Mathematical Aspects of Nonlinear Dynamical Systems

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1. Description of the field of research

Dynamical systems are systems that change as time evolves. In the present, mathematical context this means: mathematical models for such systems. These models consist of two main ingredients. The first is a state 'space': a set whose elements are the possible states of the system. The second ingredient is an evolution law, which describes how the state of the system evolves as a function of time, once an initial state is known. We usually assume that whenever the state of a system is known at a certain time t, the evolution law completely determines the state at all later times. This defines a deterministic system, as opposed to a stochastic system where the evolution is in terms of probability. Also we assume that we cannot influence the dynamics, except by choosing the initial state. Dynamical systems that do admit such interventions are the subject of systems- and control theory.

1.1. Linearity

The notion of linearity for dynamical systems usually refers to some equilibrium state, such that the only states of interest are small perturbations of this equilibrium. As an example think of water in a pond with a completely flat surface as its equilibrium, where the perturbations are small surface waves.

In such a context the system is called linear if a superposition principle holds in the following sense: for any two perturbations compatible with the law of evolution, the 'sum' is also compatible with this law. (In the example of a water surface this holds to a good approximation: two different waves can cross one another without being visibly disturbed.) In 'real' systems this linearity often holds only in first approximation.

1.2. Nonlinearity

Nonlinear systems mostly are far from equilibrium. Usually it is hard to obtain general information about the set of all possible evolutions of such systems. There are some exceptions: systems which are nonlinear, but still can be completely 'solved'. The most famous example of this is the solar system without interaction between the planets (leading to the description of the motion as given by the Kepler laws). Systems that can be analysed completely in this way are called *integrable.* Usually the dynamical behaviour of such integrable systems is not representative for that of general nonlinear systems.

In this respect integrable systems, as well as linear systems, are exceptional. However, both cases also are important for the study of the general case. Indeed, many aspects of nonlinear dynamical systems can be studied in situations obtained from linear or integrable systems by a small perturbation. An example of this is the solar system *with* interaction between the planets, where the interaction is considered small.

An early example concerning this was the work of H. Poincaré, leading to his monumental paper in the Acta Mathematica (1890) which was a first and very influential contribution to what is now called the geometric theory of dynamical systems.

1.3. Degrees of freedom

Apart from the distinction between linear and nonlinear or integrable and non-integrable, there is another distinction we want to point out. On the one hand there are systems with only a finite number of degrees of freedom, i.e., systems, the state of which is specified by a finite set of numbers. Here think of the planetary system, mechanical systems consisting only of a finite number of rigid bodies and springs, electrical circuits, etc. On the other hand there are systems with infinitely many degrees of freedom, usually systems where the specification of a state needs one or more functions. An example is given by the above water surface where the specification of the height of the surface requires a function.

It turns out that systems with infinitely many degrees of freedom in their state space often exhibit some finite dimensional structure to which, due to the law of evolution, all states converge. In such cases the dynamics essentially is that of a system of a finite number of degrees of freedom. In cases where there is no such reduction new dynamical phenomena can appear. The emphasis in the programme under description mainly is on systems with finitely many degrees of freedom.

2. Methods of research

Investigating nonlinear systems one tries to solve two types of questions. First, how to obtain information about the dynamical properties of a system, the evolution laws of which are given in terms of explicit equations. Second, what types of dynamic behaviour can be expected in typical (i.e., non-pathological) deterministic systems. Most methods, mentioned below, are used to answer both types of questions.

2.1. Analytic methods

Often it is not possible to determine analytically (i.e., without a computer) the future states of a dynamical system given its initial state. Nevertheless there are many important cases where good approximations exist by systems that one can solve in this respect. These approximating systems usually are either linear or integrable. For many years the positions of the planets have been predicted by such methods. It is remarkable how Poincaré motivated this research: "The final goal of celestial mechanics is to resolve the great problem of determining if Newton's law alone explains all astronomical phenomena. The only means of deciding is to make the most precise observations and then compare them to the calculated results." Indeed, these methods of approximation gave the required accuracy: not many years after he wrote this it was found that Mercury did not obey this 'law' — which was one of the observations leading to the general theory of relativity.

2.2. Geometrical methods: far from linear or integrable

These methods rely on abstract existence results often in the form of fixed point theorems. For example think of the theory of invariant manifolds and persistence for dynamical systems with hyperbolic subsets.

2.3. Analysis and interpretation of physical or numerical examples

A driving force in the development of the theory of dynamical systems was the desire to give a mathematical explanation of concrete examples. In recent times a number of such examples were given in the form of mathematical equations. We here mention the examples of Lorenz (1963), Hénon-Heiles (1964), Hénon (1976), Rössler (1976), etc., which were solved only numerically. See figure 1. These numerical solutions showed thoroughly unexpected patterns, requiring completely new theoretical insights for their explanation. In this context we also mention experiments in mechanical systems with resonance, e.g., Moon and Holmes (1979), the dynamics of which has much in common with that of the above Hénon system.

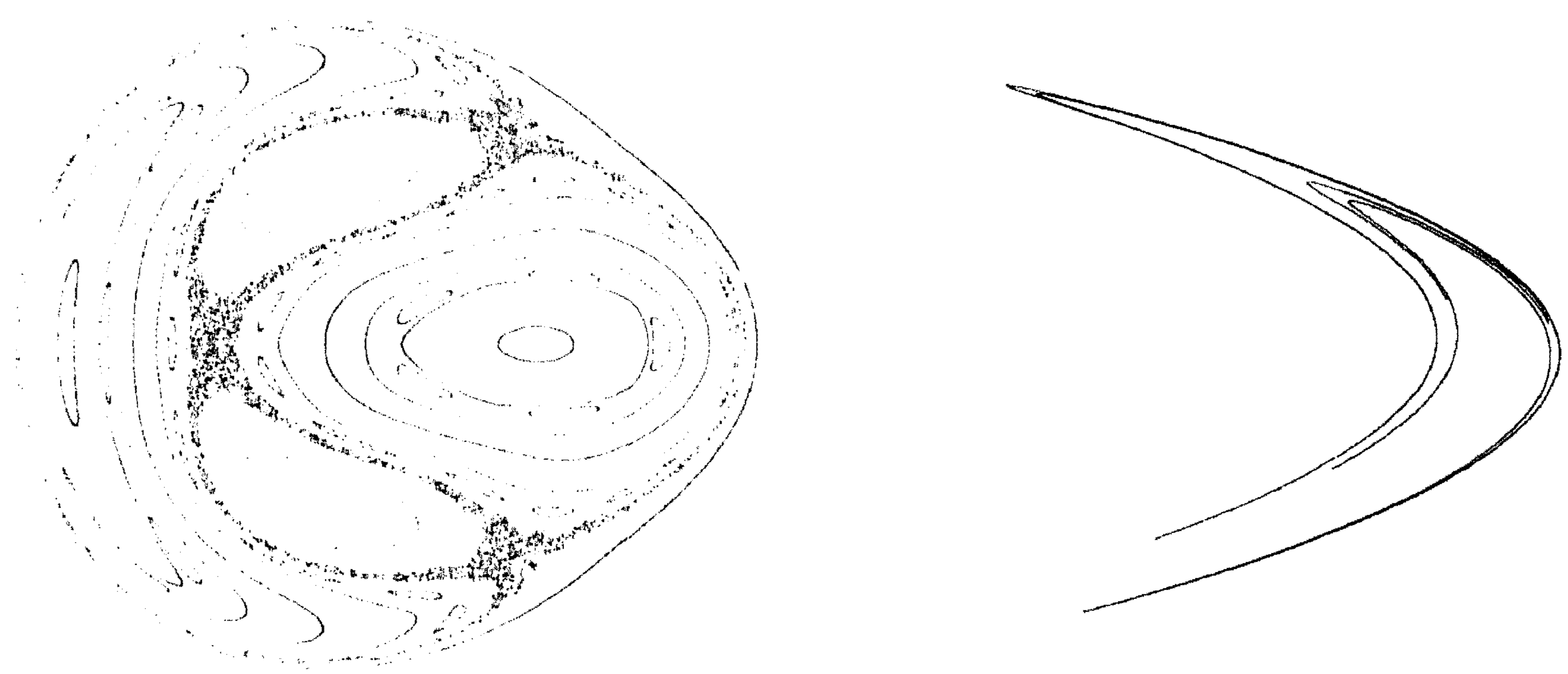

Figure 1. The conservative example of Hénon-Heiles 1964 (left) and the dissipative example of Hénon 1976 (right).

