Singularity Theory

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1. Introduction

The existence of a 'Dutch Singularity School' was first noticed by N.H. Kuiper at the *Colloque sur la monodromie* in Metz in February 1974. E.J.N. Looijenga, D. Siersma and J.H.M. Steenbrink, who were to defend their Ph.D. theses in Amsterdam that year, were present at that meeting and their work was a topic of the discussion. Five years later, this triple started the ZWO-project *Singularity Theory*, after J. Seidel invited them to set up a common activity of larger scale than usual at ZWO (predecessor of the present National Research Council NWO). Under this flag, W.A.M. Janssen, G.R. Pellikaan, D. van Straten and T. de Jong completed their Ph.D. theses and several others, such as J. Stevens and H.J.M. Sterk, were strongly influenced by its activities.

In this article we will focus on one characteristic aspect of the project: the contributions of Pellikaan, Van Straten and De Jong to the deformation and classification theory of singularities.

2. Singularities

2.1. Introduction

In the context of this article, the subject of singularity theory is the local study of complex analytic sets. Let U be an open subset of $\mathbb{C}^n$ and let $f_1, \ldots, f_k$ be holomorphic functions on U. Then the set of common zeroes of $f_1, \dots, f_k$ is called an analytic subset of U, and every analytic set is a union of such subsets. The interest in such sets arose when people realized that analytic sets can be quite rich from the topological point of view. Consider one holomorphic function f in complex variables z_1, z_2. Suppose that $f(0,0) = 0$. If one of the partial derivatives of f at $(0,0)$ is non-zero, then there exists a holomorphic function g such that (f, g) is a holomorphic coordinate system at $(0,0)$ and, hence, the analytic set $\{f = 0\}$ is similar to a linear subspace at $(0,0)$. However, if $df(0,0) = 0$, then the situation is quite different. The richness of the local topological structure can be seen by intersecting the set $\{f = 0\}$ with a small sphere centered at $(0,0)$; the result is an algebraic knot or link inside the three-dimensional sphere. In higher dimension one can construct exotic spheres in this way.

A *germ of an analytic set* at 0 in $\mathbb{C}^n$ is an equivalence class of zero-sets $\mathcal{V}(f_1, \dots, f_k)$ defined in some open neighbourhood U of $0 \in \mathbb{C}^n$. It is therefore represented as

$$\{(a_1, \dots, a_n) \in U : f_1(a_1, \dots, a_n) = \dots = f_k(a_1, \dots, a_n) = 0\}.$$

Two such sets are called equivalent if their intersections with a sufficiently small neighbourhood of 0 agree.

A more subtle notion is that of *germ of an analytic space*, also called *singularity*. Here one does not consider the zero set alone, but also the functions which define this set. For example, the space defined by the equation $x^2 = 0$ in $\mathbb{C}$ (a 'thick' point of multiplicity two) is considered to be different from a regular point (defined by $x = 0$). Equivalently, a germ of an analytic space can be defined by its 'ring of holomorphic functions', which is a quotient of the convergent power series ring $\mathbb{C}\{x_1, \dots, x_n\}$. An important example is the singular locus of a hypersurface singularity $f(x) = 0$, which is defined as an analytic space germ by the equations $f(x) = \partial_1 f(x) = \dots = \partial_n f(x) = 0$. Any analytic space has an underlying analytic set. Conversely, for any germ of an analytic set there is an associated analytic space, defined by the ideal of all functions vanishing on this analytic set. Such analytic spaces are called *reduced*.

2.2. Some terminology

The simplest case is where all the defining equations can be taken to be linear and vanishing in the origin. In this case the singularity we get we call *smooth* (its singular locus is empty). Of course, in singularity theory this case is hardly interesting. The simplest singularity which is not smooth is the A_1 singularity, given by a non-degenerate quadratic equation, for example $x_1^2 + \dots + x_n^2 = 0$. The case $n = 1$ we considered above (the fat point). Below we give real pictures for the A_1 curve singularity $xy = 0$ (see figure 1), and the A_1 surface singularity given by $xy - z^2 = 0$ (see figure 3).

