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Polynomial Splines in Two Variables

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1. Introduction

The SMC research project 'Numerical and Fundamental Aspects of Polynomial Splines in Two Variables' was focussed on the special type of functions referred to as *splines*. These functions do have profitable properties with respect to operations like interpolation, approximation and geometric modelling, which make them outstandingly suitable for applications in various fields of industrial design and numerical mathematics. For example, splines are popular tools for the description of curves and surfaces or, more general, *shapes*. In industrial design they are applied to visually represent all kinds of industrial products on the computer screen (cars, aeroplanes, ships, bottles, shoe-soles, tableware, etc.). Also the shapes of natural configurations like landscapes or earth layers can be adequately described. Not only such geometrical objects, but also functional dependencies obtained from, e.g., measurements can be easily represented: radar reflection patterns, *thermodynamic functions*, tomographic data, etc. In addition, spline functions are often used in numerical mathematics as basis functions in Rayleigh-Ritz-Galerkin processes for solution of boundary value problems for ordinary and partial differential equations.

Historically, splines were motivated as tools for interpolation due to poor behaviour of polynomials in this respect. First steps on a higher level were done by I.J. Schoenberg in the 1940's. With the advent of the computer in the 1950's/60's, the development accelerated dramatically and resulted in a

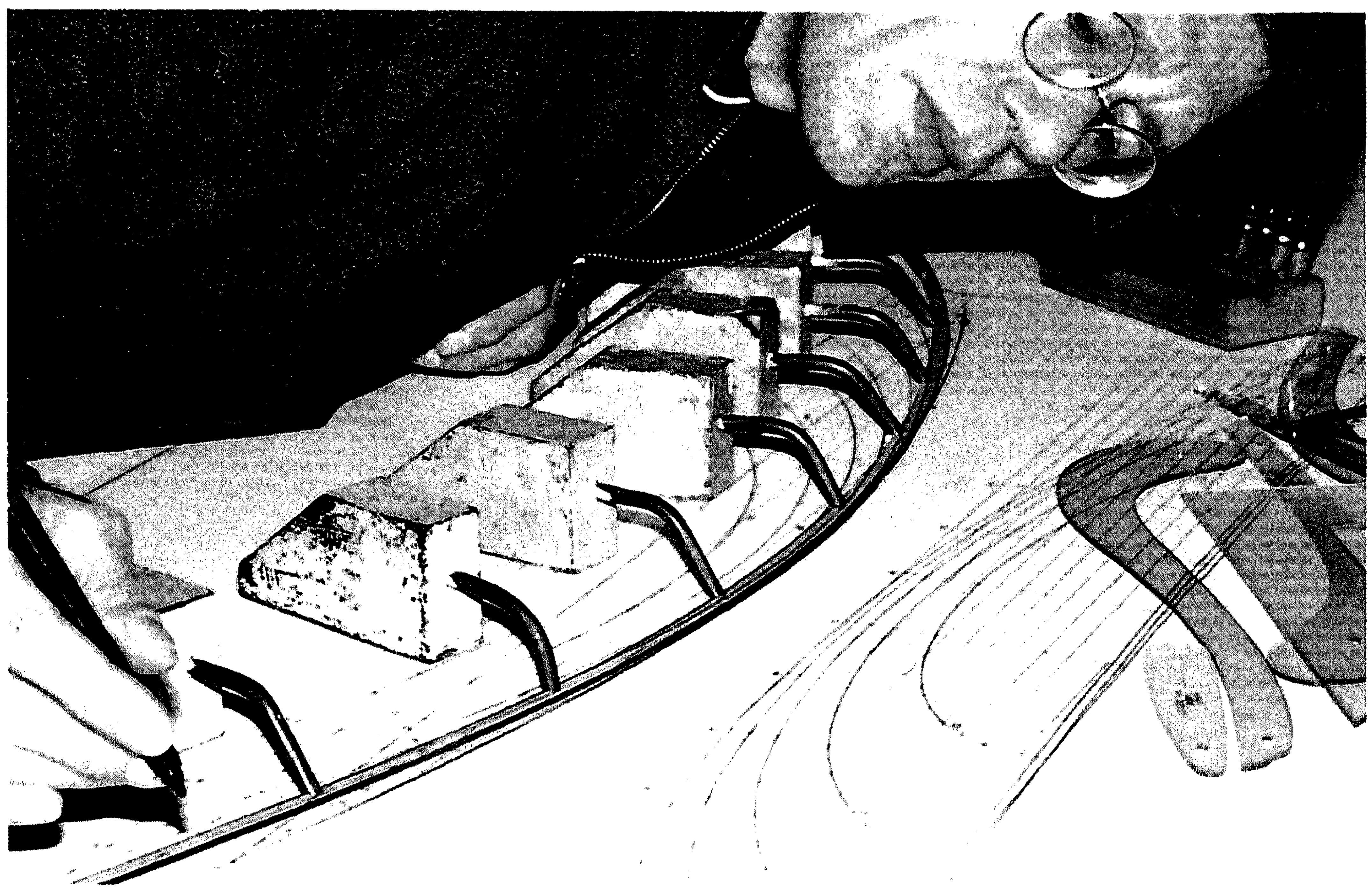

Figure 1. Originally splines were long strips made of very flexible strong material, used as an engineer's tool at the drawing-table. Nowadays the computer has almost completely taken over this design function.

fairly complete theory and practice for splines in one variable. In particular the various recursive algorithms as initiated by, e.g., C. de Boor were of great importance. Much effort also was put in the research concerning splines in two and more variables. However, notwithstanding a lot of progress, many problems are still open. For example, the problem of shape-preserving approximation is still far away from a general solution. In the current research project a number of these problems were studied in depth.

A spline function in s variables is a piecewise analytic function on its domain of definition Ω which is part of the s-dimensional space $\mathbb{R}^s$. If Ω is bounded, then the subdomains on which the spline is analytic form a finite partition of Ω. In the one-dimensional situation the points where the spline is not analytic are called 'knots'. In the current project mathematical techniques were investigated to construct spaces of spline functions having favourable properties with respect to the desired applications. Also the *computability* of such splines has been dealt with extensively. In the project the analytical parts of the splines were restricted to be of polynomial nature. This class of splines is called the class of *polynomial splines*.

