Kleene's Realizability

for Cor Baayen

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1 Introduction

1.1. The realizability interpretation of intuitionistic arithmetic was first introduced by S.C.Kleene (1945). It has turned out to be an extremely fruitful interpretation, widely applicable to axiomatic systems bases on constructive logic, and yielding interesting results such as the consistency of Church's thesis with intuitionistic formalisms. Nowadays there is not just a single notion of realizability, but a whole family of notions, which of course resemble each other in certain respects.

Here we present a streamlined development of the formalized version of Kleene's original notion. We presuppose some (not much) familiarity with intuitionistic first-order predicate logic, classical Peano arithmetic, as well as elementary recursion theory; for the rest the paper is self-contained.

For the history of the topic, see (Troelstra 1973, Dragalin 1988).

1.2. Realizability by numbers introduced by Kleene as a semantics for intuitionistic arithmetic, by defining for arithmetical sentences A a notion "the number $\mathbf{n}$ realizes A", intended to capture some essential aspects of the intuitionistic meaning of A. Here $\mathbf{n}$ is not a term of the arithmetical formalism, but an element of the natural numbers $\mathbb{N}$. The definition is by induction on the complexity of A:

- $\mathbf{n}$ realizes $t = s$ iff $t = s$ holds;
- $\mathbf{n}$ realizes $A \wedge B$ iff $\mathbf{p}_0\mathbf{n}$ realizes A and $\mathbf{p}_1\mathbf{n}$ realizes B;

- $\mathbf{n}$ realizes $A \vee B$ iff $\mathbf{p}_0\mathbf{n} = 0$ and $\mathbf{p}_1\mathbf{n}$ realizes A or $\mathbf{p}_0\mathbf{n} = 1$ and $\mathbf{p}_1\mathbf{n}$ realizes B;

- $\mathbf{n}$ realizes $A \to B$ iff for all $\mathbf{m}$ realizing A, $\mathbf{n}\bullet\mathbf{m}$ is defined and realizes B;

- $\mathbf{n}$ realizes $\neg A$ if for no $\mathbf{m}$, $\mathbf{m}$ realizes A;

- $\mathbf{n}$ realizes $\exists y\, A$ iff $\mathbf{p}_1\mathbf{n}$ realizes $A[y/\overline{\mathbf{p}_0\mathbf{n}}]$.

- $\mathbf{n}$ realizes $\forall y\, A$ iff $\mathbf{n}\bullet\mathbf{m}$ is defined and realizes $A[y/\overline{\mathbf{m}}]$, for all $\mathbf{m}$.

Here $\mathbf{p}_1$ and $\mathbf{p}_0$ are the inverses of some standard primitive recursive pairing function $\mathbf{p}$ coding $\mathbb{N}^2$ onto $\mathbb{N}$, and $\overline{\mathbf{m}}$ is the standard term $S^{\mathbf{m}}0$ (numeral) in the language of intuitionistic arithmetic corresponding to $\mathbf{m}$; $\bullet$ is partial recursive function application, i.e. $\mathbf{n}\bullet\mathbf{m}$ is the result of applying the function with code $\mathbf{n}$ to $\mathbf{m}$. (Later on we also use $\bar{m}, \bar{n}, \ldots$ for numerals.) The definition may be extended to formulas with free variables by stipulating that $\mathbf{n}$ realizes A if $\mathbf{n}$ realizes the universal closure of A.

Reading "there is a number realizing A" as "A is constructively true", we see that a realizing number provides witnesses for the constructive truth of existential quantifiers and disjunctions, and in implications carries this type of information from premise to conclusion by means of partial recursive operators. In short, realizing numbers "hereditarily" encode information about the realization of existential quantifiers and disjunctions.

1.3. Realizability, as an interpretation of "constructively true" is reminiscent of the well-known Brouwer-Heyting-Kolmogorov explanation (BHK for short) of the intuitionistic meaning of the logical connectives. BHK explains "p proves A" for compound A in terms of the provability of the components of A. For prime formulas the notion of proof is supposed to be given. Examples of the clauses of BHK are:

- p proves $A \to B$ iff p is a construction transforming any proof c of A into a proof $p(c)$ of B;

- p proves $A \wedge B$ iff $p = (p_0, p_1)$ and p_0 proves A, p_1 proves B;

- p proves $A \vee B$ iff $p = (p_0, p_1)$ with $p_0 \in \{0, 1\}$, and p_1 proves A if $p_0 = 0$, p_1 proves B if $p_0 \neq 0$.

Realizability corresponds to BHK if (a) we concentrate on (numerical) information concerning the realizations of existential quantifiers and the choices for disjunctions, and (b) the constructions considered for $\forall, \to$ are assumed to be encoded by (partial) recursive operations.

1.4. Realizability gives a classically meaningful definition of intuitionistic truth; the set of realizable statements is closed under deduction and must be consistent, since 1=0 cannot be realizable. It is to be noted that decidedly non-classical principles are realizable, for example

$$\neg\forall x[\exists y Txxy \vee \forall y \neg Txxy]$$

is easily seen to be realizable. (T is Kleene's T-predicate, which is assumed to be available in our language; $Txyz$ is primitive recursive in x, y, z and expresses that the algorithm with code x applied to argument y yields a computation with code z; U is a primitive recursive function extracting from a computation code z the result Uz.) For $\neg A$ is realizable iff no number realizes A, and realizability of $\forall x[\exists y Txxy \vee \forall y \neg Txxy]$ requires a total recursive function deciding $\exists y Txxy$, which does not exist (more about this below). In this way realizability shows how in constructive mathematics principles may be incorporated which cause it to diverge from the corresponding classical theory, instead of just being included in the classical theory.

1.5. Some notational habits adopted in this paper are: dropping of distinguishing sub- and superscripts where the context permits; saving on parentheses, e.g. for a binary predicate R applied to x, y we often write Rxy instead of $R(x, y)$ (this habit has just been demonstrated above). The symbol $\equiv$ is used for literal identity of expressions modulo renaming of bound variables. $\Rightarrow$ is used as metamathematical consequence relation, and in particular $\mathcal{A}, \mathcal{B} \Rightarrow \mathcal{C}$ expresses a rule which derives $\mathcal{C}$ from premises $\mathcal{A}, \mathcal{B}$. $\mathrm{FV}(\mathcal{A})$ is the set of free variables of expression $\mathcal{A}$.

