Centrum voor Wiskunde en Informatica

REPORTRAPPORT

MAS

Modelling, Analysis and Simulation

Modelling, Analysis and Simulation

Chaotic dynamics in hybrid systems

P.J. Collins

REPORT MAS-R0804 JUNE 2008

Centrum voor Wiskunde en Informatica (CWI) is the national research institute for Mathematics and Computer Science. It is sponsored by the Netherlands Organisation for Scientific Research (NWO).
CWI is a founding member of ERCIM, the European Research Consortium for Informatics and Mathematics.

CWI's research has a theme-oriented structure and is grouped into four clusters. Listed below are the names of the clusters and in parentheses their acronyms.

Probability, Networks and Algorithms (PNA)

Software Engineering (SEN)

Modelling, Analysis and Simulation (MAS)

Information Systems (INS)

P.O. Box 94079, 1090 GB Amsterdam (NL)
Kruislaan 413, 1098 SJ Amsterdam (NL)
Telephone +31 20 592 9333
Telefax +31 20 592 4199

ISSN 1386-3703

Chaotic dynamics in hybrid systems

ABSTRACT
In this paper we give an overview of some aspects of chaotic dynamics in hybrid systems, which comprise different types of behaviour. Hybrid systems may exhibit discontinuous dependence on initial conditions leading to new dynamical phenomena. We indicate how methods from topological dynamics and ergodic theory may be used to study hybrid systems, and review existing bifurcation theory for one-dimensional non-smooth maps, including the spontaneous formation of robust chaotic attractors. We present case studies of chaotic dynamics in a switched arrival system and in a system with periodic forcing.

2000 Mathematics Subject Classification: 34A37; 37B10; 37A40; 34A36; 37G35
Keywords and Phrases: chaotic dynamics; hybrid systems; symbolic dynamics; nonsmooth bifurcations
Note: This research was supported by the Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO) Vidi grant 639.032.408.

Chaotic Dynamics in Hybrid Systems

Pieter Collins

Centrum voor Wiskunde en Informatica, Postbus 94079
1090 GB Amsterdam, The Netherlands
Pieter.Collins@cwi.nl

Abstract

In this paper we give an overview of some aspects of chaotic dynamics in hybrid systems, which comprise different types of behaviour. Hybrid systems may exhibit discontinuous dependence on initial conditions leading to new dynamical phenomena. We indicate how methods from topological dynamics and ergodic theory may be used to study hybrid systems, and review existing bifurcation theory for one-dimensional non-smooth maps, including the spontaneous formation of robust chaotic attractors. We present case studies of chaotic dynamics in a switched arrival system and in a system with periodic forcing.

Keywords: chaotic dynamics; hybrid systems; symbolic dynamics; nonsmooth bifurcations.

Mathematics Subject Classification (2000): 34A37; 37B10; 37A40; 34A36; 37G35.

1 Introduction

A hybrid system is a dynamic system which comprises different types of behaviour. Classic examples of hybrid dynamical systems in the literature are impacting mechanical systems, for which the behaviour consists of continuous evolution interspersed by instantaneous jumps in the velocity, and dc-dc power converters, in which the behaviour depends on the state of a diode and a switch. Hybrid control systems occur when a continuous system is controlled using discrete sensors and actuators, such as thermostats and switched heating/cooling devices. Hybrid dynamics may also occur due to saturation effects on components of a system, and in idealised models of hysteresis. Finally, we mention that hybrid systems can be derived as singular limits of systems operating in multiple time-scales; indeed we may consider almost all hybrid systems to arise in this way.

From a mathematical point of view, hybrid systems typically exhibit non-smoothness or discontinuities in the dynamics, and these properties induce new dynamical phenomena which are not present in non-hybrid (i.e. smooth) systems. Most notably, hybrid systems can exhibit robust chaotic attractors, which have been conjectured not to exist for smooth systems.

This article is designed to give an introduction to hybrid systems for a specialist in dynamical systems theory, and an introduction to chaotic dynamics for an expert in hybrid systems. We cover modelling formalisms and solution concepts for hybrid systems, and discuss three of the main branches of chaotic dynamical systems theory, namely symbolic dynamics, ergodic theory and bifurcation theory. We assume that the reader is familiar with basic concepts of dynamical systems theory, including topological dynamics, ergodic theory and elementary smooth bifurcation theory. This material can be found in many of the excellent and accessible textbooks on dynamical systems, such as [24, 27, 42, 21]. The field of hybrid systems is not as mature, and many of the fundamental theoretical concepts have not yet been developed. The only introductory general textbook on hybrid systems currently available is [47], and the book [32] contains qualitative analyses of some classes of hybrid system.

The article is organised as follows. In Section 2, we give an overview of chaotic hybrid systems and introduce some representative examples. In Section 3, we give a brief introduction to hybrid systems theory. In Section 4 we discuss statistical and symbolic techniques for studying hybrid systems. In Section 5, we discuss bifurcation theory for hybrid systems. In Section 6 we present some case studies showcasing chaotic dynamics. Finally, we give some concluding remarks in Section 7.

2 Overview

We now give an informal overview of hybrid systems and chaotic dynamics, and give some motivational examples from the literature.

2.1 Hybrid systems

What exactly do we mean by a *hybrid system*? For our purposes, the following informal definition is appropriate:

a hybrid system is a dynamic system for which the evolution has a different form or structure in different parts of the state space.

Examples of hybrid system include piecewise-affine maps, differential equations with discontinuous right-hand sides, and systems in which the evolution jumps between multiple modes. The meaning of "different form or structure" is deliberately vague, and may depend on the tools we use to study the system. For example, a continuous piecewise-affine map may be considered "hybrid" when studying bifurcations, since bifurcation theory deals with the differential category, but from the point of view of topological or statistical properties it is just a single continuous function.

Within the class of all hybrid systems, we may identify *discrete-time*, *continuous-time* and *hybrid-time* systems.

Discrete-time hybrid systems are typically the easiest to study, and in applications usually arise as simplifications of continuous- or hybrid-time systems, such as the stroboscopic map of a periodically-forced oscillator or the hitting map of an impact system. Important classes of discrete-time systems in the literature include *piecewise-affine maps*, in which the dynamics is affine, $x_{n+1} = A_i x_n + b_i$ on each element P_i of a polyhedral partition of the state space. These systems can be studied by their *symbolic dynamics* in terms of the state-space partition, or by looking at *border collision bifurcations* which occur when periodic points cross the partition element boundaries, and may result in spontaneous transitions to chaos.

A continuous-time hybrid system is described by a differential equation or differential inclusion in which the right-hand side is non-smooth or discontinuous. If the right-hand side is continuous and piecewise-smooth, then it is locally Lipschitz, so local existence and uniqueness of solutions are immediate. The hybrid nature comes up when attempting to find efficient numerical methods to integrate such systems, since crossings of the switching boundary must be detected, and when considering bifurcations, since *corner-collisions* in the dynamics may lead to border collision bifurcations in time-discretisations. If the right-hand side is discontinuous, then the system can be reformulated as a differential inclusion using the Filippov solution concept [18]. Uniqueness of solutions is not guaranteed, and we shall see that this may result in *discontinuous* dependence on initial conditions due to *corner-collisions* and *grazing* phenomena, though as we shall see later, a grazing impact in a mechanical system does not induce discontinuous spacial dependence.

A *hybrid-time* system has both discrete-time and continuous-time dynamics. Hybrid-time systems naturally occur when continuous systems are controlled by actuators with a finite number of states, such as an electronic switch or a three-level induction motor, or using sensors which can only detect a finite number of states, such as a thermostat. Instantaneous transitions in the state occur when a *discrete event* is activated, causing a change in the *mode* of the system. Between discrete events the system evolves continuously. Although a discrete event causes a discontinuity in the system state, if an orbit crosses a guard set transversely, then nearby orbits undergo the same discrete event at nearly the same time, and no lasting discontinuities in the spacial dependence occur. However, a tangency of the system evolution with the activation set of a discrete event does introduce discontinuous spacial dependence, as does a situation when two discrete transitions are simultaneously activated.

The non-smooth or discontinuous dependence on initial conditions which can occur in hybrid systems is the main phenomenological difference between hybrid and non-hybrid systems. This often causes difficulties—invariant measures need not exist, topological methods either fail outright or need to be modified, and new bifurcations are seen to occur. However, these features also allow the possibility of *robust chaos*, by which we mean the presence of a chaotic attractor over an open set in parameter space; behaviour which is not seen in non-hybrid systems. Since non-smooth and discontinuous dependence on initial conditions are the key of hybrid systems, we shall pay considerable attention to determining the discontinuities and singularities of the evolution.

Discontinuous dependence on initial conditions can cause fundamental difficulties in applying existing techniques of dynamical systems theory, which were originally developed for systems without discontinuities. However, many methods can be modified to apply to either *upper-semicontinuous* or *lower-semicontinuous* systems. Hence a *regularisation* step is required to bring the system into a form which is amenable to analysis. As part of this regularisation, either existence or uniqueness of solutions is typically lost.

2.2 Chaos in hybrid systems

There are many definitions of "chaos" in the literature. We shall adopt the terminology that a system is chaotic if it has positive topological entropy. Chaotic behaviour may be *transient*, which means that the positive entropy is supported on a repelling set, or *attracting*, which means that the positive entropy is supported on a *minimal* attractor, i.e. an attractor with a dense orbit and hence no proper sub-attractors. From an applications point of view, transient chaotic behaviour is often unimportant; it is the dynamics on the attractors which is important. However, in practice it is impossible to distinguish between a very-high period limit cycle and a chaotic attractor.

It is often fairly easy to prove the existence of chaotic dynamics using techniques based on topological index theories, either the Lipschitz-Nielsen theory [6] for periodic points and the Conley index theory [34] for more general invariant sets. For interval maps, the ordering of points of a periodic orbit can be used to prove the existence of chaos, and for two-dimensional homeomorphisms, there is a rich theory based around periodic and homoclinic orbits. These tools are relevant for hybrid systems since they require only (local) continuity of the system evolution, and can be used directly for non-smooth hybrid systems, and with some modifications to piecewise-continuous systems. However, the main disadvantage of these methods is that they cannot distinguish between chaotic transients and a chaotic attractor.

The most important quantitative measure of chaos in a dynamical system is the *topological* *entropy*. It is known [51] that the topological entropy is upper-semicontinuous for the class of C^∞-smooth systems. It is also know that topological entropy is lower-semicontinuous for C^0 maps in one dimension, but not for C^∞ maps in $d \geq 2$ dimensions [35]. This means that for non-hybrid (i.e. smooth) systems, chaos cannot be spontaneously created, and for low-dimensional systems, chaos cannot be spontaneously destroyed.