Problems arise when the spline is required to be several times differentiable while the domain $\Omega \subset \mathbb{R}^s$ is arbitrarily shaped and $s > 1$. Other types of problems arise when *closed* bodies in $\mathbb{R}^3$ are required to be described using a high degree of (geometric) continuity. In this latter case one uses *parametric* spline surfaces, and the problem is connected to the fact that the surface of a closed body can not be mapped in a continuous one-to-one way into $\mathbb{R}^2$.

Finally, the problem of the *shape-preserving* description of a surface has been addressed in the project. These latter investigations were of mainly theoretical nature.

2. Polynomial splines

2.1. B-splines in one dimension

One of the basic contributions to the theory of polynomial splines in one dimension is the discovery of basis functions with compact support, the so-called B-splines, by H.B. Curry and Schoenberg in 1966 [1]. These are piecewise polynomial functions with a support of $n + 1$ consecutive subintervals, where n is the degree of the polynomial parts, and with $n - 1$ times continuously differentiable connections at the knots. A support of length $n + 1$ subintervals is the smallest possible support for non-trivial splines of degree n and of class C^{n-1}. The number $n + 1$ is called the *order* of the B-spline. The B-splines form a basis in the space of polynomial splines defined over a given partition of the considered domain $\Omega \subset \mathbb{R}$. Their importance is found in the fact of their very simple computability, which is due to the existence of a recursion relation (De Boor and M.G. Cox, 1972). This relation admits a numerically stable way of building up the higher-order B-splines, starting with B-splines of order 1. Also for differentiation and integration simple rules exist. The derivative of a B-spline can be computed as a weighted difference of two B-splines of one order lower (De Boor, 1972). An expression for the integral of a given spline was found by De Boor, T. Lyche and L.L. Schumaker (1976). With these rules the basics for practical computing with polynomial splines in one dimension are available. Further improvements could be attained by giving special attention to the specific properties of splines and their algorithms. For example, it is possible to let coincide, purposedly, several consecutive knots, which can be interpreted as admitting subintervals of length zero. At such a multiple knot the spline will have a lower order of continuity compared with the continuity order at single knots. The above mentioned recursion, however, can still be applied without any special measure. The computability is thus not affected by introducing multiple knots. Furthermore the B-splines can be normalized such that they form a partition of unity at every point in the domain. A consequence is a close relationship in shape between a spline

