and T_C are

respectively injective, surjective or bijective as maps. Two linear hybrid system realizations are *isomorphic* if there exists a linear hybrid isomorphism between them. The following proposition collects some simple properties of linear hybrid morphisms.

Proposition 1 ([11], Proposition 27 and Proposition 28, Chapter 7, page 197–198): With the notation above,

- 1) For any state $h \in \mathcal{H}_H$ of \mathcal{H} , the input-output maps induced by h and $T(h)$ are equal, i.e. $v_H(h, \cdot) = v_{H'}(T(h), \cdot)$.
- 2) If T is an isomorphism, then (H, μ) is span-reachable if and only if (H', μ') is span-reachable and (H', μ') is observable if and only if (H, μ) is observable.
- 3) T is an isomorphism, if and only if T is bijective as a map $T : \mathcal{H}_H \ni (q, x) \mapsto (T_D(q), T_{C,q}(x)) \in \mathcal{H}_{H'}$.
- 4) If T is surjective, then $\dim H' \leq \dim H$. If $\dim H = \dim H'$ and T is surjective, then T is a linear hybrid isomorphism.

The proof is very straightforward and can also be found in [11].

III. MAIN RESULTS

The goal of the section is to present the main results of the paper in a formal way. In order to do so we will have to recall some notation from Part I. Recall from Part I, Section II, Notation 2 the notation used to denote the product of system matrices of a linear hybrid system. We will start with formulating a linear-algebraic characterization of observability of linear hybrid systems. In order to do so, we will have to introduce some additional notation. Let H be a linear hybrid system of the form (1). For each discrete state $q \in Q$ define the subspace $O_{H,q}$ of \mathcal{X}_q as the intersection of the kernels of all the matrices of the form $C_{q_k} A_{q_k}^{\alpha_k+1} M_{q_k, \gamma_k, q_{k-1}} \cdots M_{q_1, \gamma_1, q_0} A_{q_0}^{\alpha_1}$, i.e.

$$O_{H,q} = \bigcap_{k \geq 0} \bigcap_{\gamma_1, \dots, \gamma_k \in \Gamma} \bigcap_{\alpha_1, \dots, \alpha_k \in \mathbb{N}} \ker C_{q_k} A_{q_k}^{\alpha_k+1} M_{q_k, \gamma_k, q_{k-1}} \cdots M_{q_1, \gamma_1, q_0} A_{q_0}^{\alpha_1} \quad (5)$$

Now we are ready to state the characterization of observability for linear hybrid systems.

Theorem 1 (Observability): H is observable if and only if

- (i) For each two discrete states $s_1, s_2 \in Q$, $s_1 = s_2$ if and only if the following two conditions hold

(a) Equality of Discrete Outputs

For all $k \geq 0$, and for any sequences of discrete inputs $\gamma_1, \dots, \gamma_k \in \Gamma$, the corresponding discrete outputs are the same, i.e. $\lambda(s_1, \gamma_1 \cdots \gamma_k) = \lambda(s_2, \gamma_1 \cdots \gamma_k)$

(b) Equality of "Generalized Markov Parameters"

For all $k \geq 0$, for all $\gamma_1, \dots, \gamma_k \in \Gamma$, for all $l = 1, \dots, k+1$, and for all $\alpha_l, \dots, \alpha_{k+1} \in \mathbb{N}$,

$$\begin{aligned} A_{q_k}^{\alpha_{k+1}} M_{q_k, \gamma_k, q_{k-1}} \cdots M_{q_l, \gamma_l, q_{l-1}} A_{q_{l-1}}^{\alpha_l} B_{q_{l-1}} = \\ A_{v_k}^{\alpha_{k+1}} M_{v_k, \gamma_k, v_{k-1}} \cdots M_{v_l, \gamma_l, v_{l-1}} A_{v_{l-1}}^{\alpha_l} B_{v_{l-1}} \end{aligned} \quad (6)$$

where $q_{l-1} = \delta(s_1, \gamma_1 \gamma_2 \cdots \gamma_{l-1})$ and $v_{l-1} = \delta(s_2, \gamma_1 \gamma_2 \cdots \gamma_{l-1})$.

(ii) For each $q \in Q$, 0 is the only element of the subspace $O_{H,q}$, i.e. $O_{H,q} = \{0\}$.

The proof of the theorem will be presented in Section IV. The intuition behind the characterization of observability is the following. Condition (ii) ensures that there are no two indistinguishable states with the same discrete state components, i.e. there are no two indistinguishable states of the form (q, x) and (q, y) with $x \neq y$. Condition (i) ensures that no two states of the form $(s_1, 0)$ and $(s_2, 0)$ are indistinguishable. It can be shown that condition (i) and condition (ii) of Theorem 1 can be checked algorithmically, see Section VI for more details.

Corollary 1: Assume that, **(1)** for each discrete state $q \in Q$, the linear system (A_q, B_q, C_q) is observable, and either **(2)** the automaton \mathcal{A}_H is observable, or **(3)** for any two distinct discrete states $s_1 \neq s_2 \in Q$, the Markov parameters of the linear systems $(A_{s_1}, B_{s_1}, C_{s_1})$ and $(A_{s_2}, B_{s_2}, C_{s_2})$ are not identical. Then H is an observable linear hybrid system.

Proof: Notice that for all discrete states $q \in Q$, the subspace $O_{H,q}$ is contained in the kernel of the observability matrix $[C_q^T, A_q^T C_q^T, \dots, A_q^{n_q-1} C_q^T]^T$. Hence, condition **(1)** implies that for all $q \in Q$, $O_{H,q} = \{0\}$, i.e., condition (ii) of Theorem 1 holds. If **(2)** or **(3)** holds, then condition (i) of Theorem 1 holds. Indeed, if **(2)** holds, then condition (a) of Theorem 1 implies that $s_1 = s_2$; if **(3)** holds, then condition (b) of Theorem 1 implies that $s_1 = s_2$. ■

Remark 1: One can construct counterexamples, which show that observability of a linear hybrid system does not imply observability of all the linear subsystems; and conversely, observability of all the linear subsystems does not imply observability of the whole hybrid system. Below we will formulate a computable algebraic characterization of span-reachability. Before we can state the theorem, we need to introduce some additional notation and terminology. Let

Φ be a set of input-output maps and assume that H is a linear hybrid system of the form (1) and assume that $\mu : \Phi \rightarrow \mathcal{H}_H$ is a map assigning initial states to each element of Φ . Recall the definition of the map $\mu_D : \Phi \rightarrow Q$ from Section II. and notice that (\mathcal{A}_H, μ_D) is a Moore-automaton realization. For each discrete state $q \in Q$ we will define the subspace $R_{H,q}$ of \mathcal{X}_q , which is analogous to the controllability subspace of linear systems, as follows. For each $f \in \Phi$, define the set $RH(f)$ of all the states $(q, x) \in \mathcal{H}_H$ such that either $(q, x) = \mu(f)$, or $x = B_q u$ for some $u \in \mathbb{R}^m$ and $q = \delta(\mu_D(f), v)$ for some sequence of discrete inputs $v \in \Gamma^*$. That is,

$$RH(f) = \{(q, x) \mid (q, x) = \mu(f) \text{ or, } \exists v \in \Gamma^*, u \in \mathbb{R}^m : q = \delta(\mu_D(f), v), x = B_q u, \}$$

With the notation above let $R_{H,q}$ be subspace of \mathcal{X}_q spanned by all the vectors

$$A_{q_k}^{\alpha_{k+1}} M_{q_k, \gamma_k, q_{k-1}} \cdots M_{q_1, \gamma_1, q_0} A_{q_0}^{\alpha_1} x, \text{ such that } q_k = q, (q_0, x) \in RH(f) \text{ for some } f \in \Phi. \text{ That is,}$$

$$R_{H,q} = \text{Span}\{A_{q_k}^{\alpha_{k+1}} M_{q_k, \gamma_k, q_{k-1}} \cdots M_{q_1, \gamma_1, q_0} A_{q_0}^{\alpha_0} x \mid f \in \Phi, \tag{7}$$

$$(q_0, x) \in RH(f), \gamma_1, \dots, \gamma_k \in \Gamma, q_k = q, q = \delta(q, \gamma_1 \cdots \gamma_k), k \geq 0\}$$

Now we are ready to state the theorem characterizing the span-reachability of linear hybrid realizations.

Theorem 2 (Span-Reachability): The linear hybrid system realization (H, μ) is span-reachable if and only if

- (i) The automaton realization (\mathcal{A}_H, μ_D) is reachable.
- (ii) For all $q \in Q$, $\dim R_{H,q} = \dim \mathcal{X}_q$.

The proof of the theorem will be presented in Section IV. The intuition behind the theorem is the following. First, in order for the linear hybrid realization to be span-reachable, we need to be able to reach every discrete state by a suitable choice of discrete inputs. Second, the continuous state-space component associated with each discrete state should contain only those states, which are really necessary. That is, the continuous-state space component \mathcal{X}_q should be the smallest vector space, which contains all the time derivatives of the state-trajectories which end in \mathcal{X}_q . It can be shown the span-reachability of linear systems can be decided algorithmically, we will discuss the issue in more detail in Section VI.

Corollary 2: If (\mathcal{A}_H, μ_D) is reachable and for each discrete state $q \in Q$, the corresponding linear system (A_q, B_q, C_q) is reachable, then (H, μ) is span-reachable.

Proof: Since the image of the controllability matrix $\text{Im}[B_q, A_q B_q, \dots, A_q^{n_q-1} B_q]$ is contained in $R_{H,q}$, it is easy to see that if (A_q, B_q, C_q) is reachable, then condition (ii) of Theorem 2 holds. By assumption, condition (i) of Theorem 2 holds, hence (H, μ) is span-reachable. ■ The concept of observability and span-reachability plays an important role in characterizing minimal linear hybrid systems. In this paper we will prove the following characterization of minimal linear hybrid system realizations.

Theorem 3 (Minimal realization): If Φ has a linear hybrid system realization, then Φ has a minimal linear hybrid system realization. If (H, μ) is a realization of Φ , then the following are equivalent;

- (i) (H, μ) is minimal,
- (ii) (H, μ) is span-reachable and it is observable,
- (iii) If (H', μ') span-reachable realization of Φ , then there exists a surjective linear hybrid system morphism $T : (H', \mu') \rightarrow (H, \mu)$.

