

System Theory – A Brief Exposition

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1. INTRODUCTION

This paper provides an introduction to system theory for a general readership of mathematicians, engineers, and other scientists. In addition, contributions of CWI's research group System and Control Theory are summarized.

In several areas of the sciences there is a need for mathematical models of phenomena that evolve in time. Such models, called *control system* or *system*, are used for control or for signal processing, and have been formulated, for example, in connection with the movement of a compact disc, the temperature in a glass furnace, the behaviour of an aircraft, the behaviour of an underwater autonomous robot, and the flow of nitrate in the human body.

A control system interacts with its environment by receiving an input signal and providing an output signal. A control system may be described by a differential equation, a difference equation, logical rules, or a combination of these as in a hybrid system. Here we restrict ourselves to system theory. Control theory, with its main concept of feedback, is only marginally touched upon.

The main problem of system theory is *realization*. This is motivated by the problem of *system identification*. The realization problem is to derive for observations a system in a recursive state space representation, explained below, and to classify all systems that represent the same observations. The

system identification problem is to construct from observations a system in a selected model class that approximates the data according to an approximation criterion. An example of such a problem is to construct a model for the flow of nitrate in the human body (see section 4). Realization and system identification theory have been developed extensively for finite-dimensional linear systems and for Gaussian systems. For other classes of systems the results are much less complete.

System theory has been developed by researchers in various disciplines, like engineering, mathematics, and econometrics, and uses many different branches of mathematics, including linear algebra, differential equations, geometry, operator theory, probability, and stochastic processes.

2. HISTORY

2.1. *Origins*

Control and signal processing problems were already intensively studied in engineering and mathematics in the 1940's, when N. Wiener and A. Kolmogorov pioneered an approach to the least-squares prediction problem. In this problem an algorithm, called a filter, is to be derived that on the *basis* of observations predicts a signal. Applications and extensions of this approach followed in the 1950's. Within engineering feedback control was developed for amplifiers and communication equipment. Within mathematics optimal control theory was studied, based on the calculus of variations, following publications of Russian mathematicians led by L.S. Pontryagin. Related developments in linear algebra, stochastic processes, information theory, and communication theory influenced researchers active at that time.

Around 1960 weaknesses and limitations of optimal control and least-squares prediction became clear. A filter that at any time needs an infinite number of past data cannot be implemented on a computer with a finite memory. Researchers in optimal control theory realized that only a limited class of problems can be solved analytically.

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Then R.E. Kalman proposed a new problem formulation for control and filtering. If finite memory, however defined, is required for implementation, then why not consider as starting point a control system with finite memory, that interacts with its environment via input and output signals? The definition of a control system is inspired by developments in computer science around 1960 with the concepts of an automaton and of a recursive function, and is based on the concept of state, as used in physics, and on a recursive structure for that state. At any time the current state and the future input of a system uniquely determine the future of the state and the output. Seen in this way engineering models, control systems, and computer algorithms, become analogous objects. System theory aims to study such objects in a unified way. As a consequence there can be a unified approach



Figure 1. R.E. Kalman. (Courtesy Springer-Verlag.)

to problems of control, communication, signal processing, and computing.

Along these lines Kalman solved a least-squares prediction and filtering problem for a control system, the solution of which, now known as the Kalman filter, is widely applied in signal processing and control. The realization problem, inspired by the definition of a control system, is to construct a recursive state space representation from observations of input-output signals. Kalman also showed that an optimal control problem for a linear system

and a quadratic criterion can be solved analytically and that the solution is dual to the Kalman filter. The shift from optimal control problems and least-squares prediction for models with infinite memory to systems with finite memory had been shown to work.

With T.S. Kuhn one may speak of a paradigm shift for control and signal processing, with enormous consequences. Control and signal processing problems now yielded solutions with finite memory that could be implemented directly and analyzed explicitly. Engineering modelling and system identification took a new turn. Results from system theory, usually through control and signal processing, are used in research areas including engineering, computer science, technology, economics, and econometrics. By now courses in systems and signals are in the undergraduate curricula of most engineering departments and of mathematics departments, and software packages with algorithms based on system theory are used in industry and in government.

