Analysis on Lie Groups

G. van Dijk

1. Introduction

The project combines two fields: analysis and the theory of Lie groups, and thus leads to a challenging enterprise. It is also a hard enterprise since ample experience in both fields is required to be successful in research. The project comprises the following closely related topics:

- Harmonic analysis and representation theory
- Special functions related to root systems
- Special chapters in functional analysis and applications to Gelfand pairs.

Below we shall briefly describe the historical development of the topics, explain its basic techniques, and discuss recent developments. We also outline the work carried out in this project in The Netherlands, which was mainly sponsored by SMC.

2. Harmonic analysis and representation theory

A main theme is the analysis of functions on a Lie group G or, more generally, on a space M on which G acts homogeneously. A good example is the Lorentz group acting on the forward light cone. In particular one is interested in the decomposition of the space $L^2(M)$ of square integrable functions on M as a sum or integral of irreducible subspaces: the so-called Plancherel decomposition. In the classical cases, where M is the real line or the circle, with its group of translations, this amounts to Fourier analysis.

If M is the sphere with its group of rotations acting on it, one obtains the decomposition in spherical harmonics.

It is a general phenomenon that G-invariant operators such as invariant differential operators leave the decomposition of $L^2(M)$ into irreducible components invariant and act by scalar multiplication on each of the components. In order to gain insight in the Plancherel decomposition of $L^2(M)$ into irreducibles, one studies

1. the fine structure of representations of G related to M, and
2. eigenfunctions and eigendistributions related to such representations.

The Plancherel decomposition has been completely determined for any real semisimple Lie group G and the associated Riemannian symmetric space, by the work of Harish-Chandra. It has not yet been determined for the interesting class of pseudo-Riemannian symmetric spaces, leaving many challenging problems to be solved. A well-known example of a pseudo-Riemannian space is the hyperboloid of one sheet in $\mathbf{R}^n$ (see figure 1), while the two sheeted hyperboloid is a standard example of a Riemannian symmetric space. In recent years important progress has been made:

(a) For pseudo-Riemannian symmetric spaces of rank one the decomposition has been obtained explicitly by the work of V.F. Molchanov, J. Faraut, and G. van Dijk and his Ph.D. students; for a rank two space an explicit decomposition was obtained by N. Bopp. S. Sano and P. Harinck were successful in the group-like case $G_{\mathbf{C}}/G_{\mathbf{R}}$.

(b) For spaces of general rank the classification of the discrete series has been achieved in the work of M. Flensted-Jensen and Oshima-Matsuki. Later G. Olafsson and B. Ørsted made a deep study of the spaces which admit a so-called holomorphic discrete series.

(c) In a very interesting, not yet completely published paper, E.P. van den Ban and H. Schlichtkrull have determined a 'partial' Plancherel theorem for the most continuous part of the spectrum.

2.1. The role of differential equations

The study of eigenfunctions and eigendistributions, which is such an important tool in harmonic analysis as described above, has its own independent interest. It provides examples of systems of partial differential equations for which one can obtain much more detailed information about the solutions than in general. Such examples include asymptotic and convergent expansions, integral formulae, meromorphic extensions and analysis of the singularities are among the phenomena which can be understood by combining general principles of the theory of differential equations (such as ellipticity, holonomicity and regular singularities) with the additional information provided by the group actions. We refer to the work of Van