2 Formalizing realizability in **HA**

2.1. In order to exploit realizability proof-theoretically, we have to formalize it. Let us first discuss its formalization in ordinary intuitionistic first-order arithmetic **HA** ("Heyting's Arithmetic"), based on intuitionistic predicate logic with equality, and containing symbols for all primitive recursive functions, with their recursion equations as axioms. Induction and successor axioms $Sx = Sy \rightarrow x = y$, $Sx \neq 0$ are present as usual.

$x, y, z, \ldots$ are numerical variables, S is successor. We use the notation $\bar{n}$ for the term $S^n 0$; such terms are called *numerals*. $\mathbf{p}_0, \mathbf{p}_1$ bind stronger than infix binary operations, i.e. $\mathbf{p}_0 t + s$ is $(\mathbf{p}_0 t) + s$. For primitive recursive predicates R, $Rt_1 \ldots t_n$ may be treated as a prime formula since the formalism contains a symbol for the characteristic function χ_R.

Now we are ready for a formalized definition of "x realizes A" in **HA**.

2.2. Definition. By recursion on the complexity of A we define $x \underline{\mathbf{rn}} A$, $x \notin \mathrm{FV}(A)$, "x numerically realizes A" :

$$
\begin{aligned}
x\,\underline{\mathbf{rn}}\,(t=s) &:= (t=s)\\
x\,\underline{\mathbf{rn}}\,(A\wedge B) &:= (\mathbf{p}_0x\,\underline{\mathbf{rn}}\,A)\wedge(\mathbf{p}_1x\,\underline{\mathbf{rn}}\,B),\\
x\,\underline{\mathbf{rn}}\,(A\to B) &:= \forall y(y\,\underline{\mathbf{rn}}\,A\to\exists z(Txyz\wedge Uz\,\underline{\mathbf{rn}}\,B)),\\
x\,\underline{\mathbf{rn}}\,\forall y\,A &:= \forall y\exists z(Txyz\wedge Uz\,\underline{\mathbf{rn}}\,A),\\
x\,\underline{\mathbf{rn}}\,\exists y\,A &:= \mathbf{p}_1x\,\underline{\mathbf{rn}}\,A[y/\mathbf{p}_0x].
\end{aligned}
$$

Note that $\mathrm{FV}(x\,\underline{\mathbf{rn}}\,A)\subset\{x\}\cup\mathrm{FV}(A)$. □

2.3. REMARKS. (i) We have omitted clauses for negation and disjunction, since in arithmetic we can take $\neg A := A\to 1=0$, $A\vee B := \exists x((x=0\to A)\wedge(x\neq 0\to B))$. If we spell out $x\,\underline{\mathbf{rn}}\,(A\vee B)$ on the basis of this definition we find:

$$x\,\underline{\mathbf{rn}}\,(A\vee B)\leftrightarrow(\mathbf{p}_0x=0\to(\mathbf{p}_0\mathbf{p}_1x)0\,\underline{\mathbf{rn}}\,A)\wedge(\mathbf{p}_0x\neq 0\to(\mathbf{p}_1\mathbf{p}_1x)0\,\underline{\mathbf{rn}}\,B),$$

(ii) The definition of realizability permits slight variations, e.g. for the first clause we might have taken

$$x\,\underline{\mathbf{rn}}'(t=s) := (x=t\wedge t=s).$$

However, it is routine to see that this variant $\underline{\mathbf{rn}}'$-realizability is *equivalent* to $\underline{\mathbf{rn}}$-realizability in the following sense: for each formula A there are two partial recursive functions ϕ_A and ψ_A such that

$$
\begin{aligned}
&\vdash x\,\underline{\mathbf{rn}}\,A\to\phi_A(x)\,\underline{\mathbf{rn}}'A\\
&\vdash x\,\underline{\mathbf{rn}}'A\to\psi_A(x)\,\underline{\mathbf{rn}}\,A.
\end{aligned}
$$

(If in the future we shall call two versions of a realizability notion equivalent, it will always be in this or a similar sense.) Similarly, if we treat $\vee$ as a primitive, the clause for $x\,\underline{\mathbf{rn}}\,(A\vee B)$ given above may be simplified to

$$x\,\underline{\mathbf{rn}}\,(A\vee B) := (\mathbf{p}_0x=0\wedge\mathbf{p}_1x\,\underline{\mathbf{rn}}\,A)\vee(\mathbf{p}_0x\neq 0\wedge\mathbf{p}_1x\,\underline{\mathbf{rn}}\,B),$$

which yields an equivalent notion of realizability.

(iii) In terms of partial recursive function application $\bullet$ and the definedness predicate $\downarrow$ ($t\downarrow$ means "t is defined"), we can write more succinctly:

$$
\begin{aligned}
x\,\underline{\mathbf{rn}}\,(A\to B) &:= \forall y(y\,\underline{\mathbf{rn}}\,A\to x\bullet y\downarrow\wedge x\bullet y\,\underline{\mathbf{rn}}\,B),\\
x\,\underline{\mathbf{rn}}\,\forall y\,A &:= \forall y(x\bullet y\downarrow\wedge x\bullet y\,\underline{\mathbf{rn}}\,B).
\end{aligned}
$$

where $t\downarrow$ expresses that t is defined (cf. next subsection). Of course, the partial operation $\bullet$ and the definedness predicate $\downarrow$ are not part of the language, but expressions containing them may be treated as abbreviations, using the following equivalences:

$$
\begin{aligned}
&t_1=t_2\leftrightarrow\exists x(t_1=x\wedge t_2=x),\\
&t_1\bullet t_2=x\leftrightarrow\exists yzu(t_1=y\wedge t_2=z\wedge Tyzu\wedge Uu=x),\\
&t\downarrow\leftrightarrow\exists z(t=z).
\end{aligned}
$$

(t_1, t_2 terms containing $\bullet$, x, y, z, u not free in t_1, t_2). However, note that the logical complexity of $A(t)$, where t is an expression containing $\bullet$, depends on the complexity of t! (On the other hand, $t\!\downarrow$ is always expressible in Σ^0_1-form.) For metamathematical investigations it is therefore more convenient to formalize realizability in a conservative extension $\mathbf{HA}^*$ of $\mathbf{HA}$ in which we can treat "$\bullet$" as a primitive. Treating $t_1 = t_2$ for partially defined t_1, t_2 as an abbreviation in a rigoorous way is possible, but involves a good deal of lengthy inductions, as demonstrated in (Kleene 1969). Since ordinary logic deals with total functions only, we first need to extend our logic to the (intuitionistic) logic of partial terms LPT, or intuitionistic E^+-logic, in the terminology of Troelstra and van Dalen(1988, 2.2.3). LPT first appeared in (Beeson 1981).