All minimal hybrid linear system realizations of Φ are isomorphic.

The proof of the above theorem can be found in Section V. One can formulate an algorithm for transforming a linear hybrid realization to a minimal one, see [11] or Section VI for details.

Corollary 3: If the automaton realization (\mathcal{A}_H, μ_D) is reachable and observable (i.e. it is minimal, see Theorem 3 in Part I, [1]), and for each discrete state $q \in Q$, the corresponding linear system (A_q, B_q, C_q) is reachable and observable (i.e. it is minimal), then (H, μ) is minimal

Proof: Follows from Corollary 2, Corollary 1 and Theorem 3. ■

Remark 2: In Example 1, Section VI we will present an example of a family of input-output maps Φ and minimal linear hybrid system realization (H, μ) of Φ , such that the automaton and the individual linear subsystems of (H, μ) are not all minimal. *In fact, it can be shown that the family of input-output maps Φ described above cannot have a linear hybrid realization where the automaton and the linear subsystems are all minimal.* Indeed, let $(\tilde{H}, \tilde{\mu})$ be a realization of Φ such that the automaton and the linear subsystems of $(\tilde{H}, \tilde{\mu})$ are all minimal. Then Corollary 3 implies that $(\tilde{H}, \tilde{\mu})$ is minimal, and hence by Theorem 3 it is isomorphic to (H, μ) . The latter means that the automaton of \tilde{H} is isomorphic to the automaton of H , and each linear subsystem of H is isomorphic to a linear subsystem of \tilde{H} . Hence the automaton and the linear subsystems of H have to be all minimal, a clear contradiction.

IV. SPAN-REACHABILITY AND OBSERVABILITY: PROOFS OF THEOREM 2 AND THEOREM 1

Proof: [Proof of Theorem 1] The core of the proof is to show the following.

(A) $v_H((s_1, 0), \cdot) = v_H((s_2, 0), \cdot)$ if and only if part (a) and part (b) of condition (i) holds.

(B) $v_H((q, x_1), \cdot) = v_H((q, x_2), \cdot)$ is equivalent to $x_1 - x_2 \in O_{H,q}$.

From (A) and (B) the proof of the theorem follows easily. In order to show it, we introduce the following notation. For any $(q, x) \in \mathcal{H}$, $u \in PC(T, \mathbb{R}^m)$, $w = \gamma_1 \gamma_2 \cdots \gamma_k \in \Gamma^*$, $\gamma_1, \dots, \gamma_k \in \Gamma$, $k \geq 0$ define the function

$$y_H((q, x), u, w, \cdot) : (t_1, \dots, t_{k+1}) \ni T^{k+1} \mapsto \\ C_{q_k} x_H((q, x), u, (\gamma_1, t_1)(\gamma_2, t_2) \cdots (\gamma_k, t_k), t_{k+1}) \in \mathbb{R}^p$$

where $q_k = \delta(q, w)$. It is easy to see that $y_H((q, x), 0, w, \cdot)$ is linear in x , and

$$y_H((q, 0), u, w, \cdot) = \\ y_H((q, x), u, w, \cdot) - y_H((q, x), 0, w, \cdot) \quad (8)$$

In fact, it is easy to see that y_H is related to the continuous-valued part of the input-output map $v_H((q, x), \cdot)$ induced by the hybrid state as follows; $y_H((q, x), u, w, \cdot)(t_1, \dots, t_{k+1}) = \Pi_{\mathbb{R}^p}(v_H((q, x), u, (\gamma_1, t_1)(\gamma_2, t_2) \cdots (\gamma_k, t_k), t_{k+1}))$. Assume now that **(A)** and **(B)** hold. Assume that H is observable. Then $v_H((s_1, 0), \cdot) = v_H((s_2, 0), \cdot)$ implies $s_1 = s_2$, and hence by **(A)** condition (i) of the theorem holds. If $v_H((q, x_1), \cdot) = v_H((q, x_2), \cdot)$, then by observability we get that $x_1 - x_2 = 0$, hence by **(B)** condition (ii) of the theorem holds. Assume that conditions (i) and (ii) hold. We will show that then H is observable. Assume that (s_1, x_1) and (s_2, x_2) are indistinguishable, that is, $v_H((s_1, x_2), \cdot) = v_H((s_2, x_2), \cdot)$. The latter equality implies that $y_H((s_1, x_1), u, w, \cdot) = y_H((s_2, x_2), u, w, \cdot)$ for all u, w . By substituting the last equality into the right-hand side of (8), we get that $y_H((s_1, 0), u, w, \cdot) = y_H((s_2, 0), u, w, \cdot)$. The latter, together with $\Pi_O \circ v_H((s_i, 0), \cdot) = \Pi_O \circ v_H((s_i, x_i), \cdot)$, $i = 1, 2$, and $\Pi_O \circ v_H((s_1, x_1), \cdot) = \Pi_O \circ v_H((s_2, x_2), \cdot)$ implies that $v_H((s_1, 0), \cdot) = v_H((s_2, 0), \cdot)$. But then $s_1 = s_2 = s$ by **(A)** and condition (i) of the theorem and thus $v_H((s, x_1), \cdot) = v_H((s, x_2), \cdot)$. Hence from condition (ii) and **(B)** it follows that $x_1 = x_2$. That is, $(s_1, x_1) = (s_2, x_2)$, and hence H contains no two distinct indistinguishable states, i.e. H is observable.

Proof of (A) For each $s_i, i = 1, 2$, let $f_i = v_H((s_i, 0), \cdot)$ and consider the following singleton set consisting of one single input-output map $\Phi_i = \{f_i\}$, $i = 1, 2$. Define the maps $\mu_i : \Phi_i \ni$

$f_i \mapsto (s_i, 0) \in \mathcal{H}$ for $i = 1, 2$. Define the map $\mu_{12} : \Phi_1 \ni f_1 \mapsto (s_2, 0)$. It is easy to see that (H, μ_i) is a realization of Φ_i , $i = 1, 2$ and that $f_1 = f_2$ if and only if (H, μ_{12}) a realization of f_1 . Thus, Φ_{s_1}, Φ_{s_2} admit a hybrid kernel representation and $y_H((s_i, 0), u, w, \cdot) = y_0^{f_i, \Phi_i}(u, w, \cdot)$ for all $u \in PC(T, \mathbb{R}^m), w \in \Gamma^*$. From Proposition 1 in Part I, [1] it follows that (H, μ_i) is a realization of Φ_i for $i = 1, 2$, and (H, μ_{12}) is a realization of Φ_1 , if and only if

- (1) Both sides of the equation in part (a) are equal to $(f_1)_D(\gamma_1 \cdots \gamma_k)$ and hence to each other, and
- (2) for every $j = 1, \dots, m$, the j th column of the matrices on both sides of the equation (6) are equal to $D^{(0, 0, \dots, 0, \alpha_1, \dots, \alpha_{k+1})} y_0^{f_1, \Phi_1}(e_j, \gamma_1 \cdots \gamma_k)$ and hence to each other.

Hence, $f_1 = f_2$ if and only if part (a) and (b) of condition (ii) hold.

Proof of (B) Notice that the $v_H((q, x_1), \cdot) = v_H((q, x_2), \cdot)$ is equivalent to $y_H((q, x_1 - x_2), 0, w, \cdot) = 0$ for all $w \in \Gamma^*$, because of (8) and linearity of $y_H((q, x), 0, w, \cdot)$ in x . Notice the following equivalence

$$y_H((q, x_1 - x_2), 0, w, \cdot) = 0 \text{ for all } w \in \Gamma^* \iff \quad (9)$$

$$D^\alpha y_H((q, x_1 - x_2), 0, w, \cdot) = 0 \text{ for all } w \in \Gamma^*, \alpha \in \mathbb{N}^{|w|+1}$$

That is, $y_H((q, x_1 - x_2), 0, w, \cdot)$ is the constant zero function for all discrete input sequences w , if for all discrete input sequences w , all the high-order derivatives of $y_H((q, x_1 - x_2), 0, w, \cdot)$ at 0 vanish. Indeed, consider the set $\Phi_{1,2} = \{f\}$, $f = v_H((q, x_1 - x_2), \cdot)$ and let $\mu_{1,2} : f \mapsto (q, x_1 - x_2)$. It is easy to see that $(H, \mu_{1,2})$ is a realization of $\Phi_{1,2}$, hence $f_C(0, w, \cdot) = y_H((q, x_1 - x_2), 0, w, \cdot)$ is analytic, and hence (9) holds. Finally, by applying Proposition 1 from Part I to $\Phi_{1,2}$ and substituting the right-hand side of the second equation in (16), Part I, [1] into (9) we get the statement of (B). ■

Proof: [Proof of Theorem 2] Recall the definition of the set $Reach_q(H, \text{Im}\mu)$ from (4) in Section II. That is, $Reach_q(H, \text{Im}\mu)$ is the linear span of all continuous states which belong to \mathcal{X}_q and which are of the form $x_H(\mu(f), u, w, t)$ for some $f \in \Phi$, u, w , and t . We will show that $R_{H,q} = Reach_q(H, \text{Im}\mu)$ for all $q \in Q$. From this the statement of the theorem follows easily.

