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Decomposition of Graphs on Surfaces

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Let $G=(V, E)$ be an Eulerian graph embedded on a triangulizable surface S. We show that E can be decomposed into closed curves $C_1, ..., C_k$ such that $\text{mincr}(G, D)=\sum_{i=1}^{k} \text{mincr}(C_i, D)$ for each closed curve D on S. Here $\text{mincr}(G, D)$ denotes the minimum number of intersections of G and D' (counting multiplicities), where D' ranges over all closed curves D' freely homotopic to D and not intersecting V. Moreover, $\text{mincr}(C, D)$ denotes the minimum number of intersections of C'' and D' (counting multiplicities), where C'' and D' range over all closed curves freely homotopic to C and D, respectively. *Decomposing* the edges means that $C_1, ..., C_k$ are closed curves in G such that each edge is traversed exactly once by $C_1, ..., C_k$. So each vertex v is traversed exactly $\frac{1}{2}\deg(v)$ times, where $\deg(v)$ is the degree of v. This result was shown by Lins for the projective plane and by Schrijver for compact orientable surfaces. The present paper gives a shorter proof than the one given for compact orientable surfaces. We derive the following fractional packing result for closed curves of given homotopies in a graph $G=(V, E)$ on a compact surface S. Let $C_1, ..., C_k$ be closed curves on S. Then there exist circulations $f_1, ..., f_k \in \mathbb{R}^E$ homotopic to $C_1, ..., C_k$ respectively such that $f_1(e)+\cdots+f_k(e)\leqslant 1$ for each edge e if and only if $\text{mincr}(G, D)\geqslant\sum_{i=1}^{k} \text{mincr}(C_i, D)$ for each closed curve D on S. Here a *circulation homotopic* to a closed curve C_0 is any convex combination of functions $\text{tr}_C \in \mathbb{R}^E$, where C is a closed curve in G freely homotopic to C_0 and where $\text{tr}_C(e)$ is the number of times C traverses e. © 1997 Academic Press

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1. INTRODUCTION

Let S be a surface. (In this paper a surface is a *triangulizable* (equivalently, metrizable) surface.) A *closed curve* on S is a continuous function $C\colon S^1 \to S$, where S^1 is the unit circle in $\mathbb{C}$. Two closed curves C and C' are called *freely homotopic*, in notation $C \sim C'$, if there exists a continuous function bringing C to C'. (That is, a continuous function $\Phi\colon S^1 \times [0, 1] \to S$ such that $\Phi(z, 0) = C(z)$ and $\Phi(z, 1) = C'(z)$ for each $z \in S^1$.)

For any pair of closed curves C, D on S, $\mathrm{cr}(C, D)$ denotes the number of intersections of C and D, counting multiplicities. That is,

$$\mathrm{cr}(C, D) := |\{(w, z) \in S^1 \times S^1 \mid C(w) = D(z)\}|. \tag{1}$$

Moreover, $\mathrm{mincr}(C, D)$ denotes the minimum of $\mathrm{cr}(C', D')$ where C' and D' range over closed curves freely homotopic to C and D, respectively. That is,

$$\mathrm{mincr}(C, D) := \min\{\mathrm{cr}(C', D') \mid C' \sim C, D' \sim D\}. \tag{2}$$

Let $G = (V, E)$ be an undirected graph embedded on S. (In this paper, a graph has a finite number of vertices and edges. We identify G with its embedding on S.) For any closed curve D on S, $\mathrm{cr}(G, D)$ denotes the number of intersections of G and D (counting multiplicities):

$$\mathrm{cr}(G, D) := |\{z \in S^1 \mid D(z) \in G\}|. \tag{3}$$

Moreover, $\mathrm{mincr}(G, D)$ denotes the minimum of $\mathrm{cr}(G, D')$ where D' ranges over all closed curves freely homotopic to D and not intersecting V:

$$\mathrm{mincr}(G, D) := \min\{\mathrm{cr}(G, D') \mid D' \sim D, D'(S^1) \cap V = \emptyset\}. \tag{4}$$

(It would seem more consistent with definition (2) if we would also allow shifting G so as to obtain G', D' in minimizing $\mathrm{cr}(G', D')$, where G' is possibly not one-to-one mapped in S. However, the following theorem implies that this would not change the minimum value.)