A few lessons can be drawn from the development of system theory. Applications of system theory algorithms provide ample evidence for the usefulness of the concept of a state space representation. A system as mathematical model must be regarded as a representation of observations. System identification must take into account the fact that for a given set of input-output signals there is in general a large equivalence class of models. Optimal control and filtering problems may not admit a solution with finite memory, however it be defined. Solutions with finite memory may be determined by turning the problem formulation around and asking for a realization of the observations in a selected class of systems with finite memory.

2.2. What is a control system?

As an example we introduce the concept of a *time-invariant finite-dimensional linear system* without much attention to mathematical finesse. Consider the system specified by the equations

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0, \quad (2.1)$$

$$y(t) = Cx(t) + Du(t), \quad (2.2)$$

where $t_0 \in \mathbb{R}$, $T = [t_0, \infty)$ is called the *time interval* of interest; $x_0 \in \mathbb{R}^n$ the *initial state*; $u : T \rightarrow \mathbb{R}^m$ the *input function*; $x : T \rightarrow \mathbb{R}^n$ the *state function*; $y : T \rightarrow \mathbb{R}^p$ the *output function*; and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, and $D \in \mathbb{R}^{p \times m}$ are matrices. As mentioned before, the main characteristic of such a system is that at any time the state and the future input uniquely determine the future of the state and of the output. The observations are formed by the input and output functions, or, alternatively, by input-output signals. The system is called *linear* because the output is a linear function of the input and the initial condition, *finite-dimensional* because the state space, \mathbb{R}^n , is a finite-dimensional vector space, and *time-invariant* because its response is the same if it starts from the same state at a later time.

The external description of such a system is specified by the following relation for input-output signals:

$$y(t) = Ce^{A(t-t_0)}x_0 + \int_{t_0}^t W(t-s)u(s)ds, \quad (2.3)$$

$$W(t) = Ce^{A(t-t_0)}B + D\delta(t-t_0), \quad W : T \rightarrow \mathbb{R}^{p \times m}, \quad (2.4)$$

where W is called the *impulse response function* and δ is the Dirac delta function. The observations of the system,

$$236 \quad \left\{ \begin{pmatrix} u \\ y \end{pmatrix} : T \rightarrow \mathbb{R}^{m+p} \mid u, y \text{ satisfy (2.3) for a } x_0 \in \mathbb{R}^n \right\}, \quad (2.5)$$

are also called the *observable behaviour* or *behaviour* of the system. A *time series* is a set of numerical values of input-output signals.

2.3. Realization theory for finite-dimensional linear systems

First realization from the impulse response function W is discussed. Consider the external representation as in (2.3). Through experimentation with a phenomenon an engineer can obtain an estimate of W . The question is then whether there exists a finite-dimensional linear system with matrices A, B, C, D such that (2.4) holds. If so, it is called a *realization* of the given external system description or of the impulse response function. The realization problem also requires the classification of all *minimal* realizations,

i.e., those where the dimension n of the state space is minimal. Other questions include: what is the characterization of the state space description of a system if its external description is either time-reversible, symmetric, or dissipative?

Kalman has derived a necessary and sufficient condition for an impulse response function to have a realization as a finite-dimensional linear system and a reachability and an observability condition for the realization to be minimal. In addition, he has provided a classification of all minimal realizations (see for details the textbook [3] by E.D. Sontag). Parameterizations for the class of minimal realizations were derived later. M. Hazewinkel and Kalman have proven that for multi-input/multi-output systems no continuous parameterization exists.

Secondly realization from input-output signals is discussed. This is very relevant in research areas with short time series or where experimentation is not permitted, such as environmental modelling, biology, economics, and econometrics. This should be contrasted with electrical engineering, where through experimentation one can obtain arbitrarily long time series. In the 1970's this problem was treated by R. Liu and L.C. Suen, and shortly afterwards by E. Emre, L.M. Silverman, and K. Glover with the term 'dynamic covers'.