Proof of $R_{H,q} \subseteq Reach_q(H, \text{Im}\mu)$. Let $w = \gamma_1 \cdots \gamma_k$, $k = |w|$, and $u = 0$ or $u = e_j$, $j = 1, \dots, m$ and consider the map

$$\begin{aligned} x_H((s, z), u, w, \cdot) : T^{k+1} \ni (t_1, \dots, t_{k+1}) &\mapsto \\ x_H((s, z), u, (\gamma_1, t_1)(\gamma_2, t_2) \cdots (\gamma_k, t_k), t_{k+1}) & \end{aligned}$$

Assume that there exists $f \in \Phi$ such that $(s, z) = \mu(f)$ and $q = \delta(s, w)$, i.e. $x_H((s, z), u, w, \underline{t})$ belongs to $Reach_q(H, \text{Im}\mu)$ for all $\underline{t} \in T^{|\omega|+1}$. Since $Reach_q(H, \text{Im}\mu)$ is a subspace of \mathcal{X}_q , and hence it is linear and closed, it implies that for all $\alpha \in \mathbb{N}^{|\omega|+1}$, the derivatives $D^\alpha x_H((s, z), e_j, w, \cdot)$ and $D^\alpha x_H((s, z), 0, w, \cdot)$ belong to $Reach_q(H, \text{Im}\mu)$ as well. Notice that (2) implies that

$$D^\alpha x_H((s, 0), e_j, w, \cdot) = D^\alpha x_H((s, z), e_j, w, \cdot) - D^\alpha x_H((s, z), 0, w, \cdot)$$

hence $D^\alpha x_H((s, 0), e_j, w, \cdot) \in Reach_q(H, \text{Im}\mu)$. Moreover, from (2) it follows that

$$D^\alpha x_H((s, z), 0, w, \cdot) = A_{q_k}^{\alpha_{k+1}} M_{q_k, \gamma_k, q_{k-1}} \cdots M_{q_1, \gamma_1, q_0} A_{q_0}^{\alpha_1} z$$

where $s_0 = q_0$ and $q_k = q$; and, if $\alpha_l > 0$ and $\alpha_{l-1} = \cdots = \alpha_1 = 0$, and $q_{l-1} = \delta(s, \gamma_1 \cdots \gamma_{l-1})$ then

$$D^\alpha x_H((s, 0), e_j, w, \cdot) = A_{q_k}^{\alpha_{k+1}} M_{q_k, \gamma_k, q_{k-1}} \cdots M_{q_l, \gamma_l, q_{l-1}} A_{q_{l-1}}^{\alpha_l - 1} B_{q_{l-1}} e_j$$

Hence, $R_{H,q}$ is spanned by vectors of the form $D^\alpha x_H((s, z), 0, w, \cdot) \in Reach_q(H, \text{Im}\mu)$ and $D^\alpha((s, 0), e_j, w, \cdot) \in Reach_q(H, \text{Im}\mu)$ for all $j = 1, \dots, m$, $(s, z) = \mu(f)$ for some $f \in \Phi$, which implies that $R_{H,q} \subseteq Reach_q(H, \text{Im}\mu)$.

Proof of $Reach_q(H, \text{Im}\mu) \subseteq R_{H,q}$. From (2) it follows that any element $x_H((s, z), u, w, t_{k+1}) \in Reach_q(H, \text{Im}\mu)$ with $w = (\gamma_1, t_1) \cdots (\gamma_k, t_k)$ is a sum of expressions

$$e^{A_{q_k} t_{k+1}} M_{q_k, \gamma_k, q_{k-1}} e^{A_{q_{k-1}} t_k} \cdots M_{q_1, \gamma_1, q_0} e^{A_{q_0} t_1} z \quad , \quad (10)$$

$$\int_0^{t_l} e^{A_{q_k} t_k} M_{q_k, \gamma_k, q_{k-1}} \cdots M_{q_l, \gamma_l, q_{l-1}} e^{A_{q_{l-1}} (t_l - s)} B_{q_{l-1}} u(s) ds \quad (11)$$

where $q_k = q$ and $q_0 = s$ and $q_{l-1} = \delta(s_0, \gamma_1 \cdots \gamma_{l-1})$. If the expressions of the form (10) and (11) belong to $R_{H,q}$, then $x_H((s, z), u, w, t_{k+1})$ belongs to $R_{H,q}$ as well. In order to prove that the values (10,11) belong to $R_{H,q}$, notice that the expression (10) is analytic in t_1, t_2, \dots, t_{k+1} and the integral expression (11) is analytic in t_1, \dots, t_k if u is constant. Consider the Taylor-series expansion of (10) and (11) for u constant. Then the Taylor-coefficients are of the form $A_{q_k}^{\alpha_{k+1}} M_{q_k, \gamma_k, q_{k-1}} \cdots M_{q_l, \gamma_l, q_{l-1}} A_{q_{l-1}}^{\alpha_l} z$ with $(q_{l-1}, z) \in RH(f)$ for some $f \in \Phi$. Each such vector belongs to $R_{H,q}$ by the definition of $R_{H,q}$. Since $R_{H,q}$ is a vector space, the finite linear combination of such vectors also belongs to $R_{H,q}$. The set $R_{H,q}$ is closed, hence their limit belongs to $R_{H,q}$ as well, which implies that (10) belongs to $R_{H,q}$ and (11) belongs to $R_{H,q}$ if u is constant. If u is piecewise-constant, then (11) is a finite sum of expressions of the form (11),

for which the function u is constant. Since all the summands belong to $R_{H,q}$, the whole sum will belong to $R_{H,q}$. Finally, if u is a general piecewise-continuous map, the integral in (11) will be a limit of integrals of the form (11), but with u being piecewise-constant. Since each element of the approximating sequence belongs to $R_{H,q}$, and $R_{H,q}$ is closed, it follows that their limit belongs to $R_{H,q}$ as well; hence the integral in (11) belongs to $R_{H,q}$. ■

V. CHARACTERIZATION OF A MINIMAL LINEAR HYBRID SYSTEM REALIZATION: PROOF OF THEOREM 3

Below we will present the proof of Theorem 3. The proof relies on the relationship between rational representations, finite Moore-automata and linear hybrid systems presented in Part I ([1]) of the current series of papers. More precisely, we will establish a number of results which relate observability and span-reachability of linear hybrid systems with observability and reachability of representations and Moore-automata realizations. These results will enable us to characterize minimal linear hybrid system realizations in terms of observability and span-reachability. In addition, we will be able to formulate a procedure for transforming an arbitrary linear hybrid realization into a minimal one.

We will start with recalling some results on minimality, observability and reachability of rational representations and Moore-automata in Subsection V-A and Subsection V-B. We will continue with recalling the relationship between linear hybrid systems, Moore-automata and rational representations in Subsection V-C. Finally, in Subsection V-D we present the proof of Theorem 3.

A. Minimality of Rational Representations

In this section we will recall some results on reachability and observability of rational representations. In addition, we will present procedures for converting a rational representation to a reachable and observable one. These results will be used for studying minimality of linear hybrid systems. In the sequel we will use the definitions and notation of Section V of Part I, [1]. Let J be an arbitrary set and let Σ be a finite set, which will be referred to as the *alphabet*. Recall the notation introduced in Section II and in Section II, Part I. and recall the definition of formal power series and their rational representations from Section V, Part I. Let $R = (\mathcal{X}, \{A_\sigma\}_{\sigma \in \Sigma}, B, C)$ be a rational representation such that B is indexed by the index set J .

Recall from Section V, Part I that the representation R is called *reachable* if $\dim W_R = \dim R$ and R is called *observable* if $O_R = \{0\}$, where W_R and O_R were defined in (17), Section V, Part I. For any subspace $W \subseteq \mathcal{X}$, the representation R is said to be *W -observable* if $W \cap O_R = \{0\}$. It is clear that if R is observable, then R is W -observable for any subspace W . We will refer to the subspace O_R as the *observability kernel* of R and to the subspace W_R as the *reachability subspace* of R . If $\dim \mathcal{X} = n$, then it can be shown (see [11] for the proof) that

$$\begin{aligned} W_R &= \text{Span}\{A_w B_j \in \mathcal{X} \mid w \in \Sigma^*, |w| \leq n, j \in J\} \text{ and} \\ O_R &= \bigcap_{w \in \Sigma^*, |w| \leq n} \ker C A_w \end{aligned} \tag{12}$$

That is, the subspaces W_R and O_R can be computed, for example, by representing them as images and kernels of finite matrices. Hence, if J is a finite set, then observability of R can be checked numerically by verifying $O_R = \{0\}$; W -observability of R can be checked numerically by verifying $O_R \cap W = \{0\}$; and reachability of R can be checked numerically by checking if $\dim W_R = n$. For more on numerical algorithms for checking reachability and observability see [11].

Let $\Psi = \{S_j \in \mathbb{R}^p \ll \Sigma^* \gg \mid j \in J\}$ be a family of formal power series indexed by J and assume that $R = (\mathcal{X}, \{A_\sigma\}_{\sigma \in \Sigma}, B, C)$ is a representation of Ψ . Similarly to linear systems or Moore automata any R can be transformed to a reachable representation of Ψ defined as follows.

Construction 1: Define $R_r = (W_R, \{A_\sigma|_{W_R}\}_{\sigma \in \Sigma}, B, C|_{W_R})$, where the linear maps $A_\sigma|_{W_R}$, $\sigma \in \Sigma$, and $C|_{W_R}$ are the restrictions of the maps $A_\sigma, \sigma \in \Sigma$ and C respectively to the subspace W_R . Notice that for all $j \in J$, B_j belongs to W_R . It follows that R_r is a well-defined representation of Ψ , and it is reachable.

The representation R can also be transformed to an observable representation of Ψ , defined as

Construction 2: $R_o = (\mathcal{X}/O_R, \{A_\sigma^o\}_{\sigma \in \Sigma}, B^o, C^o)$, where \mathcal{X}/O_R denotes the quotient space of \mathcal{X} with respect to the subspace O_R and $C^o(x + O_R) = Cx$, $B_j^o = B_j + O_R$ for all $j \in J$, and $A_\sigma^o(x + O_R) = A_\sigma x + O_R$, for all $\sigma \in \Sigma$. Here $x + O_R$ denotes the equivalence class $\{z \in \mathcal{X} \mid x - z \in O_R\}$. It follows that R_o is a well-defined representation of Ψ , and it is observable.

If R is reachable then R_o will be reachable and observable too. Recall from Theorem 2 from Section V, Part I that a representation is minimal if and only if it is reachable and observable. Hence, the following procedure transforms R to a minimal representation of the same family Ψ .

Construction 3: Transform R to a reachable representation R_r as described in Construction 1. As the next step, transform R_r to an observable representation $(R_r)_o$ using Construction 2 with $R = R_r$. The resulting representation $R_m = (R_r)_o$ of Ψ is reachable and observable, and hence minimal.

Notice that the linear spaces in (12) are computable, therefore the representations R_o and R_r defined above, and hence the minimal representation $R_m = ((R_r)_o)$ are computable from R .

B. Minimality of Moore automata

Below we will recall some of the results on observability and reachability of Moore-automata and we will show that any Moore-automaton can be transformed to a reachable and observable, and hence minimal, Moore-automaton realizing the same input-output behaviour. These results will be used for studying minimality of linear hybrid systems.