In differentiable systems, it is extremely difficult to rigorously prove the existence of a minimal attractor with "high" topological entropy; the unimodal map [3] and the Lorenz system [46] are notable exceptions. Let us consider the simplest smooth chaotic family, namely the *unimodal family* $x_{n+1} = f_a(x_n) := 1 - ax_n^2$. It is well-known that if f_a has a periodic orbit of period m which is not a power of two, then f has a chaotic set with positive topological entropy. In [3] it was shown that for a positive measure set of parameters, there exists a minimal chaotic attractor. For other parameter values, almost all points lie in the basin of a stable periodic orbit, though this orbit may have a very high period, and numerically appear to be "chaotic". However, the proof of this result is highly delicate, and it has been conjectured that there does not exist an open and dense set of smooth C^2 maps of the interval with a minimal chaotic attractor.

The situation for hybrid systems is quite different. For the non-smooth equivalent of the unimodal family, namely the family of *tent maps* $x_{n+1} := \epsilon - a|x_n|$, it is possible to spontaneously create chaos, in the form of chaotic attractors with non-vanishing topological entropy which are robust with respect to perturbation. From this point of view alone, hybrid systems are important for the study of chaotic dynamics.

The intuitive explanation for this difference between non-hybrid and hybrid systems is that to generate chaos, we need "stretching" and "folding" in the map. In one dimension, the existence of a critical point c is needed for the "folding" property, but since $f'(c) = 0$, this orbit is highly attracting, and it is difficult to get enough stretching away from the critical point to compensate.

2.3 Examples of Chaotic Hybrid Systems

We now present some examples of hybrid systems which have been extensively studied in the literature.

Electronic circuits are one of the most well-studied experimental examples of chaotic systems. Perhaps the most well-studied example is Chua's circuit [11, 10], which contains a nonlinear resistor with piecewise-linear characteristic. Another interesting example is a circuit with a hysteresis element [36, 43]. The books [48, 45] contain an overview on chaotic dynamics in electronic circuits.

From a practical perspective, the most relevant examples are the boost and buck dc-dc power converters, as shown in Figure 1. The boost power converter is used to step-up a voltage E,

Figure 1: (a) Boost dc-dc power converter. (b) Buck dc-dc power converter

and the buck power converter to step-down a voltage. The equations of motion for the boost converter are

$$\begin{aligned} S\ \text{Open},\ I \geq 0 \text{ or } V \leq E: &\quad \frac{dV}{dt} = \frac{I}{C} - \frac{V}{RC}, \quad L\frac{dI}{dt} = E - V; \\ S\ \text{Open},\ I = 0 \text{ and } V < E: &\quad \frac{dV}{dt} = -\frac{V}{RC}; \\ S\ \text{Closed}: &\quad \frac{dV}{dt} = -\frac{V}{RC}, \quad L\frac{dI}{dt} = E. \end{aligned} \tag{1}$$

When the switch is closed, the diode isolated the inductor from the capacitor. The capacitor supplies energy to the load resistance, while the power supply supplies energy to the inductor. When the switch is open, the energy in the inductor is transferred to the capacitor. However, the diode prevents the current through the inductor falling below zero; if the current reaches zero, then no energy is supplied to the circuit until the voltage at the capacitor drops below that of the power supply. The system is controlled by opening and closing the switch in response to the voltage V. Some possible switching strategies are

Duty cycle: $S = \text{Closed}$ for $t/T \bmod 1 \leq \alpha$.

Ramp switching: $S = \text{Closed}$ for $V \geq V_R$, where $V_R = V_L + (V_U - V_L)(t/T \bmod 1)$

Hysteresis: $S \to \text{Open}$ if $V \leq V_L$; $S \to \text{Closed}$ if $V \geq V_U$.

Chaotic behaviour in power converters has been extensively discussed in the literature [2, 17, 25].

Another important source of examples of chaotic hybrid systems arise in mechanics, especially the mechanics of impacting systems or systems with stick-slip behaviour caused by friction. The book [30] contains an overview of the dynamics of non-smooth mechanical systems.

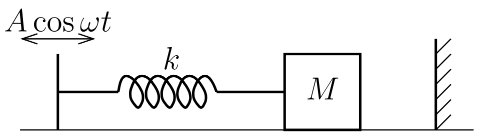

Figure 2: A simple mechanical impact oscillator.

A simple impact oscillator with chaotic dynamics [7] is given by the equations

$$\ddot{x} + \zeta\dot{x} + x = \cos(\omega t), \quad x < d;$$
$$\dot{x} \mapsto -\lambda\dot{x}, \quad x = d.$$

We let the phase ϕ be given by $\phi = t \bmod T$. Note that despite the discontinuity in the velocity at an impact, the time evolution has continuous dependence on initial conditions since the velocity reset is the identity for $\dot{x} = 0$.

A *grazing bifurcation* occurs at a parameter value for which a periodic motion of the body oscillator has an impact with zero relative velocity. The grazing bifurcation was independently discovered by Peterka [41], Whiston [50, 49] and Nordmark [37]. There have been many subsequent analyses, including [9, 19, 28, 52, 16].

One way of studying grazing phenomena is to consider the *impact map*. If (v, ϕ) are the velocity and phase of an impact, then (v', ϕ') are the velocity and phase of the next impact. The advantage of the impact map are that it is fairly easy to compute, and is derived naturally from the system. However, the impact map has the disadvantage of being discontinuous at the preimage of the grazing surface, whereas the time evolution of the system is continuous. For this reason, it may be preferable to study the stroboscopic (time T) map. A normal-form analysis shows that the grazing impact gives rise to a square-root singularity in the return map, which gives rise to many bifurcation scenarios, including period-adding and spontaneous transitions to chaos.

3 Basic Hybrid Systems Theory

In this section, we give a brief introduction to hybrid-time systems, including appropriate solution spaces, frameworks for system modelling and definition, and semantics of solution. Frequently, the appropriate definitions depend on the class of system being studied, or the properties of interest; here we give definitions which are appropriate for the study of chaotic dynamics.

3.1 Solution spaces for hybrid-time evolution

The evolution of a hybrid-time system consists of both continuous-time evolution and discrete transitions. Hence the state $x(t)$ of the system is a discontinuous function of time. We adopt the convention of taking *cadlag (continue à droit, limite à gauche)* functions, as shown in Figure 3, and let t_n be the time of the n^{th} discrete transition.

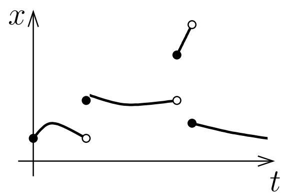

Figure 3: A cadlag solution of a hybrid-time system.

The cadlag representation of solutions is sufficient for hybrid-time systems with at most one discrete-event at any time instance. For hybrid-time systems which admit the possibility of two or more events at any time instant, the cadlag representation is not appropriate as the intermediate points are lost. Instead, we represent solutions on a *hybrid time domain* [1, 23, 12], which also records the number of discrete events which have occurred.

For continuous-time systems, an appropriate topology on solution spaces is the *compact-open* topology, with basic open sets

$$U_{(\xi,K,\epsilon)} = \{x : \mathbb{R} \to X \mid \forall t \in K,\ d(x(t), \xi(t)) < \epsilon\}. \tag{2}$$

In other words, solutions are close if they are uniformly close on compact sets.

Taking the uniform distance between solutions leads for trajectories which are close, but have slightly different event times, being considered far apart. For if

$$x_1(t) = \begin{cases} 0 & \text{if } t < t_1, \\ 1 & \text{if } t \geq t_1; \end{cases} \qquad x_2(t) = \begin{cases} \delta & \text{if } t < t_2, \\ 1+\delta & \text{if } t \geq t_2; \end{cases} \tag{3}$$

with $t_1 < t_2 < t_1 + \epsilon$, then the uniform distance between the solutions at time t with $t_1 < t < t_2$ is equal to $1 + \delta$, so $d(x_1, x_2) = 1$. This is usually inappropriate, since the distance between solutions is large even if the initial conditions are close and there are no irregularities in the behaviour.

A better topology on solutions is the *compact-open Skorohod topology* [5], originally developed for stochastic processes. The Skorohod topology allows small reparameterisations of the time domain. An equivalent topology is the *graph* topology, which is simply the Fell topology on the solution graphs. The basic open sets are:

$$U_{(\xi,K,\delta,\epsilon)} = \{x : \mathbb{R}^+ \to X \mid \forall \tau \in K,\ \exists t \in (\tau - \delta, \tau + \delta)\ d(x(t), \xi(\tau)) < \epsilon\}. \tag{4}$$

An equivalent metric description of the topology can also be formulated.

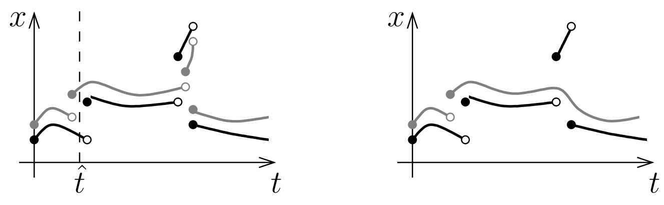

Figure 4: (a) Two solutions which are close in the hybrid Skorohod topology despite being far apart at time $\hat{t}$. (b) Two solutions which are far apart in the hybrid Skorohod topology despite the interval on which they are not close being small.

A solution $x(t)$ of a hybrid system is *Zeno* if infinitely many discrete events occur in finite time T. This means that $\lim_{n\to\infty} t_n < \infty$, where t_n is the time of the n^{th} discrete transition. Zeno behaviour in a hybrid-time model is often exhibited as chattering in the real-life system.

3.2 Modelling frameworks for hybrid systems

A commonly used framework for describing hybrid-time systems is the *hybrid automaton* framework. Informally, a hybrid automaton is based on an underlying *discrete-event system*, with *discrete modes* connected by *discrete events*. Within each discrete mode, the *continuous state* evolves under a flow until the *guard set* corresponding to a discrete event is reached. A *discrete transition* the occurs, and the discrete mode and continuous state are instantaneously updated according to a *reset* map.

The hybrid automaton framework is usually very convenient for modelling, but contains details which are superfluous for describing the dynamics. A simpler modelling framework is that of *impulse differential inclusions*, introduced in [1].