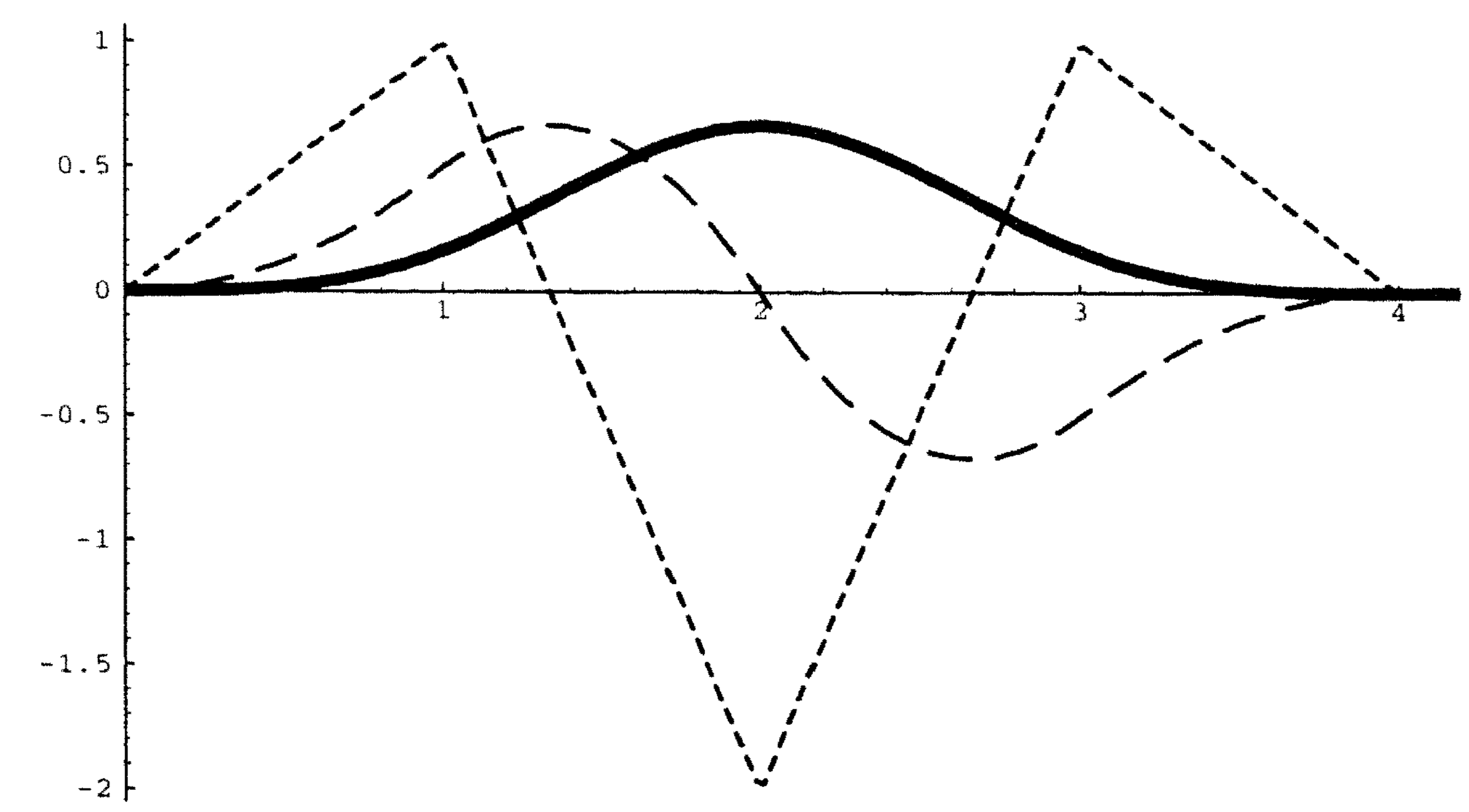

Figure 2. A cubic B-spline (order=4), and its derivatives.

$s(x)$, written as a linear combination of B-splines, and the so-called *control polygon*; this is the polygon which connects the consecutive *control points* by straight line segments. The control points are defined as the points in the x, y-plane which represent the coefficients in the B-spline expansion of $s(x)$: the y-coordinates are the values of the coefficients themselves, and the x-coordinates are the values of the coefficients in the B-spline expansion of the function x over the same knot partition. The similarity in shape allows predictable change of shape of $s(x)$ by changing coefficient values. This is of great importance for shape design purposes. For these latter purposes is also of great importance the possibility to insert additional knots at pre-selected positions, for which a number of simple and elegant algorithms exist.