In the sequel we will use the notation and terminology of Section VI Part I of the current series of papers, [1]. Recall from Section VI, Part I the concept of Moore-automaton and the notion of a Moore-automaton realization of a family of discrete input-output maps. Let Γ be the finite set of input symbols, let O be the finite set of output symbols, and let J be an arbitrary index set. Consider a finite Moore-automaton realization (\mathcal{A}, ζ) with $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ and $\zeta : J \rightarrow Q$. Recall from Section VI, Part I the definition of reachability and observability for (\mathcal{A}, ζ) . Recall the definition of the indistinguishability relation R_o ; for any two states $q_1, q_2 \in Q$, $(q_1, q_2) \in R_o$ holds, if for all $w \in \Gamma^*$, $\lambda(q_1, w) = \lambda(q_2, w)$. It can be shown (see [11], [20], [21] for the proof) that if $\text{card}(Q) = n$, then R_o can be finitely represented as

$$(q_1, q_2) \in R_o \iff \lambda(q_1, w) = \lambda(q_2, w) \text{ for all } w \in \Gamma^*, |w| \leq n \quad (13)$$

Hence, observability of \mathcal{A} can be checked algorithmically by checking if $(q_1, q_2) \in R_o \implies q_1 = q_2$ for all discrete states q_1, q_2 . Similarly, the set of all states $Q_r = \{q \in Q \mid \exists j \in J, w \in \Gamma^* : \delta(\zeta(j), w) = q\}$ reachable from $\text{Im}\zeta$ can be finitely described by

$$Q_r = \{\delta(\zeta(j), w) \mid j \in J, |w| \leq \text{card}(Q), w \in \Gamma^*\} \quad (14)$$

Hence, if J is finite, then reachability of (\mathcal{A}, ζ) can be algorithmically decided by checking if $\text{card}(Q_r) = \text{card}(Q)$.

Assume that (\mathcal{A}, ζ) is a realization of the family of input-output maps $\mathcal{D} = \{\phi_j : \Gamma^* \rightarrow O \mid j \in J\}$. Then (\mathcal{A}, ζ) can be transformed to a reachable automaton realization which realizes the same family of maps \mathcal{D} . This transformation is very similar to the classical one, see [21], [20].

Construction 4: Define the realization (\mathcal{A}_r, ζ_r) by $\mathcal{A}_r = (Q_r, \Gamma, O, \delta_r, \lambda_r)$, where the state space Q_r is just the set of all reachable states, see (14); the state-transition map is just the restriction of δ to the reachable set Q_r , i.e. for all $q \in Q_r$ and $\gamma \in \Gamma$, $\delta_r(q, \gamma) = \delta(q, \gamma)$; the readout map λ_r is the restriction of λ to Q_r , that is, for all $q \in Q_r$, $\lambda(q) = \lambda_r(q)$; finally, $\zeta_r(j) = \zeta(j)$ for all $j \in J$. It is easy to see that (\mathcal{A}_r, ζ_r) is a well-defined realization of \mathcal{D} , and it is reachable.

Similarly, the automaton realization (\mathcal{A}, ζ) can be transformed to an observable automaton realization of \mathcal{D} . This transformation is again very similar to the classical one [21], [20].

Construction 5: Define the automaton $\mathcal{A}_o = (Q^o, \Gamma, O, \delta^o, \lambda^o)$ as follows. Recall the definition of the indistinguishability relation R_o , see (13), and notice that R_o is an equivalence relation. Denote by $[q]$ the equivalence class induced by R_o which is represented by $q \in Q$. That is, $[q] = \{s \in Q \mid (s, q) \in R_o\}$. Let $Q^o = \{[q] \mid q \in Q\}$, and define the maps δ^o and λ^o as follows. For each $q \in Q$ and $\gamma \in \Gamma$ let $\delta^o([q], \gamma) = [\delta(q, \gamma)]$ and let $\lambda^o([q]) = \lambda(q)$. It is easy to see that δ^o and λ^o are well defined. Finally, define the map $\zeta_o : J \rightarrow Q^o$ by $\zeta_o(j) = [\zeta(j)]$. It is easy to see that (\mathcal{A}_o, ζ_o) is well-defined, it is a realization of \mathcal{D} and it is observable. Moreover, if (\mathcal{A}, ζ) is reachable, then (\mathcal{A}_o, ζ_o) will be reachable too.

Recall from Theorem 3, Section VI, Part I that a Moore-automaton realization is minimal if and only if it is reachable and observable. Hence, we can use the constructions above to transform (\mathcal{A}, ζ) to a minimal realization (\mathcal{A}_m, ζ_m) of \mathcal{D} .

Construction 6: First transform (\mathcal{A}, ζ) to a reachable automaton realization (\mathcal{A}_r, ζ_r) of \mathcal{D} using Construction 4. Then transform $(\mathcal{A}, \zeta) = (\mathcal{A}_r, \zeta_r)$ to a reachable and observable, i.e. minimal, realization $(\mathcal{A}_m, \zeta_m) = ((\mathcal{A}_r)_o, (\zeta_r)_o)$ of \mathcal{D} using Construction 5.

It is easy to see that the construction of (\mathcal{A}_r, ζ_r) , (\mathcal{A}_o, ζ_o) , and (\mathcal{A}_m, ζ_m) can be done by algorithmically, if J is finite and we can compare the elements of O .

C. Relationship Between Rational Representations, Moore-automata and Linear Hybrid Systems

In Part I, Section VII we presented a correspondence between rational representations and Moore-automata and linear hybrid systems. Below we will review this correspondence, which

will be used later on for providing a characterization of minimality for linear hybrid systems.

Let Φ be a set of input-output maps. Recall from Definition 1, Section III, Part I the notion of a hybrid kernel representation, and recall that if Φ has a linear hybrid realization, then Φ has a hybrid kernel representation. Therefore, we can assume that Φ has a hybrid kernel representation. Recall from Part I, Section VII, Definition 2 the definition of the *set of formal power series* Ψ_Φ associated with Φ and recall from Part I, Section VII, Definition 3 the definition *the family* \mathcal{D}_Φ of discrete input-output maps associated with Φ . In Part I, Section VII we showed that the family Φ has a realization by a linear hybrid system if and only if Ψ_Φ is rational and \mathcal{D}_Φ has a realization by a finite Moore-automaton.

More precisely, recall from Construction 1, Section VII, Part I, the definition of the *representation* $R_{H,\mu}$ associated with (H, μ) . Recall from Construction 2, Section VII, Part I the definition of *the Moore-automaton realization* $(\bar{\mathcal{A}}_H, \mu_D)$ associated with (H, μ) . In Theorem 4, Section VII, Part I we argued that if (H, μ) is a linear hybrid system realization of Φ , then $R_{H,\mu}$ is a rational representation of Ψ_Φ , and $(\bar{\mathcal{A}}_H, \mu_D)$ is a realization of \mathcal{D}_Φ . Conversely, let R be an observable representation of Ψ_Φ , and let $(\bar{\mathcal{A}}, \zeta)$ be a reachable realization of \mathcal{D}_Φ . Recall from Construction 3, Section VII, Part I the definition of the *linear hybrid system realization* $(H_{R,\bar{\mathcal{A}},\zeta}, \mu_{R,\bar{\mathcal{A}},\zeta})$ associated with R and $(\bar{\mathcal{A}}, \zeta)$. In Theorem 5, Section VII, Part I we showed that $(H_{R,\bar{\mathcal{A}},\zeta}, \mu_{R,\bar{\mathcal{A}},\zeta})$ is a realization of the original family of input-output maps Φ . We would like to remark that both the construction of $R_{H,\mu}$ and $(\bar{\mathcal{A}}_H, \mu_D)$ from (H, μ) , and the construction of $(H_{R,\bar{\mathcal{A}},\zeta}, \mu_{R,\bar{\mathcal{A}},\zeta})$ from the representation R and a Moore-automaton realization $(\bar{\mathcal{A}}, \zeta)$ can be carried out algorithmically. A short sketch of the algorithms can be found in Section VI.

D. Proof of Theorem 3

We will need a number of technical lemmas for the proof of Theorem 3. The proofs of these lemmas can be found in the appendix. The lemmas relate morphisms, and reachability and observability of automata and representations with morphisms, span-reachability and observability of linear hybrid systems. The reader is advised to review Subsections V-A, V-B and V-C and the corresponding sections of Part I, [1]. In the sequel we will use the notation and terminology of the above mentioned sections. In particular, the reader should be familiar with Definition 2, Definition 3, and Construction 1, Construction 2, and Construction 3 of Section VII, Part I, as we will use the notation introduced there without explicitly referring to it.

Lemma 1: Assume that R is an observable representation of Ψ_Φ and $(\bar{\mathcal{A}}, \zeta)$ is a reachable Moore-automaton realization of \mathcal{D}_Φ . Then $(H, \mu) = (H_{R, \bar{\mathcal{A}}, \zeta}, \mu_{R, \bar{\mathcal{A}}, \zeta})$ is span-reachable.

Lemma 2: Assume that $R = (\hat{\mathcal{X}}, \{F_\sigma\}_{\sigma \in \tilde{\Gamma}}, B, C)$ is an observable representation of Ψ_Φ and $(\bar{\mathcal{A}}, \zeta)$ is a reachable realization of \mathcal{D}_Φ . Let $(H, \mu) = (H_{R, \bar{\mathcal{A}}, \zeta}, \mu_{R, \bar{\mathcal{A}}, \zeta})$ and assume that H is of the form (1). Then $\bar{\mathcal{A}}_H = \bar{\mathcal{A}}$ and there exists a representation morphism $i_R : R_{H, \mu} \rightarrow R$, such that for any discrete state $q \in Q$ of H and for any $x \in \mathcal{X}_q$, $i_R(x) = x$.

Lemma 3: Assume that (H, μ) is a linear hybrid realization and assume that H is of the form (1). Then the following holds. (H, μ) is span-reachable if and only if $R_{H, \mu}$ is reachable and $(\bar{\mathcal{A}}_H, \mu_D)$ is reachable. (H, μ) is observable if and only if $\bar{\mathcal{A}}_H$ is observable and $R_{H, \mu}$ is \mathcal{X}_q -observable for all $q \in Q$.

Lemma 4: If R is an observable representation of Ψ_Φ and $(\bar{\mathcal{A}}, \zeta)$ is a minimal realization of \mathcal{D}_Φ , then $(H, \mu) = (H_{R, \bar{\mathcal{A}}, \zeta}, \mu_{R, \bar{\mathcal{A}}, \zeta})$ is an observable and span-reachable realization of Φ .