Definition 1. An impulse differential inclusion is a tuple $H = (X, D, F, G, R)$ where

- The *state space* X is a differential manifold;
- $D \subset X$ is the *domain* or *invariant*;
- $\dot{x} \in F(x)$ is a differential inclusion defining the *flow* or *dynamic* $\Phi : X \times \mathbb{R} \rightrightarrows X$;
- $G \subset X$ is the *guard set* or *activation*;
- $R : X \rightrightarrows X$ is the *reset* relation.

Here, we use the notation $X \rightrightarrows Y$ to denote a multiple-valued map from X to Y.

A solution of an impulse differential inclusion is a cadlag function $x : \mathbb{R}^+ \to X$ with finitely or infinitely many discontinuities which occur at times $t_1, t_2, \ldots$ such that

1. between event times, we have $x(t) \in D$ and $x(t)$ is absolutely continuous with $\dot{x}(t) \in F(x(t))$ almost everywhere.
2. at event times times, we have $x^-(t_i) \in G$ and $x(t_i) \in R(x^-(t_i))$.

where $x^-(t_i) := \lim_{t \nearrow t_i} x(t)$.

Notice that if $x(t) \in D^\circ \cap G$, then both continuous evolution and a discrete transition are possible, hence the evolution is multivalued or indeterminate. As we shall see in the next section, the solutions of an arbitrary impulse differential inclusion may have irregularities which need to be tamed, giving rise to different solution concepts.

Henceforth we make the following simplifying assumptions on our hybrid systems with respect to the general framework of Definition 1:

- The guard set G is a subset of the boundary of the domain D.
- The continuous dynamics is given by a locally Lipschitz differential equation $\dot{x} = f(x)$.
- The guard set G is partitioned into subsets G_i such that the reset map $r_i := r|_{G_i}$ is single-valued and continuous.

In the hybrid automaton framework, the sets G_i correspond to activation sets for different discrete events.

Given a hybrid time system, we can define the *return map* which takes an initial point to the point We alternatively define the *hitting map* as the set of points which can be reached by a discrete transition followed by continuous evolution into a guard set.

3.3 Solution concepts

Many techniques of dynamical systems rely on the solutions having continuous or smooth dependence on initial conditions. As previously mentioned, the evolution of a hybrid system may not have continuous dependence on initial conditions. Further, this property is lost in hybrid systems in the following situations, which are depicted in Figure 5

- A solution of the differential equation $\dot{x} = f(x)$ crosses ∂D at a point not in G. At this time, no further evolution is possible and the system is said to be *blocking*.
- A solution of the differential equation touches ∂D at a point of G but does not leave D. At this time, both a discrete transition and further continuous evolution may be allowed.
- A solution of the differential equation reaches a point at which the reset map r is discontinuous. At this point, continuous dependence on initial conditions is lost.

However, it is often sufficient to have *semicontinuous* dependence on initial conditions, giving rise to two different *semantics of evolution*.

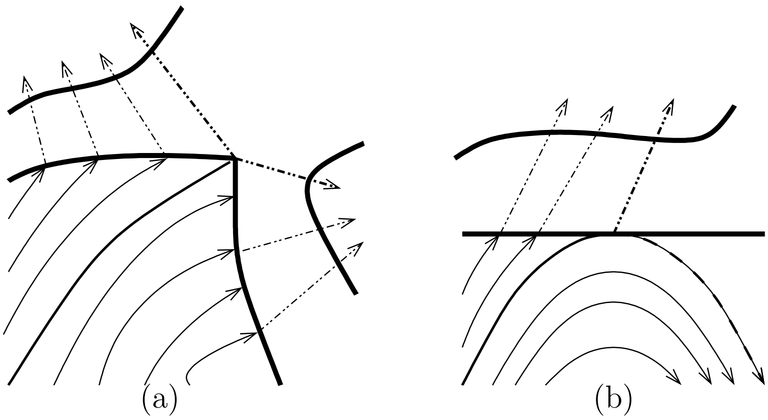

Figure 5: (a) Discontinuity of solutions due to multiply-enabled transitions (corner collision). (b) Discontinuity of solutions due to tangency with the guard set.

For *upper semantics*, we assume that at a tangency with the guard set, then both a discrete transition and continuous evolution are possible. Further, if the continuous evolution reaches a point in $\overline{G}_i \cap \overline{G}_j$, then both resets r_i and r_j are possible. Hence the system evolution is multivalued or nondeterministic, but under these semantics, the limit of a sequence of solutions is also a solution, and the solution set varies upper-semicontinuously with the system parameters [22].

Further, it is possible to effectively compute over-approximations to the set of points which can be reached from a given initial set [13, 20].

For *lower semantics*, we assume that at a tangency with the guard set, at a discontinuity point of the reset map, then no further evolution is possible. Hence solutions which exist for all time only exist on the set of initial conditions from which further evolution does not reach a discontinuity point. Under fairly mild conditions on the reset map, finite-time evolution is defined on an open set of initial conditions, and solutions vary continuously on this set. This property is useful for topological techniques based on index theory. Under the same conditions, infinite-time evolution is defined on a G_δ set of initial conditions, which is dense by the Baire category theorem.

3.4 Dependence on initial conditions in continuous time

We have seen that for hybrid-time systems, discontinuous dependence on initial conditions occurs at tangencies with the guard set and on the boundary of activation sets for different discrete events. However, discontinuities in the evolution may also occur in continuous-time hybrid systems.

Given a differential equation $\dot{x} = f(x)$ with discontinuous right-hand side, or a differential inclusion $\dot{x} \in F(x)$, the *Filippov regularisation* of F is the function

$$\hat{F}(x) = \bigcap_{\epsilon \to 0} \overline{\mathrm{conv}} F(N_\epsilon(x)). \tag{5}$$

The Filippov regularisation of F is an upper-semicontinuous multivalued function with closed, convex values.

Theorem 2. *If $F : \mathbb{R}^n \to \mathbb{R}^n$ is an upper-semicontinuous multivalued function with compact, convex values, then for every $x_0 \in \mathbb{R}^n$ there exists an absolutely continuous function $x : [0, T) \to X$ such that $x(0) = x_0$ and $\dot{x}(t) \in F(x(t))$ a.e.*

Additionally, the set of solutions is a closed set in the compact-open topology, and the set of points reachable from a given x_0 at time $t > 0$ is closed.

Hence Filippov solution concept gives existence of solutions for arbitrary differential equations, possibly at the expense of introducing nondeterminism.

Filippov solutions are useful when a discontinuity set of the right-hand side is attracting from both sides, since one obtains *sliding* orbits. Using an explicit hybrid model, one would obtain Zeno or chattering behaviour, as the solution would constantly switch from one mode to the other.

In some circumstances, the set of Filippov solutions may be larger than one would obtain using a hybrid-time model with explicit mode switching. Consider the generic situations shown in Figure 6. In (a), orbits which reach the sliding surface have the same future behaviour, and leave the sliding surface by the indicated trajectory. In (b) orbits which reach the sliding surface from below cross it immediately, *except* for the indicated orbit. Using the classical Filippov solution concept, the grazing orbit may slide along the discontinuity surface, even though this is unstable, and leave the switching hypersurface at any time. The evolution is nondeterministic, and any point and continues nondeterministically into the shaded region. Using a mode-switching solution concept, the grazing orbit either switches immediately into the upper region, or continues in the lower region. Which solution is more appropriate depends on the system being modelled. In a system in which the discontinuity of the right-hand side is an approximation to a fast-varying function, then the Filipov solution concept is appropriate, since it captures approximations to the solution of the original system. If the discontinuity of the

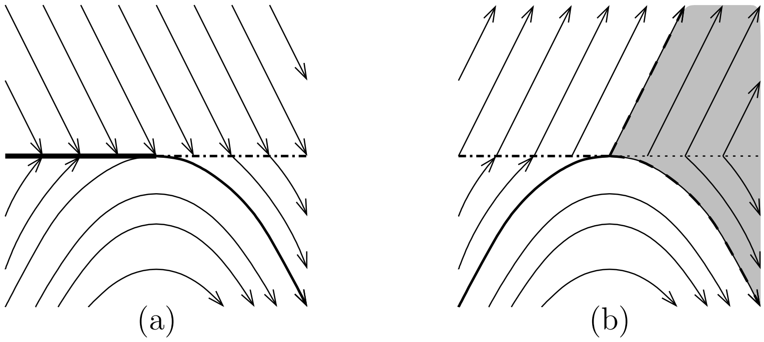

Figure 6: (a) Grazing at a sliding mode causes instability. (b) Discontinuity on a sliding mode.

right-hand side is due to a state-dependent switching, then a mode-switching solution concept is more appropriate, since the system is either in one mode or the other.

Whichever solution concept is used in (b), the solution varies discontinuously with initial condition. In contrast, the solution in (a) varies continuously with initial conditions. This is because one side of the switching hypersurface is attracting.

Theorem 3. *Let $\dot{x} = f(x)$ be a system with discontinuous right-hand side, and let M be a codimension-1 switching boundary. Suppose that at every point of M, at least one side is strictly attracting. Then the evolution across M is continuous, and for every initial point there exists a unique Filippov solution.*

A special case of grazing behaviour occurs in impact oscillators.

Definition 4. An *impact oscillator* is a dynamical system such that that is $\dot{x} = f(x)$ for $g(x) \geq 0$, and $x' = r(x)$ if $g(x) = 0$ and $f(x) \cdot \nabla g(x) < 0$. where $g : M^- \to M^+$ is such that $g(x) \to x$ as $x \to M^0$, where $M_0 = \{x \in X \mid g(x) = 0 \text{ and } f(x) \cdot \nabla g(x) = 0\}$, $M^- = \{x \in X \mid g(x) = 0 \text{ and } f(x) \cdot \nabla g(x) < 0\}$ and $M^+ = \{x \in X \mid g(x) = 0 \text{ and } f(x) \cdot \nabla g(x) > 0\}$.

Theorem 5. *Let (f, g, r) define an impact oscillator. Then under the identification $x \sim r(x)$ on $M^- \times M^+$, the evolution is continuous.*

A similar situation to that shown in Figure 6 occurs at corner collisions, as shown in Figure 7.