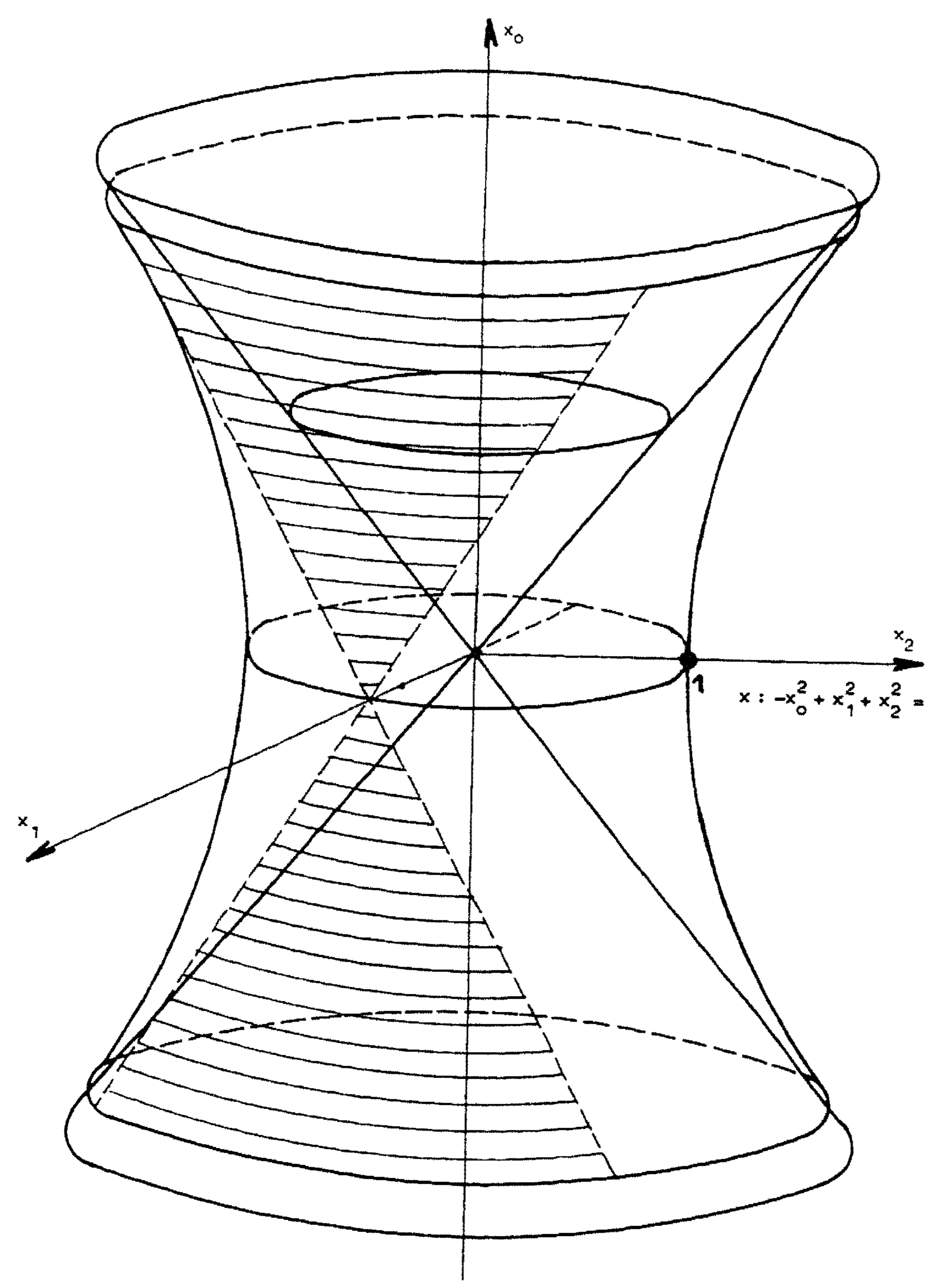

Figure 1. A pseudo-Riemannian space.

den Ban-Schlichtkrull on boundary values and Helgason's conjecture, and to the work of Kolk-Varadarajan on Lorentz invariant distributions on the light cone. Also the work on rank one spaces, mentioned above, provides a good example of the strength of the role of differential equations in harmonic analysis.

3. SPECIAL FUNCTIONS RELATED TO ROOT SYSTEMS

The theory of special functions is very closely related to the representation theory of Lie groups: most special functions appear as matrix coefficients of representations of special Lie groups. This connection was already known in the previous century, when the various special functions were introduced, for instance by C.F. Gauss. However, it was H. Weyl who emphasized this connection more clearly in his basic work on representations of semisimple Lie groups. It turns out to be fruitful in two ways: on the one hand, results from special function theory, analytically derived without knowledge of group theory, are needed to answer some of the questions of harmonic analysis on Lie groups; on the other hand, the group theoretic interpretation of the special functions suggests results, which would probably never have been discovered without this interpretation.