Lemma 5: Assume that (H, μ) is a span-reachable realization and H is of the form (1), R is observable and $(\bar{\mathcal{A}}, \zeta)$ is reachable. If $T : R_{H, \mu} \rightarrow R$ is a representation morphism and $\phi : (\bar{\mathcal{A}}_H, \mu_D) \rightarrow (\bar{\mathcal{A}}, \zeta)$ is an automaton morphism, then there exists a surjective linear hybrid morphism $(T_D, T_C) : (H, \mu) \rightarrow (H_{R, \bar{\mathcal{A}}, \zeta}, \mu_{R, \bar{\mathcal{A}}, \zeta})$ such that $T_{C, q}x = Tx$ for all $x \in \mathcal{X}_q$ and $q \in Q$, and $\phi = T_D$.

Lemma 6: Assume (H, μ) is a realization of Φ , and assume that H is of the form (1). Then there exists a span-reachable linear hybrid realization (H_r, μ_r) of Φ , such that $\dim H_r \leq \dim H$; and $\dim H_r = \dim H$ if and only if H is span-reachable .

Proof: [Proof of Theorem 3] First, we will show that if Φ has a linear hybrid system realization, then $\bar{\Phi}$ has a linear hybrid system realization satisfying (iii). Indeed, if Φ has a linear hybrid system realization, then Theorem 1 in Part I, [1] implies that Ψ_Φ has a representation and \mathcal{D}_Φ has a Moore-automaton realization. By Theorem 2 in Part I, [1] and Theorem 3 in Part I, [1] we can pick a minimal representation R of Ψ_Φ and a minimal realization $(\bar{\mathcal{A}}, \zeta)$ of \mathcal{D}_Φ . Then by Lemma 4 the linear hybrid system realization $(H_f, \mu_f) = (H_{R, \bar{\mathcal{A}}, \zeta}, \mu_{R, \bar{\mathcal{A}}, \zeta})$ is an observable and span-reachable realization of Φ . We will call the realization (H_f, μ_f) the *free realization of* Φ . We will show that (iii) holds for (H_f, μ_f) . Indeed, let (H, μ) is a span-reachable realization of Φ and assume that H is of the form (1). Then $R_{H, \mu}$ is reachable and $(\bar{\mathcal{A}}_H, \mu_D)$ is reachable. By Theorem 3 in Part I, [1] and Theorem 2 in Part I, [1] there exists surjective morphisms $T : R_{H, \mu} \rightarrow R$ and $\phi : (\bar{\mathcal{A}}_H, \mu_D) \rightarrow (\bar{\mathcal{A}}, \zeta)$. Then by Lemma 5 there exists a surjective linear

hybrid system morphism $S_{H,\mu} = (\phi, T_C) : (H, \mu) \rightarrow (H_f, \mu_f)$ such that $T_C x = Tx$ for all $x \in \mathcal{X}_q, q \in Q$.

Below we will show that **(iii)**, **(ii)**, and **(i)** are equivalent. This also implies that (H_f, μ_f) is minimal, since (H_f, μ_f) satisfies **(iii)**. Since (H_f, μ_f) exists whenever Φ has a linear hybrid system realization, we get that if Φ has a linear hybrid system realization, then Φ has a minimal linear hybrid system realization.

(iii) \implies **(i)**. Assume that (H, μ) satisfies (iii). Assume now that $(\tilde{H}, \tilde{\mu})$ is a realization of Φ . Then by Lemma 6 there exists a span-reachable realization (H_r, μ_r) of Φ , such that $\dim H_r \leq \dim \tilde{H}$. Therefore, there exists a surjective linear hybrid morphism $T : (H_r, \mu_r) \rightarrow (H, \mu)$. Then Proposition 1 implies that $\dim H \leq \dim H_r \leq \dim \tilde{H}$. Hence, (H, μ) is a minimal realization of Φ .

(ii) \implies **(iii)** Let (H, μ) be any span-reachable and observable realization of Φ and assume that H is of the form (1). We will show that **(iii)** holds for (H, μ) . Consider the surjective linear hybrid morphism $S_{H,\mu} = (\phi, T_C) : (H, \mu) \rightarrow (H_f, \mu_f)$ existence of which was proved above. We will show that $S_{H,\mu}$ is injective and thus it is a linear hybrid isomorphism. From the fact that $S_{H,\mu}$ is an isomorphism it follows that the inverse linear hybrid morphism $S_{H,\mu}^{-1}$ is also an isomorphism. For any span-reachable realization (H', μ') of Φ , there exists a surjective linear hybrid morphism $T : (H', \mu') \rightarrow (H_f, \mu_f)$, which implies that $S_{H,\mu}^{-1} \circ T : (H', \mu') \rightarrow (H, \mu)$ is a surjective linear hybrid morphism. Hence, (H, μ) satisfies **(iii)**.

The proof that $S_{H,\mu}$ is an isomorphism goes as follows. Since $S_{H,\mu}$ is surjective, it is enough to show that $S_{H,\mu}$ is injective. First we will show that ϕ is injective. If (H, μ) is observable, then $(\bar{\mathcal{A}}_H, \mu_D)$ is observable by Lemma 5. Observability of $\bar{\mathcal{A}}_H$ implies that ϕ is injective. Indeed, if $q_1, q_2 \in Q$, then $\bar{\lambda}(q_i, w) = \tilde{\lambda}(\phi(q_i), w)$, $i = 1, 2$ for all $w \in \Gamma^*$, where $\tilde{\lambda}$ denotes the readout map of $\bar{\mathcal{A}}$ and $\bar{\lambda}$ denotes the readout map of $\bar{\mathcal{A}}_H$. Hence, if $\phi(q_1) = \phi(q_2)$, then $\bar{\lambda}(q_1, w) = \bar{\lambda}(q_2, w)$ for all $w \in \Gamma^*$, which by observability of $\bar{\mathcal{A}}_H$ implies that $q_1 = q_2$. Next, we will show that T_C is injective. Denote by Q^f the set of discrete states of H_f and denote by \mathcal{X}_s^f the continuous state-space component of H_f associated with the discrete state $s \in Q^f$ of H_f . Recall that the linear map T_C is of the form $T_C : \bigoplus_{q \in Q} \mathcal{X}_q \rightarrow \bigoplus_{s \in Q^f} \mathcal{X}_s^f$. Assume that $x_i \in \mathcal{X}_{q_i}, i = 1, 2$ and $T_C x_1 = T_C x_2$. Since $T_C x_i \in \mathcal{X}_{\phi(q_i)}^f, i = 1, 2, T_C x_1 = T_C x_2$ implies that $\phi(q_1) = \phi(q_2)$. Since ϕ is injective, we get that that $q_1 = q_2 = q$ and $x_1, x_2 \in \mathcal{X}_q$. We will show that $x_1 = x_2$. By Lemma 5, observability of (H, μ) implies that $R_{H,\mu}$ is \mathcal{X}_q observable.

Assume that $R_{H,\mu} = (\mathcal{X}, \{M_\sigma\}_{\sigma \in \tilde{\Gamma}}, \tilde{B}, \tilde{C})$ and assume that $R = (\tilde{\mathcal{X}}, \{F_\sigma\}_{\sigma \in \tilde{\Gamma}}, G, H)$. Recall that T_C is the restriction of the representation morphism $T : R_{H,\mu} \rightarrow R$ to $\bigoplus_{q \in Q} \mathcal{X}_q$. Since T is a representation morphism and $x_1 - x_2 \in \mathcal{X}_q \subseteq \mathcal{X}$, it follows that $\tilde{C}M_w(x_1 - x_2) = HF_w T_C(x_1 - x_2)$ for all $w \in \tilde{\Gamma}^*$. Hence, if $T_C(x_1 - x_2) = 0$, then $\tilde{C}M_w(x_1 - x_2) = 0$ for all $w \in \tilde{\Gamma}^*$, that is, $x_1 - x_2 \in O_{R_{H,\mu}} \cap \mathcal{X}_q$. Since $R_{H,\mu}$ is \mathcal{X}_q -observable, i.e. $O_{R_{H,\mu}} \cap \mathcal{X}_q = \{0\}$, it follows that $x_1 - x_2 = 0$, i.e. $x_1 = x_2$.

(ii) \implies (i) Let (H, μ) be a span-reachable and observable realization of Φ . Let $S_{H,\mu} : (H, \mu) \rightarrow (H_f, \mu_f)$ be the isomorphism, existence of which was shown above. Then $\dim H = \dim H_f$, thus (H, μ) is minimal.

(i) \implies (ii) Let (H, μ) a minimal realization of Φ . From Lemma 6 it follows that (H, μ) has to be span-reachable. Indeed, if (H, μ) is not span-reachable, then by Lemma 6 there exists a span-reachable realization (H_r, μ_r) of Φ such that $\dim H_r < \dim H$, which is a contradiction. Hence there exists a surjective $T : (H, \mu) \rightarrow (H_f, \mu_f)$. But (H_f, μ_f) and (H, μ) are both minimal, thus $\dim H = \dim H_f$. Then Proposition 1 implies that T is a linear hybrid isomorphism, and since (H_f, μ_f) is observable, Proposition 1 implies that (H, μ) is observable too.

Finally, the statement of the theorem on isomorphism of all minimal linear hybrid realizations can be shown as follows. If (H_i, μ_i) , $i = 1, 2$ are two minimal realizations of Φ , then (H_i, μ_i) are both span-reachable and observable. But from the proof of **(ii) \implies (iii)** it follows that (H_i, μ_i) are both isomorphic to (H_f, μ_f) , and hence to each other. \blacksquare

The proof above implies the following.

Corollary 4: If R is a minimal representation of Ψ_Φ and $(\bar{\mathcal{A}}, \zeta)$ is a minimal realization of \mathcal{D}_Φ , then $(H_{R,\bar{\mathcal{A}},\zeta}, \mu_{R,\bar{\mathcal{A}},\zeta})$ is a minimal realization of Φ .