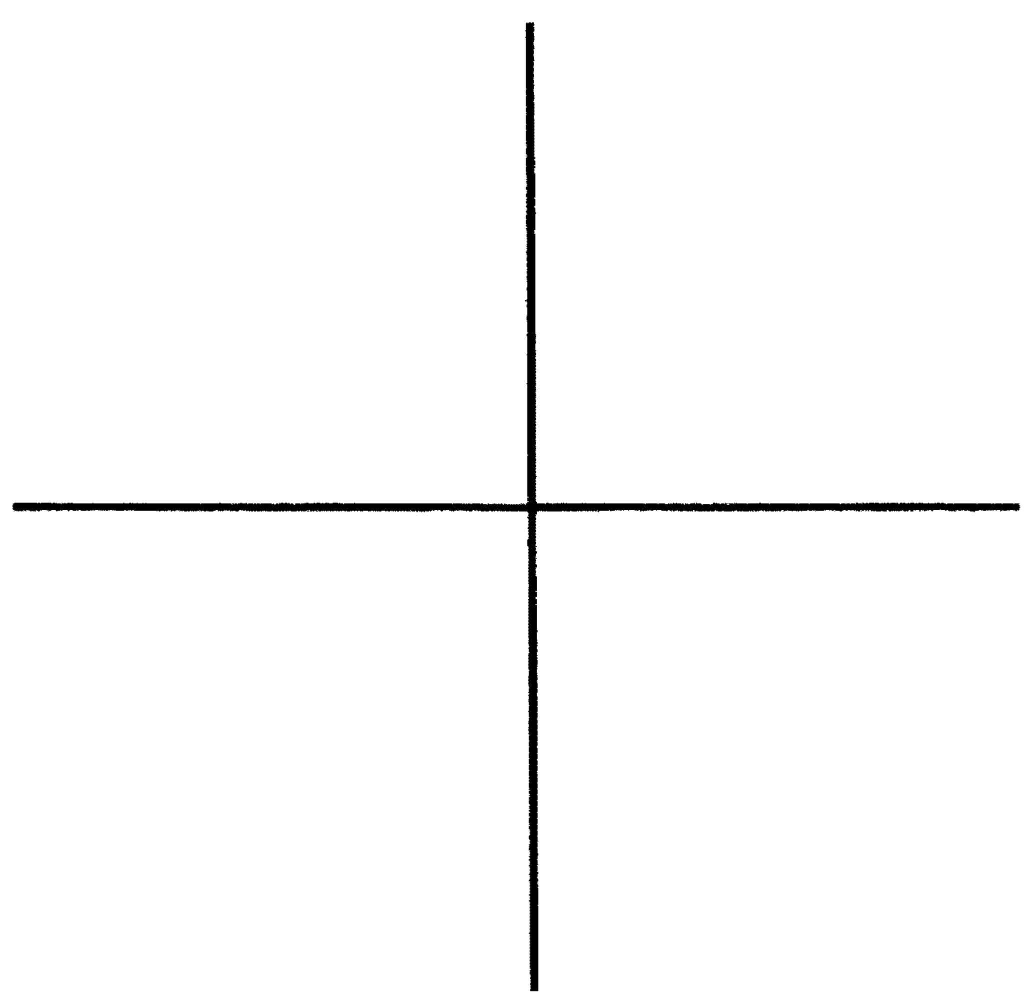

Figure 1.

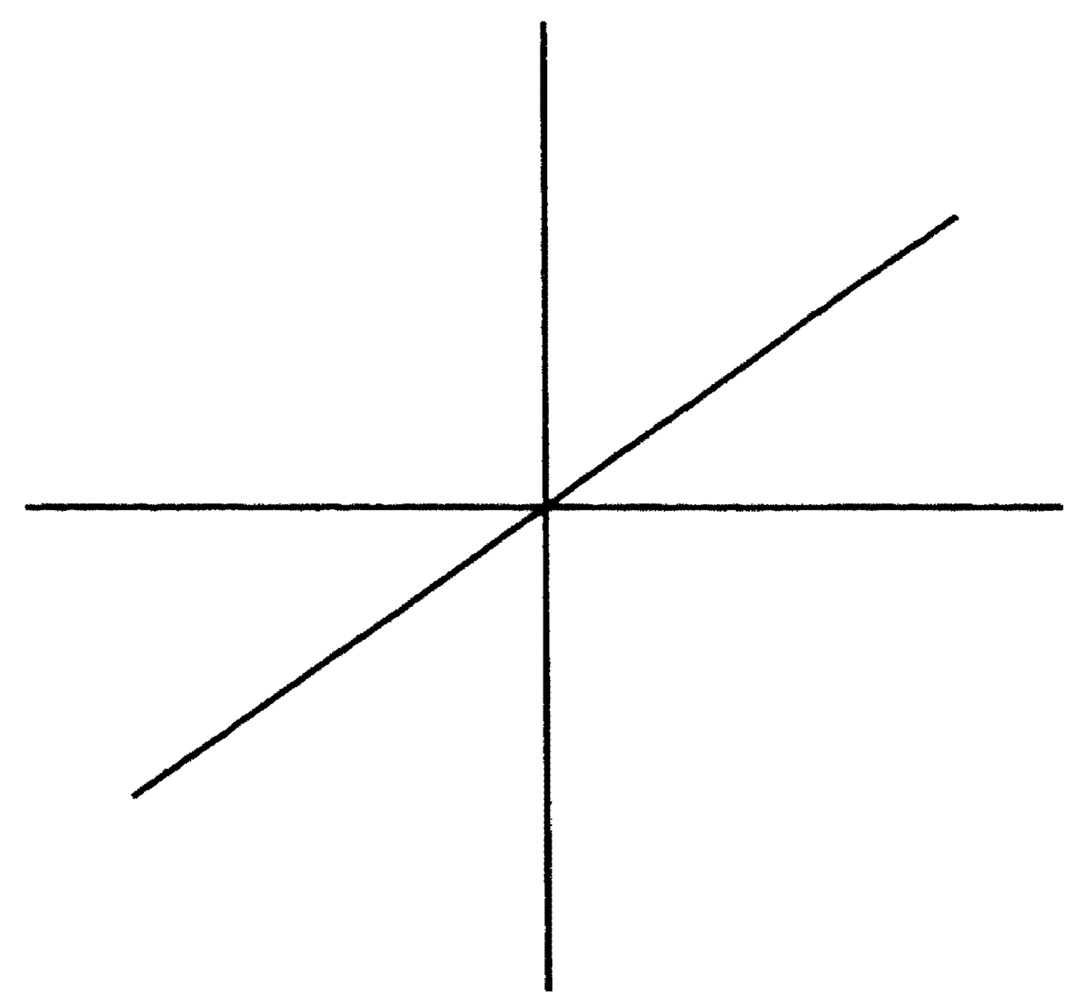

Figure 2.