The earlier, but fundamental, work of B. van der Pol at the Philips Natuurkundig Laboratorium (1920 and later) was aimed at the understanding of the dynamics of electrical circuits with nonlinear elements (vacuum tubes).

2.4. Analysis of geometric examples
There have been a number of highly important examples of dynamical systems, not given by explicit equations, but by a geometric description. From this one could prove mathematically the possibility of certain types of dynamic behaviour. In particular, the horseshoe map (Smale 1965) and related

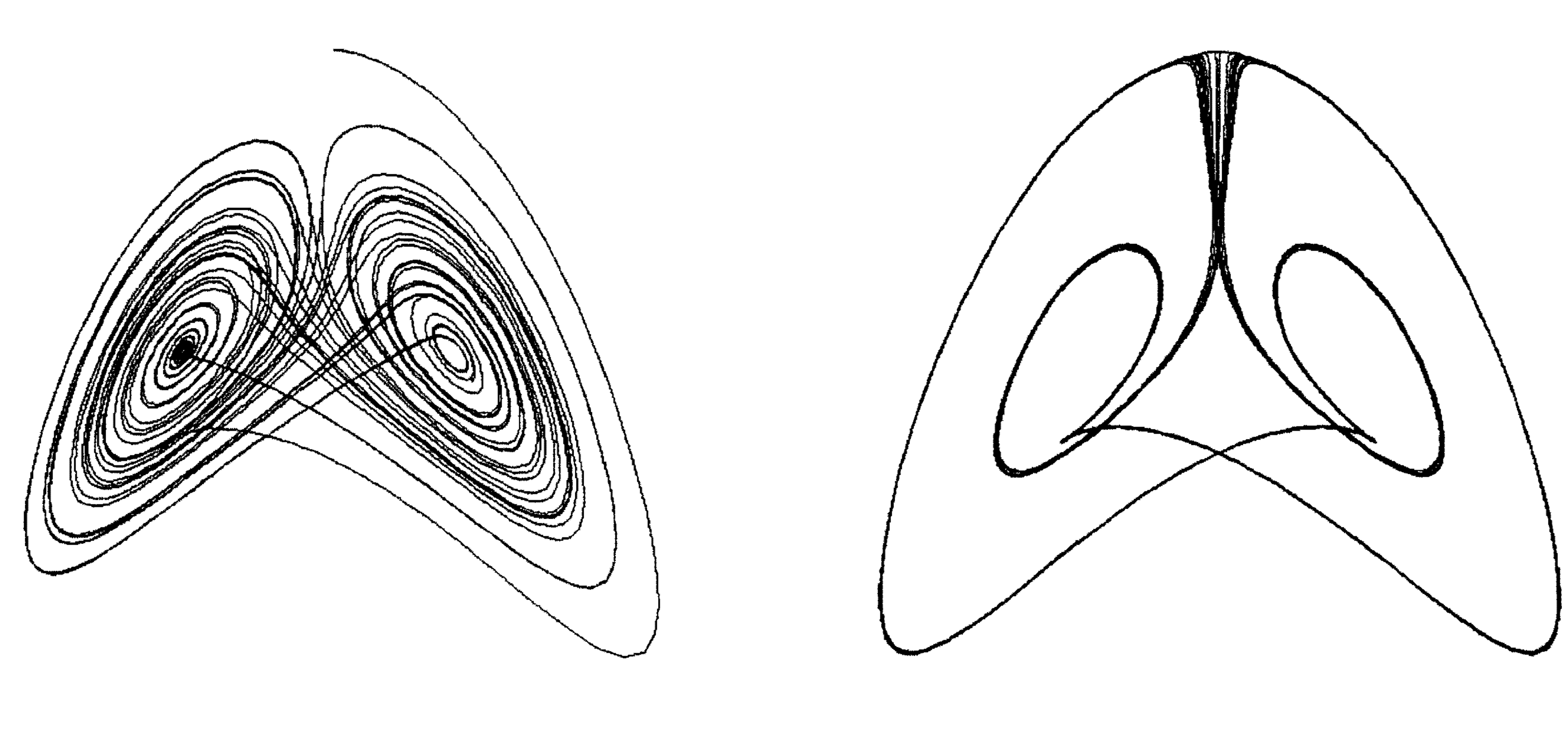

Figure 2. The dissipative example of Lorenz 1963 (left) and a modification, e.g., Palis and Takens 1993 (right).

examples showed that a deterministic system could simulate random behaviour like coin tossing. This horseshoe example isolates the essentials of the homoclinic 'webs' to be discussed below in §3.2.

In this context we consider the existence of so-called chaotic attractors in systems described by differential equations. Interestingly, this existence has not yet been proven mathematically for any system given in terms of explicit equations (without parameters). Nonetheless the geometric examples prove that there must be differential equations, say of polynomial form, which do exhibit this chaotic behaviour. See figure 2.

A similar remark holds for conservative systems used for modelling the dynamics in the world of frictionless mechanics. Here the chaos-question is whether regions of positive measure in the state space are densely filled by single orbits, another fact which is strongly suggested by computer simulations. See figure 1. The mathematical affirmation of this, even for simple systems, is open for at least 30 years and there seems to be no hope in the near future.

These geometric examples are typically related to the second class of questions mentioned before: What kind of dynamic behaviour can one expect in typical deterministic dynamical systems?

3. New concepts

The investigations of nonlinear dynamical systems opened our eyes to new notions, relevant to the description of the different types of dynamic behaviour. We mention the most important ones.

3.1. Chaos

With chaos, or chaotic dynamics, a type of deterministic dynamics is meant which looks like random. This phenomenon is displayed by systems where the evolution, following a typical initial state, is very sensitive to perturbations of this state. In fact, usually such perturbations grow exponentially. The behaviour of these systems can only be predicted over a short period: after this the uncertainty concerning the initial state, possibly combined with the round-off errors in the calculations, make further prediction impossible. A well-known example is the impossibility to predict the weather over a period longer than typically a couple of weeks.

3.2. Fractals

Fractals are self-similar objects. This means that any magnification of a fractal shows the same large-scale structures as present originally. Such structures can show up as attracting sets, i.e., sets to which typical evolutions are attracted in the case of chaos. See figure 3.