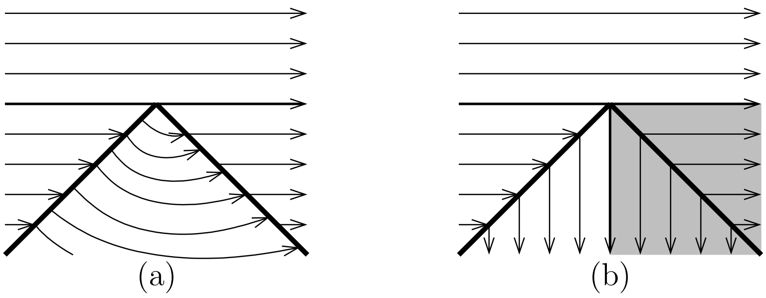

Figure 7: (a) A corner collision causing non-smoothness. (b) A corner collision causing discontinuity in the evolution.

We may obtain continuous (but non-smooth) evolution, or may obtain discontinuities in the evolution, the exact nature of which depends on whether we use Filippov semantics or switching semantics. The following result gives conditions under which a corner collision does not induce discontinuities in the system evolution.

Theorem 6. *Let $\dot{x} = f(x)$ be a system with discontinuous right-hand side, let $g : X \to \mathbb{R}^2$ be such that $\nabla g_i(x) \neq 0$ if $g_i(x) = 0$. Let $X^- = \{x \mid g(x) < 0\}$ and $X^+ = \{g_1(x) > 0 \vee g_2(x) > 0\}$. Let $M^C = \{g(x) = 0\}$ Suppose that $f(x) \cdot \nabla g_1(x) > 0$ and $f(x) \cdot \nabla g_2(x) < 0$ for all $x \in X^C$. Then evolution is continuous in a neighbourhood of X^C.*

Proof. In a neighbourhood of X^C, the time spent in X^0 is continuous, and tends to 0 as $x \to X^C$. Hence $\Phi_t(x) = \Phi^+_{t_3} \circ \Phi^-_{t_2} \circ \Phi^+_{t_1}(x)$ with t_1, t_2, t_3 continuous functions of x and $t_2 \to 0$ as $x \to X_C$. □

4 Symbolic Dynamics and Invariant Measures

Symbolic dynamics is potentially a powerful tool to study hybrid systems, since these have a naturally-defined partition of the state space into the discrete modes. Since symbolic dynamics is most naturally defined for discrete-time systems, in this section, we assume that we are considering a discrete-time hybrid system, possibly originating as a time-discretisation of a continuous- or hybrid-time system.

Given a finite collection of sets $\{R_s \subset X : s \in S\}$, which need not be disjoint or cover X, we say that a sequence $(s_0, s_1, s_2, \ldots)$ is an *itinerary* for an orbit $(x_0, x_1, x_2, \ldots)$ of a discrete-time system f if $x_k \in R_{s_k}$ for all k. The closure of the set of allowed itineraries of a system f is called the *shift space* of f, denoted Σ_f. The shift space of f is invariant under the *shift map* $\sigma : S^{\mathbb{N}} \to S^{\mathbb{N}}$ defined by $\sigma(s_0, s_1, s_2, \ldots) = (s_1, s_2, \ldots)$. The main aim of symbolic dynamics is to compute the set of itineraries and/or the shift space.

If the R_s are mutually disjoint compact sets, then every point has at most one itinerary, and if f is continuous, then the set of itineraries itself is closed. Further, if we define $R_{s_0,s_1,\ldots,s_k} = \{x \in X \mid f^i(x) \in R_i \,\forall i = 0, \ldots, k\}$, then $(s_0, s_1, s_2, \ldots)$ is an itinerary for f if, and only if, every $R_{s_j,\ldots,s_k}$ is nonempty. Hence it is possible to compute over-approximations to Σ_f by starting with the entire space $S^{\mathbb{N}}$ and removing all sequences which contain a *forbidden* word, that is, a word $(s_j, \ldots, s_k)$ with $R_{s_j,\ldots,s_k} = \emptyset$.

In many applications, the sets R_s are not disjoint, but form a *topological partition* of X, which means that $X = \bigcup_{s \in S} \overline{R^\circ_s}$ and $R^\circ_{s_1} \cap R^\circ_{s_2} = \emptyset$ if $s_1 \neq s_2$. In this case, we obtain different shift spaces depending on whether the sets R_s to be open or closed. However, if the sets R_s are closed, we often obtain too many itineraries, since for example, a fixed point $p \in \partial R_s \cap \partial R_{s'}$ would have any sequence with $s_i \in \{s, s'\}$ as an itinerary, so it is usually preferable to consider itineraries with respect to R°_s and take the closure in $S^{\mathbb{N}}$ to obtain the shift space. If $\vec{x}$ is an orbit, then we say $\vec{s}$ is a *limit itinerary* for $\vec{x}$ if there exist orbits $\vec{x}_i$ with itineraries $\vec{s}_i$ such that $\vec{x}_i \to \vec{x}$ and $\vec{s}_i \to \vec{s}$

If f has the property that the preimage of an open and dense set is dense, then $\bigcap_{i=0}^{\infty} f^{-i}(R^\circ)$, where $R^\circ = \bigcup\{R^\circ_s \mid s \in S\}$ is a G_δ set, and hence is dense by the Baire Category Theorem. Therefore, for a dense set of points, the itinerary exists and is unique.

Computing under-approximations to the shift space is usually much more difficult than computing over-approximations. This is because although we can deduce that $s^{\mathbb{N}}$ is not an itinerary of f if $R_s \cap f^{-1}(R_s) = \emptyset$, we cannot deduce that $s^{\mathbb{N}}$ *is* an itinerary of f even if $R_{s,s} \neq \emptyset$, since we may have $R_{s,s,s} = \emptyset$. The most important methods for proving that an itinerary exists are based Lefschetz and Nielsen fixed-point theory, and the Conley index theory, all of which can be used to prove the existence of periodic itineraries $s_{n+i} = s_i$.

In one dimension, it is easier to compute infinitely many periodic orbits using covering relations. If I, J are intervals, we say that I f-covers J if $f(I) \supset J$. Using the intermediate value theorem, we can show that if $I_0, I_1, I_2, \ldots$ is a sequence of intervals and I_k covers I_{k+1} for all k, then there exists a point x such that $f^k(x) \in I_k$ for all k. Further, if $I_{n+k} = I_k$ for all k, then x can be chosen such that $f^n(x) = x$.

4.1 Piecewise-continuous systems

Let $f : X \to X$ a single-valued, piecewise-continuous function. Let $\mathcal{P} = \{P_s \mid s \in S\}$ be a locally-finite topological partition of X such that f is continuous when restricted to each P_s°, and that $f|_{P_s^\circ}$ extends over each P_s to a continuous function f_s. Let $\partial\mathcal{P} = \bigcup\{\partial P \mid P \in \mathcal{P}\}$ and $\mathcal{P}^\circ = \bigcup\{P^\circ \mid P \in \mathcal{P}\}$ Define $\bar{f} : X \rightrightarrows X$ by $\bar{f}(x) = \bigcup\{f_s(x) \mid x \in P_s\}$, and assume that $\bar{f}(x) \supset f(x)$ (notice that $\bar{f}(x) = f(x)$ unless f is discontinuous at x. We may also define f° by $f^\circ := f|_{\bigcup\{P_s^\circ \mid s \in S\}}$.

The function $\bar{f}$ is a finite-valued upper-semicontinuous over-approximation to f obtained by taking all accumulation points of the graph of f. By *upper-semicontinuous*, we mean $\bar{f}^{-1}(A)$ is closed whenever A is closed. Consequently, the set of itineraries of $\bar{f}$ is an over-approximation to the set of itineraries of f.

The function f° is a single-valued partially-defined lower-semicontinuous under-approximation to f obtained by discarding all values of f at discontinuity points. By *lower-semicontinuous*, we mean that $(f^\circ)^{-1}(U)$ is open whenever U is open. taking all accumulation points of the graph of f. By *upper-semicontinuous*, we mean $\bar{f}^{-1}(A)$ is closed whenever A is closed.

4.2 Computing over-approximations to the shift space

If f is not continuous, computing over-approximations of the set of itineraries is more complicated. For if $f(R_s) \cap R_{s'} = \emptyset$ but $\overline{f(R_s)} \cap R_{s'} \neq \emptyset$, it may be extremely difficult to show that the word (s, s') is forbidden. However, if we take the upper-semicontinuous over-approximation of f, then we can compute itineraries in a similar way to the continuous case, though a little care is needed over the definitions.

We define

$$R_{s_0,s_1,\ldots,s_k} = \{x \in X \mid \exists x_0, x_1, \ldots, x_k \text{ such that } x = x_0, x_i \in P_i \text{ and } x_i \in f(x_{i-1})\}. \quad (6)$$

We can compute the sets $R_{s_0,s_1,\ldots,s_k}$ by the recurrence relation

$$R_{s_0,s_1,\ldots,s_k} = R_{s_0} \cap f^{-1}(R_{s_1,\ldots,s_k}). \quad (7)$$

We can then define a finite-type shift on S by taking disallowed words

$$\{s_0, s_1, \ldots, s_k \mid R_{s_0,\ldots,s_k} = \emptyset\}. \quad (8)$$

By disallowing successively more words, we can construct a sequence of finite type shifts converging to Σ_f.

Theorem 7. $(s_0, s_1, \ldots) \in \Sigma_f \iff \forall k,\ X_{s_0,s_1,\ldots,s_k} \neq \emptyset$.

For many hybrid systems, the state space X is disconnected, with the components $\{X_q \mid q \in Q\}$ corresponding to the discrete modes of the system. In this case, by taking the upper-semicontinuous over-approximation $\bar{f}$ to f, we can compute over-approximations to the set of allowed sequences of discrete events. An example is given in Section 6.

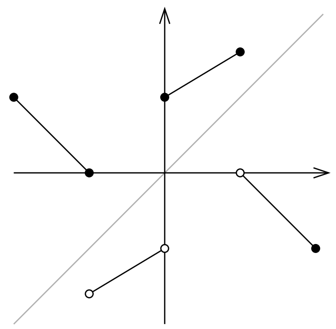

Figure 8: A piecewise-continuous interval map.

4.3 Computing under-approximations to the shift map

We can compute lower approximations to the shift space by attempting to compute periodic orbits. We recall the *Lefschetz fixed point index*, which for each triple (f, X, U) where $f : X \to X$ is a continuous map and $U \subset X$ is an open set such that $\operatorname{fix} f \cap \partial U = \emptyset$, assigns an index $\operatorname{ind}(f, X, U) \in \mathbb{Z}$ such that if $\operatorname{ind}(f, X, U) \neq \emptyset$, then f has a fixed point in U. Further, the index is local, which means that it depends only on the values of f on $\overline{U}$.