Remark 3 (Minimization of Linear Hybrid Systems): We can use the following procedure for transforming a linear hybrid system realization to a minimal one. Assume that (H, μ) is a linear hybrid system realization of Φ . Construct the rational representation $R_{H,\mu}$ and compute the Moore-automaton realization $(\bar{\mathcal{A}}_H, \mu_D)$. Transform the representation $R_{H,\mu}$ of Ψ_Φ to a minimal rational representation R of Ψ_Φ , using Construction 3 from Subsection V-A. Transform the Moore-automaton realization $(\bar{\mathcal{A}}_H, \mu_D)$ of \mathcal{D}_Φ to a minimal Moore-automaton realization (\mathcal{A}, ζ) of \mathcal{D}_Φ , using Construction 6 from Subsection V-B. Finally, construct the linear hybrid realization $(H_{R,\mathcal{A},\zeta}, \mu_{R,\mathcal{A},\zeta})$, as described in Construction 3, Section VII, Part I. Then Corollary 4 ensures that that $(H_{R,\mathcal{A},\zeta}, \mu_{R,\mathcal{A},\zeta})$ is a minimal realization of Φ .

The construction above can be implemented, the details will be presented in Section VI.

VI. ALGORITHMS FOR MINIMIZATION, OBSERVABILITY AND SPAN-REACHABILITY

The purpose of this section is to give a flavour of the algorithmic aspects of minimality of linear hybrid systems. We do not intend to describe the algorithms in full detail. The interested reader can find a more detailed account on the topic in [11], [10]. Note that the algorithms which will be outlined below have been implemented. Assume that Φ is finite and let (H, μ) be a realization of Φ . We will be interested in the following problems.

Computing the representation $R_{H,\mu}$. It is easy to see that Construction 1, Section VII, Part I, [1] for constructing $R_{H,\mu}$ can be implemented as a numerical algorithm.

Computing the automaton $\bar{\mathcal{A}}_H$. Recall from Part I, Section VII, Construction 2, [1] the definition of the automaton $\bar{\mathcal{A}}_H$ associated with the linear hybrid system H . It is clear that the only difficulty in computing the automaton $\bar{\mathcal{A}}_H$ from H is that its output space $O \times \bar{O}$ is infinite. However, for our purposes it is sufficient to compute another automaton $\bar{\mathcal{A}}_{H,N}$ which has a *finite* output space. More precisely, let $M(N)$ be the number of all words over $\tilde{\Gamma}$ of length at most N . The automaton $\bar{\mathcal{A}}_{H,N} = (Q, \Gamma, \mathbb{R}^{M(N) \times m} \times O, \delta, \lambda_N)$ has the same state-space and state-transition maps as \mathcal{A}_H , but its readout map is of the form $\lambda_N(q) = ((Z_{q,1}^N, \dots, Z_{q,m}^N), \lambda(q))$ for all $q \in Q$, where $Z_{q,j}^N$ denotes the vectors of *finite length* formed by the values $Z_{q,j}(s)$ for all words $s \in \tilde{\Gamma}^*$ of length at most N . Here $Z_{q,j}$ is the formal power series defined in (21), Section VII, Part I, [1]. It can be shown (see [11] for the proof) that the following holds.

Proposition 2 ([11]): If we choose N so that $\max\{\text{card}(Q), \text{card}(Q)m + \sum_{q \in Q} n_q\} \leq N$, then $\bar{\mathcal{A}}_{H,N}$ is observable (reachable) if and only if $\bar{\mathcal{A}}_H$ is observable (reachable). Moreover, if (H, μ) is a realization of Φ , then $(\bar{\mathcal{A}}_{H,N}, \mu_D)$ will be a realization of the indexed set of input-output maps $\mathcal{D}_{\Phi,N}$, where $\mathcal{D}_{\Phi,N}$ the indexed set of input-output maps which was defined before Proposition 2, Section VIII, Part I.

Checking observability, span-reachability and minimality of (H, μ) . Using Lemma 3 we can check observability or span-reachability of (H, μ) as follows. We compute $(\bar{\mathcal{A}}_{H,N}, \mu_D)$ and we compute $R_{H,\mu}$ as described above. From Proposition 2 and Lemma 3 it follows that in order to check observability of (H, μ) , we need to check if $R_{H,\mu}$ is \mathcal{X}_q -observable for all $q \in Q$ and if $(\bar{\mathcal{A}}_{H,N}, \mu_D)$ is observable. The first step is equivalent to checking $O_{R_{H,\mu}} \cap \mathcal{X}_q = \{0\}$ for all $q \in Q$, where $O_{R_{H,\mu}}$ is the observability subspace O_R of $R = R_{H,\mu}$ as defined in (12);

the second step is equivalent checking that for all $q_1, q_2 \in Q$, $(\lambda_N(q_1, w) = \lambda_N(q_2, w), w \in \Gamma^*, |w| \leq \text{card}(Q)) \implies q_1 = q_2$. Proposition 2 and Lemma 3 also imply that in order to check span-reachability of (H, μ) , we have to check if $R_{H,\mu}$ is reachable and $(\mathcal{A}_{H,N}, \mu_D)$ is reachable. That is, we have to check if $\dim W_{R_{H,\mu}} = \dim R_{H,\mu}$, where $W_{R_{H,\mu}}$ is the reachability subspace W_R of $R = R_{H,\mu}$ as defined in (12); and we have to check if $\text{card}(Q_r) = Q$, where Q_r is the set of states reachable from $\text{Im}\mu$, defined in (14) for $(\mathcal{A}, \zeta) = (\mathcal{A}_{H,N}, \mu_D)$. For checking minimality of (H, μ) we have to check if (H, μ) is span-reachable and observable. Hence, we just have to combine the algorithms outlined above.

Minimizing (H, μ) . We compute the representation $R_{H,\mu}$ and the Moore-automaton realization $(\bar{\mathcal{A}}_{H,N}, \mu_D)$. We transform $R = R_{H,\mu}$ to a minimal representation R_m by using Construction 3, Subsection V-A. We transform $(\bar{\mathcal{A}}_{H,N}, \mu_D)$ to a minimal realization $(\bar{\mathcal{A}}, \zeta)$ by using Construction 6 in Subsection V-B. It is easy to see that if we apply the steps of Construction 3, Section VII, Part I to $(\bar{\mathcal{A}}, \zeta)$ and R_m , then we can compute a linear hybrid realization $(H, \mu) = (H_{R_m, \bar{\mathcal{A}}, \zeta}, \mu_{R_m, \bar{\mathcal{A}}, \zeta})$. From Proposition 2, Section VIII, Part I it follows that (H, μ) will be a minimal realization of Φ .

To demonstrate the procedure above, consider the following numerical example.

Example 1: Recall from Example 1 from Part I, Section II-B the definition of the linear hybrid system realization (H, μ) and the definition of the family of input-output maps Φ . Using the algorithms sketched above (H, μ) can be transformed to the following minimal realization (H_m, μ_m) of Φ . H_m is of the form

$$H_m = H^m = (\mathcal{A}^m, \mathbb{R}^m, \mathbb{R}^p, (\mathcal{X}_q^m, A_q^m, B_q^m, C_q^m)_{q \in Q^m}, \{M_{\delta^m(q,\gamma),\gamma,q}^m \mid q \in Q^m, \gamma \in \Gamma\})$$

where the automaton \mathcal{A}^m is of the form: $\mathcal{A}^m = (Q^m, \Gamma, O, \delta^m, \lambda^m)$ where $Q^m = \{q_1, q_2, q_3\}$, $\Gamma = \{a, b\}$ and the transition function δ^m is defined by $\delta^m(q_1, z) = q_1$ for $z = a, b$, $\delta^m(q_2, a) = q_3$, $\delta^m(q_3, b) = q_2$, $\delta^m(q_2, b) = q_2$, $\delta^m(q_3, a) = q_3$. The readout map is of the form $\lambda^m(q_1) = o$, $\lambda^m(q_2) = d$, $\lambda^m(q_3) = g$. The linear systems and the reset maps are of the following form.

$$A_{q_1}^m = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, B_{q_1}^m = \begin{bmatrix} 0 \\ 0 \\ 0.71 \end{bmatrix}, C_{q_1}^m = [1 \quad 1 \quad 1.41], M_{q_1, b, q_1}^m = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, M_{q_1, a, q_1}^m = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$A_{q_2}^m = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, B_{q_2}^m = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, C_{q_2}^m = [1 \quad 0], M_{q_3, a, q_2}^m = [0 \quad 1], M_{q_2, b, q_2}^m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_{q_3}^m = [-1], B_{q_3}^m =$$

$$[0], C_{q_3}^m = [0], M_{q_3, a, q_3}^m = [1], M_{q_2, b, q_3}^m = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \text{ The map } \mu_m : \Phi \rightarrow \mathcal{H}_{H_m} \text{ is defined as } \mu_m(f_1) = (q_1, [0 \quad 1 \quad 0]^T)$$

and $\mu_m(f_2) = (q_2, [0 \quad 0]^T)$. Notice that none of the linear subsystems of H_m is minimal. Consider the sub-automata $\mathcal{A}_1 = (Q_1, \Gamma, O, \delta_1, \lambda_1)$ and $\mathcal{A}_2 = (Q_2, \Gamma, O, \delta_2, \lambda_2)$ of \mathcal{A}^m , where $Q_1 = \{q_1\}$ and $Q_2 = \{q_2, q_3\}$

and $\delta_i(q, \gamma) = \delta^m(q, \gamma)$, $\lambda_i(q) = \lambda^m(q)$, for all $q \in Q_i$, $i = 1, 2$. Consider the subsystems H_i , $i = 1, 2$ of H_m formed by the discrete states belonging to Q_i , $i = 1, 2$ and the corresponding linear systems and reset maps. That is, for $i = 1, 2$, $H_i = (\mathcal{A}_i, \mathbb{R}^m, \mathbb{R}^p, (\mathcal{X}_q^m A_q^m, B_q^m, C_q^m)_{q \in Q_i}, \{M_{\delta^m(q, \gamma), \gamma, q} \mid q \in Q_i, \gamma \in \Gamma\})$. Define the maps $\mu_i : \{f_i\} \mapsto \mu_m(f_i)$, $i = 1, 2$, where f_1, f_2 are the input-output maps defined above. Then it is easy to see that for $i = 1, 2$ the system (H_i, μ_i) is a minimal realization of f_i , but none of the linear subsystems of H_m , and hence of H_i , $i = 1, 2$, are minimal. Hence, (H_m, μ_m) and (H_i, μ_i) , $i = 1, 2$ represent examples of minimal linear hybrid realizations such that none of the linear subsystems is minimal.