3 Intuitionistic predicate logic with partial terms LPT

3.1. Variables are supposed to range over the objects of the domain considered, so always denote; arbitrary terms need not denote, so we need a predicate $\mathbf{E}$, expressing definedness; $\mathbf{E}t$ reads "t denotes" or "t is defined". Instead of $\mathbf{E}t$ we shall write $t\!\downarrow$, in the notation commonly used in recursion theory.

If we also have equality in our logic, and read $t = s$ as "t and s are both defined and equal", we can express $t\!\downarrow$ as $t = t$.

3.2. The following axiomatization is a convenient (but not canonical) choice for arguments proceeding by induction on the length of formal deductions:

L1 $\quad A \to A,$
L2 $\quad A,\ A \to B \Rightarrow B,$
L3 $\quad A \to B,\ B \to C \Rightarrow A \to C,$
L4 $\quad A \wedge B \to A,\ A \wedge B \to B,$
L5 $\quad A \to B,\ A \to C \Rightarrow A \to B \wedge C,$
L6 $\quad A \to A \vee B,\ B \to A \vee B,$
L7 $\quad A \to C,\ B \to C \Rightarrow A \vee B \to C,$
L8 $\quad A \wedge B \to C \Rightarrow A \to (B \to C),$
L9 $\quad A \to (B \to C) \Rightarrow A \wedge B \to C,$
L10 $\quad \bot \to A,$
L11 $\quad B \to A \Rightarrow B \to \forall x\, A \ \ (x \notin \mathrm{FV}(B)),$
L12 $\quad \forall x\, A \wedge t\!\downarrow \to A[x/t] \ \ (t \text{ free for } x \text{ in } A),$
L13 $\quad A[x/t] \wedge t\!\downarrow \to \exists x\, A \ \ (t \text{ free for } x \text{ in } A),$
L14 $\quad A \to B \Rightarrow \exists x\, A \to B \ \ (x \notin \mathrm{FV}(B))$

where $t\!\downarrow := t = t$. For equality we have (F function symbol, R relation symbol of the language):

$$\mathrm{EQ} \quad \begin{cases} \forall xy(x = y \to y = x), \quad \forall xyz(x = y \wedge y = z \to x = z), \\ \forall \vec{x}\vec{y}(\vec{x} = \vec{y} \wedge F\vec{x}\!\downarrow \to F\vec{x} = F\vec{y}), \quad \forall \vec{x}\vec{y}(R\vec{x} \wedge \vec{x} = \vec{y} \to R\vec{y}) \end{cases}$$

Basic predicates and functions of the language are assumed to be strict:

STR $F(t_1, \ldots, t_n)\downarrow \to t_i\downarrow, \quad R(t_1, \ldots, t_n) \to t_i\downarrow$

Note that this logic reduces to ordinary first-order intuitionistic logic if all functions are total, i.e. $\forall\vec{x}(f\vec{x}\downarrow)$, since then $t\downarrow$ for all terms t.

For the notion *"equally defined and equal if defined"* introduced by

$$t \simeq s := (t\downarrow \vee s\downarrow) \to t = s,$$

we can prove the replacement schema for arbitrary formulas A

$$t \simeq s \wedge A[x/t] \to A[x/s].$$

4 Conservativeness of defined functions

Relative to the logic of partial terms, the following conservative extension result is easily proved. Let Γ be a theory based on LPT, such that

$$\Gamma \vdash A(\vec{x}, y) \wedge A(\vec{x}, z) \to y = z.$$

Then we may introduce a symbol ϕ_A for a partial function with axiom

$$\mathrm{Ax}(\phi_A)(\vec{x}, y) \leftrightarrow y = \phi_A(\vec{x}).$$

The conservativeness of this addition can be proved in a straightforward syntactic way; the easiest method, however, uses completeness for Kripke models, see Troelstra and van Dalen (1988, 2.7).

Let Γ^* consist of Γ and all substitution instances of the axiom schemata w.r.t. the extended language, and let $\phi(\Gamma^*)$ be the result of systematically eliminating the function symbol ϕ_A from the elements of Γ, and assume $\phi(\Gamma^*)$ to be provable from Γ, then the conservative extension result still holds in the form: "$\Gamma^* + \mathrm{Ax}(\phi_A)$ is conservative over Γ".

This extended result applies to $\mathbf{HA}^*$ defined below, since eliminating the symbol for partial recursive function application from instances of induction yields instances of induction in the language of $\mathbf{HA}$.

5 Formalizing elementary recursion theory in $\mathbf{HA}^*$

5.1. $\mathbf{HA}^*$ is the conservative extension of $\mathbf{HA}$, formulated in the intuitionistic logic of partial terms, with a primitive binary partial operation $\bullet$ of partial recursive function application. $t_1 \bullet t_2 \bullet t_3 \ldots$ abbreviates $(\ldots((t_1 \bullet t_2) \bullet t_3)\ldots)$ (association to the left).

Note that strictness entails in particular $t \bullet t'\downarrow \to t\downarrow \wedge t'\downarrow$ for the application operation. Of course we have to require totality for the primitive recursive functions; it suffices to demand $0\downarrow$, $Sx\downarrow$. In all other case the primitive recursive functions satisfy equations with $=$, characterizing them inductively in terms of functions introduced before (e.g. $x + 0 = x$, $x + Sy = S(x + y)$). By induction one can then prove $Fx_1 \ldots x_n\downarrow$ for each primitive recursive function symbol F.