In figure 2 the normalized cubic B-spline over the set of knots $\{0, 1, 2, 3, 4\}$ is shown (solid line), together with its first derivative (dashed line) and second derivative (dotted line). In figure 3 a cubic spline is depicted over the same set of knots, together with its control polygon. Due to multiplicity of the boundary-knots, the first and last control points coincide with the begin point and end point, respectively, of the curve.

From a theoretical point of view the relation that exists between divided differences of polynomial half-space functions (truncated power functions) and B-splines is of great importance. Properties of B-splines can be derived from this relation in an elegant way. This opens perspectives with respect to the research to B-splines in more than one dimension if the notion of 'divided differences' can be extended to more dimensions in a suitable way. In the present project this item was an important subject of research.

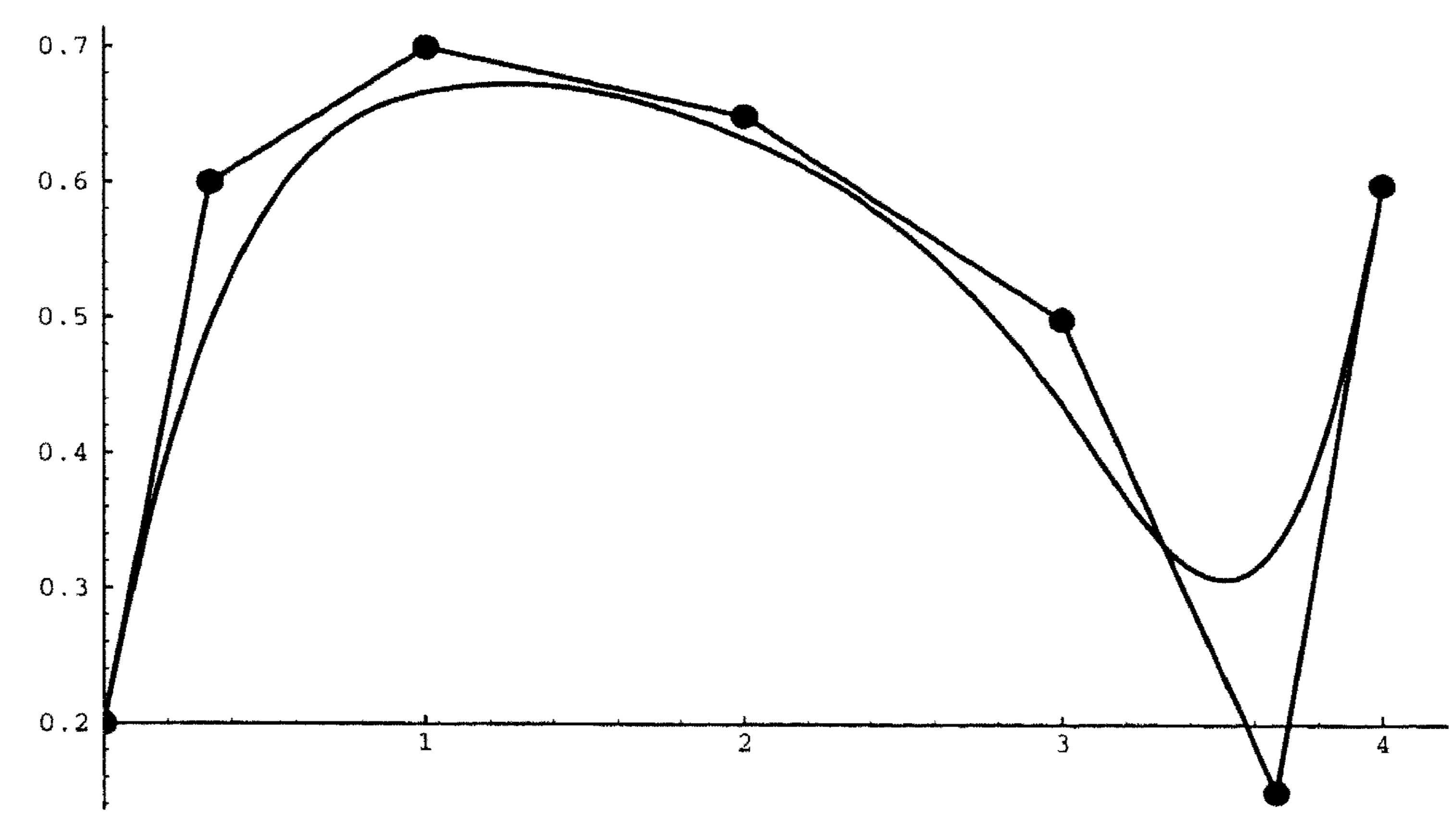

Figure 3. A cubic spline and its control polygon.