In the behavioural approach to system theory, proposed by J.C. Willems and developed by him and co-workers, realization from input-output signals is generalized. In this approach the observation vector is not a priori distinguished into an input and output signal. In many engineering problems the distinction is clear because of a causality relation, but in other problems, for example in econometrics (e.g., income and consumption of households), this is often not a priori the case.

2.4. Stochastic realization of stationary Gaussian processes

Kalman also proposed a definition of a stochastic control system. Consider three discrete-time stationary stochastic processes: an input, a state, and an output process. They form a stochastic control system if for all $t \in T$ the conditional probability distribution of the next state and the current output, $(x(t+1), y(t))$, given the past of the state, output, and input process, depends only on the current state and the current input, $(x(t), u(t))$. If the probability distribution is Gaussian or normal and if only the conditional mean of this distribution depends linearly on $(x(t), u(t))$, then the processes satisfy the following relations

$$x(t+1) = Ax(t) + Bu(t) + Mv(t), \quad x(t_0) = x_0, \quad (2.6)$$

$$y(t) = Cx(t) + Du(t) + Nv(t), \quad (2.7)$$

where v is a Gaussian white noise process, i.e., a sequence of independent random variables each of which has a Gaussian probability distribution function. The system specified by this representation is said to be a *Gaussian stochastic control system* and a *Gaussian system* in case there is no input process.

Furthermore, Kalman formulated the weak stochastic realization problem for stationary Gaussian processes. It was motivated by an analysis of the Kalman filter. A stationary Gaussian process is said to have a *stochastic realization* if there exists a Gaussian system such that the output process equals the given process in distribution. P. Faurre and co-workers, in cooperation with Kalman, have given a characterization of a minimal stochastic realization, classified them, and also analyzed a parameterization of the class of stochastic realizations.

A. Lindquist and G. Picci (see [1]), and G. Ruckebusch, have solved the strong Gaussian stochastic realization problem, in which the output process must equal the given process almost surely. This problem is best studied in geometric terms in which a stationary Gaussian process is associated with a subspace of a Hilbert space.

Stochastic realization theory forms the theoretical foundation of signal processing. Prediction problems were treated in the 1940's. By now signal processing includes several techniques based on system theory, such as prediction, filtering, smoothing, interpolation, and image processing.

2.5. Realization theory—extensions

The realization theory formulated for finite-dimensional linear systems and for Gaussian systems has been generalized to many other classes of systems. Only a few of these generalizations will be mentioned below. For each mathematical structure the concept of a system must be defined anew.

Algebraic generalizations are linear systems over modules, rings, and finite fields. Research in these directions was initiated by Kalman with contributions by M.L.J. Hautus, E.W. Kamen, and E.D. Sontag. Linear systems over finite fields are used as mathematical models in coding theory and have recently drawn new interest. The realization problem for systems in algebraic structures as groups, semigroups, and algebras, is essentially the problem of finding irreducible representations of input-output maps. A special case of current interest is realization of positive linear systems that is motivated by, for example, problems in biomathematics, chemical engineering, and economics. The realization problem for this class is unsolved and requires further study of polyhedral cones and positive linear algebra.

Other systems for which the realization problem has been studied include: linear systems with functions in Hilbert spaces (by P.A. Fuhrmann), systems in which the dynamics is specified by polynomials (by Sontag, in cooperation with Kalman), bilinear systems, a system in a differential geo-

metric context (described in terms of vector fields), and specific classes of nonlinear systems (by H. Sussmann, M. Fliess, and B. Jacubczyk) and of mechanical systems, such as Hamiltonian systems. In computer science the concept corresponding to a system is an input-output automaton, a Petri net, or a process algebra. In the automata literature, the realization problem has been solved by A. Nerode. A generalization of the concept of state for systems with functions taking values in arbitrary sets was formulated by Willems in terms of the conditional independence relation for sets. In this definition the current state and the input signal make the past and the future of the state and output signal conditionally independent.

A multi-parameter system is a system in which the time axis has been generalized to an arbitrary index set or to a partially ordered set. A picture may be modeled as a two-parameter system. The concept of state of such a system may be phrased in terms of the conditional independence relation of sets. The realization problem for this class has been studied in connection with image processing.