A formalization of elementary recursion theory in $\mathbf{HA}^*$ can be given by using Kleene's index method in combination with the theory of elementary inductive definitions in arithmetic (Troelstra and van Dalen 1988, 3.6, 3.7). The idea behind this formalization is the following: one gives an elementary inductive definition of the relation $\Omega := \{(n, x, m) : x{\bullet}m \simeq n\}$. An elementary inductive definition of a predicate P_A is given by a predicate $A(X, z)$ in the language of $\mathbf{HA}^*$ extended with an extra predicate variable X, such that A is in a class $\mathcal{P}$ generated by the following clauses:

- all arithmetical formulas are in $\mathcal{P}$;
- $Xt \in \mathcal{P}$ for all numerical terms t;
- $\mathcal{P}$ is closed under $\wedge, \vee, \exists$ and bounded universal quantification $\forall x < t$ with $x \notin \mathrm{FV}(t)$.

The predicate P_A then satisfies

$$\forall x(A(P_A, x) \to P_A(x), \quad \text{and} \quad \forall x(A(Q, x) \to Qx) \to \forall x(P_A(x) \to Qx),$$

for all predicates Q definable in $\mathbf{HA}^*$ extended with P_A. Predicates introduced by elementary inductive definitions are in fact explicitly definable in arithmetic, and the principles for P_A stated above are provable in arithmetic.

This leads to a smooth formalization of elementary recursion theory; in particular we obtain the smn-theorem, the recursion theorem (Kleene's fixed-point theorem): for some primitive recursive ϕ

$$\forall \vec{x}\vec{y}z(\phi(z, \vec{x}){\bullet}(\vec{y}) \simeq z{\bullet}(\vec{x}, \vec{y}))$$

(where $(\vec{u})$ is some standard encoding of the sequence $\vec{u}$), the Kleene normal form theorem, etc. Moreover, by the normal form theorem, every partial recursive function is definable by a term of the language of $\mathbf{HA}^*$.

5.2. NOTATION. If t is a term in the language of $\mathbf{HA}^*$, then $\Lambda x.t$ is a canonically chosen code number for t as a partial recursive function of x, uniformly in the other free variables; by the smn-theorem we may therefore assume $\Lambda x.t$ to be primitive recursive in $\mathrm{FV}(t) \setminus \{x\}$. $\Lambda x_1 \ldots x_n.t$ abbreviates $\Lambda x_1(\Lambda x_2 \ldots (\Lambda x_n.t) \ldots)$. □

We note the following

5.3. LEMMA. *In* $\mathbf{HA}^*$ *the* Σ^0_1*-formulas of* $\mathbf{HA}$ *are equivalent to prime formulas of the form* $t = t$ *for suitable* t*, and each formula* $t = s$ *is equivalent to a* Σ^0_1*-formula of* $\mathbf{HA}$.

PROOF. Systematically using the equivalences mentioned above transforms any formula $t = s$ of $\mathbf{HA}^*$ into a Σ^0_1-formula of $\mathbf{HA}$. Conversely, let a Σ^0_1-formula be given; by the normal form results of recursion theory, we can write this in the form $\exists z T(\bar{n}, \langle \vec{x} \rangle, z)$ for a numeral $\bar{n}$; this is equivalent to $\bar{n}{\bullet}\langle \vec{x} \rangle = \bar{n}{\bullet}\langle \vec{x} \rangle$. □

We are now ready to formalize $x \underline{\mathrm{rn}}\, A$ directly in $\mathbf{HA}^*$.

6 Formalizing $\underline{\mathtt{rn}}$-realizability in $\mathbf{HA}^*$

6.1. Definition. $x \,\underline{\mathtt{rn}}\, A$ is defined by induction on the complexity of A, $x \notin \mathrm{FV}(A)$.

$$\begin{aligned}
x \,\underline{\mathtt{rn}}\, P &:= P \wedge x{\downarrow} \text{ for } P \text{ prime},\\
x \,\underline{\mathtt{rn}}\, (A \wedge B) &:= \mathbf{p}_0 x \,\underline{\mathtt{rn}}\, A \wedge \mathbf{p}_1 x \,\underline{\mathtt{rn}}\, B,\\
x \,\underline{\mathtt{rn}}\, (A \to B) &:= \forall y (y \,\underline{\mathtt{rn}}\, A \to x{\bullet}y \,\underline{\mathtt{rn}}\, B) \wedge x{\downarrow},\\
x \,\underline{\mathtt{rn}}\, \forall y\, A &:= \forall y (x{\bullet}y \,\underline{\mathtt{rn}}\, A),\\
x \,\underline{\mathtt{rn}}\, \exists y\, A &:= \mathbf{p}_1 x \,\underline{\mathtt{rn}}\, A[y/\mathbf{p}_0 x].
\end{aligned}$$

We also define a combination of realizability with truth, $x \,\underline{\mathtt{rnt}}\, A$; the clauses are the same as for $\underline{\mathtt{rn}}$, the clause for implication excepted, which now reads:

$$x \,\underline{\mathtt{rnt}}\, (A \to B) := \forall y (y \,\underline{\mathtt{rnt}}\, A \to x{\bullet}y \,\underline{\mathtt{rnt}}\, B) \wedge x{\downarrow} \wedge (A \to B). \qquad \square$$

6.2. Remarks. (i) $t \,\underline{\mathtt{rn}}\, A$ is $\exists$-free (i.e. does not contain $\exists$) for all A. Note that, by our definition of $\vee$ in terms of the other operators, $\exists$-free implies $\vee$-free.

(ii) The clauses "$\wedge\, x{\downarrow}$" have been added for the cases of prime formulas and implications, in order to guarantee the truth of part (i) of the following lemma.

(iii) For negations we have $x \,\underline{\mathtt{rn}}\, \neg A \leftrightarrow \forall y (\neg y \,\underline{\mathtt{rn}}\, A) \wedge x{\downarrow}$, and $x \,\underline{\mathtt{rn}}\, \neg\neg A \leftrightarrow \forall y (\neg y \,\underline{\mathtt{rn}}\, \neg A) \wedge x{\downarrow} \leftrightarrow \forall y \neg \forall z \neg (z \,\underline{\mathtt{rn}}\, A) \wedge x{\downarrow} \leftrightarrow \neg\neg \exists z (z \,\underline{\mathtt{rn}}\, A) \wedge x{\downarrow}$.
The following lemmas are easily proved by induction on A.