We show the following theorem. It was proved for the projective plane by Lins [2] and for compact *orientable* surfaces by Schrijver [3]. (Our present proof is much simpler than that in [3], but uses a lemma on minimizing intersections of closed curves proved in [1].)

THEOREM. *Let $G = (V, E)$ be an Eulerian graph embedded on a triangulizable surface S. Then the edges of G can be decomposed into closed curves $C_1, \ldots, C_k$ such that*

$$\mathrm{mincr}(G, D) = \sum_{i=1}^{k} \mathrm{mincr}(C_i, D) \tag{5}$$

for each closed curve D on S.

Here a graph is *Eulerian* if each vertex has even degree. (We do not assume connectedness of the graph.) Moreover, *decomposing* the edges into $C_1, ..., C_k$ means that each edge is traversed by exactly one C_i, and by that C_i exactly once.

Note that the inequality $\geqslant$ in (5) trivially holds, for *any* decomposition of the edges into closed curves $C_1, ..., C_k$: by definition of $\text{mincr}(G, D)$, there exists a closed curve $D' \sim D$ in $S \backslash V$ such that $\text{mincr}(G, D) = \text{cr}(G, D')$, and hence

$$\text{mincr}(G, D) = \text{cr}(G, D') = \sum_{i=1}^{k} \text{cr}(C_i, D') \geqslant \sum_{i=1}^{k} \text{mincr}(C_i, D). \tag{6}$$

The content of the theorem is that there exists a decomposition attaining equality.

In Section 3 we give a proof of the Theorem, and in Sections 4 and 5 we derive applications, including a 'homotopic circulation theorem'.

2. MAKING CURVES MINIMALLY CROSSING BY REIDEMEISTER MOVES

The basic tool in our proof is the following result of de Graaf and Schrijver [1]. Denote by $\text{cr}(C)$ the number of self-intersections of C. That is,

$$\text{cr}(C) := \tfrac{1}{2} \left| \{ (w, z) \in S^1 \times S^1 \mid C(w) = C(z), w \neq z \} \right|. \tag{7}$$

Moreover, $\text{mincr}(C)$ denotes the minimum of $\text{cr}(C')$ where C' ranges over all closed curves freely homotopic to C:

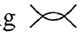
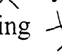
$$\text{mincr}(C) := \min\{ \text{cr}(C') \mid C' \sim C \}. \tag{8}$$

Let $C_1, ..., C_k$ be a system of closed curves on S. We call $C_1, ..., C_k$ *minimally crossing* if

$$\begin{array}{lll} \text{(i)} & \text{cr}(C_i) = \text{mincr}(C_i) & \text{for each } \; i = 1, ..., k; \\ \text{(ii)} & \text{cr}(C_i, C_j) = \text{mincr}(C_i, C_j) & \text{for all } \; i, j = 1, ..., k \;\text{ with }\; i \neq j. \end{array} \tag{9}$$

We call $C_1, ..., C_k$ *regular* if $C_1, ..., C_k$ have only a finite number of (self-) intersections, each being a crossing of only two curve parts. (That is, each point of S traversed twice by the $C_1, ..., C_k$ has a disk-neighbourhood on which the curve parts are topologically two crossing straight lines.)

In [1] we showed:

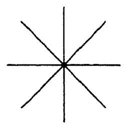
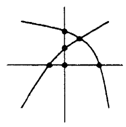
> Any regular system of closed curves on a triangulizable surface S can be transformed to a minimally crossing system by a series of "Reidemeister moves": replacing  by  (*type* 0); replacing  by (*type I*); replacing  by  (*type II*); replacing  by  (*type III*). (10)

The pictures in (10) represent the intersection of the union of $C_1, ..., C_k$ with a closed disk on S. So no other curve parts than the ones shown intersect such a disk.

It is important to note that in (10) we do not allow to apply the operations in the reverse direction—otherwise the result would follow quite straightforwardly with the techniques of simplicial approximation.