Stochastic realization theory of Gaussian processes has also been generalized, for example to diffusion processes in analogy with statistical mechanics and quantum mechanics. A finite stochastic system may be defined analogously to a Gaussian system for a finite-valued process with a finite-state Markov process. In signal processing it is called a *hidden Markov model* and in automata theory a *probabilistic automaton*. The stochastic realization problem for this class, already studied in the 1960's, is still unsolved, as is the case for counting and jump processes.

An investigation is needed of the stochastic realization problem for stochastic control systems with partial observations. The concepts of information state and of information system should be studied in the framework of exponential families of distribution functions.

Kalman's definition of a stochastic system can be reformulated in terms of the conditional independence relation of probability theory, stating that at any time the current state and the input process make the past and the future of the state and the output process conditionally independent. Multi-parameter stochastic realization problems in connection with random fields are under investigation.

A generalization in another direction is the factor analysis model. In this model for random variables the factor, corresponding to the state, makes two or more variables conditionally independent. This generalization of the concept of state is very interesting. R. Frisch, who received the Nobel prize in economics, proposed this model as an alternative for the model used in least squares estimation. Kalman has pointed out its relevance for economic modelling and contributed to the associated stochastic realization problem. The problem is unsolved. The stochastic system corresponding to a factor analysis model is termed an *errors-in-variables* model or a dynamic factor