6.3. Lemma. *(Definedness of realizing terms; Substitution Property) For* $\mathrm{R} \in \{\underline{\mathtt{rn}}, \underline{\mathtt{rnt}}\}$

(i) $\vdash t \,\mathrm{R}\, A \to t{\downarrow}$,

(ii) $(x \,\mathrm{R}\, A)[y/t] \equiv x \,\mathrm{R}\, (A[y/t])$ $(x \notin \mathrm{FV}(A) \cup \mathrm{FV}(t), y \not\equiv x)$.

Proof. By induction on the complexity of A. Let e.g. $t \,\underline{\mathtt{rn}}\, \exists y A$, then $\mathbf{p}_1 t \,\underline{\mathtt{rn}}\, A[y/\mathbf{p}_0 t]$, hence by induction hypothesis $\mathbf{p}_1 t{\downarrow}$, and so by strictness $t{\downarrow}$. $\square$

6.4. Lemma. $\mathbf{HA}^* \vdash t \,\underline{\mathtt{rnt}}\, A \to A$.
A similar lemma holds for all combinations of realizability with truth (i.e. realizabilities with $\underline{\mathtt{t}}$ in their mnemonic code) we shall encounter in the sequel; we shall not bother to state it explicitly in the future. We can readily prove that realizability is sound for $\mathbf{HA}^*$:

7 Soundness

7.1. Theorem. *(Soundness theorem)*

$$\mathbf{HA}^* \vdash A \Rightarrow \mathbf{HA}^* \vdash t \,\underline{\mathtt{rn}}\, A \wedge t \,\underline{\mathtt{rnt}}\, A$$

for a suitable term t with $\mathrm{FV}(t) \subset \mathrm{FV}(A)$.

PROOF. The proof proceeds by induction on the length of derivations; that is to say, we have to find realizing terms for the axioms, and for the rules we must show how to find a realizing term for the conclusion from realizing terms for the premises. We check some cases.

L5. Assume $t \,\underline{\mathtt{rn}}\, (A \to B)$, $t' \,\underline{\mathtt{rn}}\, (A \to C)$, and let $x \,\underline{\mathtt{rn}}\, A$; then $\mathbf{p}(t{\bullet}x, t'{\bullet}x) \,\underline{\mathtt{rn}}\, (B \wedge C)$, so $\Lambda x.\mathbf{p}(t{\bullet}x, t'{\bullet}x) \,\underline{\mathtt{rn}}\, (A \to B \wedge C)$.

L14. Assume $t \,\underline{\mathtt{rn}}\, (A \to B)$, $x \notin \mathrm{FV}(B)$, and let $y \,\underline{\mathtt{rn}}\, \exists x\, A$, then $\mathbf{p}_1 y \,\underline{\mathtt{rn}}\, A[x/\mathbf{p}_0 y]$, hence $t[x/\mathbf{p}_0 y]{\bullet}(\mathbf{p}_1 y) \,\underline{\mathtt{rn}}\, B$, so $\Lambda y.t[x/\mathbf{p}_0 y]{\bullet}(\mathbf{p}_1 y) \,\underline{\mathtt{rn}}\, (\exists x\, A \to B)$.

Of the non-logical axioms, only induction requires attention. Suppose

$$x \,\underline{\mathtt{rn}}\, (A[y/0] \wedge \forall y(A \to A[y/Sy])).$$

Then

$$\mathbf{p}_0 x \,\underline{\mathtt{rn}}\, A[y/0], \quad z \,\underline{\mathtt{rn}}\, A \to (\mathbf{p}_1 x){\bullet}y{\bullet}z \,\underline{\mathtt{rn}}\, A[y/Sy].$$

So let t be such that

$$t{\bullet}0 \simeq \mathbf{p}_0 x, \quad t{\bullet}(Sy) \simeq (\mathbf{p}_1 x){\bullet}y{\bullet}(t{\bullet}y).$$

The existence of t follows either by an application of the recursion theorem, or is immediate if closure under recursion has been built directly into the definition of recursive function. It is now easy to prove by induction that t realizes induction for A. □

A statement weaker than soundness is $\vdash A \Rightarrow \vdash \exists x(x \,\underline{\mathtt{rn}}\, A)$; we might call this *weak soundness*. We can also prove a stronger version of soundness:

7.2. THEOREM. *(Strong Soundness Theorem) For closed A*

$$\mathbf{HA}^* \vdash A \Rightarrow \mathbf{HA}^* \vdash \bar{n} \,\underline{\mathtt{rn}}\, A \wedge \bar{n} \,\underline{\mathtt{rnt}}\, A \quad \textit{for some numeral } \bar{n}.$$

PROOF. Let $\mathbf{HA}^* \vdash A$; from the soundness theorem we find a term t such that

$$t \,\underline{\mathtt{rn}}\, A, \text{ hence } t{\downarrow}.$$

$t{\downarrow}$, i.e. $t = t$ is equivalent to a Σ^0_1-formula of **HA**, say $\exists x(s = 0)$, and **HA** proves only true Σ^0_1-formulas, from which we see that $t = \bar{n}$ must be provable in $\mathbf{HA}^*$ for some numeral $\bar{n}$. Similarly for $\underline{\mathtt{rnt}}$. □

7.3. REMARK. If one formalizes the proof of the soundness theorem, it is easy to see that there are primitive recursive functions ψ, ϕ such that

$$\mathbf{HA} \vdash \mathrm{Prf}(x, \ulcorner A \urcorner) \to \mathrm{Prf}(\phi(x), \mathrm{Sub}(\ulcorner y \,\underline{\mathtt{rn}}\, A \urcorner, y, \psi(x)))$$

where "Prf" is the formalized proof-predicate of $\mathbf{HA}^*$, $\ulcorner \xi \urcorner$ is the gödelnumber of expression ξ, and $\mathrm{Sub}(\ulcorner B \urcorner, x, \ulcorner s \urcorner)$ is the gödelnumber of $B[x/s]$.