This and related occurrences of fractals in dynamical systems have drawn

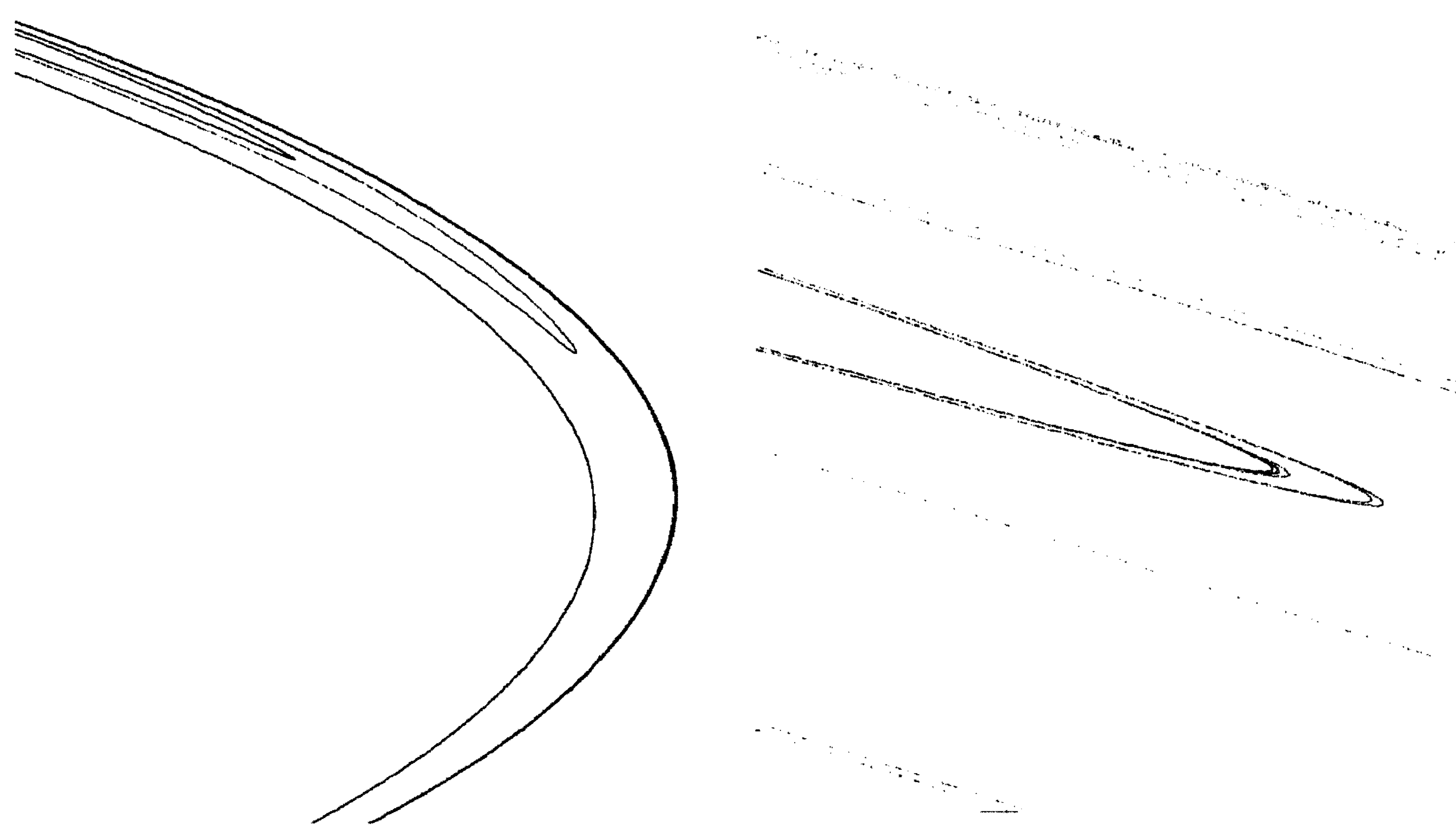

Figure 3. Magnifications in the Hénon attractor.

a lot of attention, not in the least because of the beautiful pictures that can be provided in this way. Especially the group of H. Peitgen (University of Bremen) has been very successful in this respect.

Still it is only fair to say that the relation between chaotic dynamics and fractals is rather complicated. In particular the often heard suggestion that every chaotic attractor also is fractal, is not true. Of course, it is neither true that every fractal is a chaotic attractor ...

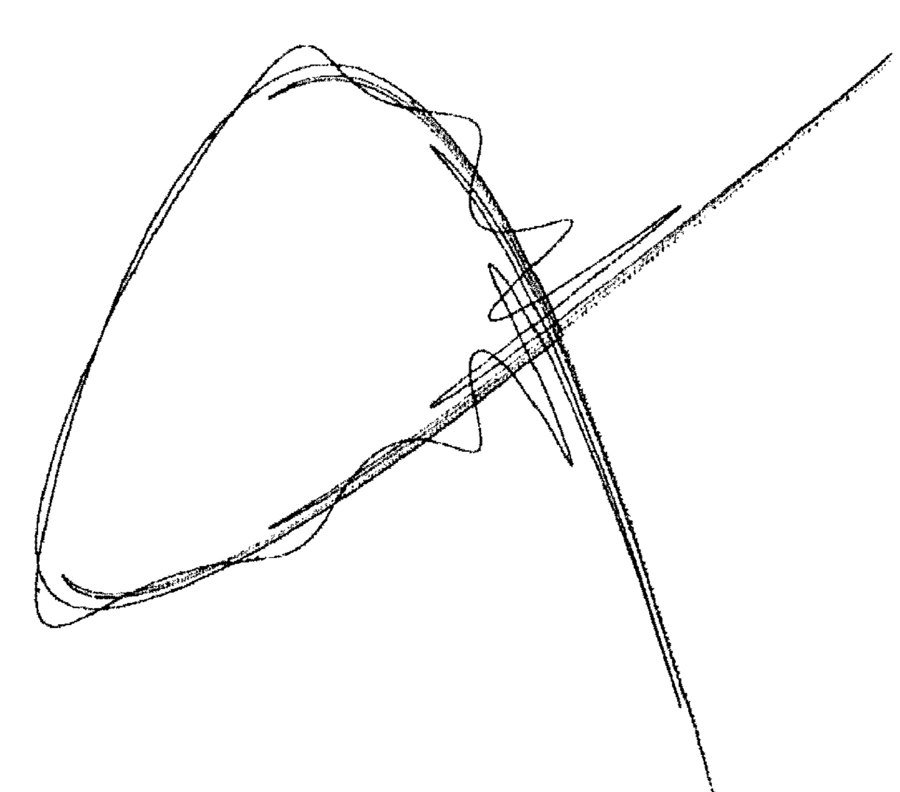

Figure 4. A homoclinic web.

One way in which fractal structures appear in dynamical systems is by the 'web' of stable and unstable separatrices in the presence of a homoclinic intersection. Let us briefly explain this. The stable separatrix of a state p, or point in the state space, consists of all points approaching p as time goes on. Reversing time gives the analogous notion of the unstable separatrix. Intersections of two such separatrices belonging to the same point p is a *homoclinic* intersection. In figure 4 we show such a web for a dynamical system with two degrees of freedom.