The above singularities are examples of *hypersurface* singularities, singularities which can be given by one equation. Hypersurface singularities in turn are examples of *complete intersection* singularities: here the number of functions needed to describe the singularity is equal to the *codimension* of the singularity. One of the simplest examples of a singularity which is not a complete intersection is the union of the coordinate-axes in three-space (see figure 2). Here one needs *three* equations: $xy = xz = yz = 0$, whereas the codimension is two.

Even if one is just interested in smooth spaces, it might be interesting,

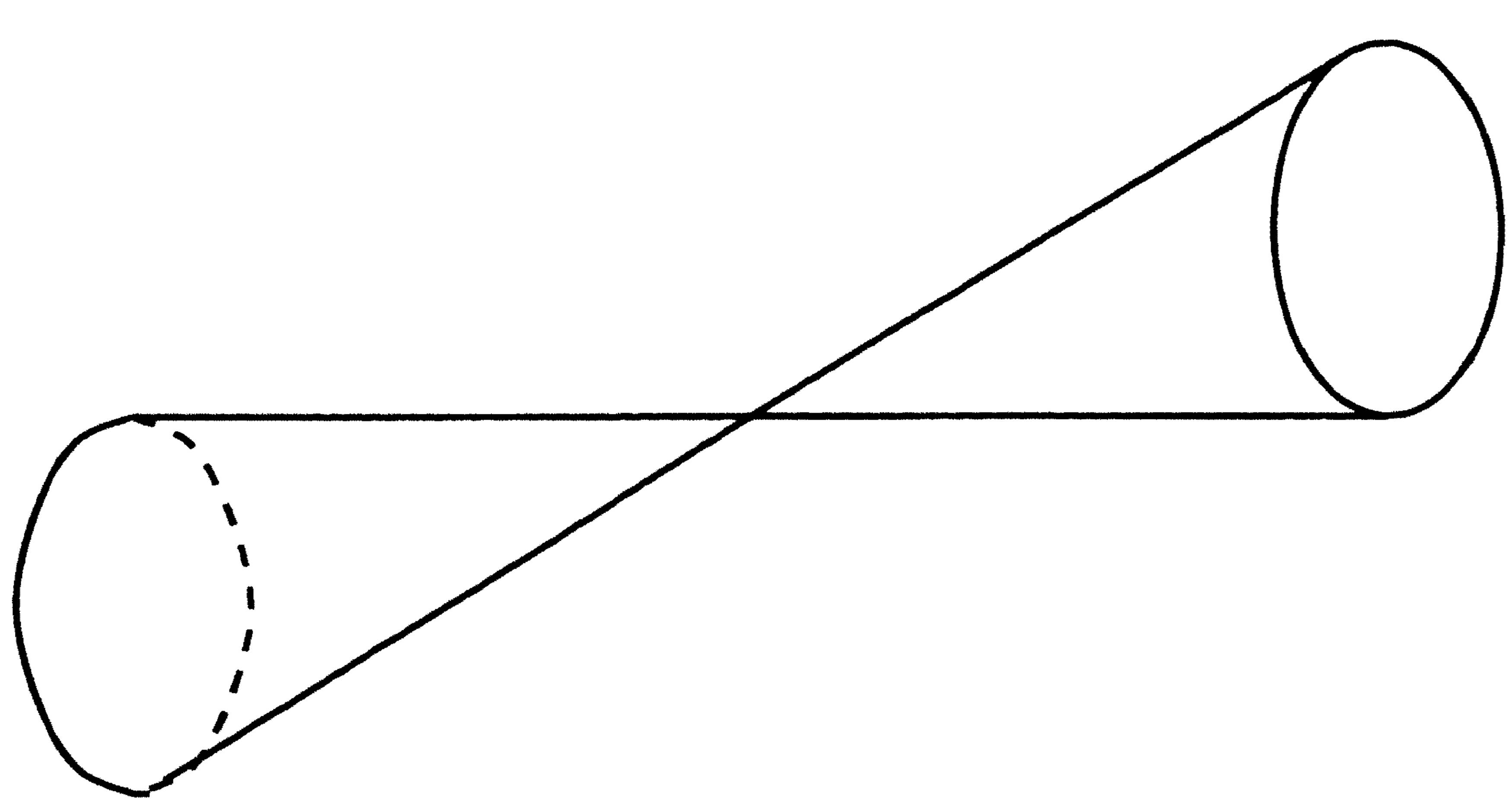

Figure 3.

handy and even 'necessary' to study spaces with singularities. For example, classically smooth curves (Riemann surfaces) were studied by taking a model in the projective plane, which always exists by the theorem of the primitive element. The big advantage of studying plane models is of course that they can be given by just *one* equation. The price one has to pay is that the plane model in general must have singularities. This can be seen for instance by the genus formula. For a *smooth* plane curve with genus g and degree d one has the relation

$$g = \frac{(d-1)(d-2)}{2}$$

from which it follows that a curve of genus two does not have a smooth plane model. Similarly, surfaces were studied by taking a model in $\mathbb{P}^3$. Here one even has to allow non-isolated singularities, i.e., the set of points where the surface is not smooth is a curve itself.