2.2. B-splines in more dimensions

B-splines in more dimensions have been topic of research already since several years. An obvious extension to the higher dimensional situation is obtained by constructing *tensor products* of univariate splines. This type of extension is of practical advantage in the sense that it allows the use of repeated univariate algorithms. An obvious drawback, however, is the very limited flexibility with respect to the shape of the domain and the density distribution of the set of knots. A more general extension arises from the notion of *polyhedral spline.* The definition of B-splines, based upon this notion, is strongly geometrical: B-splines in s dimensions are defined as functions, the values of which are proportional to the volumes of corresponding intersections through polyhedra in higher-dimensional spaces. The first actual construction of a B-spline on the basis of this geometrical principle and with a *simplex* chosen for the polyhedron, was performed by De Boor (1976) [2]. Later on other types of polyhedra were used, resulting in the construction of, e.g., *box splines* and *cone splines.* The tensor product splines, mentioned above, could be interpreted as special cases of box splines. Recurrence relations were found soon (L.A. Micchelli, 1980), guaranteeing the relatively simple computability of the splines.

The possibilities for practical use of simplex B-splines were for the first time extensively investigated by R.H.J. Gmelig Meyling (1986). These B-splines appeared to be suitable for high quality approximations of functions in two variables over arbitrary finite domains. However, the computing effort appeared to be high. A further improvement of computing efficiency is needed in order to render these splines really useful for practice. In figure 4 a bivariate quadratic simplex B-spline of class C^1 is shown. Its support