Fairly recent is the development of a special function theory in several variables, as suggested by the theory of special functions on symmetric spaces of higher rank. This research was initiated by T.H. Koornwinder in his Ph.D. thesis, by working out some rank two examples. It is remarkable that this theory admits a perfect generalization of the one variable case. The differential equation, the Rodrigues formula, the beta-function integral of Euler, all have a perfect analogue, which can be calculated explicitly. In particular the multivariable beta-function integral had attracted some attention before, by the work of A. Selberg, F. Dyson, M.L. Mehta and by the Macdonald conjectures. These conjectures were solved because of the work of G.J. Heckman and E.M. Opdam, who used the multivariable hypergeometric function theory associated with root systems. This is a general feature: the role of the Lie group is taken by the root system of its Lie algebra. So the group structure is lost but some connection remains, namely by means of the root system. This leads naturally to a much more algebraic and geometric theory of special functions. In figure 2 we give, only as an illustration of the complexity of root systems, a partial table of its Satake diagrams, which contains detailed information for the theory.

Many questions remain open, of which the most natural ones are: is there a good spectral theory (suggested by Harish-Chandra's Plancherel formula for symmetric spaces), and what about q-analogues (as started by Macdonald)? The second question is at the moment a quickly developing subject, because of the connection with *quantum groups*. We refer to work by M. Noumi, Koornwinder and H.T. Koelink. As to the first question, recently a new tool was developed in the study of special functions associated with root systems by C.F. Dunkl, by the introduction of his so-called Dunkl operators. The original philosophy to study equivalents of the radial parts of Laplace operators (by regarding the root multiplicities as parameters), has the disavantage that in general more than one Laplace operator exists whose form is unknown. Dunkl's operators have explicit expressions, commute and are related to Laplace operators of Cartan motion groups. Later I. Cherednik has adapted these operators for the group case as well. This new impulse has provided answers to question one. We refer to work of Opdam, Heckman and M.F.E. De Jeu. Dunkl's operators have nowadays a worldwide interest.

4. Special chapters in functional analysis and applications to Gelfand pairs

The decomposition of Hilbert spaces as integrals of irreducible component spaces involves aspects of direct integral theory, that can be clarified by