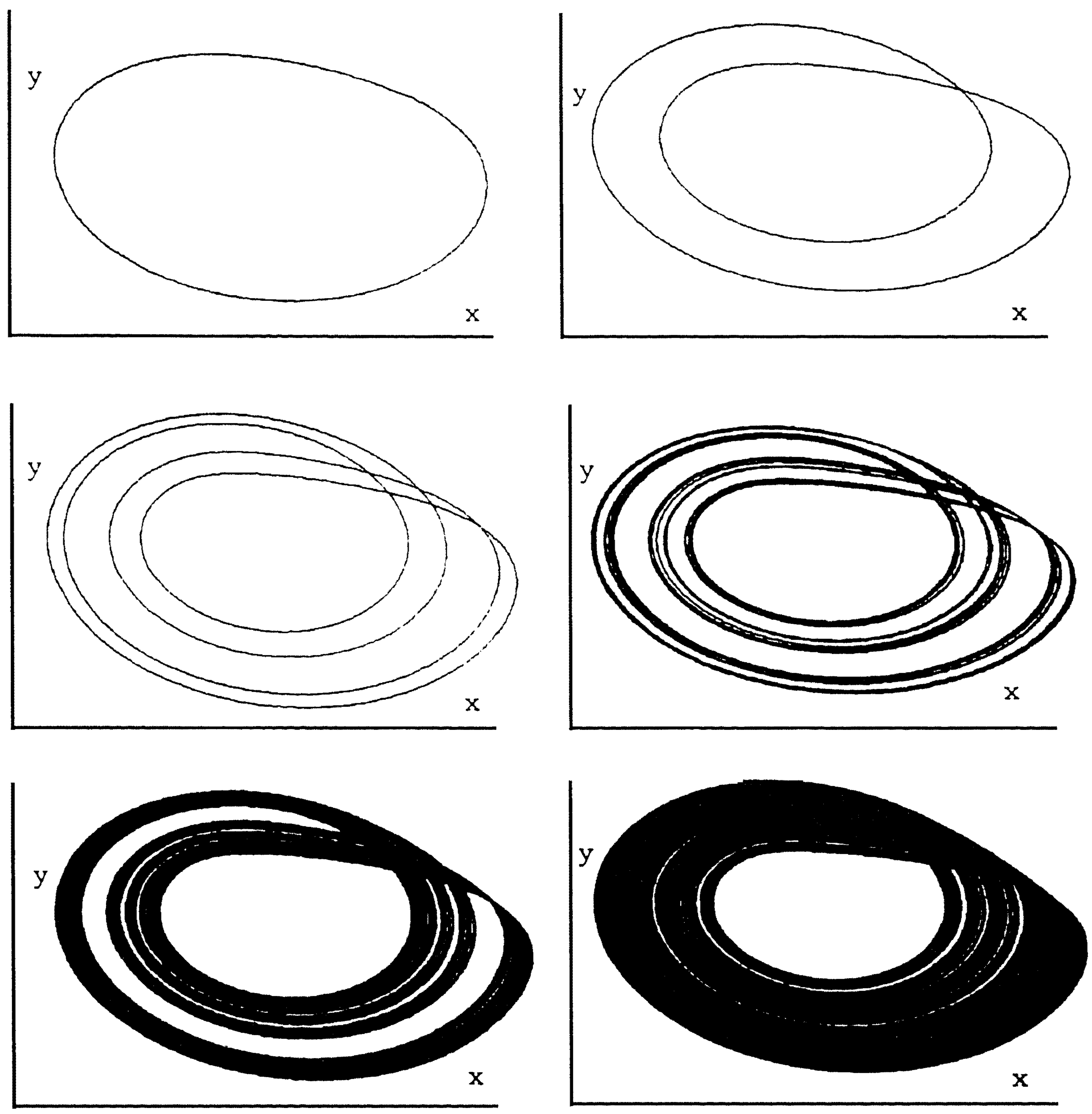

Figure 5. A sequence of bifurcations leading to complicated dynamics.

These webs (or parts of them) are strongly related to chaos in both the dissipative and the conservative context, but only in exceptional cases precise mathematical results on this have been obtained, cf. Benedicks and Carleson [4], and Palis and Takens [19].

3.3. Bifurcations

A bifurcation is a transition between different dynamical regimes. For example think of a dynamical system, the dynamics of which can be changed by tuning one or more dials. Although this may somewhat look like 'input variables' in control theory, the situation is quite different: the dials are to be fixed when the evolution law is working. It turns out that the occurrence of certain bifurcations implies the presence of other bifurcations. This imposes a complicated hierarchy on the world of bifurcations.

Instances of this include infinite bifurcation sequences leading from stationary to chaotic dynamics. One example is the Feigenbaum sequence, where an infinite repetition of period doublings occurs. This and other examples show a strong persistence: if one perturbs the evolution law a bit, the whole infinite succession of bifurcations remains qualitatively the same. See figure 5.

4. A history

In the above exposition we already mentioned some historical aspects. Here we give a somewhat more systematic description.

4.1. From Newton to the 19th century

Through the Newtonian laws (Principia Mathematica Philosophiæ Naturalis, 1687) it became possible to treat many problems of the dynamics of mechanical systems in a mathematical way. The corresponding analysis was given in terms of explicit solutions (e.g., vibrations) or approximations (such as lunar and planetary motion). These approximations were only known to be reliable, as a description of the motion, over a restricted time interval.

4.2. The stability of the solar system

The stability of the solar system has been considered in many different forms. The main point is that information is asked concerning the dynamics of the (Newtonian) solar system, valid for the whole future, so over an infinite interval of time.

The conservation laws for the energy and the (angular) momentum give some information about the infinite future. But even taking these conservation laws into account, the following still is conceivable. Due to the interaction of planets one of these, say the earth, systematically gains energy (which the others are losing), and finally escapes from the solar system. In this sense the solar system could be unstable. This stability problem was posed by Weierstraß, and became part of the problems in 1885 set for a prize by king Oscar of Sweden. Poincaré won this prize, not by establishing stability, but because of the new insights he revealed, showing the complexity of the problem. This was the content of the paper in the Acta of 1890 mentioned before.

This work of Poincaré can be considered the starting point of what is now called the geometric theory of dynamical systems.

4.3. The theory of nonlinear oscillations

In the beginning of this century, due to the growing electronic technology, there was much interest in nonlinear electronic circuits and their oscillations. This was the subject of the fundamental work of Van der Pol (around 1920), which was later continued by Cartwright, Littlewood and Levinson (around 1950). These developments inspired Smale and his co-workers when they extended the geometric approach of Poincaré (around 1965). This extension made it possible to formulate and partly solve the basic questions behind this, that are nowadays associated to the chaotic dynamics in nonlinear oscillations.

4.4. Theory of bifurcations

The theory of bifurcations was also initiated by Poincaré. Here one investigates how the qualitative properties of a dynamical system can change as a function of one or more parameters. Later contributions are due to Andronov and co-workers around 1940, who started a systematic study of the hierarchy (based on the idea of co-dimension) of bifurcations of mechanical systems, and Hopf (1942) who investigated the transition from stationary to oscillatory behaviour inspired by questions about turbulence in the motion of fluids. Later on, R. Thom made this hierarchy of co-dimensions the basis of his general ideas on morphology and catastrophy theory [23].

Afterwards, when the importance of chaotic dynamics was discovered, one of the main questions in bifurcation theory became how transitions to chaos take place in a persistent (or typical) way. The most well-known scenario's are the transition via quasi-periodic motion by Ruelle and Takens [20] and the transition via period doubling due to Feigenbaum [9]. See figure 5.

4.5. KAM-theory

KAM-theory, around 1960 initiated by Kolmogorov, Arnol'd and Moser, deals with the persistent occurrence of quasi-periodicity in dynamical systems. This is a kind of periodicity with more than one frequency involved. Its first interest was in the conservative systems modelling classical mechanics. The context of KAM-theory again is perturbation theory: it deals with nearly integrable systems such as the solar system, see [2,3].

Concerning the stability of the solar system, this conservative KAM-theory guarantees that positive (Liouville) measure in the state-space is swept out by orderly, quasi-periodic orbits. For the stability problem this means that there is positive probability that the 'actual' evolution of the solar system is quasi-periodic, which certainly would imply stability.

In general, however, it is expected that both the quasi-periodic and the chaotic regime have positive measure. See figure 1. This coexistence of order and chaos makes it hard to infer stability from these qualitative considerations for an explicit initial state and King Oscar's question is still open . . .

A related point of interest is that for nearly integrable systems with finitely many degrees of freedom ergodicity does not hold. Ergodicity roughly means that all evolutions in the long run come everywhere in the state space. Since the quasi-periodic orbits yield measure theoretically nontrivial invariant sets, ergodicity does not hold for the systems under consideration. In the case of an infinite number of degrees of freedom often an Ergodic Hypothesis is postulated and it is an unsolved problem to understand the limiting processes involved.

Later on, KAM-theory also became important for 'dissipative' systems depending on external parameters. Here again generally coexistence of quasi-periodic order and chaos holds. As stated earlier, this means that quasi-periodicity can indeed be a transient stage in a sequence of bifurcations from order to chaos. The behaviour of quasi-periodic attractors under variation of parameters was studied by, e.g., Broer, Huitema, Takens and Braaksma [5].

5. Main themes of the programme

We now turn to the NWO-SMC programme 'Mathematical aspects of nonlinear dynamical systems', which was carried out the last couple of years. The research projects involved here easily can be traced back to the history sketched above.

5.1. Resonance phenomena

Let a dynamical system be given, mechanical or otherwise, containing several oscillatory parts that are somehow linked. Then the term 'resonance' refers to an exceptional, though often strong, interaction between two or more of such parts. The simplest interaction involves the equality of frequencies of two oscillatory parts, but also other simple arithmetic relations between frequencies can occur. This is a classical setting for bifurcation theory: small changes of parameters may tune away from resonance, bringing about drastic changes of the dynamics. The programme contains several activities in this area. This research is relevant for technological applications.

Parametric resonance (Broer, Hoveijn (postdoc), Levi (guest from RPI, Troy NY)). A model problem is the following. Consider a pendulum, its point of suspension oscillating vertically. Suppose that we can change the corresponding period at will, so that we may consider it as a parameter. Here a strong form of so-called parametric resonance occurs whenever the period is near the period or near half the period of the pendulum itself. It turns out that for certain values of the system parameters the equilibrium of the pendulum becomes unstable!