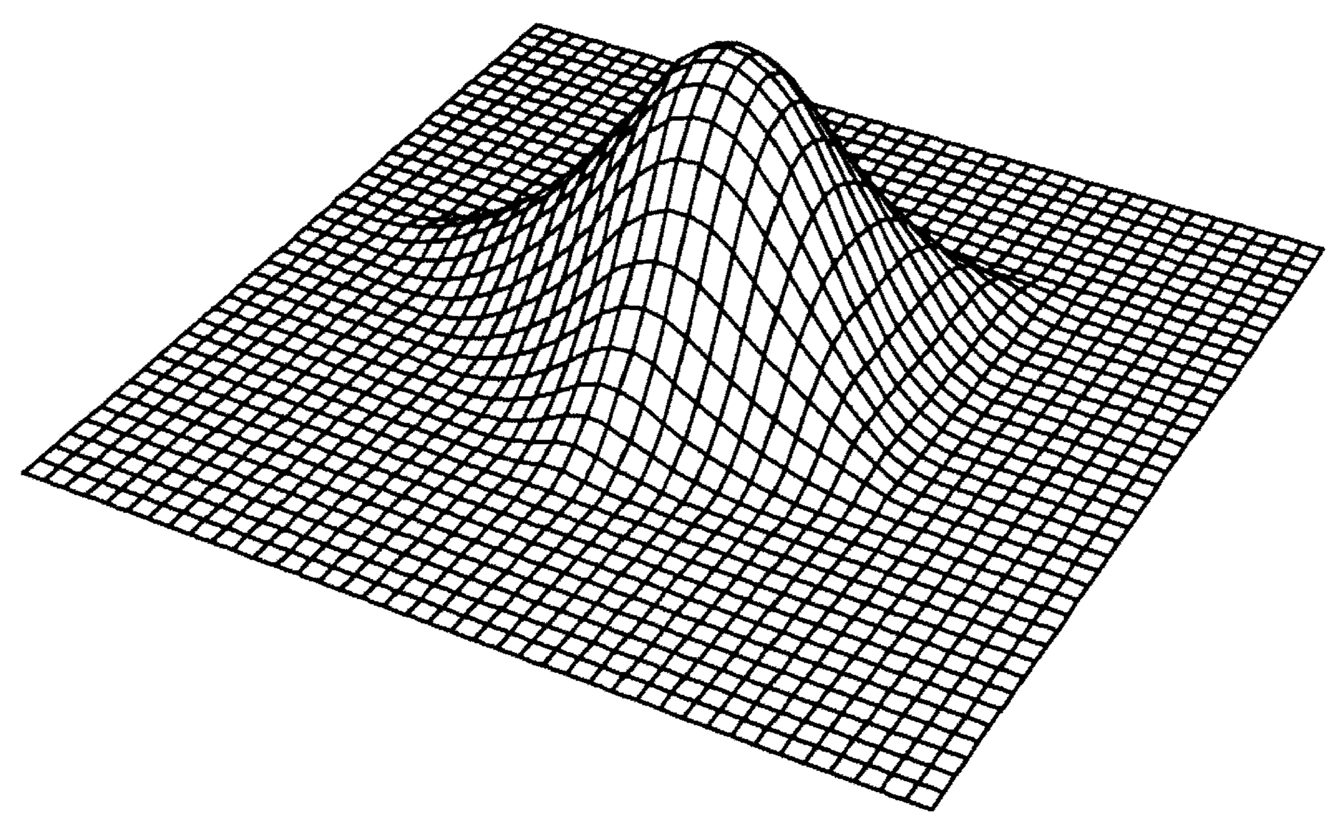

Figure 4. A bivariate quadratic simplex B-spline.

is the convex hull of 5 knots in the plane.

Another extension of splines to more than one dimension, which is not based on polyhedra in higher-dimensional spaces, uses a *triangulation* of the domain and the definition of *Bernstein polynomials* on each of the elements of this triangulation. Using Bernstein polynomials allows in a relatively simple way the construction of a surface of class C^1, or even class C^2, by imposing side conditions on the control points. Pioneering work has been done by Schumaker (1979 and later), in particular with respect to the dimensions of such spline spaces [3]. Also for these splines the practical utility was investigated extensively by Gmelig Meyling (1986).

3. THE RESEARCH IN THE PROJECT

The aim was to consider polynomial splines in two variables. A major part of the fundamental research was devoted to bivariate simplex splines. A first step was the generalization of the notion of univariate divided differences to the higher-dimensional situation. This generalization is based upon a pointwise evaluation of a multivariate function. Next, the simplex spline is expressed as the multivariate divided difference of a generalized polynomial half-space function. The properties of the multivariate simplex splines could be derived from the properties of the multivariate divided differences. Also new proofs were formulated for a number of already known results.

Much attention was paid to the computability of the simplex spline. Using as starting point a publication of E.T. Cohen among others (1987), in which an alternative recurrence relation was presented for evaluation of multivariate simplex splines, new recurrence relations were found for directional derivatives and for inner products of simplex splines. The numerical prop-