Singularities also occur naturally in the study of so-called *minimal models* of smooth algebraic varieties. Minimal models are known to exist for curves and surfaces for a long time. It was discovered by Mori and Reid, that for a good notion of minimal models for higher dimensional varieties, one has to allow *singularities* on the minimal model. Another interesting motivation is the study of exotic spheres. Exotic spheres are differential manifolds homeomorphic but not diffeomorphic to the standard sphere. Interesting examples of these appear as links of singularities. (The link of an analytic set is the intersection of a suitable representative of this analytic set with a small sphere.) For example, the link of the singularity defined by the equation

$$x_1^5 + x_2^3 + x_3^2 + x_4^2 + x_5^2 = 0$$

is an exotic sphere of dimension seven.

3. Deformations of singularities

One way to study singularities started off with the book of J. Milnor [2]. He considered hypersurface singularities, defined by a holomorphic function f. Take the ball B_ϵ with center 0 and radius ϵ in $\mathbb{C}^n$ and a disc D_η with center 0 and radius η in $\mathbb{C}$, such that $0 < \eta << \epsilon << 1$. One of the main results of Milnor is that the map

$$f : B_\epsilon \cap f^{-1}(D_\eta \backslash \{0\}) \to D_\eta \backslash \{0\}$$

is a C^∞ fibration. The 'general fibre' $f = t$ for $t \in D_\eta \backslash \{0\}$ is called the Milnor fibre. In case we have an *isolated* singularity, the Milnor fibre is homotopy equivalent to a finite wedge of spheres of dimension $n-1$; the number of those spheres is called the Milnor number of the singularity.

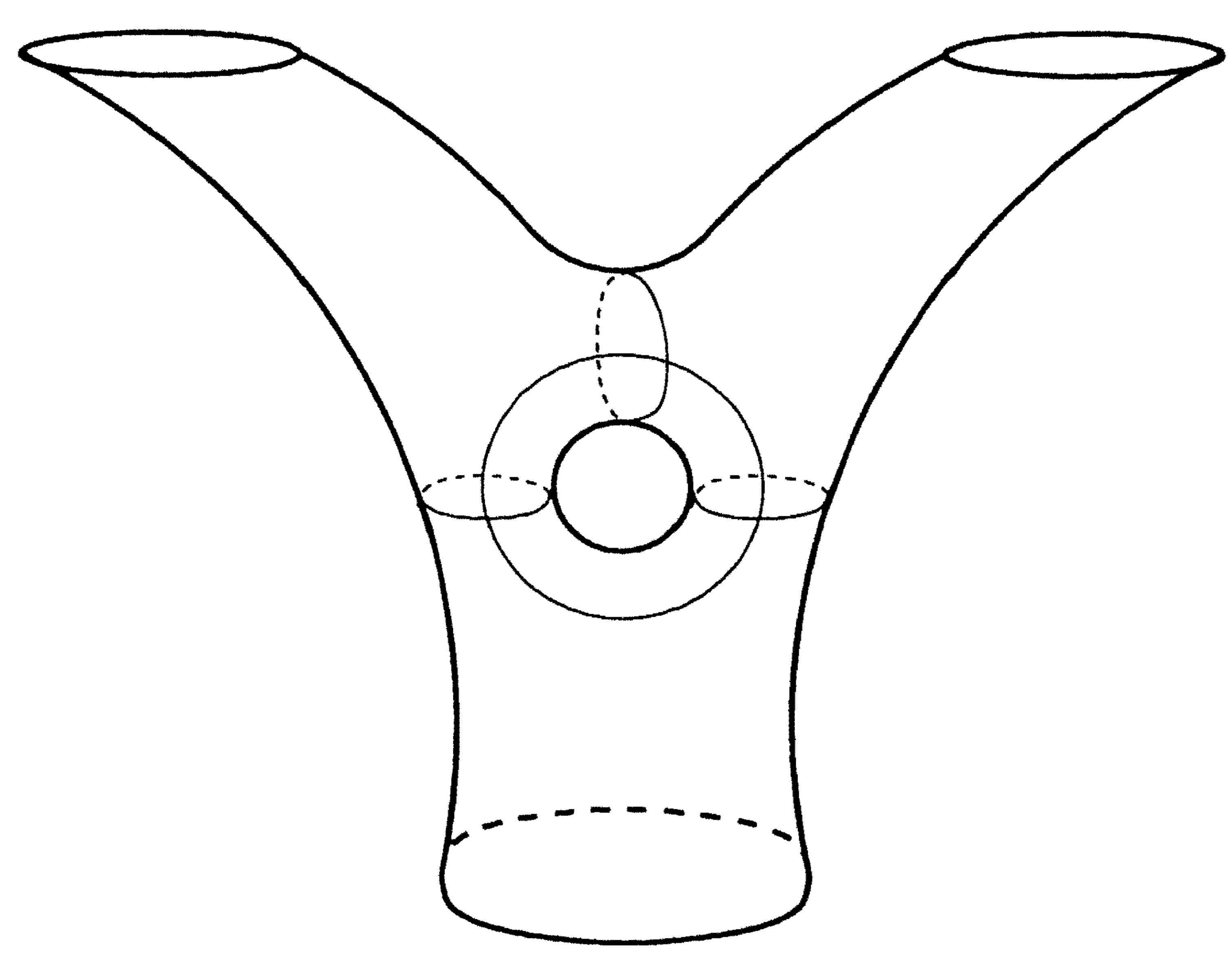

Figure 4.