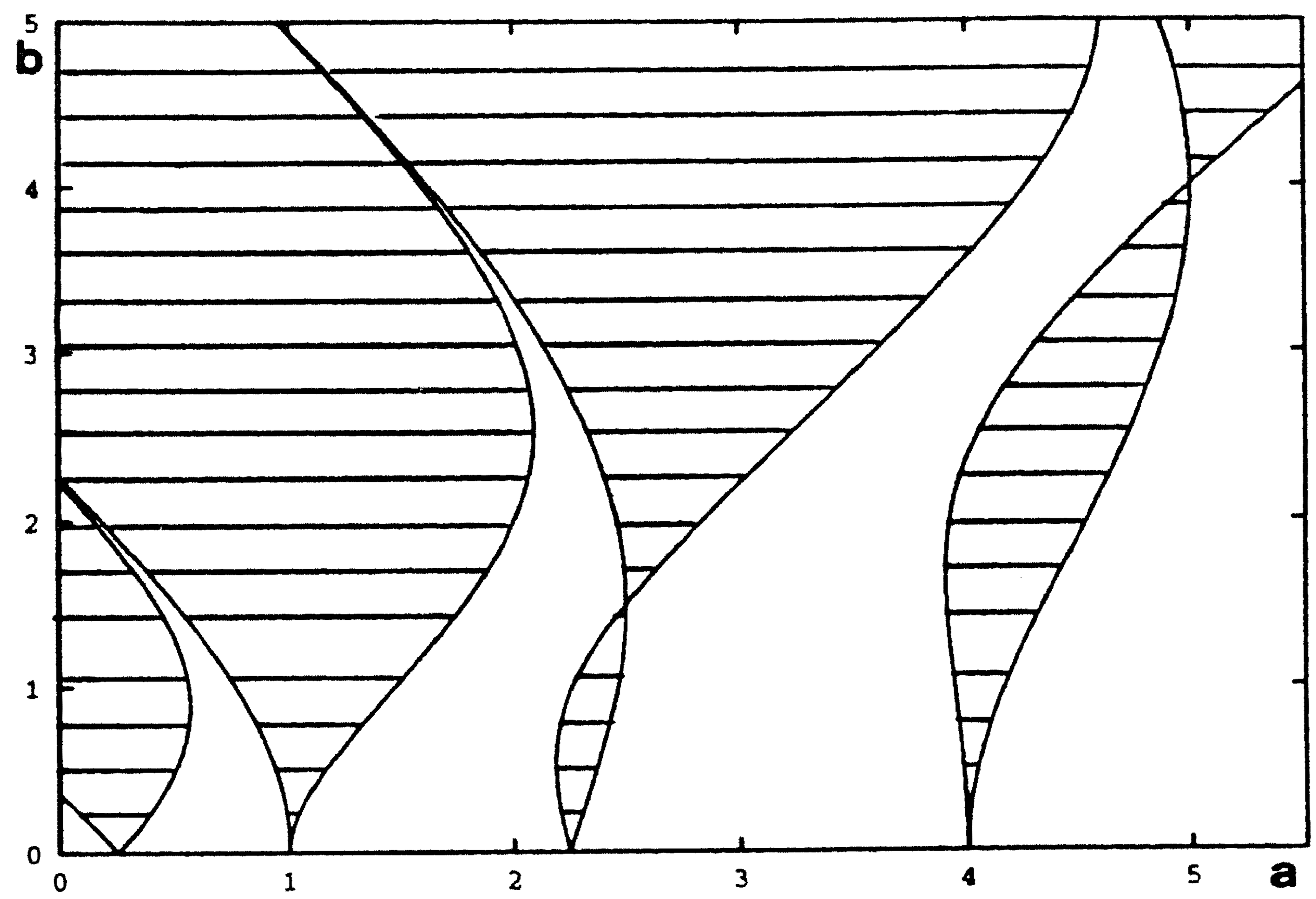

Figure 6. Pockets in the stability diagram of Square Hill's Equation.

One problem is how the dynamical possibilities are organized in parameter space (which here happens to be a plane). Often the regions of instability form tongues, sometimes exhibiting so-called instability-pockets, see figure 6. This is a complicated matter that has been the subject of research since the 1920's. The research of Broer and Levi [6] has contributed to the geometric insight in this.

A variation of this problem occurs when two of such pendulums are coupled by a weak string. If the resonance is such that the sum of the natural frequencies of the pendulums equals the frequency of the forcing, the system is stable only for parameter values in a narrow tongue. The geometric understanding of this phenomenon was enhanced by Hoveijn and Ruijgrok [14].

The fattened Arnol'd family (Broer, Simó & Tatjer (guests from the University of Barcelona), Viana (guest from IMPA, Rio de Janeiro)). V.I. Arnol'd is one of the leading members of the dynamical systems community. For a better understanding of — among other things — resonant dynamics, he has introduced a model system operating on a circle, which by now is well-understood. It seems that extensions of this model to the plane may play a central role in bifurcations to chaos, related to homoclinic points. As more

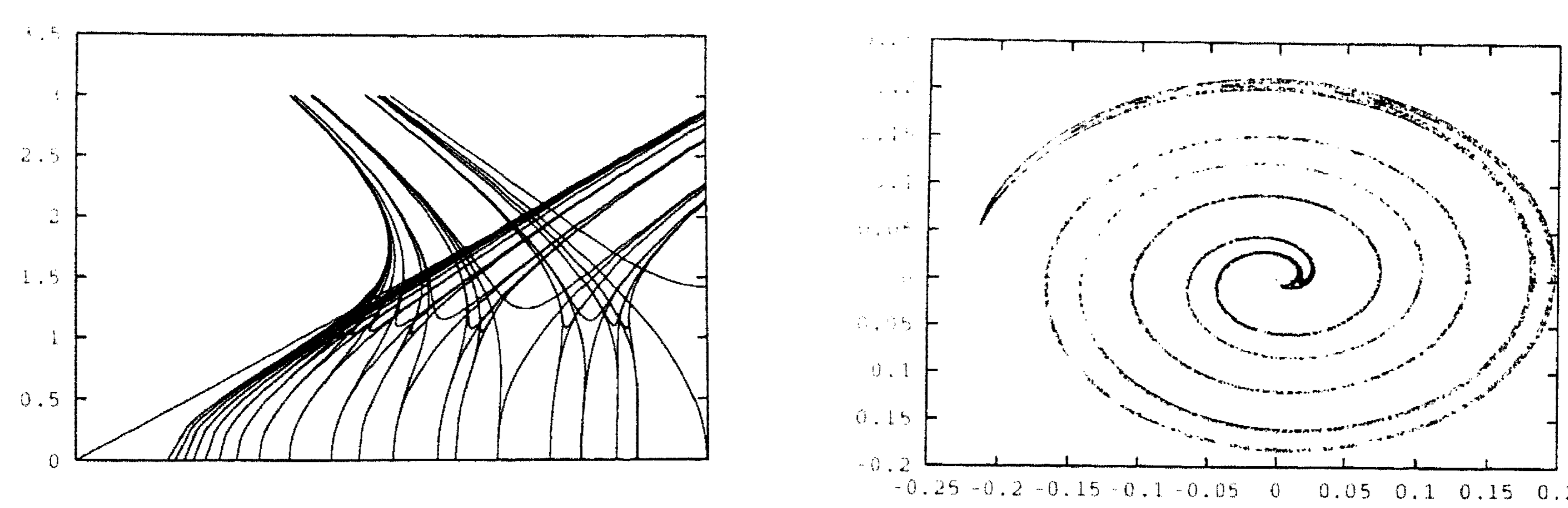

Figure 7. Complexity in the stability diagram (left) and a global Viana-attractor in the fattened Arnol'd family (right).

often, this 'fattened' model itself also turns out to contain a lot of chaos. These phenomena are strongly related to resonance, in particular where the resonance is about to disappear and where several resonance areas start to interact. Viana did theoretical work on this when visiting Groningen. Simó and Tatjer, together with Broer contributed to the understanding of this in a computer assisted way. Among other things, phenomena predicted by Viana's theory were found, see figure 7. Compare [24].