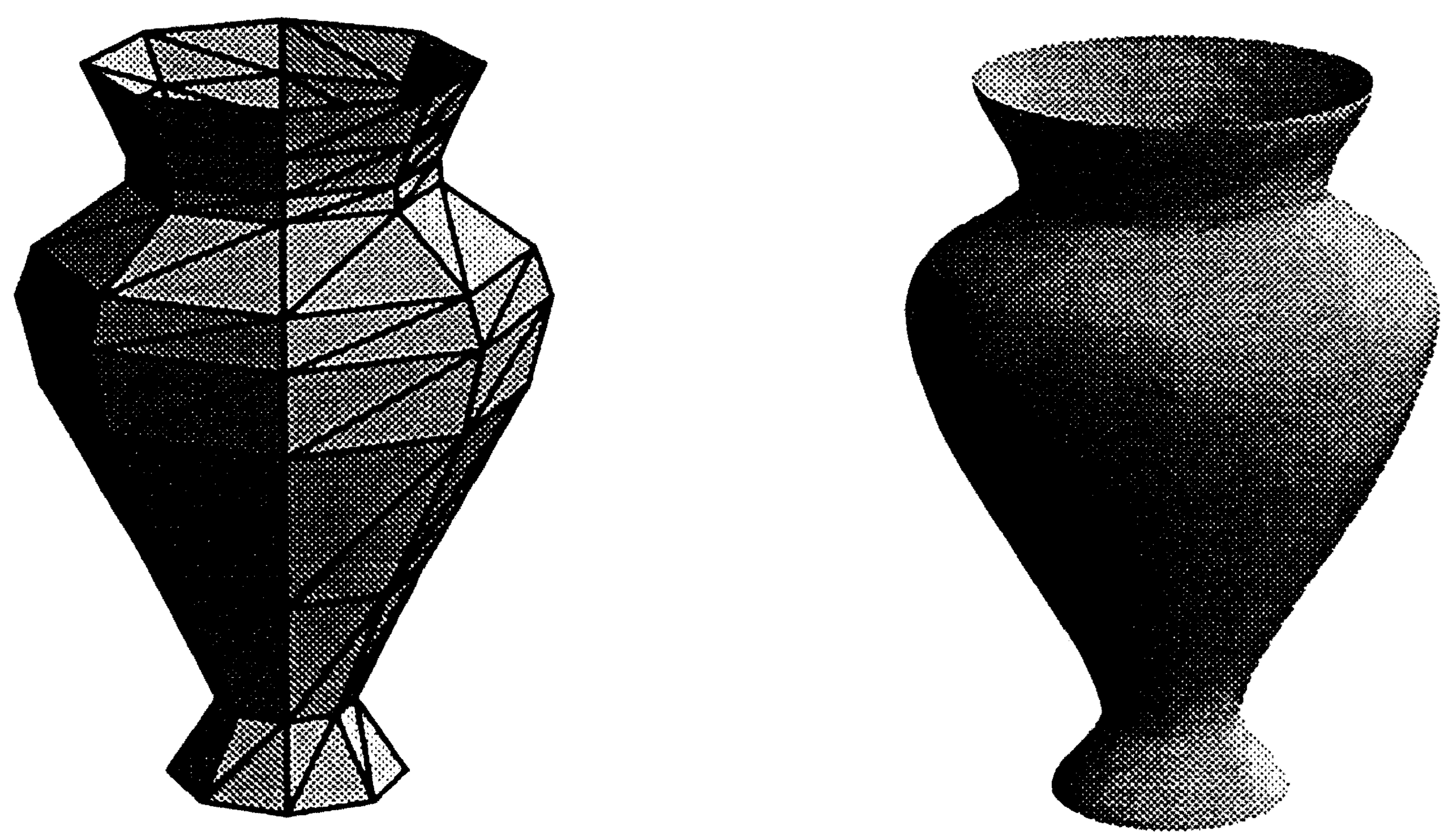

Figure 5. A triangulation (left) and a quintic interpolant.

erties of these new algorithms are investigated (stability, computing effort) and compared with the older algorithms. It appears that progress has been made in particular with respect to the complexity of the algorithms.

An alternative for the computation of simplex splines is based on the concept of *subdivision.* In the case of box splines this is an accepted and practically very useful technique, due to its efficiency and simplicity. For the case of simplex splines little was known about this topic. For this reason the notion of *discrete simplex spline* is introduced and some properties are derived. Discrete B-splines arise when continuous B-splines, defined with respect to a given net of knots, are expressed as linear combinations of continuous B-splines which are defined with respect to another net of knots in the same domain. The latter net usually is a refinement of the first net. It will then be obvious that discrete B-splines take a central position in subdivision processes. The investigations have led to the formulation of an algorithm for subdivision of simplex splines.

Another topic of research was the smooth interpolation of scattered data in three-dimensional space, using spline surfaces. A suitable method was designed using *degenerated* triangular Bézier-Bernstein patches. This degeneration has to be understood as a multiplicity of some of the control points. The necessity of the use of degenerated patches is a direct consequence of the imposed requirements: (1) the method should be *local*, (2) the patches should be *polynomial*, and (3) the geometric continuity should be of *order 1* at least. Locality means that only local information is used to construct the accessory local part of the surface. Results were obtained for the polynomial degrees 4 and 5. It appeared that the method is suitable for the description of closed bodies as well. In figure 5 an example is given of an object which is described by using degenerated quintic polynomial patches and which is of geometric continuity class C^1. The data set coincides with the vertices of the triangulation.

Finally, attention was given to the smooth approximation and interpolation of convex functions, preserving the convexity. These investigations were mainly of theoretical nature. One result that was obtained is a proof that, whenever a finite-dimensional approximation space is used, the use of *local* operators for interpolation in general will not result in preservation of convexity. Thus an interpolation method which is such that preservation of convexity is guaranteed must be *global.*

The major part of the above research was done by M. Neamtu in the framework of his Ph.D. thesis.

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