There is a simple formula for computing the Milnor number μ, as the $\mathbb{C}$-dimension of the algebra $\mathbb{C}\{x_1, \ldots, x_n\}/(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$.

Let us consider the example of the D_4-singularity $x^3 - y^3 = 0$. One can take here even $\epsilon = \infty$ and $\eta = 2$, so the Milnor fibre can be given by $x^3 - y^3 = 1$. Therefore, the Milnor fibre is an elliptic curve with the three points at infinity removed. Topologically the Milnor fibre is as in figure 4.

We see that one can retract the Milnor fibre on the four drawn circles. If one now makes a graph with vertices corresponding to these circles and edges corresponding to their points of intersection, one obtains the Dynkin diagram of type D_4!

One can ask in general whether for a given isolated singularity X there exists a flat one-parameter family $X_T \to T$ (T is a small disc in $\mathbb{C}$) such that for $t = 0$ one has the original singularity X and for $t \neq 0$ the fibre is *smooth.* Here 'flat' is a technical notion (t, a parameter on T, is to be a non-zero divisor on the space X_T; this insures that much information of the zero-fibre can be read off the general fibre. It implies for instance that each component of X_T maps surjectively to the parameter space T, explaining the word 'flat'). For complete intersection singularities, every small perturbation of the functions defining the singularity gives a flat one-parameter deformation, but for non-complete intersections the situation is much more complicated. For instance, the space defined by the equations

$$xy - t = xz - t = yz - t = 0$$

is *not* a flat one-parameter deformation of the coordinate axes in $\mathbb{C}^3$. In general it is not clear that one can give non-trivial (i.e., not isomorphic to a product) deformations of a singularity at all! Indeed there exist examples of singularities which are rigid, i.e., admit only trivial deformations. But there exist also examples of singularities, the easiest one being the cone over a rational curve in $\mathbb{P}^4$, due to H. Pinkham which admit two one-parameter deformations for which the general fibres are smooth, but not homeomorphic!

4. Classification of singularities

One can try to classify singularities up to holomorphic coordinate changes. This goal is too ambitious in general, but a beginning of the classification of hypersurface singularities was made by V.I. Arnol'd, R. Thom, Mather and Siersma in the early 1970's. The list starts with the simple singularities:

$$A_k : x_1^{k+1} + x_2^2 + \ldots + x_n^2 = 0$$

$$D_k : x_1^2 x_2 + x_2^{k-1} + x_3^2 + \ldots + x_n^2 = 0; \qquad k \geq 4$$

$$E_6 : x_1^4 + x_2^3 + x_3^2 + \ldots + x_n^2 = 0$$

$$E_7 : x_1 x_2^3 + x_1^3 + x_3^2 + \ldots + x_n^2 = 0$$

$$E_8 : x_1^5 + x_2^3 + x_3^2 + \ldots + x_n^2 = 0.$$

Here 'simple' is a technical term, meaning more or less that the singularity can only deform in a finite number of isomorphism classes of other singularities. The labels A, D, E come from the Dynkin diagrams of simple Lie groups. In fact, there exist at least 15 different characterisations of these simple singularities.

A peculiar aspect of Arnold's classification is that singularities tend to appear in series. We quote Arnol'd [1]: 'Although the series undoubtedly exist, it is not at all clear what a series of singularities is'. And: 'It is only clear that the series are associated with singularities of infinite multiplicity' (non-isolated singularities). Indeed, in the example of A_k and D_k singularities one can formally put $k = \infty$ to get the non-isolated singularities of figure 5.