Generic 1:4 resonance (Broer, Krauskopf (Ph.D.-student), Takens). This is a study of all possible dynamical consequences of a loss of stability, associated to a frequency ratio of 1 to 4 (or 3 to 4, which can be reduced to the same problem). A corresponding study was carried out for all other integer ratio's in the 1970's by Arnol'd, Bogdanov, Carr, Khorozov and Takens, but the present case, which is much more complicated than the others, is still not completely solved. There is a conjecture by Arnol'd [3] which, when true, gives a complete solution of this case. In the present project a combination of analytical and numerical methods were used to study and visualize the consequences of the conjecture in terms of the bifurcation structure in a three-dimensional parameter space; see figures 8 and 9. In combination with the study of a certain singularity that acts as an organizing centre, this is convincing evidence in favour of the Arnol'd conjecture; see Krauskopf [15,16].

Resonance in adiabatically forced Hamiltonian systems (Huveneers (Ph.D.-student), Verhulst). Here we have a resonance problem in the context of conservative or frictionless mechanics with two parameters, one detuning the resonance and one related with the average energy. It turns out that here one can successfully transform to the quantummechanical formalism. Although

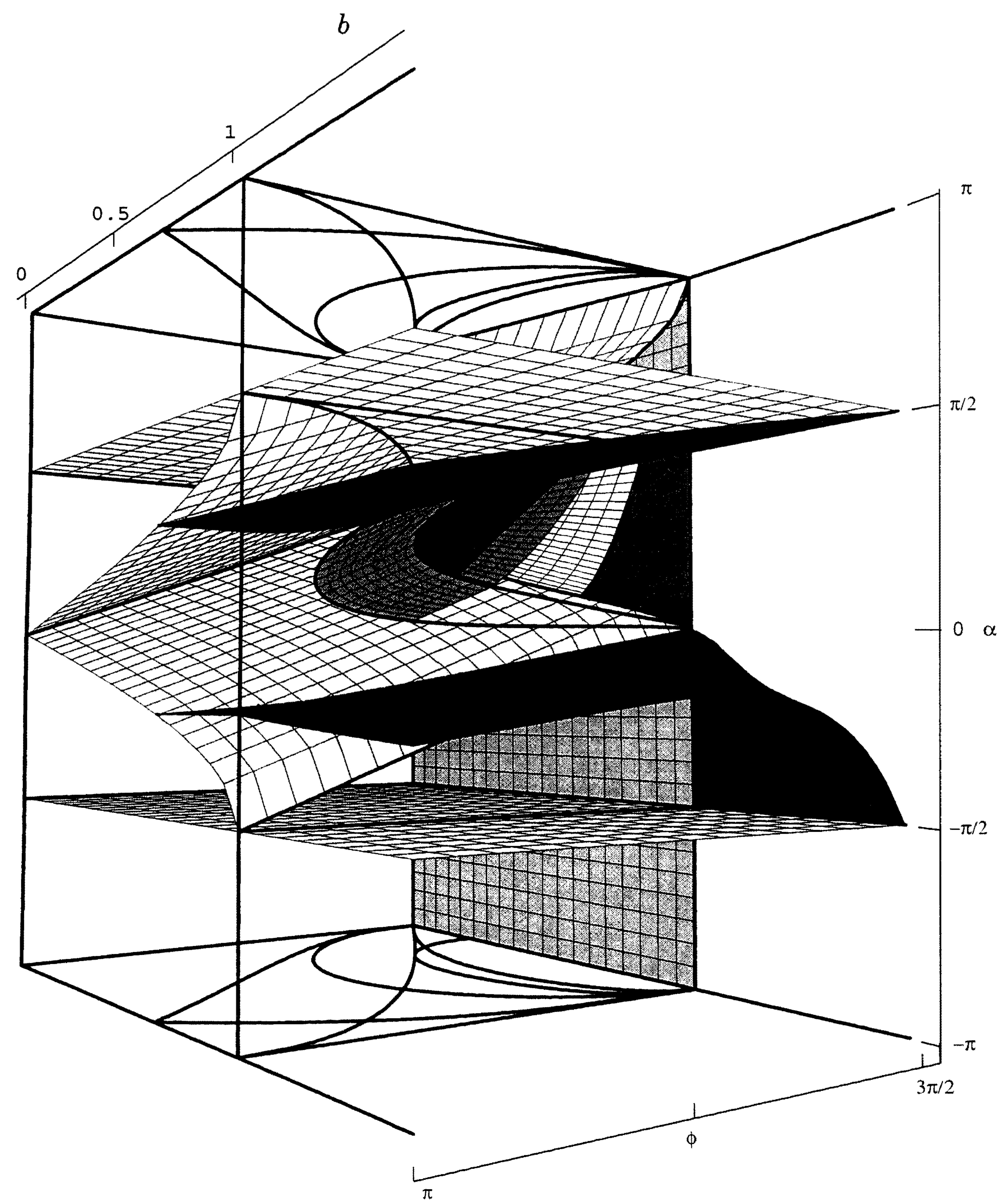

Figure 8. Bifurcation structure in 3D parameter space related to the 1:4 resonance.

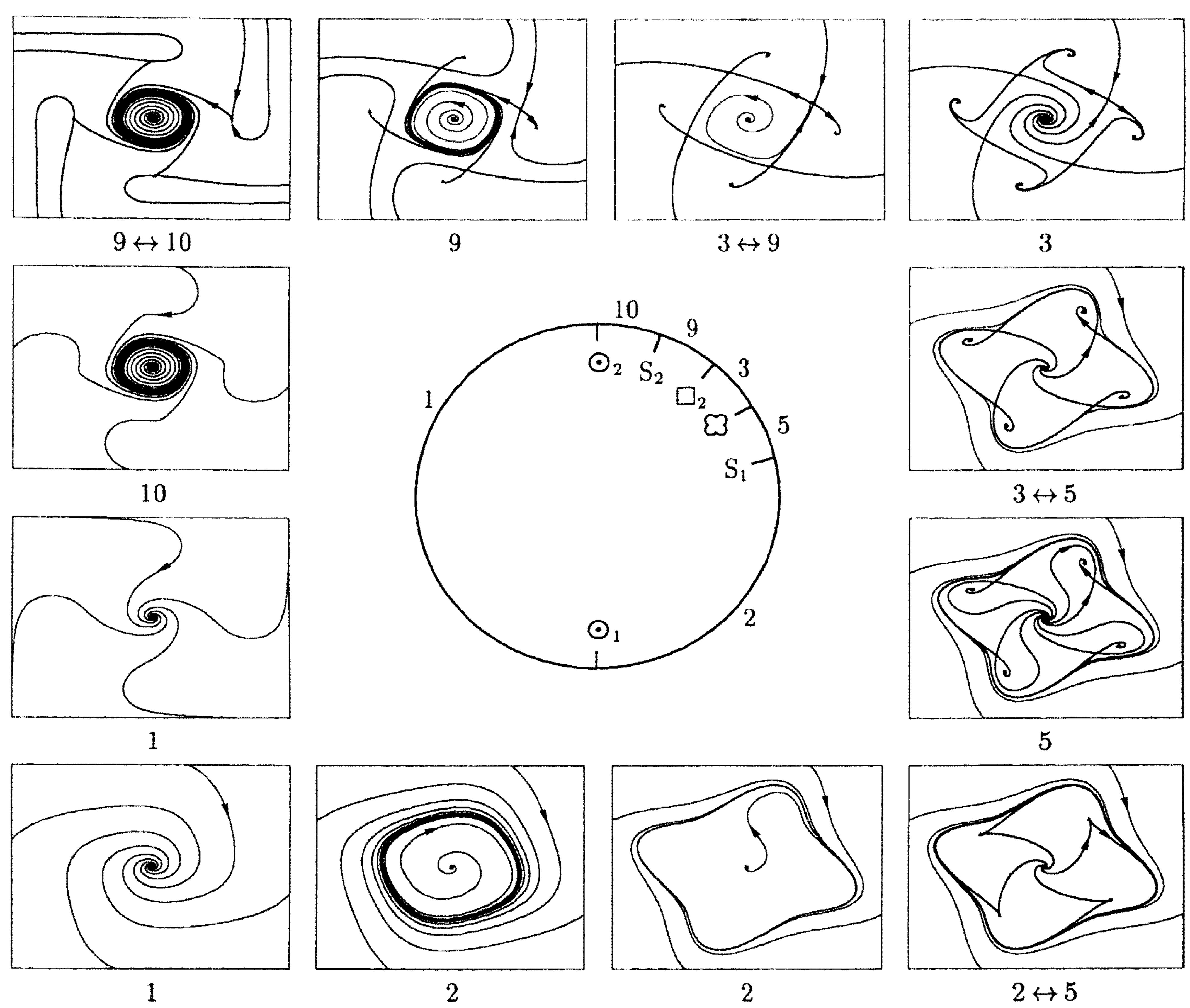

Figure 9. A sequence of phase portraits near 1:4 resonance.

this change of formalism changes the context from nonlinear and finite-dimensional to both linear and infinite-dimensional, the bifurcations can still be interpreted. The bifurcations are analysed in terms of the invariant subspaces of the associated infinite-dimensional Hilbert space.