3. PROOF OF THE THEOREM

I. We may assume that each vertex v of G has degree at most 4. If v would have a degree larger than 4, we can replace G in a neighbourhood of v like

 by 

This modification does not change the value of mincr(G, D) for any D. Moreover, closed curves decomposing the edges of the modified graph satisfying (5), directly yield closed curves decomposing the edges of the original graph satisfying (5).

II. For any graph G embedded on S with each vertex having degree 2 or 4, we define the *straight decomposition* of G as the regular system of closed curves $C_1, ..., C_k$ such that $G = C_1 \cup \cdots \cup C_k$. So each vertex of G of degree 4 represents a (self-)crossing of $C_1, ..., C_k$.

Up to some trivial operations, such a decomposition is unique, and conversely, it uniquely describes G. Moreover, any Reidemeister move applied to $C_1, ..., C_k$ carries over a modification of G. So we can speak of Reidemeister moves applied to G.

Note that:

> if G' arises from G by one Reidemeister move of type III, then mincr$(G', D) =$ mincr(G, D) for each closed curve D. (11)

III. We call any graph $G = (V, E)$ that is a counterexample to the theorem with each vertex having degree at most 4 and with a minimal number of faces, a *minimal counterexample*.

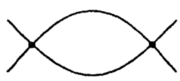
From (11) it directly follows that:

> if G' arises from a minimal counterexample G by one Reidemeister move of type III, then G' is a minimal counterexample again. (12)

Moreover one has:

> if G is a minimal counterexample, then no Reidemeister move of type 0, I or II can be applied to G. (13)

For suppose that a Reidemeister move of type II can be applied to G. Then G contains the following subconfiguration:



Replacing this by:

would give a smaller counterexample (since the function $\text{mincr}(G, D)$ does not change by this operation), contradicting the minimality of G.

One similarly sees that no Reidemeister move of type 0 or I can be applied.

IV. We finish the proof by showing that the straight decomposition $C_1, ..., C_k$ of any minimal counterexample G satisfies (5)—which is a contradiction to the fact that we have a counterexample.

Choose a closed curve D. We may assume that $D, C_1, ..., C_k$ form a regular system. By (10) we can apply Reidemeister moves so as to obtain a minimally crossing system $D', C'_1, ..., C'_k$.

By (12) and (13) we did not apply Reidemeister moves of type 0, I or II to $C_1, ..., C_k$. Hence by (11) for the graph G' obtained from the final $C'_1, ..., C'_k$ we have $\text{mincr}(G', D) = \text{mincr}(G, D)$. So

$$\begin{aligned}\text{mincr}(G, D) = \text{mincr}(G', D) \leqslant \text{cr}(G', D') &= \sum_{i=1}^{k} \text{cr}(C'_i, D') \\ &= \sum_{i=1}^{k} \text{mincr}(C'_i, D') = \sum_{i=1}^{k} \text{mincr}(C_i, D). \qquad (14)\end{aligned}$$

Since the converse inequality holds by (6), we have (5). ∎

4. A COROLLARY ON LENGTHS OF CLOSED CURVES

Using surface duality we obtain as in [3] the following. If G is a graph embedded on a surface S and C is a closed curve in G, then $\text{minlength}_G(C)$ denotes the minimum length of any closed curve $C' \sim C$ in G. (The *length* of C' is the number of edges traversed by C', counting multiplicities.)

COROLLARY 1. *Let $G=(V, E)$ be a bipartite graph embedded on a compact surface S and let $C_1, ..., C_k$ be closed curves in G. Then there exist closed curves $D_1, ..., D_t$ on $S \backslash V$ such that each edge of G is crossed by exactly one D_j and by this D_j only once and such that*

$$\text{minlength}_G(C_i) = \sum_{j=1}^{t} \text{mincr}(C_i, D_j) \tag{15}$$

for each $i = 1, ..., k$.

Proof. Let

$$d := \max\{\text{minlength}_G(C_i) \mid i = 1, ..., k\}. \tag{16}$$

We can extend G to a bipartite graph L embedded on S, so that each face of L is an open disk. By inserting d new vertices on each edge of L not occurring in G, we obtain a bipartite graph H satisfying $\text{minlength}_H(C_i) = \text{minlength}_G(C_i)$ for each $i = 1, ..., k$.