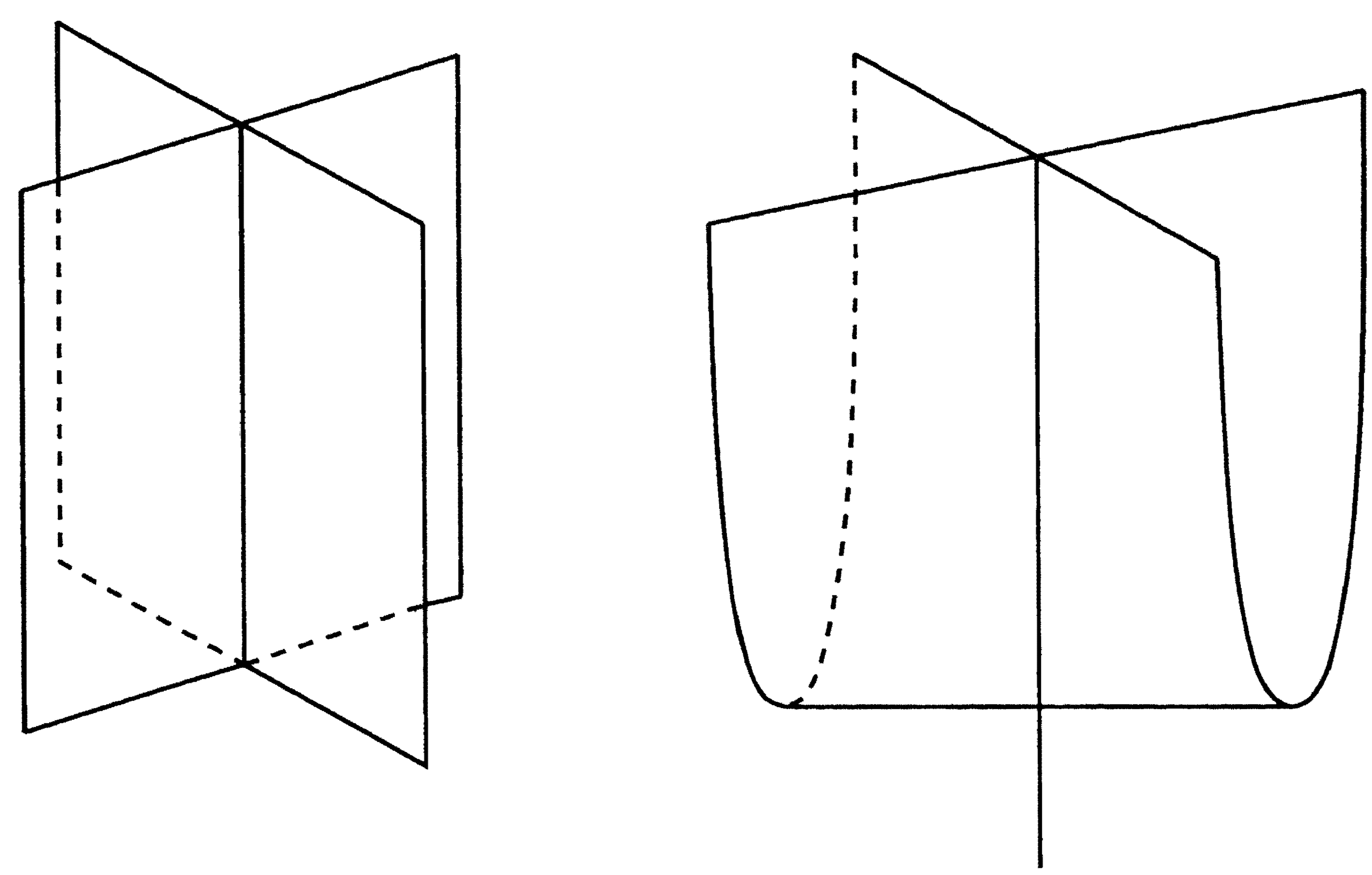

Figure 5.

5. THE PROJECT SINGULARITY THEORY

5.1. Hypersurface singularities

In this section we discuss some of the results obtained in the SMC-project Singularity Theory. Inspired by the remarks of Arnol'd, Siersma and later his student Pellikaan started to study the simplest types of non-isolated singularities: hypersurface singularities with one-dimensional singular locus and transverse type A_1. Transverse type A_1 means that if one takes a transverse slice at the general point of the singular locus the intersection is an isolated A_1 singularity. For example, in figure 6 the first singularity has transverse type A_1 whereas the second has not.

One of the goals of Siersma and Pellikaan was to understand the topology of the Milnor fibre of such singularities. In the case of an isolated singularity, this can be done by deforming the defining function to a Morse function, i.e., a function with only singularities of type A_1. The number of Morse points appearing is just the Milnor number. This 'morsification method' was generalized to the case that the reduced singular locus is a complete intersection, and led to so-called admissible deformations. Loosely speaking,

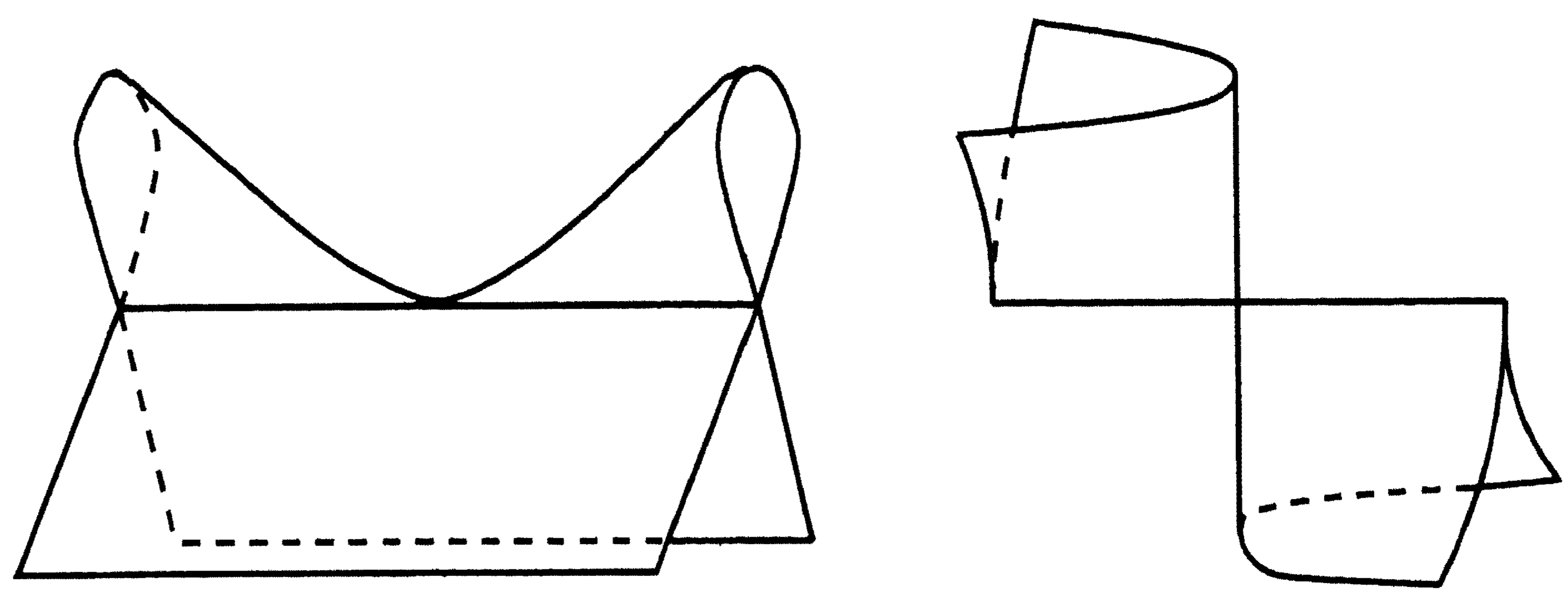

Figure 6.

these are deformations of the pair (f, Σ) where Σ is the singular locus of f.