5.2. Symmetry

Many natural systems exhibit some form of symmetry, which then determines the bifurcations, i.e., the drastic changes in the behaviour one can typically expect. Another reason for studying symmetric systems is that they are often integrable and hence can be used as a first step to study (approximate) more complicated systems. This holds in particular for the symmetric systems arising from truncation of higher order terms, in combination with normal form procedures.

Coupled Josephson junctions (Van Gils, Krupa, Tchistiakov (Ph.D.-student)). This project is concerned with the dynamical properties of a number of identical Josephson junctions, each two of them coupled in the same way. A Josephson junction is a gadget from the theory of superconductivity, modelled by a pendulum with oscillating point of suspension. This present system is invariant under any permutation of the Josephson junctions. The main question, which is related with the applications of these systems, is under which conditions the different junctions operate in a synchronous way. The investigations, which centre around a homoclinic bifurcation, are carried out both by analytical and numerical means.

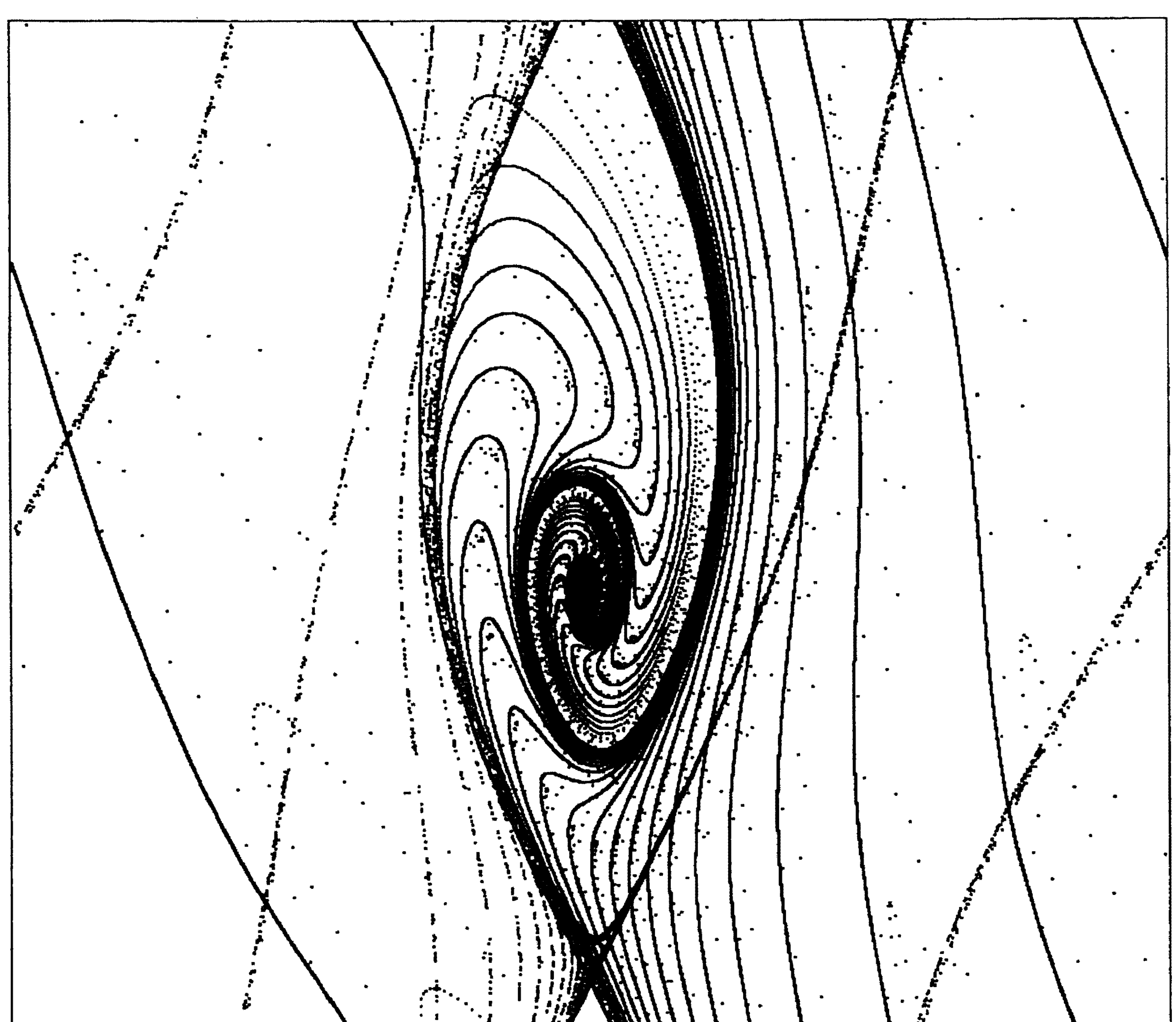

Figure 10. Chaos in the skew Hopf bifurcation.

Skew Hopf bifurcation (Broer, Takens, Wagener (Ph.D.-student)). This project started with the investigation of a transition to chaos in the presence of symmetry, which showed the possibility of mixed spectra for chaotic systems, see Broer and Takens [7]. Mixed spectra were observed in experiments, but no persistent mathematical example was known. The present investigations are concerned with the question what happens to this transition if the symmetry gets (slightly) broken. This needs a generalization of the KAM-theory of quasi-periodic motions. On the other hand, computer simulations indicate that new types of attractors are formed when the symmetry is broken. See figure 10.

Resonance and Symmetry (Hoveijn (postdoc)). Resonant systems can be considered as small perturbations of symmetric systems in the sense that their Taylor series have a formal symmetry, which is inherited by truncations. Therefore, in the setting of resonance it is quite natural to consider symmetric systems.

The presence of a symmetry group makes it possible to lower the dimension of the system by considering the dynamics on the orbit space of the group. For Hamiltonian systems one often can even lower the number of degrees of freedom. This approach raises some interesting problems. In general orbit spaces will have singularities. The first problem is to characterize these just from the symmetry group at hand. In many cases these singularities turn out to be rational, which facilitates this task. The second problem is to determine the global structure of the orbit space. This question is harder, involving real algebraic geometry. Apart from determining the nature of the orbit space there is the question of defining a dynamical system on a phase space with singularities.

For Hamiltonian systems many results have been obtained by Lerman & Sjamaar [17] and Arms et al. [1] where the symmetry group is a Lie group with linear action. More detailed results for particular resonant systems with two degrees of freedom were already found by Churchill et al. [8]. For these systems the singular reduction method is very powerful because here there are no global problems and the singularities are simple. Singular reduction for resonant Hamiltonian systems with more than two degrees of freedom is a subject of further research, for partial results see Hoveijn [13].

In systems of this form one often finds an interplay between various structures, such as a symplectic structure and a symmetry group. Research inspired by the coupled pendulums example led to the characterisation of infinitesimally reversible symplectic matrices.

One-dimensional dynamics (Van Strien, Kozlovsky (Ph.D.-student)). The most complete, and profound, mathematical results concern dynamical systems with only one degree of freedom. Although this is a rather restricted class it has applications in biology and in fluid dynamics. The dynamics of these one-dimensional systems can be extremely complicated, but still is mathematically well understood. See De Melo and Van Strien [18]. The aim of this project is to try and extend the one-dimensional results to higher dimensions. In the special case of the Hénon map this already turned out to be possible, compare the fundamental work of Benedicks and Carleson [4].