Consider a surface dual graph H^* of H. Since H is bipartite, H^* is Eulerian. Hence by the Theorem, the edges of H^* can be decomposed into closed curves $D_1, ..., D_t$ such that

$$\text{mincr}(H^*, C) = \sum_{j=1}^{t} \text{mincr}(D_j, C) \tag{17}$$

for each closed curve C. Now for each $i = 1, ..., k$, $\text{mincr}(H^*, C_i) = \text{minlength}_H(C_i) = \text{minlength}_G(C_i)$, and (15) follows. ∎

In [3] an example is given showing that we cannot replace $C_1, ..., C_k$ by the set of *all* closed curves occurring in G. However, the proof above also gives that we *can* replace $C_1, ..., C_k$ by the set of all closed curves if G is cellularly embedded (i.e., each face is an open disk)—in that case we do not need to extend G to L and H.

5. A HOMOTOPIC CIRCULATION THEOREM

By linear programming duality (Farkas' lemma) we derive from Corollary 1 the following 'homotopic circulation theorem'—a fractional packing theorem for cycles of given homotopies in a graph on a compact surface.

Let $G=(V, E)$ be a graph embedded on a compact surface S. For any closed curve C in G and any edge e of G let $\mathrm{tr}_C(e)$ denote the number of times C traverses e. So $\mathrm{tr}_C \in \mathbb{R}^E$.

Call a function $f: E \to \mathbb{R}$ a *circulation* (of value 1) if f is a convex combination of functions tr_C. We say that f is *freely homotopic* to a closed curve C_0 if we can take each C freely homotopic to C_0.

Note that if f is a circulation freely homotopic to C_0, then for each closed curve D on $S\setminus V$ one has (denoting by $\mathrm{cr}(e, D)$ the number of times D intersects edge e):

$$\sum_{e \in E} f(e)\,\mathrm{cr}(e, D) \geqslant \mathrm{mincr}(C_0, D). \tag{18}$$

This follows from the fact that (18) holds for $f := \mathrm{tr}_C$ for each C freely homotopic to C_0 (as $\sum_{e \in E} \mathrm{tr}_C(e)\,\mathrm{cr}(e, D) = \mathrm{cr}(C, D) \geqslant \mathrm{mincr}(C_0, D)$), and hence also for any convex combination of such functions.

COROLLARY 2 (Homotopic Circulation Theorem). *Let $G=(V, E)$ be an undirected graph embedded on a compact surface S and let $C_1, ..., C_k$ be closed curves on S. Then there exist circulations $f_1, ..., f_k$ such that f_i is freely homotopic to $C_i (i=1, ..., k)$ and such that $\sum_{i=1}^{k} f_i(e) \leqslant 1$ for each edge e, if and only if for each closed curve D on $S\setminus V$ one has*

$$\mathrm{cr}(G, D) \geqslant \sum_{i=1}^{k} \mathrm{mincr}(C_i, D). \tag{19}$$

Proof: *Necessity*. Suppose there exist circulations $f_1, ..., f_k$ as required. Let D be a closed curve on $S\setminus V$. Then by (18):

$$\begin{aligned} \mathrm{cr}(G, D) &= \sum_{e \in E} \mathrm{cr}(e, D) \\ &\geqslant \sum_{e \in E} \mathrm{cr}(e, D) \sum_{i=1}^{k} f_i(e) \\ &= \sum_{i=1}^{k} \sum_{e \in E} f_i(e)\,\mathrm{cr}(e, D) \\ &\geqslant \sum_{i=1}^{k} \mathrm{mincr}(C_i, D). \end{aligned} \tag{20}$$

Sufficiency. Suppose (19) is satisfied for each closed curve D on $S\backslash V$. Let $I := \{1, ..., k\}$, and let K be the convex cone in $\mathbb{R}^I \times \mathbb{R}^E$ generated by the vectors

$$\begin{aligned} &(\varepsilon_i; \mathrm{tr}_C) \qquad (i \in I;\ C \text{ closed curve in } G \text{ with } C \sim C_i); \\ &(0_I; \varepsilon_e) \qquad (e \in E). \end{aligned} \tag{21}$$

Here ε_i denotes the ith unit basis vector in $\mathbb{R}^I$ and ε_e denotes the eth unit basis vector in $\mathbb{R}^E$. Moreover, 0_I denotes the all-zero vector in $\mathbb{R}^I$.