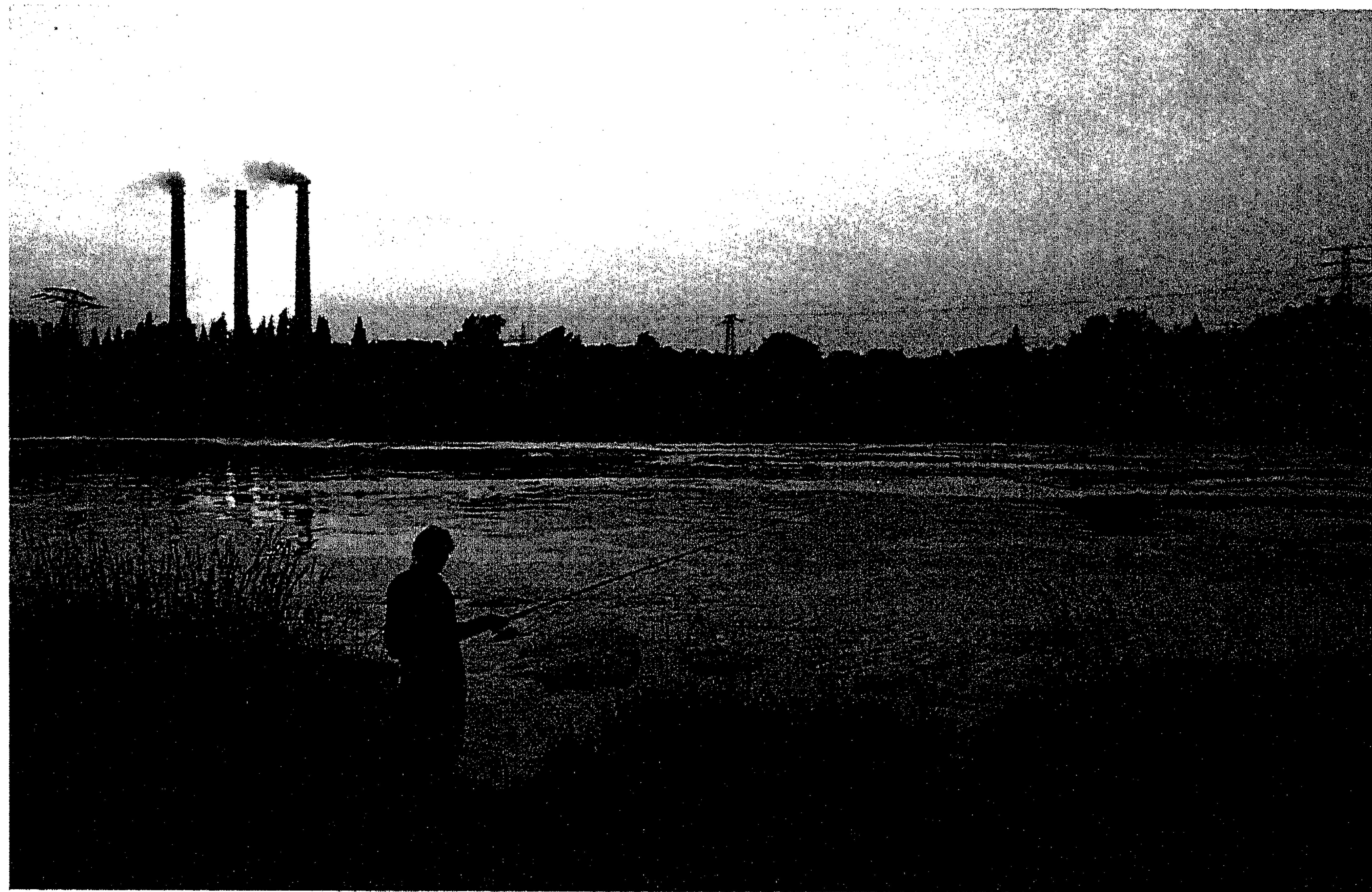


Figure 2. Environmental problems will motivate future research in system identification. (Photo: James Nachtwey—Magnum Photos. Courtesy ABC Press.)

system. The realization problem for this class is studied in system theory and econometrics by G. Picci and M. Deistler.

2.6. System identification

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As stated in the Introduction, the system identification problem is to construct from observations a control system in a chosen model class that best fits the observations according to a specified approximation criterion. A procedure for system identification is: (1) Selection of a model class based on a priori information; (2) Input design, experimentation, and data collection; (3) Parameterization of the model class based on realization theory and a check of its identifiability; (4) Approximation, selection of a control system; and (5) Evaluation of the quality of the selected system. The selection of the model class is often based on domain modelling, for example on physical laws, on chemical reaction kinetics, and on economic or physiological modelling. In step (3) of the procedure realization theory is used exclusively. A textbook on system identification is that of L. Ljung [2].

System identification has been well developed for the classes of finite-dimensional linear systems and for Gaussian stochastic systems. For the

approximation criterion use is made of the least-squares criterion or the likelihood function. The problem is largely solved for single-input/single-output linear systems, but still not satisfactory for multi-input/multi-output linear systems. The most effective solutions are based on realization theory. The most promising approach is presently the so-called subspace method, based on stochastic realization theory and numerical linear algebra.

System identification problems for nonlinear systems have been studied for a long time in engineering and econometrics. The relation between realization theory for nonlinear systems and system identification problems for the same class remains to be explored.

3. CWI CONTRIBUTIONS

3.1. Stochastic realization and system identification

The research by J.H. van Schuppen in stochastic realization theory is motivated by system identification, signal processing, and control for counting and jump processes. The stochastic realization problem for finite-valued processes is investigated in cooperation with G. Picci. The current bottleneck is the characterization of minimal realizations of finite stochastic systems. Solution of this problem leads to a factorization problem for positive matrices. The closely related realization problem for deterministic positive linear systems is currently investigated by J.M. van den Hof and Van Schuppen. The stochastic realization problem in terms of σ -algebras, as well as for the factor analysis model have been treated.

Motivated by the engineering practice of using Gaussian white noise as input signal, a stochastic realization problem for a Gaussian stochastic control system has been formulated and solved. Parameter estimation problems for counting processes were treated by P.J.C. Spreij. Recently A.A. Stoorvogel and Van Schuppen investigated the approximation problem for Gaussian stochastic systems using information theoretic criteria.