This approach is strongly related to the analysis of the fattened Arnol'd family mentioned above.

5.3. Numerical tools and visualization

There is now software to analyse bifurcations of explicitely given systems (DS tool, AUTO and LOCBIF), but much needs to be done to integrate the possibilities of these programs, and to combine them with the normal form algorithms. The dynamical systems laboratory (DSL) at CWI has been active in this direction and has supported applications of this software, e.g., for the 1:4 resonance mentioned earlier, but also for investigations in population dynamics. The objects visualized are usually of dimension two, or at most three.

Other problems arise when studying higher-dimensional objects, like invariant manifolds. Both the calculation of these objects and their visualization require new methods.

Development of software and applications to population dynamics at DSL (Sanders, Kuznetsov, Levitin (NWO-visitor), Lisser, Kirkilonis, Hantke (Postgraduate)). One of the main projects in this group is the development of a new program CONTENT combining the advantages of the different now existing programs. There is an intensive cooperation with authors of previous programs, like Doedel (author of AUTO) and Kuznetsov (author of LOCBIF). Another, related theme is the computation of normal forms with Mathematica-based software. Partially these activities were also supported by the NWO priority programme 'Nonlinear Systems'.

From the applications outside mathematics we mention the bifurcation analysis of structured populations.

Computation and vizualization of invariant manifolds (Broer, Osinga (Ph.D. student), Vegter). The problem is to develop programs that numerically compute invariant manifolds, using (normal) hyperbolicity. A first contribution to this, by Homburg, Osinga and Vegter [12], computes the stable and unstable manifolds of a stationary point. The corresponding algorithm is based on the existence proofs of both Perron and Hadamard.

Research is in progress concerning a general normally hyperbolic case, where variations of the graph transform are employed. The corresponding algorithm is very suitable for continuation purposes and fills a gap in the

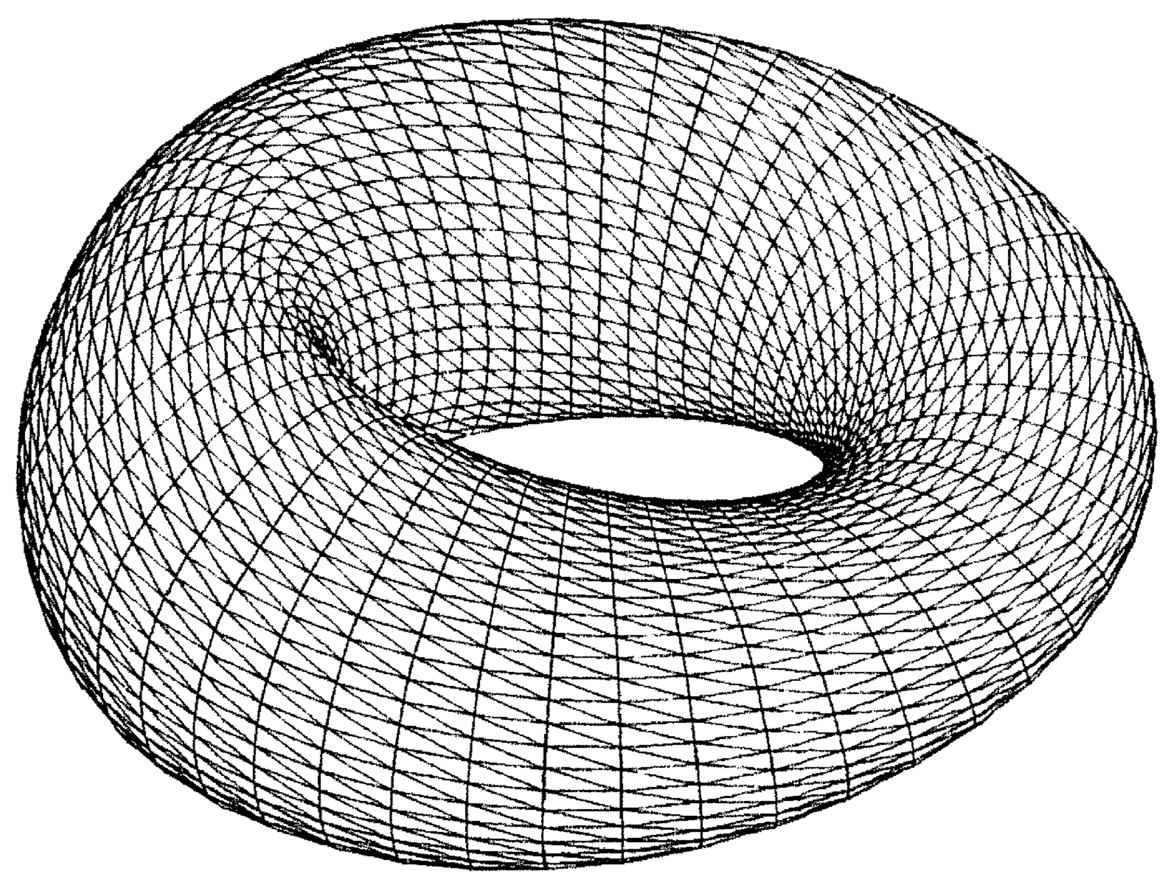

Figure 11. Invariant 2-torus for a fattening of the Thom-automorphism.

existing software. As an example of an invariant 2-torus in a 3-dimensional diffeomorphism, produced by this method, see figure 11.

Since the algorithms described here are based on a theoretical existence proof, analytic error bounds can be obtained while the methods can be used to provide computer assisted proofs of all kinds of dynamical features.

5.4. KAM-theory

Further development of KAM-theory turns out to be of general interest for the research involved. Within the scope of the present programme we mention the above project of the 'skew Hopf bifurcation', where KAM-theory has to be extended to investigate the persistence and bifurcation of certain quasi-periodic attractors. Also similar quasi-periodic bifurcations in conservative systems have to be studied as they occur, for instance, in the rigid body dynamics, see below.

One other project financed by FOM/SMC (via the National Mathematical Physics Community) deals with the Ergodicity Problem mentioned before (Broer, Van Enter, de Jong (Ph.D.-student), Takens, Winnink). Indeed, it considers a concrete infinite-dimensional lattice system as a limit of finite degree of freedom systems, investigating the fate of the quasi-periodic motions in the limiting process.

Again in the context of conservative systems there is a project, financed by Groningen University, dealing with two and three quasi-periodic motions of a rigid body which is a perturbation of the Euler top. (Broer, Cushman (Utrecht University), Hanßmann (Ph.D.-student)). One tool for this problem is normal form theory, yielding an approximate system with a 2-torus symmetry. The reduced (slow) system in two-dimensions can be studied in its own right by singularity theory. This leads to quasi-periodic motions with two and three frequencies (including some bifurcations) in the integrable approximation. After this a KAM perturbation theory has to be carried out. See Hanßmann [10,11].

Finally we mention the manuscript of a book by Broer, Huitema (PTT-research) and Sevryuk (guest from Russian Academy of Sciences, Moscow). This is a survey of KAM-theory in classes of systems determined by the preservation of a given structure. Examples are given by the classes of conservative or dissipative systems mentioned before. Another example consists of reversible systems related to a given involution. The involution takes evolutions to evolutions, reversing the time-parametrization. Especially the minimal amount of parameters needed for persistence of quasi-periodic motions is of interest.

5.5. Methods and applications of nonlinear time series analysis

This area, which was not included in the present programme, but in the NWO priority programme 'Nonlinear Systems', is based on concepts from the theory of nonlinear dynamical systems. The idea is the following: chaotic systems behave like random systems, but is it possible to distinguish the two just by observing the dynamics? The answer turns out to be positive, but it requires new methods of time series analysis. Compare Takens [21,22]. These methods are also relevant for systems that are not completely deterministic. At this moment there is a project analysing the statistical aspects of these new methods (Borovkova (Ph.D.-student), Dehling, Takens) and there are two experimental groups applying these methods: the group of Van den Bleek et al. at Delft University of Technology applying this to the problems of design and operation of fluid bed reactors, and the group of De Goede et. al. at Leiden University applying this to physiological time series (EEG and ECG).

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