Although generally there are infinitely many vectors (21), K is finitely generated. This can be seen by observing that we can restrict the vectors $(\varepsilon_i; \mathrm{tr}_C)$ in the first line of (21) to those that are minimal with respect to the usual partial order $\leqslant$ on $\mathbb{Z}^I_+ \times \mathbb{Z}^E_+$ (with $(x, y) \leqslant (x', y') \Leftrightarrow x_i \leqslant x'_i$ for all $i \in I$ and $y_e \leqslant y'_e$ for all $e \in E$). They form an 'antichain' in $Z^I_+ \times \mathbb{Z}^E_+$ (i.e., a set of pairwise incomparable vectors), and since each antichain in $\mathbb{Z}^I_+ \times \mathbb{Z}^E_+$ is finite, K is finitely generated.

We must show that the vector $(1_I; 1_E)$ belongs to K. Here 1_I and 1_E denote the all-one sectors in $\mathbb{R}^I$ and $\mathbb{R}^E$, respectively. By Farkas' lemma, it suffices to show that each vector $(d; l) \in \mathbb{Q}^I \times \mathbb{Q}^E$ having nonnegative inner product with each of the vectors (21), also has nonnegative inner product with $(1_I; 1_E)$. Thus let $(d; l) \in \mathbb{Q}^I \times \mathbb{Q}^E$ have nonnegative inner product with each vector among (21). This is equivalent to:

$$\begin{aligned} &\text{(i)} \quad d_i + \sum_{e \in E} l(e)\, \mathrm{tr}_C(e) \geqslant 0 \quad (i \in I;\ C \text{ closed curve in } G \text{ with } C \sim C_i); \\ &\text{(ii)} \quad\qquad\qquad l(e) \geqslant 0 \quad (e \in E). \end{aligned} \tag{22}$$

Suppose now that $(d; l)^T (1_I; 1_E) < 0$. By increasing l slightly, we may assume that $l(e) > 0$ for each $e \in E$. Next, by blowing up $(d; l)$ we may assume that each entry in $(d; l)$ is an even integer.

Let G' be the graph arising from G by replacing each edge e of G by a path of length $l(e)$. That is, we insert $l(e) - 1$ new vertices on e. Then by (22)(i),

$$-d_i \leqslant \mathrm{minlength}_{G'}(C_i) \tag{23}$$

for each $i \in I$. Since G' is bipartite, by Corollary 1 there exist closed curves $D_1, ..., D_t$ not intersecting any vertex of G' such that each edge of G' is intersected by exactly one D_j and only once by that D_j and such that

$$\mathrm{minlength}_{G'}(C_i) = \sum_{j=1}^{t} \mathrm{mincr}(C_i, D_j) \tag{24}$$

for each $i \in I$. So

$$l(e) = \sum_{j=1}^{t} \mathrm{cr}(e, D_j) \tag{25}$$

for each edge e of G. Hence (19), (23) and (24) give

$$\begin{aligned}
\sum_{e \in E} l(e) &= \sum_{j=1}^{t} \sum_{e \in E} \mathrm{cr}(e, D_j) \\
&= \sum_{j=1}^{t} \mathrm{cr}(G, D_j) \geqslant \sum_{j=1}^{t} \sum_{i=1}^{k} \mathrm{mincr}(C_i, D_j) \\
&= \sum_{i=1}^{k} \sum_{j=1}^{t} \mathrm{mincr}(C_i, D_j) \\
&= \sum_{i=1}^{k} \mathrm{minlength}_{G^*}(C_i) \geqslant -\sum_{i=1}^{k} d_i.
\end{aligned} \tag{26}$$

So $(d; l)^T (1_I; 1_E) \geqslant 0$. ∎

In [3] it is shown that generally we cannot take the f_i 0, 1-valued, even not if certain "parity conditions" hold.

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