In fact, the whole implication is provable even in primitive recursive arithmetic. But the statement expressing a formalized version of the *strong* completeness theorem:

$$\mathrm{Prf}(x, \ulcorner A \urcorner) \to \mathrm{Prf}(\phi(x), \ulcorner \overline{\psi(x)}\, \underline{\mathtt{rn}}\, A \urcorner)$$

(A closed, for suitable provably recursive ϕ, ψ) is not provable in **HA** (see 10.6).

The following lemma will be used in the sequel, but is also interesting in its own right:

7.4. LEMMA. *(Self-realizing formulas) For $\exists$-free formulas, canonical realizers exist, that is to say for each $\exists$-free A we have in* $\mathbf{HA}^*$

(i) $\vdash \exists x(x\, \underline{\mathtt{rn}}\, A) \to A$,

(ii) $\vdash A \to t_A\, \underline{\mathtt{rn}}\, A$ *for some term* t_A *with* $\mathrm{FV}(t_A) \subset \mathrm{FV}(A)$.

(iii) A formula A is provably equivalent to its own realizability, i.e. $A \leftrightarrow \exists x(x\, \underline{\mathtt{rn}}\, A)$), iff A is provably equivalent to an existentially quantified $\exists$-free formula.

(iv) Realizability is idempotent, i.e. $\exists x(x\, \underline{\mathtt{rn}}\, \exists y(y\, \underline{\mathtt{rn}}\, A)) \leftrightarrow \exists x(x\, \underline{\mathtt{rn}}\, A)$; in fact,even $\exists x(x\, \underline{\mathtt{rn}}\, (A \leftrightarrow \exists y(y\, \underline{\mathtt{rn}}\, A)))$ holds.

PROOF. Take $t_{s=s'} := 0$, $t_{A \wedge B} := \mathbf{p}(t_A, t_B)$, $t_{\forall x A} := \Lambda x.t_A$, $t_{A \to B} := \Lambda x.t_B$ ($x \notin \mathrm{FV}(t_B)$), and prove (i) and (ii) by simultaneous induction on A. (iii) and (iv) are immediate corollaries. □

7.5. REMARK. An observation of practical usefulness is the following. For any definable predicate with canonical realizers (i.e. a predicate A definable by an $\exists$-free formula) we obtain an equivalent realizability if we read restricted quantifiers $\forall x(A(x) \to \dots)$ and $\exists x(A(x) \wedge \dots)$ as quantifiers $\forall x{\in}A$, $\exists x{\in}A$ over a new domain with realizability clauses copied from numerical quantification, i.e.

$$x\, \underline{\mathtt{rn}}\, \forall y{\in}A.B := \forall y{\in}A(x{\bullet}y\, \underline{\mathtt{rn}}\, B) \wedge x{\downarrow},$$
$$x\, \underline{\mathtt{rn}}\, \exists y{\in}A.B := \mathbf{p}_1 x\, \underline{\mathtt{rn}}\, B[x/\mathbf{p}_0 x] \wedge A(\mathbf{p}_0 x).$$

In short, we may simply forget about the canonical realizers.

8 AXIOMATIZING PROVABLE REALIZABILITY

8.1. As we have seen already in the introduction, realizability validates more than what is provable in **HA**; in fact, we can formally prove realizability of in $\mathbf{HA}^*$ an intuitionistic version of Church's thesis:

CT_0 $\forall x \exists y\, A(x, y) \to \exists z \forall x(A(x, z{\bullet}x) \wedge z{\bullet}x{\downarrow})$.

CT_0 is certainly not *provable in* **HA**, since it is in fact refutable in classical arithmetic. This version of Church's thesis is in fact a combination of the well-known version which states "Each humanly computable function is recursive" and the intuitionistic reading of $\forall x \exists y A(x, y)$ which states that there is a method

for constructing, for each given x, a y such that $A(x,y)$. Such a method describes a humanly computable function.

We now ask ourselves: is there a reasonably simple axiomatization (by a few axiom schemata say) of the formulas provably realizable in **HA** ? The answer is yes, the provably realizable formulas can be axiomatized by a generalization of CT_0, namely *"Extended Church's Thesis"*:

$$\mathrm{ECT}_0 \forall x(Ax \to \exists y\, Bxy) \to \exists z \forall x(Ax \to z{\bullet}x{\downarrow} \wedge B(x, z{\bullet}x)) \quad (A \ \exists\text{-free}).$$

8.2. LEMMA. *Each instance of* ECT_0 *is* $\mathbf{HA}^*$*-realizable.*
PROOF. Suppose

$$u \,\underline{\mathtt{rn}}\, \forall x(Ax \to \exists y Bxy)$$

Then $\forall xv(v \,\underline{\mathtt{rn}}\, Ax \to u{\bullet}x{\bullet}v \,\underline{\mathtt{rn}}\, \exists y Bxy)$, and since A is $\exists$-free, in particular $\forall x(Ax \to u{\bullet}x{\bullet}t_A \,\underline{\mathtt{rn}}\, \exists y Bxy)$, so $\forall x(Ax \to \mathbf{p}_1(u{\bullet}x{\bullet}t_A) \,\underline{\mathtt{rn}}\, B(x, \mathbf{p}_0(u{\bullet}x{\bullet}t_A))$. Then it is straightforward to see that

$$\mathbf{p}(\Lambda x.\mathbf{p}_0(u{\bullet}x{\bullet}t_A), \Lambda xv.\mathbf{p}(0, \mathbf{p}_1(u{\bullet}x{\bullet}t_A)))$$

realizes the conclusion. □

REMARK. The condition "A is $\exists$-free" in ECT_0 cannot be dropped: applying unrestricted ECT_0 to $Ax := \exists z Txxz \vee \neg\exists z Txxz$, $Bxy := (y = 0 \wedge \exists z Txxz) \vee (y = 1 \wedge \neg\exists z Txxz)$ yields a contradiction. In fact, this example can be used to show that even unrestricted $\mathrm{ECT}_0!$ fails ($\mathrm{ECT}_0!$ is like ECT_0 except that $\exists y$ in the premise is replaced by $\exists! y$; $\exists! y$ means "there is a unique y such that").

