The Moduli Project, 1981-1988

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1. Some early history

1.1. Introduction

The term 'moduli' was introduced by B. Riemann in 1857:

> '... *es hängt also eine Klasse von Systemen gleichverzweigter* $2p+1$*-fach zusammenhangender Funktionen und die zu ihr gehörende Klassen algebraischer Gleichungen von* $3p-3$ *stetig veränderlichen Grössen ab, welche die Moduln dieser Klasse genannt werden sollen.*'
> B. Riemann—*Theorie der Abel'schen Funktionen*, Journ. reine angew. Math. (Crelle), 54 (1857), pp. 115-155, (see p. 134).

In mathematics the word 'moduli' has various meanings. In our context, however, it only occurs in plural and refers to the essential parameters on which certain algebraic structures depend (usually in a continuous way). Used in this sense the term stems from Riemann, who introduced it in his study of Riemann surfaces. He described these as coverings of the (Riemann) sphere and proved that the number of essential parameters equals $3g-3$ for a Riemann surface of genus $g \geq 2$ (equivalently, for an algebraic curve of that genus). The genus g is a discrete invariant which only assumes non-negative integral values, whereas the (complex) $3g-3$ parameters vary continuously. The word 'moduli' indicates the (number of) parameters on which a geometric structure like a Riemann surface depends. Frequently these 'moduli' themselves satisfy algebraic equations and, hence, can be identified with the points of an algebraic variety, called the 'moduli space'.

Elliptic curves form an example. An elliptic curve E is characterized—up to isomorphism over an algebraically closed field—by the invariant $j(E)$ and hence corresponds with a point on the affine line $\mathbb{A}^1$. The set of (isomorphism classes of) elliptic curves corresponds with the affine line. Generalization of this to elliptic curves endowed with an additional structure, such as a point of a certain order on the curve, leads to 'modular curves'. The geometry and number theory of modular curves were extensively studied in the beginning of this century by F. Klein and others.

The notion of elliptic curve can be generalized by considering algebraic curves of higher genus ($g > 1$) and by considering group varieties of higher dimension ($g > 1$). This in turn leads to generalizations of the modular curves mentioned above: the moduli space $\mathcal{M}_g$ of algebraic curves of genus g and the moduli space $\mathcal{A}_g$ of abelian varieties of dimension g (with a polarization). Both generalizations and both types of moduli space are in the center of present-day mathematics. However, the study of these spaces was (and is) far from easy and the theory was developed only with great difficulty. Here O. Teichmüller's work on the moduli of algebraic curves was important.

1.2. History in a nutshell

A full historic overview of the period 1860-1960 would take far more space than is available here. Apart from some intermediate results, in 1960—remarkably enough—still no satisfactory algebraic-geometric theory of moduli spaces was developed. The reason is that it is not only important to know that the parameters or moduli satisfy certain equations, but as much that these equations are universal. However, tackling that problem—and even formulating it properly—required the revolutionary conceptual apparatus, called the theory of *schemes*, introduced by A. Grothendieck in the 1960's in algebraic geometry.

His fundamental work, building on results by A. Weil, O. Zariski, J.-P. Serre and many others, allows a unified treatment of complex geometry and number theory: in short, algebraic geometry in all its aspects. Grothendieck introduces the concept of 'representable functor' and shows that the fundamental problem of moduli is to determine whether certain functors are representable. From this viewpoint all properties of moduli spaces acquire a 'modular' interpretation, which enables in principle a far better understanding of this class of varieties than of other ones.

Characteristic for this transition period is J.-I. Igusa's work [2] on the moduli (over $\mathbb{Z}$) of genus 2 curves and the way it was received in the 'French' world of mathematics. P. Samuel starts his Séminaire Bourbaki lecture as follows:

'*Signalons aussitôt que le travail d'Igusa ne résoud pas, pour les courbes de genre 2, le 'problème des modules' tel qu'il a été posé par Grothendieck à diverses reprises dans ce Séminaire.*' P. Samuel—*Invariants arithmétiques des courbes de genre 2.*' Sém. Bourbaki 14 (1961/62), Exp. 226, Décembre 1961.

D. Mumford takes up Grothendieck's challenge in trying to construct the moduli spaces of curves and abelian varieties (in the sense of Grothendieck) [1]. For many years he is the innovator and great stimulator for the development of the basics of moduli and succeeds in drawing many researchers into this field. It is fair to say that Mumford learned us, among other things, to handle Grothendieck's new, formidable conceptual apparatus.

Following this fundamental work, including the compactification of $\mathcal{M}_g$ jointly with P. Deligne, harvesting started with the theorem of Harris and Mumford (1982), stating that for g sufficiently large the moduli spaces $\mathcal{M}_g$ are 'of general type'. It was the first major success. The concept of moduli spaces has spread ever since over large parts of present-day mathematics and has fully proved its value.