If we define $P^{\circ}_{s_0,s_1,\ldots,s_{k-1}}$ analogously to in Section 4.2, then f^k is continuous on $P^{\circ}_{s_0,s_1,\ldots,s_{k-1}}$ and indeed extends to a continuous function $f_{s_0,s_1,\ldots,s_{k-1}} := f_{s_{k-1}} \circ \cdots \circ f_{s_1} \circ f_{s_0}$ on $P_{s_0,s_1,\ldots,s_{k-1}}$. Hence for any open set U in X such that $U \subset P_{s_0,s_1,\ldots,s_{k-1}}$, we can define the fixed-point index of $f_{s_0,s_1,\ldots,s_{k-1}}$ over U. Then if $\operatorname{ind}(f_{s_0,s_1,\ldots,s_{k-1}}, P_0, U) \neq \emptyset$, then f has a periodic orbit with itinerary $s_0, s_1, \ldots, s_{k-1}, s_0, s_1, \ldots$.

Just as for continuous functions, the methods presented here can only be used to deduce the existence of finitely many periodic orbits. However, since the functions $f_{s_0,s_1,\ldots,s_{k-1}}$ are continuous on $P_{s_0,s_1,\ldots,s_{k-1}}$, we can in principal use advanced topological methods to approximate the dynamics. Again, the one-dimensional case is much easier. Using the regularisation of f, we can show that if $f_{s_i}(P_{s_i}) \supset P_{s_{i+1}}$ for all i, then there exists an orbit $(x_0, x_1, x_2, \ldots)$ with $x_i \in P_{s_i}$ for all i.

4.4 Ergodic theory and statistical behaviour

We now try to give a probabilistic description of a hybrid system by finding an invariant probability measure for its return map. If $f : X \to X$ is a single-valued map, a measure μ on X is *invariant* under f if $\mu(f^{-1}(A)) = \mu(A)$ for all measurable sets A. Any continuous map on a compact metric space has an invariant probability measure.

It is known that for piecewise-expanding maps of the interval, there exists an absolutely-continuous invariant measure [29]. A major generalisation of this result is that certain piecewise monotone-convex mappings also have an absolutely continuous invariant measure [4]. In higher dimensions the situation is considerably more complicated, though for a generic class of piecewise-expanding maps, there do exist absolutely-continuous invariant measures [14, 15].

The following example shows that discontinuous maps of the interval need not have an invariant probability measure.

Example 8. Let

$$f(x) = \begin{cases} -1 - x \text{ if } -2 \leq x \leq -1; \\ x/2 - 1 \text{ if } -1 < x < 0; \\ x/2 + 1 \text{ if } 0 \leq x \leq 1; \\ 1 - x \text{ if } 1 < x \leq 2, \end{cases}$$

as shown in Figure 8. Then every orbit starting in $[-2, 2]$ converges to the sequence $(0^+, 1^+, 0^-, -1^-, 0^+, \dots)$ but the sequence $(0, 1, 0, -1, 0, \dots)$ cannot be an orbit of and single-valued map.

The difficulty in the above example is that the the natural "invariant" measure would assign nonzero weight to the discontinuity point. In cases where an absolutely-continuous invariant measure exists, the discontinuity points have measure zero and therefore cause no difficulties.

To obtain an invariant measure for general piecewise-continuous maps, we can lift the map to the product of the state space and the symbol space.

Let f be piecewise-continuous, with $f_s := f|_{P_s}$ continuous on each element P_s of a topological partition $\mathcal{P}$, and such that f_s extends continuously over $\overline{P}_s$. Let Σ_f be the shift space of f with respect to the partition elements P_s°.

For each itinerary $\vec{s}$, let $X_{\vec{s}}$ be the set of points with itinerary or limit itinerary $\vec{s}$, and define $\widehat{X} := \bigcup_{\vec{s} \in \Sigma_f} \{X_{\vec{s}} \times \vec{s}\}$ with the inherited product topology. Then $\widehat{X}$ is compact if X is compact, and f lifts to a continuous function $\widehat{f} : \widehat{X} \to \widehat{X}$.

There must therefore always exist an invariant measure $\widehat{\mu}$ for $\widehat{f}$. Further, define $\mu(A) := \widehat{\mu}(\pi^{-1}(A))$, where $\pi(x, \vec{q}) = x$. We call μ a *shift-invariant measure* for f. If $\widehat{\mu}(\partial\mathcal{P}) = 0$, then μ is an invariant measure for f.

We therefore have the following simple theorem

Theorem 9. *If f is piecewise-continuous, then f has a shift-invariant measure.*

5 Bifurcation theory for non-smooth maps

In this section we describe the most important *border-collision bifurcations* for one-dimensional piecewise-smooth maps. The analysis of these bifurcations is considerably simpler than the analysis of bifurcations in three-dimensional flows, but provides insight into the higher-dimensional cases. In particular, the nonsingular border-collision bifurcations provide a model for corner-collision bifurcations in continuous- and hybrid-time systems, and the border-collision bifurcations with a square-root singularity provide a model for grazing bifurcations. and use these to study *corner-collision* and *grazing* bifurcations in continuous- and hybrid-time systems. In both cases, we consider the continuous and discontinuous cases separately.

For more detailed exposition of bifurcations in non-smooth systems, see the book [53].

5.1 Continuous border-collision bifurcations

The border-collision bifurcation can occur in systems with continuous evolution, such as piecewise-affine maps. Border-collision bifurcations were observed in [26, 44, 31, 39, 38]; here we follow the exposition in [40].

Let $f_\epsilon : \mathbb{R} \to \mathbb{R}$ be a continuous piecewise smooth map with parameter ϵ whose derivative is discontinuous at 0, as shown in Figure 9. The simplest example of a *border-collision* bifurcation occurs when $f_\epsilon(0) = 0$. Assume further that f_ϵ is differentiable in ϵ and $c = df_\epsilon(0)/d\epsilon > 0$ at $\epsilon = 0$. We let $a = \lim_{x \nearrow 0} f_0'(x)$ and $b = -\lim_{x \searrow 0} f_0'(x)$, and assume $0 < a < 1 < b$.

Now for $\epsilon > 0$ small, we have $f_\epsilon(0) = c\epsilon + O(\epsilon^2) > 0$, and $f_\epsilon^2(0) = c(1-b)\epsilon - O(\epsilon)^2 < 0$. Further, if $x_\epsilon = \xi\epsilon + O(\epsilon^2)$ for $\xi \leq 0$, then $f_\epsilon(x) = (c + a\xi)\epsilon + O(\epsilon^2) > x$. Taking $I_\epsilon = [f_\epsilon^2(0), f_\epsilon(0)]$,

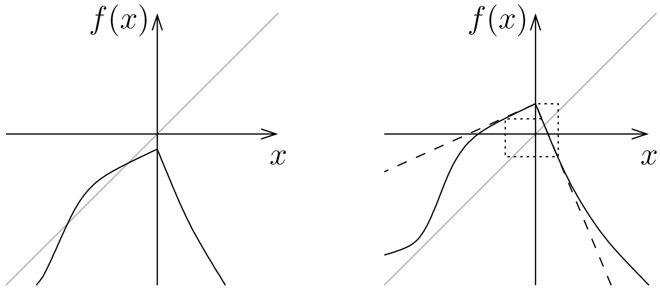

Figure 9: A border collision bifurcation for a non-smooth map.

we see that $f_\epsilon(I_\epsilon) \subset I_\epsilon$. Hence for $\epsilon > 0$ small, the dynamics is contained in an interval of size $O(\epsilon)$ about 0. The linearization of $f_\epsilon(x)$ about $x = 0$ is therefore a good approximation to f_ϵ in I_ϵ.

It can be rigorously shown that linearising at $x = 0$ yields a *normal form* of the bifurcation as an affine map

$$F_{a,b,\epsilon}(x) = \begin{cases} \epsilon + ax \text{ if } x \leq 0; \\ \epsilon - bx \text{ if } x \geq 0, \end{cases} \qquad \text{with} \quad 0 < a < 1 \text{ and } b > 0. \tag{9}$$

as shown in Figure 10. For simplicity, we henceforth only consider the linearised map (9).

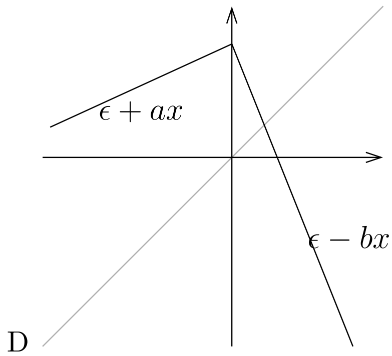

Figure 10: Near a border collision bifurcation and for a piecewise-affine map.

If $\epsilon < 0$, then $x_p := \epsilon/(1-a) \leq 0$ is a fixed point, and since $F_{a,b,\epsilon}(x) \leq \epsilon$ for all x, all orbits converge to x_p. If $\epsilon = 0$, then 0 is a fixed point, which is stable if $b < 1$ and one-sided unstable if $b > 1$. If $\epsilon > 0$ and $0 < b < 1$ then $x_0 = \epsilon/(1+b)$ is a stable fixed point.

Note that $f(0) = 1$ and $f(1) = 1 - b < 0$, and that if $x < 0$, then $x < f(x) < 1$. Hence all orbits eventually enter the interval $[0, 1]$.

The interesting case is $\epsilon > 0$ and $b > 1$.

Note that by the coordinate transformation $x \mapsto x/\epsilon$, we can scale ϵ to equal 1; we define $F_{a,b} := F_{a,b,\epsilon}$. Taking $c = 0$, the critical point, we see that $F_{a,b}(0) = 1$, $F^2_{a,b}(0) = 1 - b < 0$, and $F^3_{a,b}(0) = 1 + a - ab$. Let $I_0 = [1 - b, 0]$ and $I_1 = [0, 1]$. Clearly $F_{a,b}(I_1) = I_0 \cup I_1$. Then if $1 < b < 1 + 1/a$, we have $F_{a,b}(1 - b) = 1 + a - ab > 1 + a - (1 + a) = 0$, so $f(I_0) \subset I_1$. Then for all $x \in [1 - b, 1]$, we have $(f^2)'(x)$ is either $-ab$ or b^2, so if $b > 1/a$, then $|(f^2)'(x)| > 1$.