An example of a deformation which is not admissible is the deformation of A_∞ to A_k, given by the equation $yz - tx^{k+1} = 0$. For the special fibre the singular locus is a line, but for the general fibre the singular locus is just one point. Therefore this does not induce a flat deformation of the singular locus, and the deformation is not admissible. The deformation suggested by figure 7 however, is admissible.

Using these admissible deformations Siersma and Pellikaan proved a theorem on the homotopy type of the Milnor fibre for hypersurface singularities whose singular locus is a complete intersection with transverse type A_1. Except for some special cases, the homotopy type turns out to be a wedge of spheres (as in the isolated singularity case), and a formula for the number

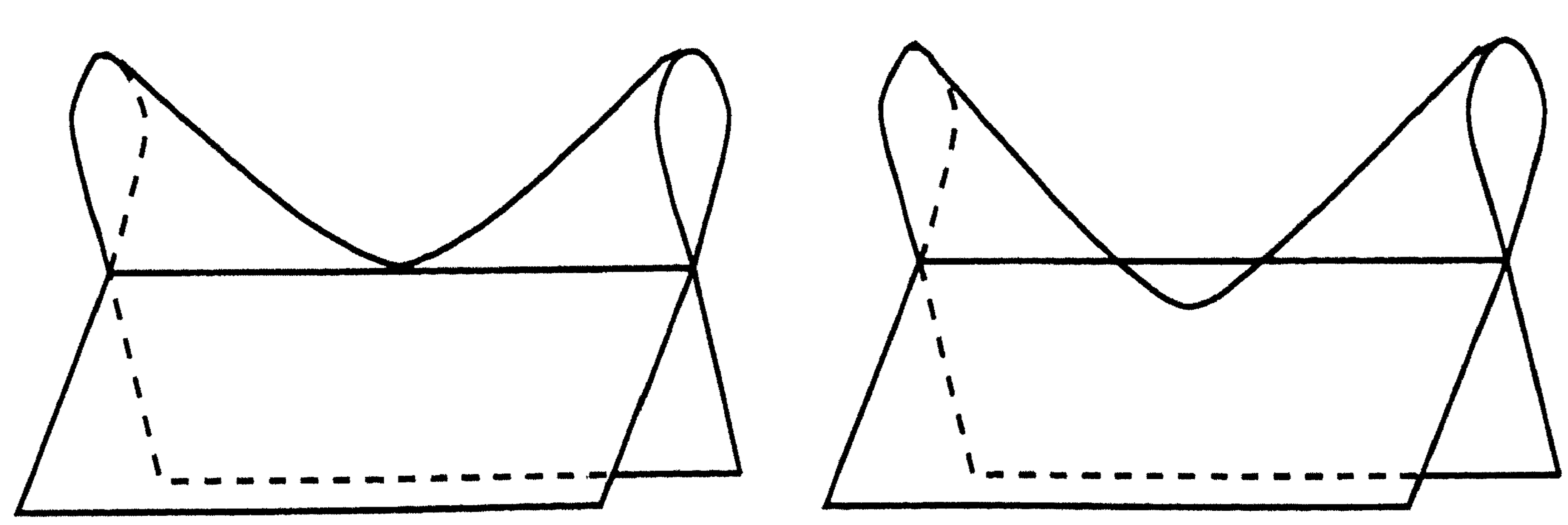

Figure 7.