Evidence for this is abundant. E. Witten showed in the mid-1980's that the moduli spaces of curves are of fundamental importance in theoretical physics. Here the Riemann surface appears as a 'dressed-up' version of the Feynman diagram in physics. Interestingly, physical intuition has led to the amazing 'Witten conjectures' concerning the cohomology of the moduli spaces of curves (proved by Kontsevich a few years afterwards). Moduli spaces are also central in the recent proof of Fermat's Last Theorem by A. Wiles.

Present research also extends to other moduli spaces than those of curves and abelian varieties. In particular we are witnessing an explosive growth of research on moduli of vector bundles.

Looking back we must admit that at that time none of us foresaw such a spectacular development and success for the theory of moduli.

1.3. Intercity Seminar

This seminar has its origins in a seminar started around 1958 under the name 'schovenclub'. This owed its existence to the inspiring personality of N.H. Kuiper who felt that Dutch mathematicians should become acquainted with the concept of 'sheaf'. Thus a platform for cooperation was established that has shown a great vitality up to the present day. The seminar's permanent aim was the discussion of new developments, usually in geometry and algebra, and later including number theory. During the period 1958-1981 there was no fixed structure and subjects varied. For example, starting in the late 1970's E.J.N. Looijenga, C.A.M. Peters and J.H.M. Steenbrink initiated joint activities in the area of complex geometry and singularities, and at the instigation of F. Oort the arithmetic aspect of geometry was also emphasized. Often new and unpublished results were presented, which enabled young researchers to familiarize with new aspects, sometimes long before publication in international journals. This included difficult and deep results, which were not easy to master.

In 1980-1981 the Intercity Seminar focused on modular curves. This proved to be a nursery for new talent and a source of intensive cooperation. Around the same time the Dutch research structure in mathematics was enhanced by the creation by NWO of 'Landelijke Werkgemeenschappen', managed by SMC. Within this framework two projects: 'Moduli' and 'Singularities', were initiated. These projects have guided for many years—jointly and alternately—the activities of what was called the 'Intercity Seminar'.

2. The Moduli Project

The project was jointly proposed by G.B.M. van der Geer, F. Oort and C.A.M. Peters; H.W. Lenstra Jr. and J.P. Murre acted as advisors. Research results included three Ph.D. theses by L.N.M. van Geemen (1985; cum laude), C.F. Faber (1988) and J. Top (1989). The project's applicants, all experts in algebraic geometry, differed in education and interests, which turned out to be quite an advantage. Researchers in the project included, apart from the three project leaders, several graduate students and senior researchers. Collaboration formed one of the most fascinating aspects for each of us. An ongoing avalanche of new results in the field—a pleasant surprise—was gratefully exploited and frequently led to the set-up of new research.

Over the years the project gave us the opportunity to invite several mathematicians for short visits as well as for longer stays. These stimulating visits have led to a broad spectrum of research, useful developments and interesting publications.

Characteristic for the Moduli Project was that it naturally emerged from an existing and well-functioning collaboration between the members of a small, enthousiastic group of mathematicians. The initiators' foremost concern was to stimulate the field of algebraic geometry with NWO-support. Inviting eminent mathematicians from abroad was considered as important as appointing promising graduate students, certainly in a time of ever shrinking academic budgets (when, indeed, will that stop?).

In the sections below we briefly describe the work of the three Ph.D. theses completed in the framework of the Moduli Project. This work was, in the focus of international developments at the time, of high quality, and in all cases proved to be a stepping-stone to further research, presently in full swing.

3. THE SCHOTTKY PROBLEM

The period mapping (Torelli mapping)

$$j : \mathcal{M}_g \to \mathcal{A}_{g,1}$$

assigns to an (isomorphism class of an) algebraic curve C its principally polarized Jacobian $(\mathrm{Jac}(C), \Theta_c)$. This mapping is injective at geometric points, as Torelli proved in 1914 (over the complex numbers). The closure of the image of this mapping