We have therefore shown that

1. $F_{a,b}([1 - b, 1]) \subset [1 - b, 1]$, and
2. $|(F^2_{a,b})'(x)| > 1$ for all $x \in [1 - b, 1]$.

3. $[1-b,1] = \bigcap_{n=0}^{\infty} F_{a,b}^n(U)$ for all bounded $U \supset [1-b,1]$.

Hence $[1-b,1]$ is a minimal chaotic attractor for $F_{a,b}$; in particular, $F_{a,b}$ has no stable periodic orbits and strictly positive topological entropy.

Since this situation occurs for any $\epsilon > 0$ regardless of the value of ϵ, we have a bifurcation to a robust chaotic attractor. Note that the entropy of the attractor is bounded away from zero, but the size of the attractor is ϵb.

Following [40] we see that the map $F_{a,b}$ has positive entropy and may have a chaotic attractor. Similar windows exist in which the system has a chaotic attractor with k pieces, separated by periodic orbits.

Note that for smooth interval maps, the entropy varies continuously with the parameters. Here, the entropy jumps *discontinuously* at the border-collision bifurcation. By considering the change in a and b, we can *rigorously* prove the existence of a chaotic attractor with high entropy in a generic two-parameter family of maps. Since f_ϵ is unimodal, the symbolic dynamics is determined by the kneading theory [33].

5.2 Singular border-collision bifurcation

In an impact oscillator, grazing the impact set causes a square root singularity in the evolution. If this occurs on a periodic orbit, we have a *grazing bifurcation*. A normal form for the grazing bifurcation is given by

$$f(x) = \begin{cases} \epsilon + ax \text{ if } x \leq 0; \\ \epsilon - b\sqrt{x} \text{ if } x \geq 0, \end{cases} \tag{10}$$

as shown in Figure 11. Note that unlike the affine border collision, we cannot scale away the bifurcation parameter ϵ without affecting the form of the square root term:

$$F(y) = \begin{cases} 1 + ay \text{ if } y \leq 0; \\ 1 - (b/\sqrt{\epsilon})\sqrt{y} \text{ if } y \geq 0, \end{cases} \tag{11}$$

where $y = x/\epsilon$. We therefore prefer to work with the original form (10).

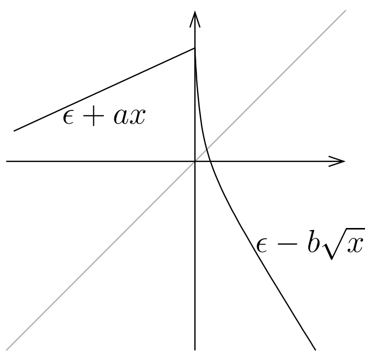

Figure 11: A border collision bifurcation for a map with a square-root singularity.

We again look for conditions under which there exists a chaotic attractor. It is easy to see that the interval $[-b\sqrt{\epsilon} + \epsilon, \epsilon]$ is globally attracting. There is a single fixed point $p = (1 + 2\epsilon - \sqrt{1+4\epsilon})/2\epsilon$, so $p \sim \epsilon^2$ for small ϵ. We also have have $f'(\epsilon) = -1/2\sqrt{\epsilon}$.

Now, if $-\sqrt{\epsilon} < x < 0$, then $f^n(x)$ is first greater than 0 for $n \sim \log(c - x)$, so if we let $n(x)$ be the minimum n such that $F^{n(x)}(x) > 0$, then $(f^{n(x)})'(x) \geq 1/(c+x)$ for some constant c depending only on a. Hence for ϵ sufficiently small, there must be a chaotic attractor of f for

$x \in [-\sqrt{\epsilon}+\epsilon, \epsilon]$. This is a one-piece attractor if $0 < 1 - a + a/\sqrt{\epsilon} < q \sim \epsilon$, which is impossible for small ϵ. Indeed, as $\epsilon \to 0$, the critical point spends increasingly long in $[\epsilon - \sqrt{\epsilon}, 0]$, and the kneading theory shows that the topological entropy approaches $\log 2$.

5.3 Discontinuous border-collision bifurcation

We now consider a discontinuous border-collision bifurcation of a stable fixed-point.

$$f(x) = \begin{cases} ax + \epsilon \text{ if } x \le 0; \\ bx - c \text{ if } x \ge 0, \end{cases} \tag{12}$$

as shown in Figure 12.

Assume $a < 1$, and $a^N b > 1$ for some least integer $N \ge 0$. Assume further that $\epsilon < 1/b$. Then $f(0^-) = \epsilon$, $f(\epsilon) = b\epsilon - 1 < 0$, and $f(b\epsilon - 1) > b\epsilon - 1$. If $f^i(x) < 0$ for $0 \le i < n$, then a closed form for $f^n(x)$ is $f^n(x) = a^n x + \epsilon(1-a^n)/(1-a)$. Since $x \ge b\epsilon - 1$, we have $f^n(x) \ge a^n b\epsilon - a^n + \epsilon(1-a^n)/(1-a)$, so $f^n(x) > 0 \iff \epsilon(a^n b(1-a) + 1 - a^n) > a^n(1-a)$.

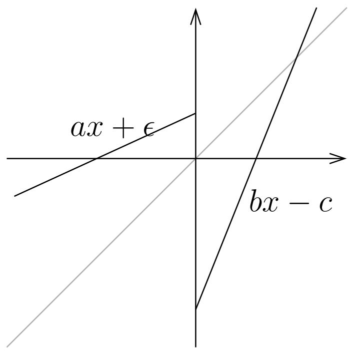

Figure 12: A border collision bifurcation for a discontinuous affine map.

We again look for conditions under which there exists a chaotic attractor. It is easy to see that the interval $[b\epsilon - 1, \epsilon]$ is globally attracting. However, since expansion only occurs on the interval $[0, \epsilon]$, and this interval maps to $[-1, b\epsilon - 1]$, for small ϵ, the contraction for $x < 0$ outweighs the expansion for $x > 0$. Hence, the the bifurcation, the fixed point first jumps to a periodic orbit, and this periodic orbit may then split up into a chaotic attractor as ϵ increases. Hence spontaneous chaos does not occur at this bifurcation.

5.4 Discontinuous singular border-collision bifurcation

We now consider a discontinuous border-collision bifurcation of a stable fixed-point with a square-root singularity.

$$f(x) = \begin{cases} ax + \epsilon \text{ if } x \le 0; \\ b\sqrt{x} - c \text{ if } x \ge 0, \end{cases} \tag{13}$$

as shown in Figure 13.

Let $d = 0$, the discontinuity point, and suppose $0 < a, \epsilon < 1$ Then $f(0^-) = \epsilon$, $f(\epsilon) = \sqrt{\epsilon} - 1 < 0$, and $f'(x) > 1/2\sqrt{\epsilon}$ for $x > 0$. Similarly to the case studied in Section 5.3, a point $x < 0$ becomes positive if

$$f^n(x) = a^n x + \epsilon(1-a^n)/(1-a) \tag{14}$$

which takes at most $n = \log(1 - x(1-a)/\epsilon)/\log(1/a)$ steps. Since $x > -1$, we find $n \sim -\log \epsilon$ for fixed a. Hence the derivative of the return map is $(a^n)/2\sqrt{\epsilon} \sim \sqrt{\epsilon}$ for small ϵ, and

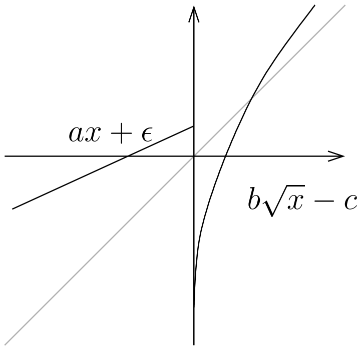

Figure 13: A border collision bifurcation for a map with a discontinuity.

so the singularity in the derivative is not sufficient to compensate for the discontinuity, and the bifurcation causes high-period periodic orbits which may later break-up to give a chaotic attractor.

6 Case studies

6.1 Switched queueing/arrival systems

The following switched arrival system was first considered in [8], and later in the book [32].

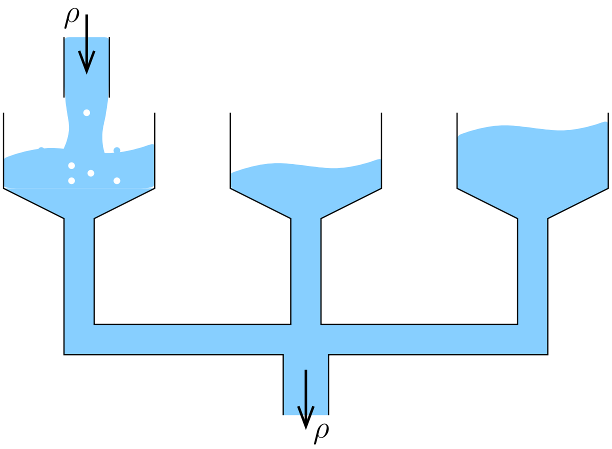

Figure 14: A switched arrival system.

Tanks T_i, $i = 1, 2, 3$ containing volume x_i of fluid with constant outflows ρ_1, ρ_2 and ρ_3 can be filled by a single pipe with inflow $\rho_{\text{in}} = \rho_{\text{out}} := \rho_1 + \rho_2 + \rho_3$. There are three modes q_i corresponding to filling tank T_i. Since the total volume is preserved, we have $x_1 + x_2 + x_3 = x_{\text{tot}}$.

A simple switching law is to switch to filling tank T_i whenever $x_i = 0$. The dynamics of the system is shown in Figure 15. If the system begins in mode q_i with $x_i = 0$, then the system switches to mode q_j over mode q_k if tank T_j empties first, which occurs if $x_j/\rho_j < x_k/\rho_k$. Hence the return map f is defined on the sets $I_i := \{(x, q_i) \mid x_i = 0\}$. Under the return map we have $f(I_1) \supset I_2 \cup I_3$, $f(I_2) \supset I_3 \cup I_1$ and $f(I_3) \supset I_1 \cup I_2$. Hence any sequence of mode switches is possible.