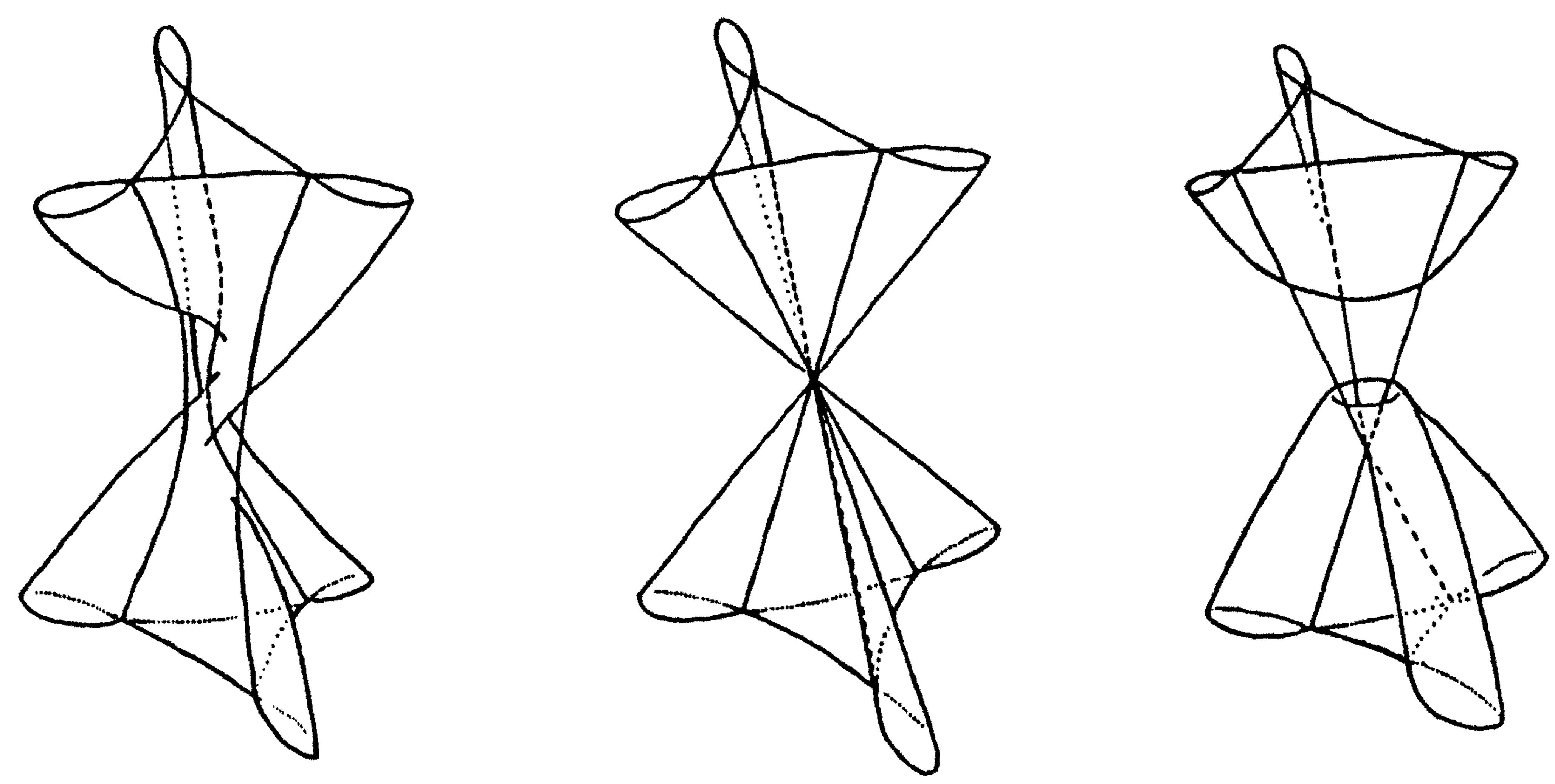

Figure 8.

of those was given by De Jong. Van Straten proved similar formulas using differential forms. Later on De Jong extended these results to certain cases of hypersurface singularities whose singular locus is a line and whose transverse type is a simple isolated singularity.

5.2. Weakly normal surface singularities

In his Ph.D. thesis, Van Straten studied weakly normal surfaces. Important examples of these are surfaces which are obtained as generic projections of smooth (or even normal) surfaces in $\mathbb{C}^3$. Such a surface has a singular locus which is a curve with isolated singular points itself. The structure of the surface near these special points is investigated by 'improving' them, i.e., replacing them by certain curve configurations, analogous to the process of resolution of isolated singularities by blowing-up. Van Straten generalized many results from the theory of normal surface singularities to this class of non-isolated surface singularities, and complemented in this way the knowledge obtained by Pellikaan. Also this approach led to a rich treasure of examples, obtained from a rough classification of weakly normal surface singularities by the structure of their improvements.

5.3. Admissible deformations

Pellikaan gave the following very interesting example of an admissible deformation: consider the hypersurface singularity given by the equation $(xy)^2 + (yz)^2 + (zx)^2 = 0$: the cone over a quartic curve in the complex projective plane with three A_1 singularities (see figure 8). This hypersurface singularity is a projection of the cone over the rational normal curve of degree 4. Pellikaan wrote down explicitely two essentially different admissible deformations, as in figure 8. It turned out that they correspond exactly to the two different deformations of the cone over the rational normal curve of degree 4 discovered by Pinkham! Note that the surface in Pellikaan's example is a generic projection of the cone over the rational normal curve of degree 4.

Inspired by Pellikaan's example, De Jong and Van Straten started to develop the following program: given a normal surface singularity, project it to $\mathbb{C}^3$ to obtain a weakly normal surface, given by an equation $f(x, y, z) = 0$ and with singular locus Σ. Try to determine which deformations of the projected surface are obtained as projections of deformations. Surprisingly, the deformations they found were precisely the *admissible deformations* of (f, Σ), which were introduced by Siersma and Pellikaan. The advantage of course is that one needs just one equation to describe the projection, the disadvantage being having to allow non-isolated singularities. This *projection method* has been very fruitful: the base space of a semi-universal deformation of rational quadruple points could be determined, in spite of the fact that equations for these singularities have never been written down. Using the projection method one also sees that in series of singularities (which we still do not know what they are) deformation theory behaves well, i.e., for two members of a series, it is easy to compare the deformations of one with the other. Further applications of this projection method are still being discovered.

References

1. V.I. Arnol'd (1981). *Singularity Theory, Selected Papers.* Lecture Note Ser. 53, Cambridge Univ. Press, Cambridge.
2. J. Milnor (1968). *Singular Points of Complex Hypersurfaces.* Annals of Math. Studies 61, Princeton Univ. Press.