To demonstrate experimentally that the input-output behavior of (H_m, μ_m) is the same as that of (H, μ) we have carried out simulations; we tested the response of the system on ten switching scenarios under generated white noise continuous input. As the theory predicts, the responses of (H, μ) and (H_m, μ_m) are almost identical, the small discrepancy can be attributed to the presence of numerical errors in the computation. For an illustration see Figure 1.

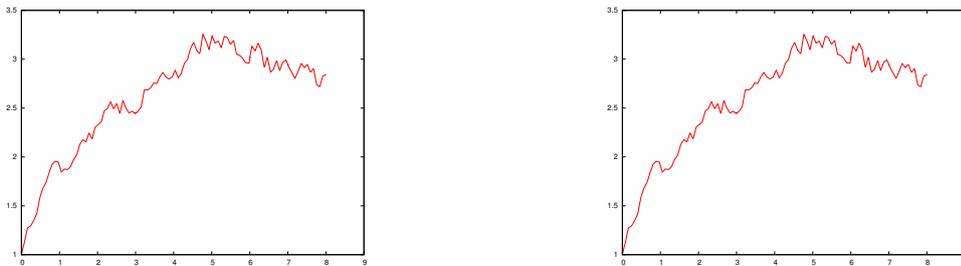


Fig. 1. The value of the input-output map f_1 for white noise continuous input and timed sequence of discrete inputs $(b, 1)(a, 2)(a, 3)(b, 1), 1$. The left-hand side figure shows the continuous response of the original system H from the initial state $\mu(f_1)$, the right-hand side figure shows the continuous response of the system H_m from the initial state $\mu_m(f_1)$

VII. CONCLUSIONS AND FUTURE WORK

Linear algebraic conditions for observability and span-reachability, along with a characterization of minimality of linear hybrid systems were presented. We also showed that if a family of input-output maps has a realization by a linear hybrid system, then it also has a realization by a minimal linear hybrid system. Moreover, any linear hybrid system realization can be transformed to a minimal linear hybrid system realization which realizes the very same family of input-output maps as the original one. Topics of further research include realization theory for piecewise-affine systems on polytopes, and general non-linear hybrid systems without guards.

We hope that the presented results will be useful for model reduction and identification of hybrid systems. The results of the paper indicate, that simply combining linear identification

methods with ways of estimating the discrete state might fail even for relatively simple classes of hybrid systems. Indeed, from Remark 2 and Example 1 in Section VI it follows *that there exists an input-output map f , such that f has a realization by a linear hybrid system, but it cannot be realized by a linear hybrid system for which the automaton and all the linear subsystems are minimal. Hence, it is impossible to identify a linear hybrid system which realizes f , by estimating the automaton based on the discrete outputs and then estimating the linear systems for each discrete state.*

Acknowledgement. The research described in this paper was carried out during the Mihály Petreczky's stay as a PhD student at Centrum voor Wiskunde en Informatica (CWI) in Amsterdam, The Netherlands. The authors thank Pieter Collins and Luc Habets, their colleagues at CWI.

REFERENCES

- [1] M. Petreczky and J. H. van Schuppen, "Realization theory for linear hybrid systems, Part I: Existence of a realization," Centrum voor Wiskunde en Informatica (CWI), Amsterdam, Tech. Rep., 2008.
- [2] J. Willems and J. Polderman, *An Introduction to Mathematical Systems Theory: A Behavioral Approach*. Springer Verlag, New York, 1998.
- [3] A. van der Schaft and H. Schumacher, *An Introduction to Hybrid Dynamical Systems*. Springer-Verlag London, 2000.
- [4] M. Petreczky, "Realization theory for linear switched systems," in *Proceedings MTNS2004*, 2004.
- [5] —, "Realization theory of linear and bilinear switched systems: A formal power series approach," CWI, Tech. Rep. MAS-R0403, 2005.
- [6] —, "Realization theory for bilinear switched systems," in *Proceedings ECC-CDC2005*, 2005.
- [7] —, "Realization theory for bilinear hybrid systems," in *Proceedings MMAR 2005*, 2005.
- [8] —, "Hybrid formal power series and their application to realization theory of hybrid systems," in *Proc. MTNS2006*, 2006.
- [9] —, "Realization theory for discrete-time piecewise-affine hybrid systems," in *Proc. MTNS2006*, 2006.
- [10] —, "Realization theory for linear and bilinear hybrid systems," CWI, Tech. Rep. MAS-R0502, 2005.
- [11] —, "Realization theory of hybrid systems," Ph.D. dissertation, Vrije Universiteit, Amsterdam, 2006, available at <http://www.cwi.nl/~mpetrec>.
- [12] G. Pola, A. J. van der Schaft, and M. D. Di Benedetto, "Bisimulation theory of switching linear systems," in *Proceedings CDC2004*, 2004.
- [13] A. J. van der Schaft, "Bisimulation of dynamical systems," in *Hybrid Systems: Computation and Control*, ser. LNCS 2933, 2004.
- [14] G. Lafferriere, G. J. Pappas, and S. Sastry, "Hybrid systems with finite bisimulations," in *Hybrid Systems V*, ser. LNCS 1567, 1997.
- [15] E. Haghverdi, P. Tabuada, and G. J. Pappas, "Bisimulation relations for dynamical, control, and hybrid systems," *Theoretical Computer Science*, vol. 342, no. 2-3, pp. 229–261, 2005.

- [16] A. van der Schaft, “Equivalence of dynamical systems by bisimulation,” *IEEE Transactions on Automatic Control*, vol. 49, pp. 2160–2172, 2004.
- [17] G. Pola, A. van der Schaft, and M. D. Di Benedetto, “Equivalence of switching linear systems by bisimulation,” *International Journal of Control*, vol. 79, no. 1, pp. 74–92, 2006.
- [18] J. Berstel and C. Reutenauer, *Rational series and Their Languages*. Springer-Verlag, 1984.
- [19] W. Kuich and A. Salomaa, in *Semirings, Automata, Languages*, ser. EATCS Monographs on Theoretical Computer Science. Springer-Verlag, 1986.
- [20] F. Gécseg and I. Peák, *Algebraic theory of automata*. Akadémiai Kiadó, Budapest, 1972.
- [21] S. Eilenberg, *Automata, Languages and Machines*. Academic Press, New York, London, 1974.
- [22] B. Jakubczyk, “Realization theory for nonlinear systems, three approaches,” in *Algeb. & Geom. Methods in Nonlin. Contr. Theory*, 1986.
- [23] E. D. Sontag, “Realization theory of discrete-time nonlinear systems: Part I – the bounded case,” *IEEE Transaction on Circuits and Systems*, vol. CAS-26, no. 4, April 1979.
- [24] A. Isidori, *Nonlinear Control Systems*. Springer Verlag, 1989.

APPENDIX

Proof: [Proof of Lemma 2] Since $(\bar{\mathcal{A}}, \zeta)$ is reachable and \mathcal{A} coincides with $\bar{\mathcal{A}}$ with the exception of the readout map, we get that $(\mathcal{A}, \zeta) = (\mathcal{A}_H, \mu_D)$ is reachable. By substituting formula (24) in Part I, [1] into the formula (25) in Part I, [1] it is easy to see that $\mathcal{X}_q = R_{H,q}$ for all $q \in Q$, hence by Theorem 2 (H, μ) is span-reachable. ■

Proof: [Proof of Lemma 3] Assume that H is of the form (1), and assume that μ is of the form $\mu : \Phi \rightarrow \mathcal{H}_H$. Consider the representation $R_{H,\mu}$ from (19) in Part I, [1]. For each pair $q_1, q_2 \in Q$ of discrete states define the subset of words $R(q_1, q_2) \subseteq \tilde{\Gamma}^*$ as follows. A word $v \in \tilde{\Gamma}^*$ of the form $v = e^{\alpha_1} \gamma_1 e^{\alpha_2} \gamma_2 \cdots e^{\alpha_k} \gamma_k e^{\alpha_{k+1}}$ belongs to $R(q_1, q_2)$, if and only if $\delta(q_2, \gamma_1 \cdots \gamma_k) = q_1$. From (21) in Part I, [1] it follows that

$$R_{H,q} = \text{Span}\{M_s M_e M_w \tilde{B}_{f,j}, M_v \tilde{B}_f \mid s, v \in \tilde{\Gamma}^*, f \in \Phi, \\ j = 1, \dots, m, w \in \Gamma^*, w e s, v \in R(q, \mu_D(f))\}$$

which implies that $W_{R_{H,\mu}} \cap \mathcal{X}_q = R_{H,q}$ for all $q \in Q$. It is easy to see that $R_{H,\mu}$ is reachable if and only if $\bigoplus_{q \in Q} \mathcal{X}_q = \bigoplus_{q \in Q} W_{R_{H,\mu}} \cap \mathcal{X}_q = \bigoplus_{q \in Q} R_{H,q}$ and the equality below holds.

$$\mathbb{R}^{|Q|m} = \text{Span}\{e_{q,j} \mid j = 1, \dots, m, \\ q = \delta(\mu_D(f), w), w \in \Gamma^*, f \in \Phi\} \quad (15)$$

Since $e_{q,j}$, $q \in Q$, $j = 1, \dots, m$ are linearly independent, it follows that (15) holds if and only if (\mathcal{A}_H, μ_D) is reachable. The latter is equivalent to $(\bar{\mathcal{A}}_H, \mu_D)$ being reachable. That is, $R_{H,\mu}$

is reachable and if and only if $\bigoplus_{q \in Q} \mathcal{X}_q = \bigoplus_{q \in Q} R_{H,q}$, or, equivalently $\mathcal{X}_q = R_{H,q}$ for all $q \in Q$, and (\mathcal{A}_H, μ_D) is reachable, i.e. (H, μ) is span-reachable. Since span-reachability of (H, μ) implies reachability of (\mathcal{A}_H, μ_D) and hence reachability of $(\bar{\mathcal{A}}_H, \mu_D)$, we have proven the first equivalence of the lemma.