Since the system on average spends time ρ_i/ρ filling tank i, an invariant measure for the flow is given by $2\rho_i\lambda/\rho$, where λ is Lesbesgue measure. An invariant measure for the return map f

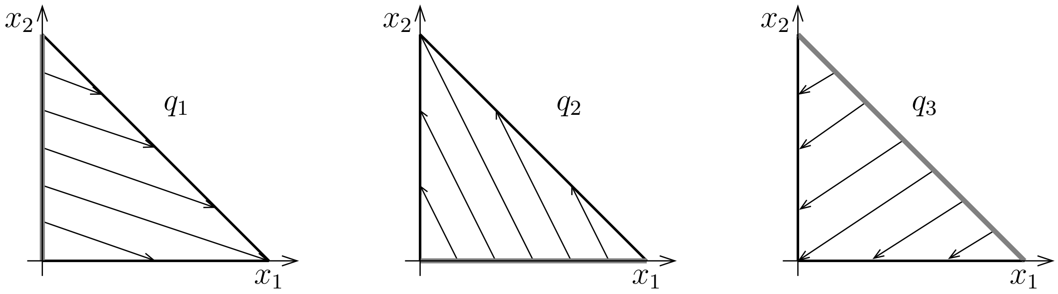

Figure 15: A simple switching law with Zeno behaviour.

is given by a measure which is uniform on each I_i, with

$$\mu(I_i) = \frac{1}{2}\frac{\rho_i(\rho - \rho_i)}{\rho_1\rho_2 + \rho_2\rho_3 + \rho_3\rho_1}.$$

Using this, we can deduce that the average switching time is

$$T_{\text{av}} = \frac{1}{4}\frac{\rho_1 + \rho_2 + \rho_3}{\rho_1\rho_2 + \rho_2\rho_3 + \rho_3\rho_1} = \frac{1}{2}\frac{\rho}{\rho^2 - \rho_1^2 + \rho_2^2 + \rho_3^2}.$$

If all inflows are equal, this yields $3/4\rho$, and if one inflow is twice the other two, this yields $4/5\rho$. Compare this with a regular cyclic switching strategy with an average switching time of $1/3\rho$.

A major problem with this switching law is that if two tanks are both close to being empty, then we switch rapidly between them, and if two tanks become empty at exactly the same time, then the system deadlocks due to Zeno behaviour. We therefore seek a switching law with a lower average number of switches.

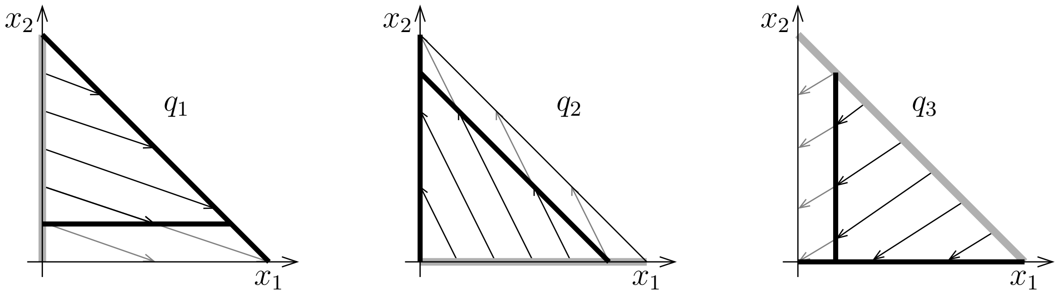

Figure 16: A switching law with without Zeno behaviour.

A modified switching law is to switch preferentially from tank T_1 to tank T_2, from T_2 to T_3, and from T_3 to T_1. We accomplish this by switching from mode q_i to mode q_{i+1} if x_{i+1} drops below a non-zero threshold ξ_{i+1}. This succeeds in avoiding Zeno behaviour, since if x_1 and x_2 are both low in mode q_3, the system switches to mode q_1 before x_1 reaches 0, and then immediately to mode q_2 if x_2 is small. The system then remains away from mode q_3 until both x_1 and x_2 have recovered. (See Figure 16.)

To obtain a return map f in the form of a self map on the sets I_i defined above, we take the state after switching to mode q_i and flow backwards until $x_i = 0$. The resulting map is shown in Figure 17. Notice that we do not now have $f(I_1) \supset I_3$, as it is not possible to switch from mode q_1 to mode q_2 with a value of x_1 greater than $x_{\text{tot}} - \xi_2$. As a result, the symbolic dynamics will not include all transition sequences, but sequences with a large number of repetitions of two modes will be cut.

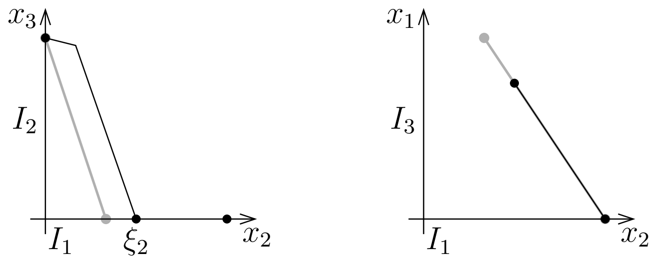

Figure 17: The return map from interval I_1 in mode q_1 for threshold-controlled preferred switching compared with switching when empty (light).

6.2 Control systems with periodic forcing

We finally consider a simple example of a control system with periodic forcing, where the control objective is to keep some value within a certain bound.

$$\dot{x} = k(a + b \sin \omega t - x) + u \tag{15}$$

Where u is some input. Taking period $T = 2\pi/\omega$ and $\phi = t \bmod T$, we obtain an autonomous system with two degrees of freedom.

We assume x is some quantity which we need to control below some safe threshold x_{MAX} by means of an safety system described by the input u, which can take values $u_{\text{OFF}} = 0$ and $u_{\text{ON}} < 0$. Without any control i.e. $u = 0$, there is a unique globally asymptotically stable periodic orbit,

$$x = a + \frac{b}{1 + \omega^2/k^2} \sin(\omega t - \alpha) \text{ where } \alpha = \tan^{-1}(\omega/k) \tag{16}$$

We consider a number of switching strategies, which illustrate the various bifurcation scenarios mentioned in Section 5.

First, consider the switching law:

$$s \rightsquigarrow \text{ON if } x \geq x_{\text{MAX}} \text{ and } \phi < \phi_L; \quad s \rightsquigarrow \text{OFF if } \phi \geq \phi_U. \tag{17}$$

The control is turned on if x becomes too high, but the phase is less than a critical value; the rationale being that if the phase is above the critical value, then the maximum value of x will only be slightly higher than x_{MAX}. The system is turned off at a fixed time ϕ_U. As shown in Figure 18, this leads to a discontinuous corner collision if $\dot{x} > 0$ and $(x, \phi) = (x_{\text{MAX}}, \phi_L)$, and a discontinuous grazing if $\dot{x} = 0$ and $\phi < \phi_L$ when $x = x_{\text{MAX}}$. The bifurcations indicated in Section 5 occur if the corner collision or grazing occur on the periodic orbit.

An alternative control law is given by

$$s \rightsquigarrow \text{ON if } x \geq x_{\text{MAX}}; \quad s \rightsquigarrow \text{OFF if } x < a + b \sin \omega t. \tag{18}$$

The control is turned on when $x \geq x_{\text{MAX}}$, and is turned off when the external forcing $a + b \sin \omega t$ is sufficiently low that x would decrease without the input u. This leads to a continuous grazing bifurcation scenario, as depicted in Figure 18(c), since we always have $\dot{x} = 0$ immediately after the control is turned off.

Hysteresis switching is a commonly used technique to control a variable within bounds and avoid overly fast switching. The control law is given by

$$s \rightsquigarrow \text{ON if } x \geq x_{\text{MAX}}; \quad s \rightsquigarrow \text{OFF if } x < x_{\text{OFF}}. \tag{19}$$

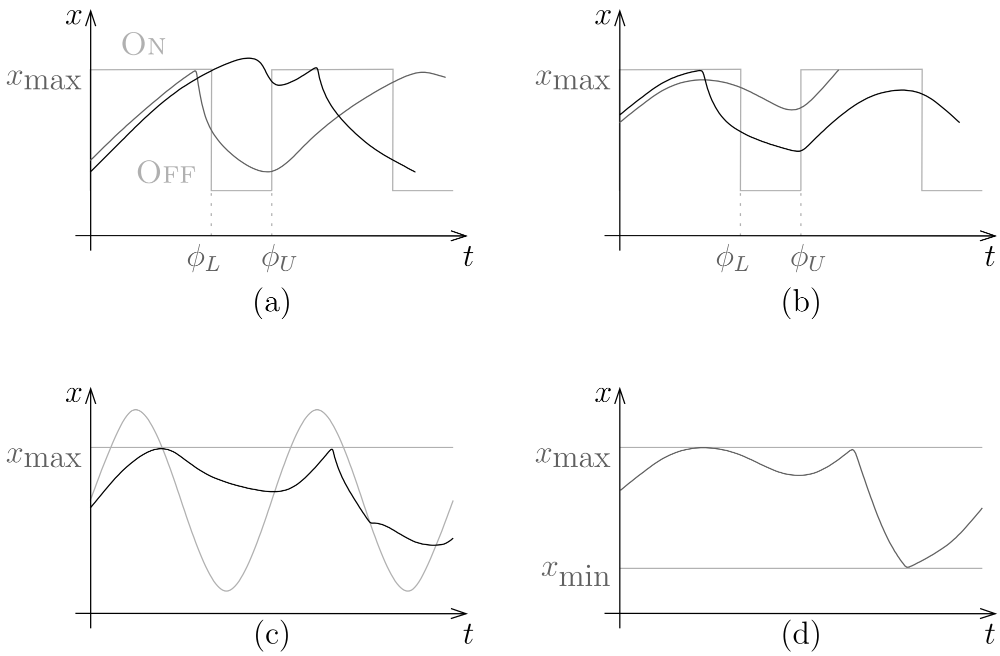

Figure 18: (a,b) Discontinuous corner collision and discontinuous grazing in a switched control system. (c) Continuous grazing. (d) Hysteresis switching.

Assuming $-u_{\text{ON}}$ is sufficiently large, the system turns before the end of forcing period at time T. This gives rise to a discontinuous square-root singularity.

Another possible control law is a switching law with fixed hold,

$$s \rightsquigarrow \text{ON if } x \geq x_{\text{MAX}}; \quad s \rightsquigarrow \text{OFF after time } \tau. \tag{20}$$

This always gives rise to a stable periodic orbit, since the switching does not introduce stretching between nearby orbits.

7 Conclusions

In this article, we have considered chaotic dynamics in low-dimensional hybrid systems. We have seen that the key feature of such systems is discontinuous or non-differentiable spacial dependence, which allows for the formation of robust chaotic attractors. We have seen that discontinuous hybrid systems can be regularised to give shift-invariant measures, and that it is possible to effectively compute approximations to the symbolic dynamics. We have also considered bifurcations in non-smooth systems arising from corner collisions and grazing, and shown that these features can spontaneously generate chaos. Finally, we have illustrated these features using examples from hybrid control systems.