8.3. THEOREM. *(Characterization Theorem for* $\underline{\mathtt{rn}}$*-realizability)*

(i) $\mathbf{HA}^* + \mathrm{ECT}_0 \vdash A \leftrightarrow \exists x(x \,\mathtt{R}\, A)$ *for* $\mathtt{R} \in \{\underline{\mathtt{rn}}, \underline{\mathtt{rnt}}\}$,

(ii) For closed A, $\mathbf{HA}^* + \mathrm{ECT}_0 \vdash A \Leftrightarrow \mathbf{HA}^* \vdash \bar{n} \,\underline{\mathtt{rn}}\, A$ *for some numeral* $\bar{n}$.

PROOF. (i) is proved by a straightforward induction on A. The crucial case is $A \equiv B \to C$; then $B \to C \leftrightarrow (\exists x(x \,\underline{\mathtt{rn}}\, B) \to \exists y(y \,\underline{\mathtt{rn}}\, C))$ (by the induction hypothesis) $\leftrightarrow \forall x(x \,\underline{\mathtt{rn}}\, B \to \exists y(y \,\underline{\mathtt{rn}}\, C))$ (by pure logic) $\leftrightarrow \exists z \forall x(x \,\underline{\mathtt{rn}}\, B \to z{\bullet}x \,\underline{\mathtt{rn}}\, C)$ (by ECT_0, since $x \,\underline{\mathtt{rn}}\, B$ is $\exists$-free) $\equiv \exists z(z \,\underline{\mathtt{rn}}\, (B \to C))$.

(ii). The direction $\Rightarrow$ follows from the strong soundness theorem plus the lemma; $\Leftarrow$ is an immediate consequence of (i). □

Curiosity prompts us to ask which formulas are classically provably realizable, i.e. provably realizable in first-order Peano Arithmetic **PA**, which is just **HA** with classical logic. The answer is contained in the following

8.4. PROPOSITION. $\mathbf{PA} \vdash \exists x(x \underline{\mathtt{rn}}\, A) \Leftrightarrow \mathbf{HA} + \mathrm{M} + \mathrm{ECT}_0 \vdash \neg\neg A$,
where M *is* Markov's principle*:*

M $\quad \forall x(A \vee \neg A) \wedge \neg\neg\exists x\, A \to \exists x\, A.$

PROOF. Let $\mathbf{PA} \vdash \exists x(x \underline{\mathtt{rn}}\, A)$, and let B be a negative formula (i.e. a formula in the $\wedge, \forall, \to$-fragment) such that $\mathbf{HA} + \mathrm{M} \vdash x \underline{\mathtt{rn}}\, A \leftrightarrow B(x)$. Then $\mathbf{PA} \vdash \neg\forall x\neg(x \underline{\mathtt{rn}}\, A)$, and since **PA** is conservative over **HA** for negative formulas (in consequence of Gödel's negative translation), also $\mathbf{HA} \vdash \neg\forall x\neg B$, i.e. $\mathbf{HA} + \mathrm{M} \vdash \neg\neg\exists x(x \underline{\mathtt{rn}}\, A)$, and thus it follows that $\mathbf{HA} + \mathrm{M} + \mathrm{ECT}_0 \vdash \neg\neg A$. The converse is simpler. □

9 EXTENSIONS OF $\mathbf{HA}^*$

9.1. For suitable sets Γ of extra axioms, we may replace $\mathbf{HA}^*$ in the soundness and characterization theorem by $\mathbf{HA}^* + \Gamma$. Weak soundness and the characterization theorem require for all $A \in \Gamma$

(1) $\quad \mathbf{HA}^* + \Gamma \vdash \exists x(x \underline{\mathtt{rn}}\, A)$.

Soundness requires for all $A \in \Gamma$

(2) $\quad \mathbf{HA}^* + \Gamma \vdash t \underline{\mathtt{rn}}\, A$ for some term t,

and Strong Soundness requires (2) and in addition: $\mathbf{HA}^* + \Gamma$ proves only true Σ^0_1-formulas.

9.2. EXAMPLES

(a) For Γ any set of $\exists$-free formulas soundness and the characterization theorem extend. If $\mathbf{HA}^* + \Gamma$ proves only true Σ^0_1-formulas, strong soundness holds. The next two examples permit characterization and strong soundness.

(b) Let $\prec$ be a primitive recursive well-ordering of $\mathbb{N}$, provably total and linear in $\mathbf{HA}^*$; for Γ we take all instances of *transfinite induction over* $\prec$:

$\mathrm{TI}(\prec)\forall y(\forall x{\prec}y\, A \to A[x/y]) \to \forall x A.$

(c) Γ is the set of instances of Markov's principle (cf. the last proposition in 8). In fact, in the presence of CT_0, which is valid under realizability, Γ may be replaced by a single axiom:

$$\forall xy(\neg\neg\exists z Txyz \to \exists z Txyz).$$

It is also worth noting that in the presence of M, we can use the following variant of ECT_0 which is equivalent to ECT_0:

$\mathrm{ECT}'_0\forall x(\neg A \to \exists y Bxy) \to \exists z\forall x(\neg A \to z{\bullet}x{\downarrow} \wedge B(x, z{\bullet}y))$.