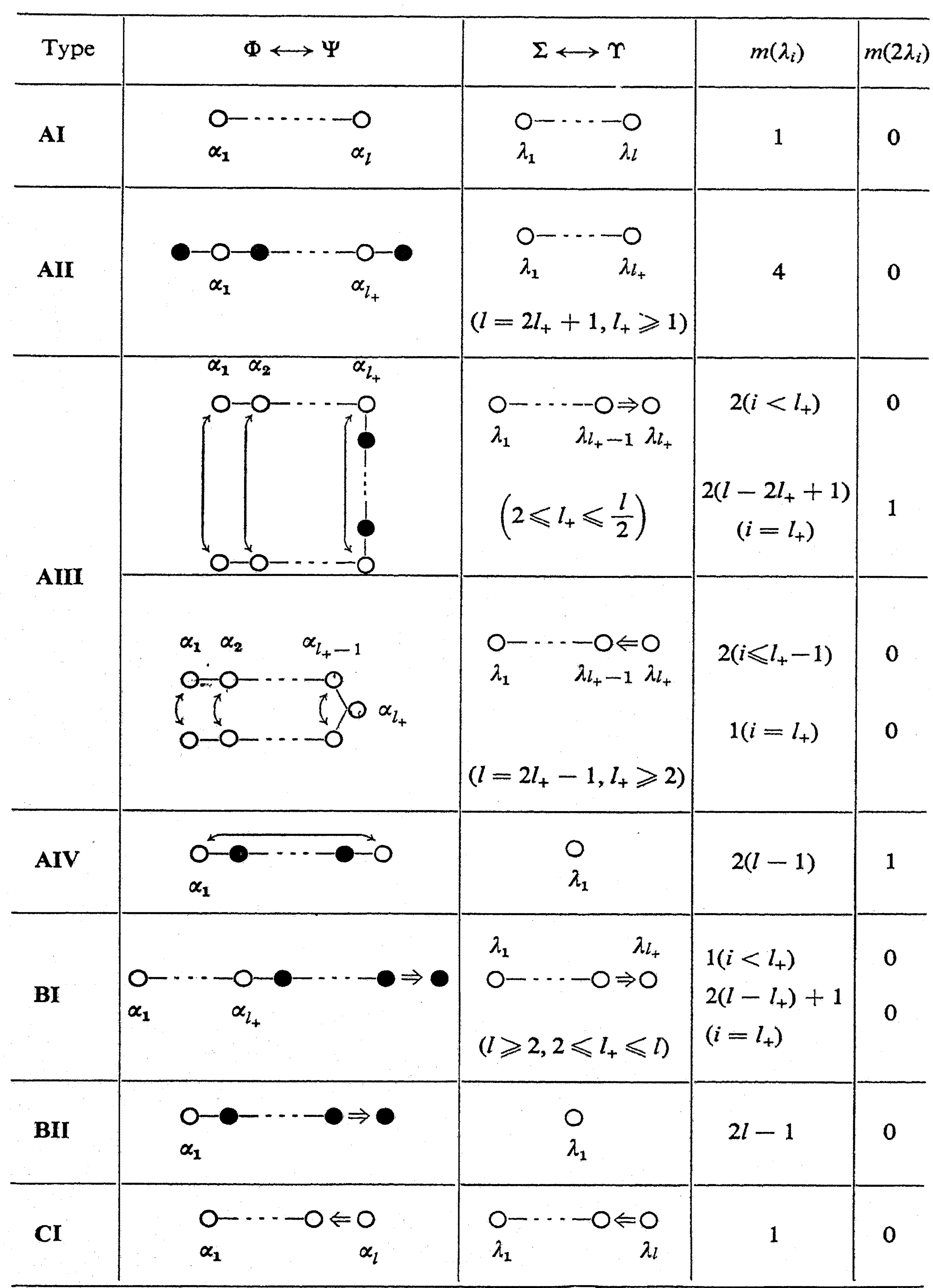

Figure 2. Satake diagrams of root systems [5].

establishing the link with the integral representation theory of G. Choquet and certain generalizations of it, due to E.G.F. Thomas. The pair (G, H) is said to be a Gelfand pair if the cone of H-invariant distributions of positive type on G is simplicial. These and other characterizations are relevant for the concrete determination of Gelfand pairs. Gelfand pairs are named after I.M. Gelfand and have initially only been studied in the case of compact H. A classical example of a Gelfand pair is a pair (G, H) such that G/H is a Riemannian symmetric space, e.g., $G = \mathrm{SL}(n, \mathbf{R})$, $H = \mathrm{SO}(n, \mathbf{R})$. These pairs are well studied and lead to a very beautiful theory, which is mainly due to Harish-Chandra and S. Helgason. A more general situation, involving finite-dimensional representations of H has been studied by H. van der Ven. In the general situation where H is noncompact, a nice result has been obtained by Van Dijk, studying rank one pseudo-Riemannian pairs. These pairs are Gelfand pairs, with the exception of the pair $(\mathrm{SO}_0(1,n), \mathrm{SO}_0(1,n-1))$. The Plancherel formula for the space associated with this pair has multiplicity two in the continuous spectrum.

5. Concluding remarks

One of the most interesting new lines of research of the last ten years in the field on Lie groups is without any doubt the second item: special functions related to root systems. Several new topics of research come to The Netherlands from other countries, mostly from the United States. However, this topic is to a great extent really Dutch, with pioneering work by Koornwinder, Heckman and Opdam. This does not happen very often. It is something to be proud of.

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