References

[1] Jean-Pierre Aubin, John Lygeros, Marc Quincampoix, Shankar Sastry, and Nicolas Seube. Impulse differential inclusions: a viability approach to hybrid systems. *IEEE Trans. Automat. Control*, 47(1):2–20, 2002.

[2] Soumitro Banerjee, M.S. Karthik, Guohui Yuan, and James A. Yorke. Bifurcations in one-dimensional piecewise smooth maps—theory and applications in switching circuits. *IEEE Trans. Circuits Systems*, 47(3):389–394, 2000.

[3] M. Benedicks and L. Carleson. On iterations of $1 - ax^2$ on $(-1, 1)$. *Ann. of Math. (2)*, 122:1–25, 1985.

[4] Christopher Bose, Véronique Maume-Deschamps, Bernard Schmitt, and Sujin Shin. Invariant measures for piecewise convex transformations of an interval. *Studia Math.*, 152(3):263–297, 2002.

[5] Mireille Broucke and Ari Arapostathis. Continuous selections of trajectories of hybrid systems. *Systems Control Lett.*, 47(2):149–157, 2002.

[6] Robert Brown. *The Lefschetz Fixed Point Theorem*. Scott, Foresman and Company, Glenview, Illinois, 1971.

[7] Chris Budd. Non-smooth dynamical systems and the grazing bifurcation. In *Nonlinear mathematics and its applications*, pages 219–235. Cambridge University Press, Cambridge, 1996.

[8] C. Chase, J. Serrano, and P.J. Ramadge. Periodicity and chaos from switched flow systems: Contrasting examples of discretely controlled continuous systems. *IEEE Trans. Automatic Control*, 38:70–83, 1993.

[9] Wai Chin, Edward Ott, Helena E. Nusse, and Celso Grebogi. Grazing bifurcations in impact oscillators. *Phys. Rev. E (3)*, 50(6):4427–4444, 1994.

[10] Leon O. Chua, Motomasa Komuro, and Takashi Matsumoto. The double scroll family. I. Rigorous analysis of bifurcation phenomena. *IEEE Trans. Circuits and Systems*, 33(11):1097–1118, 1986.

[11] Leon O. Chua, Motomasa Komuro, and Takashi Matsumoto. The double scroll family. I. Rigorous proof of chaos. *IEEE Trans. Circuits and Systems*, 33(11):1072–1097, 1986.

[12] Pieter Collins. Generalised hybrid trajectory spaces. In *Proceedings of the 17th International Symposium on the Mathematical Theory of Networks and Systems, Kyoto, Japan, July 24-28, 2006*, pages 2101–2109, 2006.

[13] Pieter Collins and John Lygeros. Computability of finite-time reachable sets for hybrid systems. In *Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference 2005*, pages 4688–4693, New York, 2005. IEEE Press.

[14] William J. Cowieson. Stochastic stability for piecewise expanding maps in $\mathbf{R}^d$. *Nonlinearity*, 13(5):1745–1760, 2000.

[15] William J. Cowieson. Absolutely continuous invariant measures for most piecewise smooth expanding maps. *Ergodic Theory Dynam. Systems*, 22(4):1061–1078, 2002.

[16] M. di Bernardo, C. J. Budd, and A. R. Champneys. Normal form maps for grazing bifurcations in n-dimensional piecewise-smooth dynamical systems. *Phys. D*, 160(3-4):222–254, 2001.

[17] M. Di Bernardo, F. Garofalo, L. Iannelli, and F. Vasca. Bifurcations in piecewise-smooth feedback systems. *Internat. J. Control*, 75(16-17):1243–1259, 2002. Switched, piecewise and polytopic linear systems.

[18] A. F. Filippov. *Differential equations with discontinuous righthand sides*, volume 18 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1988. Translated from the Russian.

[19] Mats H. Fredriksson and Arne B. Nordmark. Bifurcations caused by grazing incidence in many degrees of freedom impact oscillators. *Proc. Roy. Soc. London Ser. A*, 453(1961):1261–1276, 1997.

[20] Yan Gao, John Lygeros, and Marc Quincapoix. The reachability problem for uncertain hybrid systems revisited: a viability theory perspective. In *Hybrid systems: computation and control*, volume 3927 of *Lecture Notes in Comput. Sci.*, pages 242–256. Springer, Berlin, 2006.

[21] Robert Gilmore and Marc Lefranc. *The topology of chaos*. Wiley-Interscience [John Wiley & Sons], New York, 2002. Alice in Stretch and Squeezeland.

[22] R. Goebel and A. R. Teel. Solutions to hybrid inclusions via set and graphical convergence with stability theory applications. *Automatica J. IFAC*, 42(4):573–587, 2006.

[23] Rafal Goebel, Joao Hespanha, Andrew R. Teel, Chaohong Cai, and Ricardo Sanfelice. Hybrid systems: Generalized solutions and robust stability. In *Proceedings of the Symposium on Nonlinear Control Systems*. Elsevier, 2004.

[24] John Guckenheimer and Philip Holmes. *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, volume 42 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1990. Revised and corrected reprint of the 1983 original.

[25] D. Hamill, J. Deane, and D. Jeffries. Modelling of chaotic dc-dc converters by iterated nonlinear mappings. *IEEE Trans. Power Electron.*, 7:25–36, 1992.

[26] Shunji Ito, Shigeru Tanaka, and Hitoshi Nakada. On unimodal linear transformations and chaos. I, II. *Tokyo J. Math.*, 2(2):221–239, 241–259, 1979.

[27] Anatole Katok and Boris Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1995. With a supplementary chapter by Katok and Leonardo Mendoza.

[28] M. Kunze and T. Küpper. Qualitative bifurcation analysis of a non-smooth friction-oscillator model. *Z. Angew. Math. Phys.*, 48(1):87–101, 1997.

[29] A. Lasota and James A. Yorke. On the existence of invariant measures for piecewise monotonic transformations. *Trans. Amer. Math. Soc.*, 186:481–488 (1974), 1973.

[30] Remco I. Leine and Henk Nijmeijer. *Dynamics and bifurcations of non-smooth mechanical systems*, volume 18 of *Lecture Notes in Applied and Computational Mechanics*. Springer-Verlag, Berlin, 2004.

[31] Yu. L. Maĭstrenko, V. L. Maĭstrenko, and L. O. Chua. Cycles of chaotic intervals in a time-delayed Chua's circuit. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 3(6):1557–1572, 1993.

[32] Alexey S. Matveev and Andrey V. Savkin. *Qualitative theory of hybrid dynamical systems.* Control Engineering. Birkhäuser, 2000.

[33] John Milnor and William Thurston. On iterated maps of the interval. In *Dynamical systems (College Park, MD, 1986–87)*, volume 1342 of *Lecture Notes in Math.*, pages 465–563. Springer, Berlin, 1988.

[34] Konstantin Mischaikow and Marian Mrozek. Conley index. In *Handbook of dynamical systems*, volume 2, pages 393–460. North-Holland, Amsterdam, 2002.

[35] Michał Misiurewicz. On non-continuity of topological entropy. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 19:319–320, 1971.

[36] Robert W. Newcomb and Nevine El-Leithy. Chaos generation using binary hysteresis. *Circuits Systems Signal Process.*, 5(3):321–341, 1986.

[37] A.B. Nordmark. Non-periodic motions caused by grazing incidence in an impact oscillator. *J. Sound Vib.*, 145:279–297, 1991.

[38] Helena E. Nusse, Edward Ott, and James A. Yorke. Border-collision bifurcations: An explanation for observed bifurcation phenomena. *Phys. Rev. E*, 49(2):1073–1076, Feb 1994.

[39] Helena E. Nusse and James A. Yorke. Border-collision bifurcations including "period two to period three" for piecewise smooth systems. *Phys. D*, 57(1-2):39–57, 1992.

[40] Helena E. Nusse and James A. Yorke. Border-collision bifurcations for piecewise smooth one-dimensional maps. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 5(1):189–207, 1995.

[41] F. Peterka and J. Vacík. Transition to chaotic motion in mechanical systems with impacts. *J. Sound Vibration*, 154(1):95–115, 1992.

[42] Clark Robinson. *Dynamical systems.* Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, second edition, 1999. Stability, symbolic dynamics, and chaos.

[43] Toshimichi Saito and Shinji Nakagawa. Chaos from a hysteresis and switched circuit. *Phil. Trans. R. Soc. Lond. A*, 353:47–57, 1995.

[44] F. Takens. Transitions from periodic to strange attractors in constrained equations. In *Dynamical systems and bifurcation theory (Rio de Janeiro, 1985)*, volume 160 of *Pitman Res. Notes Math. Ser.*, pages 399–421. Longman Sci. Tech., Harlow, 1987.

[45] Chi Kong Tse. *Chomplex behaviour of switching power converters.* Power Electronics and Applications. CRC Press, Boca Raton, 2004.

[46] Warwick Tucker. The Lorenz attractor exists. *C. R. Acad. Sci. Paris Sér. I Math.*, 328(12):1197–1202, 1999.

[47] Arjan van der Schaft and Hans Schumacher. *An introduction to hybrid dynamical systems.* Number 251 in Lecture notes in control and information sciences. Springer, London, 2000.

[48] M. A. van Wyk and W.-H. Steeb. *Chaos in electronics*, volume 2 of *Mathematical Modelling: Theory and Applications.* Kluwer Academic Publishers, Dordrecht, 1997.

[49] G. S. Whiston. Global dynamics of a vibro-impacting linear oscillator. *J. Sound Vibration*, 118(3):395–424, 1987.

[50] G. S. Whiston. The vibro-impact response of a harmonically excited and preloaded one-dimensional linear oscillator. *J. Sound Vibration*, 115(2):303–319, 1987.

[51] Yosef Yomdin. Volume growth and entropy. *Israel J. Math.*, 57(3):285–300, 1987.

[52] Guohui Yuan, Soumitro Banerjee, Edward Ott, and James A. Yorke. Border-collision bifurcations in the buck converter. *IEEE Trans. Circuits Systems I Fund. Theory Appl.*, 45(7):707–716, 1998.

[53] Zhanybai T. Zhusubaliyev and Erik Mosekilde. *Bifurcations and chaos in piecewise-smooth dynamical systems*, volume 44 of *World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises*. World Scientific Publishing Co. Inc., River Edge, NJ, 2003.