3.2. Linear systems

Systems are modeled by a variety of methods including black-box identification and the use of physical laws. The classical input/output framework, see (2.1) and (2.2), which dominates control theory is less appropriate in the modelling context, and has to be replaced by a setting in which all external variables are treated on an equal footing. This point of view, recently emphasized in particular by Willems, leads to new questions for realization theory. First-order representations of other types than the standard input/state/output form (2.1) and (2.2) are used, and one needs to analyze the minimality conditions for such representations. As a basis for the notion of equivalence of representations, the transfer function is replaced by the ‘behaviour’, which is the set of all trajectories (in some given func-

tion space) admitted by the system equations. A study of minimality and equivalence for general first-order representations of linear systems was undertaken at CWI by J.M. Schumacher together with M. Kuijper and has led to several journal papers and a book. In the approach based on behaviours, the choice of a function space has an impact on the notion of equivalence. The equivalence notion that is obtained from working with the space of so-called 'impulsive-smooth distributions', was studied in a joint effort of A.H.W. Geerts and Schumacher. Recent work of M.S. Ravi and J. Rosenthal in the U.S.A. and of V. Lomadze in the Republic of Georgia has made clear that the set of generalized linear systems obtained in this way provides the long-sought smooth compactification of the class of standard linear systems of a fixed McMillan degree. This issue is currently being further explored in joint work of Ravi, Rosenthal, and Schumacher.

4. SYSTEM IDENTIFICATION OF NITRATE FLOW IN THE HUMAN BODY

4.1. Structural identifiability from input-output signals

In biology and mathematics the class of compartmental systems is frequently used. A physiological model of a living organism may consist of several compartments with more or less homogeneous concentrations of material. The compartments interact by processes of transportation and diffusion. In biology there often is prior knowledge on the structure of the model. Therefore the class of compartmental systems is related to the class of structured linear systems, in which the system is structured by physical laws. Before estimating the parameters, it should be examined whether the parameterization is structurally identifiable, i.e., whether the parameters can in principle be determined *uniquely* from the data. Conditions for structural identifiability from the impulse response follow directly from realization theory. J.M. van den Hof has investigated structural identifiability from input-output signals with unknown initial condition for both finite-dimensional linear systems and positive linear systems.

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4.2. Example

As an example of a system identification problem we consider a model for the uptake and dispersion of nitrate in the human body. In the model class four compartments are considered: nitrate (NO_3^-) in the stomach, the body pool, and the saliva, and nitrite (NO_2^-) in the saliva, as shown in figure 3.

The model may be described by the following differential equations:

$$\begin{aligned} \dot{x}_1 &= -K_a x_1 + \frac{b}{V_s} x_3 + u_1, & \dot{x}_2 &= K_a x_1 - (K_2 + K_T) x_2 + K, \\ \dot{x}_3 &= K_2 x_2 - (K_1 + \frac{b}{V_s}) x_3, & \dot{x}_4 &= K_1 x_3 - \frac{b}{V_s} x_4, \end{aligned}$$