Next, we will show that (H, μ) is observable if and only if $\bar{\mathcal{A}}_H$ is observable and $R_{H,\mu}$ is \mathcal{X}_q observable for all $q \in Q$. It is easy to see that part (i) of Theorem 1 is equivalent to $[(\lambda(q_1, w) = \lambda(q_2, w), w \in \Gamma^* \text{ and } Z_{q_1,j} = Z_{q_2,j}, j = 1, \dots, m) \iff q_1 = q_2]$, which, in turn, is equivalent to $[(\bar{\lambda}(q_1, w) = \bar{\lambda}(q_2, w), \forall w \in \Gamma^*) \iff q_1 = q_2]$. But the latter expression is equivalent to $(\bar{\mathcal{A}}_H, \mu_D)$ being observable. That is, part (i) of Theorem 1 is equivalent to observability of $(\bar{\mathcal{A}}_H, \mu_D)$. Consider part (ii) of Theorem 1. From (22) in Part I, [1] it follows that for each $q \in Q$, $O_{H,q} = O_{R_{H,\mu}} \cap \mathcal{X}_q$. That is, part (ii) of Theorem 1 is equivalent to $\mathcal{X}_q \cap O_R = \{0\}$ for all $q \in Q$, i.e. R is \mathcal{X}_q -observable for each $q \in Q$. Hence, by Theorem 1 (H, μ) is observable if and only if $(\bar{\mathcal{A}}_H, \mu_D)$ is observable and $R_{H,\mu}$ is \mathcal{X}_q observable for each $q \in Q$. ■

Proof: [Proof of Lemma 2] Assume that H is of the form (1) and μ is of the form $\mu : \Phi \rightarrow \mathcal{H}_H$, and assume that $R_{H,\mu}$ is of the form (20) in Part I, [1]. Recall that $\mathcal{X}_q \subseteq \hat{\mathcal{X}}$, hence the identity map $i_q : \mathcal{X}_q \ni x \mapsto x \in \hat{\mathcal{X}}$ is well defined. Define $i_R : \bigoplus_{q \in Q} \mathcal{X}_q \oplus \mathbb{R}^{Nm} \rightarrow \hat{\mathcal{X}}$ as follows. Let $i_R(x) = i_q(x) = x$ for all $x \in \mathcal{X}_q$, $q \in Q$, and let $i_R(e_{q,j}) = F_w B_{f,j}$ such that $\delta(\zeta(f), w) = q$ for some $w \in \Gamma^*$. We will show i_R is well defined. First, since $(\tilde{\mathcal{A}}, \zeta)$ is reachable, for each $q \in Q$ there exists $f \in \Phi$ and w such that $\delta(\zeta(f), w) = q$. If (q, v) is also such a pair that $q = \delta(\zeta(g), v)$, then $F_w B_{f,j} = F_v B_{g,j}$. Indeed, in this case $\bar{\lambda}(\zeta(f), w) = \bar{\lambda}(q) = \bar{\lambda}(\zeta(g), v)$, hence $w \circ Z_{f,l} = v \circ Z_{g,l}$, $l = 1, \dots, m$. But for each $s \in \tilde{\Gamma}^*$, $C F_s F_w B_{f,j} = (w \circ Z_{f,j})(s) = (v \circ Z_{g,j})(s) = C F_s F_v B_{g,j}$. Since R is observable, we get that $F_w B_{f,j} = F_v B_{g,j}$. Hence, we have shown that i_R is a well defined linear map. It is easy to show that the other conditions for i_R being a representation morphism hold as well. ■

Proof: [Proof of Lemma 5] In order to fix the notation, assume that $H_{R,\bar{\mathcal{A}},\zeta} = (\tilde{\mathcal{A}}, \mathbb{R}^m, \mathbb{R}^p, (\tilde{\mathcal{X}}_q, \tilde{\mathcal{A}}_q, \tilde{\mathcal{B}}_q, \tilde{\mathcal{C}}_q)_{q \in \tilde{Q}}, \{\tilde{M}_{\delta(q,\gamma),\gamma,q} \mid q \in \tilde{Q}, \gamma \in \Gamma\}, \bar{\mathcal{A}} = (\tilde{Q}, \Gamma, O \times \bar{O}, \tilde{\delta}, \tilde{\lambda}), \tilde{\mathcal{A}} = (\tilde{Q}, \Gamma, O, \tilde{\delta}, \Pi_O \circ \tilde{\lambda}), \text{ and } R = (\tilde{\mathcal{X}}, \{F_\sigma\}_{\sigma \in \tilde{\Gamma}}, G, S)$, where $G = \{G_j \in \tilde{\mathcal{X}} \mid j \in I_\Phi\}$. Assume that $R_{H,\mu}$ is of the form (20) in Part I, [1]. It is easy to see that ϕ can be viewed as an automaton morphism $\phi : (\mathcal{A}_H, \mu_D) \rightarrow (\tilde{\mathcal{A}}, (\mu_{R,\bar{\mathcal{A}},\zeta})_D)$. Next, we have to show that T_C satisfies the requirements for a linear hybrid morphism. First, we have to show that $T_C(x) = Tx \in \tilde{\mathcal{X}}_{\phi(q)}$ for each

$q \in Q, x \in \mathcal{X}_q$. Indeed, by Lemma 3 we get that $R_{H,\mu}$ is reachable. Hence each $x \in \mathcal{X}_q$ is a linear combination of elements of the form $M_s \tilde{B}_f$ and $M_z M_e M_v \tilde{B}_{f,l}$ with $f \in \Phi$ and $l = 1, \dots, m$ and $s, v, e, z \in R(q, \mu_D(f))$, where $R(q_1, q_2)$ is the set defined in the proof of Lemma 3. Since ϕ is an automaton map, it follows that if $s, v, e, z \in R(q, \mu_D(f))$, then $s, v, e, z \in R(\phi(q), \zeta(f))$. Notice that $T(M_s \tilde{B}_f) = F_s G_f$ and $T(M_z M_e M_v \tilde{B}_{f,l}) = F_z F_e F_v G_{g,l}$. Hence, by the definition of $\tilde{\mathcal{X}}_{\phi(q)}$, $T(M_z M_e M_v \tilde{B}_{f,l})$ and $T(M_s \tilde{B}_f)$ belong to $\tilde{\mathcal{X}}_{\phi(q)}$. Hence $T_C(x) \in \tilde{\mathcal{X}}_{\phi(q)}$. It is easy to see that all the other conditions for (ϕ, T_C) being a linear hybrid morphism hold too. The map ϕ is surjective, since $(\bar{\mathcal{A}}, \zeta)$ is reachable; and it is easy to show that T_C is surjective. Hence, (ϕ, T_C) is indeed a surjective linear hybrid morphism. \blacksquare

Proof: [Proof of Lemma 4] Assume that H is of the form (1) and assume that $R_{H,\mu}$ is as in (20) in Part I, [1]. Assume that $R = (\hat{\mathcal{X}}, \{F_\sigma\}_{\sigma \in \tilde{\Gamma}}, G, S)$. Consider the map i_R from Lemma 2 and notice that for each $x \in \mathcal{X}_q$, $\tilde{C} M_w x = S F_w i_R(x) = S F_w x$. Hence, $O_{R_{H,\mu}} \cap \mathcal{X}_q \subseteq O_R \cap \mathcal{X}_q = \{0\}$, since R is observable. That is, $R_{H,\mu}$ is \mathcal{X}_q observable for each $q \in Q$. The realization (H, μ) is span-reachable by construction and $(\bar{\mathcal{A}}_H, \mu_D) = (\bar{\mathcal{A}}, \zeta)$ is observable by the assumption of the lemma. Thus, by Lemma 3, (H, μ) is span-reachable and observable. \blacksquare

Proof: [Proof of Lemma 6] Let $(\mathcal{A}_r = (Q_r, \Gamma, O, \delta_r, \lambda), \mu_D)$ be the reachable automaton realization constructed from (\mathcal{A}_H, μ_D) as described in Subsection V-B. For each $q \in Q_r$ let $\mathcal{X}_q^r = R_{H,q}$. Since $n_q^r = \dim \mathcal{X}_q^r < +\infty$, we can identify \mathcal{X}_q^r with $\mathbb{R}^{n_q^r}$. Define the linear maps $A_q^r : \mathcal{X}_q^r \rightarrow \mathcal{X}_q^r$, $C_q^r : \mathcal{X}_q^r \rightarrow \mathbb{R}^p$ and $M_{\delta_r(q,\gamma),\gamma,q}^r : \mathcal{X}_q^r \rightarrow \mathcal{X}_{\delta_r(q,\gamma)}^r$, $\gamma \in \Gamma$ as the restriction of the linear maps corresponding to the matrices $A_q, C_q, M_{\delta(q,\gamma),\gamma,q}$ to the subspace $R_{H,q}$, that is, for all $x \in \mathcal{X}_q$ $A_q^r x = A_q x$, $C_q^r x = C_q x$, and $M_{\delta_r(q,\gamma),\gamma,q}^r x = M_{\delta(q,\gamma),\gamma,q} x$ for all $\gamma \in \Gamma$. It is clear that we can identify A_q^r , C_q^r , and $M_{\delta_r(q,\gamma),\gamma,q}^r$ with a $n_q^r \times n_q^r$, $p \times n_q^r$ and $n_{\delta_r(q,\gamma)}^r \times n_q^r$ matrix respectively. Let $B_q^r \in \mathbb{R}^{n_q^r \times m}$ be the matrix of the linear map $\mathbb{R} \ni u \mapsto B_q u \in R_{H,q}$. Define H_r as $(\mathcal{A}_r, \mathbb{R}^m, \mathbb{R}^p, (\mathcal{X}_q^r, A_q^r, B_q^r, C_q^r)_{q \in Q_r}, \{M_{\delta_r(q,\gamma),\gamma,q}^r \mid q \in Q_r, \gamma \in \Gamma\})$ and define μ_r as $\mu_r(f) = (q, x)$ if $\mu(f) = (q, x)$, where we view $x \in R_{H,q}$ as an element of \mathcal{X}_q . It follows from Theorem 2 that (H_r, μ_r) is span-reachable. Using Proposition 1 in Part I, [1] it is easy to see that (H_r, μ_r) is a realization of Φ . Finally, $\text{card}(Q) \leq \text{card}(Q_r)$ and $\dim R_{H,q} \leq n_q$, which implies $\dim H_r \leq \dim H$; in addition, $\dim H_r = \dim H$ implies that $\text{card}(Q) = \text{card}(Q_r)$ and $\dim R_{H,q} = n_q$ for all $q \in Q$, which means (\mathcal{A}, μ_D) is reachable and $R_{H,q} = \mathcal{X}_q$, $q \in Q$, i.e. (H, μ) is span-reachable. \blacksquare