$$(j(\mathcal{M}_g))^c =: \mathcal{J}_g \subset \mathcal{A}_{g,1}$$

is usually called the Torelli locus or Jacobi locus. For $g \geq 4$ this yields a lower-dimensional subvariety in $\mathcal{A}_{g,1}$. Riemann had asked for a characterization of this subvariety of the 'periods'. F. Schottky, in 1888 for $g = 4$, and F. Schottky and H. Jung in 1909 for general g, indicated relations which were expected to characterize the Jacobi locus. When Van Geemen [3] started to work on this classical problem, only partial results were known. His main result states that *the Jacobi locus is a component of the Schottky locus for all* g. His proof contains an induction on g, the most important idea being to intersect the Schottky locus with the boundary of the moduli space, in a blown-up compactification of the moduli space $\mathcal{A}_g$. Modestly, Van Geemen notices that this idea was already present implicitly in work by Schottky and by F. Frobenius in 1888 and 1889, but he deserves credit for being the first to understand the argumentation and to carry out the proof. His work links to very different methods developed for the same problem by G.E. Welters, E. Arbarello and C. De Concini, as well as by T. Shiota in connection with the Novikov conjecture (stating that the Jacobi locus is described by solutions of the Kadomtsev-Petviashvili differential equations).

These elegant and fine results on a classical problem are indicative for Van Geemen's insight in geometry which he combined with an extensive arsenal of techniques in algebraic geometry. Clearly he vastly profited from his participation in the Moduli Project and—vice versa—the project from his insight and dedication.

4. CHOW RINGS OF MODULI SPACES OF CURVES

Studying varieties frequently involves the use of a cohomology theory. However, the ring of cycles modulo rational equivalence on that variety provides a finer invariant.

Mumford was one of the first to apply this method to moduli spaces of curves [6]. Since the moduli space is usually singular, $\mathbb{Q}$-coefficients have to be used. Moreover, since a regular covering was lacking at that time (it was discovered only much later), even the definition of a Chow ring is not obvious in this case. Mumford founded this theory and he computed the Chow ring for the moduli space of curves of genus 2.

In his Ph.D. thesis [4], Faber studies the Chow rings of a variety of moduli spaces of curves. His main result is the complete computation of the Chow ring $\mathcal{A}^*(\overline{\mathcal{M}_3})$ of a compactification of the moduli space of curves of genus 3. This space has a natural interpretation as the union of subspaces, like the boundary $\overline{\mathcal{M}_3} - \mathcal{M}_3$, the hyperelliptic locus $\mathcal{H}_3$, and the moduli space $(\mathbb{P}^{14} - \Delta)/PGL(3)$ of plane, non-singular curves of degree 4. In this way generators of the various Chow groups of $\overline{\mathcal{M}_3}$ can be given. Now the difficult part starts: which are the relations between the obvious generators? Faber invents a fascinating method and applies it with virtuosity. In order to find relations between classes of cycles of codimension k on this 6-dimensional variety $\overline{\mathcal{M}_3}$, 'test objects' in codimension $6-k$ are selected, and all intersection products between cycles and test objects are calculated, until from that the structure of the Chow ring follows. When shortly afterwards E. Witten proposed his conjectures concerning the intersection numbers, these could be successfully tested with Faber's results for $g = 3$.

5. L-SERIES IN GEOMETRY

Whereas the Ph.D. theses by Van Geemen and Faber treated typically geometric problems, Top's work [5] is closer to number theory, although geometry is frequently drawn upon as motivation or as a tool. L-series are central here. These show up, for example, in connection with cycles on an abelian threefold. Ceresa proved that the cycle $C - C^-$ for a generic curve of genus 3 yields a non-torsion class. Top links this with a conjecture by S. Bloch concerning zero's of an L-series; his work provides evidence for this conjecture.

Top also considers L-series occurring in the theory of Siegel modular forms. In this difficult field understanding of the GL_2 case is dawning, but other cases still seem far beyond our grasp. Top works out examples, illustrating a conjecture by H. Yoshida about Siegel modular forms of weight 2 belonging to $Sp(\mathbb{Z})$—an area now in full swing.

6. WHAT HAPPENED AFTERWARDS

These developments, from the 'schovenclub' through the 'intercity seminar' to Moduli and Singularities , have generated a group of Dutch mathematicians well-versed in algebraic geometry. Abroad our young researchers, in collaboration and at meetings, now easily match world level, as is evidenced by their addresses at important conferences and their usually easy acquisition of good positions (abroad!) in these difficult times. Our research is linked, in depth and in diversity, with the work of the best people in the field, and is published in the leading journals. The breeding ground, to which the Moduli Project belonged, proved to be very valuable.

Collaboration did not stop when the Moduli Project terminated. Arithmetic aspects were elaborated in a project 'Arithmetic Algebraic Geometry', and geometric aspects of moduli theory are addressed in a project studying moduli of curves and of Riemann surfaces. Thus collaboration started in the Moduli Project has borne fruit and scientific methods developed in the project are used and expanded.

We expect moduli spaces to gain significance and to be a focal point in the research of the coming decades. As in all good mathematics, closer study generates more questions than solutions. Assessing the project's impact after ten of twenty years may yield an even more positive picture than at present.

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