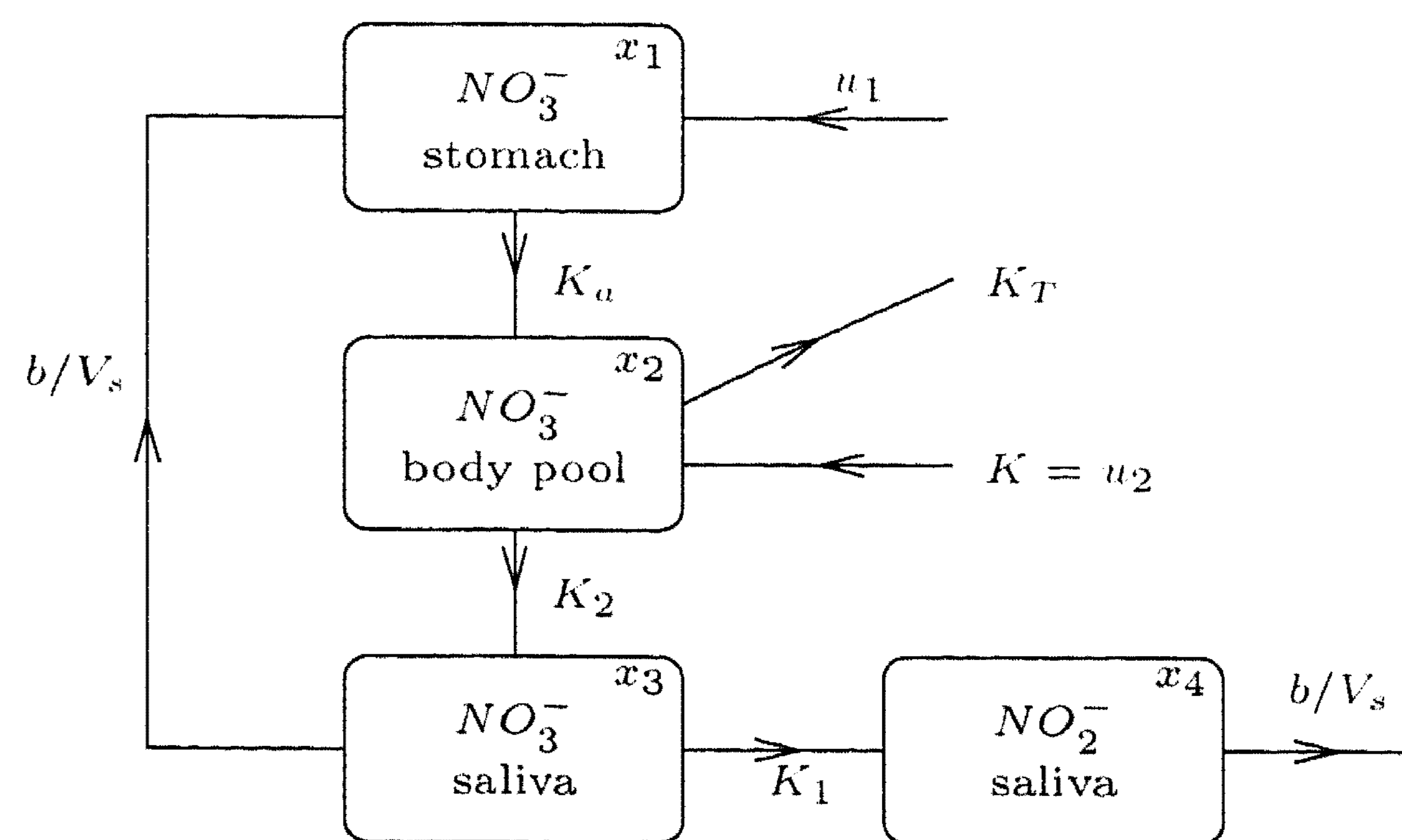


Figure 3. Nitrate model.

in which x_1, x_2 , and x_3 denote the amount of NO_3^- in the stomach, the body pool, and the saliva, respectively, and x_4 denotes the amount of NO_2^- in the saliva; u_1 denotes the uptake of nitrate. The remaining variables are constants. The constants K, K_T , and V_d are assumed to be known. The unknown parameters are K_a, K_2, K_1, b, V_s , and the initial condition x_0 . One can observe the concentration of NO_3^- in the body pool and the saliva, and the concentration of NO_2^- in the saliva, i.e., we can observe $x_2/V_d, x_3/V_s$, and x_4/V_s . The model has been developed by the National Institute of Public Health and Environmental Protection (RIVM).

The theory developed by Van den Hof for structural identifiability from input-output signals with a nonzero initial condition provides conditions on the inputs u_1 and K such that the unknown parameters K_a, K_2, K_1, b, V_s , and the initial condition x_0 can be uniquely determined from the observations.

5. CONCLUDING REMARKS

System theory has proven to be extremely useful for engineering, mathematics, and other areas of the sciences, in particular for control and signal processing. The concept of a control system, and the results of realization theory and system identification are widely applied in industry, commerce, and government.

System theory will in the future be motivated by new problems of engineering and the sciences. Solution of these problems will become urgent through the technological development and through the demands for increased living standards. There may also be a shift away from electrical and mechanical engineering to information processing. Realization problems motivated by information processing may therefore receive relatively

more interest. A realization approach is also needed for team and game problems. In such decision and control problems there are two or more decision makers with different observations. System theory has many open problems.

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