(d) An extension of another kind is obtained if we enrich the language with constants for inductively defined predicates, e.g. the tree predicate Tr. Intuitively, Tr is the least set containing the (code of the) single-node tree (i.e. $\langle\,\rangle \in \mathrm{Tr}$), and with every recursive sequence of tree codes $n{\bullet}0, n{\bullet}1, \ldots, n{\bullet}m, \ldots$ in Tr, Tr also contains a code for the infinite tree having the trees with codes $n{\bullet}m$ as immediate subtrees, namely $\mathbf{p}(1, n)$. Thus if

$$A(X, x) := (x = 0) \vee (\mathbf{p}_0 x = 1 \wedge \forall m(\mathbf{p}_1 x{\bullet}m \in X))$$

we have

$$A(\mathrm{Tr}, x) \to x \in \mathrm{Tr},$$
$$\forall x(A(\lambda y.B, x) \to B[y/x]) \to \forall x \in \mathrm{Tr}.B[y/x]$$

for all B in the language extended with the new primitive predicate Tr. Then we can extend $\underline{\mathbf{rn}}$-realizability simply by putting

$$x\,\underline{\mathbf{rn}}\,(t \in \mathrm{Tr}) := t \in \mathrm{Tr}.$$

Let us check that the soundness theorem extends. $A(\mathrm{Tr}, x)$ is equivalent to an $\exists$-free formula, so its realizability implies its truth, and $x \in \mathrm{Tr}$ follows. As to the schema, assume

$$u\,\underline{\mathbf{rn}}\,\forall x(A(\lambda y.B, x) \to B[y/x]), \text{ or}$$
$$u\,\underline{\mathbf{rn}}\,\forall x[(x = 0 \to B(0)) \wedge (\mathbf{p}_0 x = 1 \wedge \forall y B(\mathbf{p}_1 x{\bullet}y) \to Bx)].$$

So

$$\mathbf{p}_0(u{\bullet}0){\bullet}(0,0)\,\underline{\mathbf{rn}}\,B(0),$$
$$\mathbf{p}_1(u{\bullet}x){\bullet}v\,\underline{\mathbf{rn}}\,B(x) \text{ if } \mathbf{p}_0 x = 1 \text{ and } v\,\underline{\mathbf{rn}}\,(\mathbf{p}_0 x = 1 \wedge \forall y B(\mathbf{p}_1 x{\bullet}y).$$

Assume $\forall y(e{\bullet}(\mathbf{p}_1 x{\bullet}y)\,\underline{\mathbf{rn}}\,B(\mathbf{p}_1 x{\bullet}y))$, $\mathbf{p}_0 x = 1$. Then

$$v = \mathbf{p}(0, \Lambda y.e{\bullet}(\mathbf{p}_1 x{\bullet}y))\,\underline{\mathbf{rn}}\,(\mathbf{p}_0 x = 1 \wedge \forall y B(\mathbf{p}_1 x{\bullet}y)).$$

Therefore

$$\text{if } \mathbf{p}_0 x = 1 \text{ and } \forall y(e{\bullet}(\mathbf{p}_1 x{\bullet}y)\,\underline{\mathbf{rn}}\,B(\mathbf{p}_1 x{\bullet}y))$$
$$\text{then } \mathbf{p}_1(u{\bullet}x){\bullet}(0, \Lambda y.e{\bullet}(\mathbf{p}_1 x{\bullet}y))\,\underline{\mathbf{rn}}\,B(x).$$

Now we construct by the recursion theorem an e such that

$$e{\bullet}x \simeq \begin{cases} \mathbf{p}_0(u{\bullet}0){\bullet}0 \text{ if } x = 0, \\ \mathbf{p}_1(u{\bullet}x){\bullet}\mathbf{p}(0, \Lambda y.e{\bullet}(\mathbf{p}_1 x{\bullet}y)) \text{ if } \mathbf{p}_0 x = 1, \\ \text{undefined otherwise.} \end{cases}$$

We then prove by induction on Tr that $\forall x \in \mathrm{Tr}(e{\bullet}x\,\underline{\mathbf{rn}}\,B(x))$. This is straightforward. This example is capable of considerable generalization, namely to arithmetic enriched with constants for predicates introduced by iterated inductive definitions of higher level; see e.g. Buchholz, Feferman, Pohlers and Sieg (1981, IV, section 6).

The examples just mentioned also permit extension of $\underline{\mathbf{rnt}}$-realizability. We end the section with some applications of $\underline{\mathbf{rn}}$- and $\underline{\mathbf{rnt}}$-realizability.

10 Applications

10.1. Proposition. *(Consistency and inconsistency results)*

(i) $\mathbf{HA}^* + \mathrm{ECT}_0$ *is consistent relative to* $\mathbf{HA}^*$ *(and hence also relative to* $\mathbf{PA}$*).*

(ii) $\neg\forall x(A \vee \neg A)$, $\neg(\forall x \neg\neg B \to \neg\neg\forall x B)$ *are consistent with* $\mathbf{HA}^*$ *for certain arithmetical* A, B.

(iii) The schema "Independence of Premise"

$$\mathrm{IP} \quad (\neg A \to \exists z B) \to \exists z(\neg A \to B)$$

is not derivable in $\mathbf{HA}^* + \mathrm{CT}_0 + \mathrm{M}$*; in fact,* $\mathbf{HA}^* + \mathrm{IP} + \mathrm{CT}_0 + \mathrm{M} \vdash 1 = 0$.

Proof. (i) Immediate from the characterization theorem.

(ii) is a corollary of the realizability of CT_0: take $A \equiv \exists y T x x y$, $B \equiv \exists y T x x y \vee \neg\exists y T x x y$.

(iii) By M, $\neg\neg\exists y T x x y \to \exists z T x x z$; apply IP to obtain $\forall x \exists z(\neg\neg\exists y T x x y \to T x x z)$, then by CT_0 there is a total recursive F such that $\neg\neg\exists y T x x y \to T(x, x, Fx)$, and this would make $\exists y T x x y$ recursive in x. □

We next give an example of a conservative extension result.

10.2. Definition. $\mathrm{CC}(\underline{\mathtt{rn}})$ (the $\underline{\mathtt{rn}}$-Conservative Class) is the class of formulas A such that whenever $B \to C$ is a subformula of A, then B is $\exists$-free. □

10.3. Lemma. *For* $A \in \mathrm{CC}(\underline{\mathtt{rn}})$ *we have* $\vdash \exists x(x \,\underline{\mathtt{rn}}\, A) \to A$.

Proof. By induction on the structure of A. Consider the case $A \equiv B \to C$; then B is $\exists$-free, so there is a t_B such that $\vdash B \to t_B \,\underline{\mathtt{rn}}\, B$. Assume B and $x \,\underline{\mathtt{rn}}\, (B \to C)$, then $x{\bullet}t_B{\downarrow} \wedge x{\bullet}t_B \,\underline{\mathtt{rn}}\, C$, hence by the induction hypothesis C; therefore $(x \,\underline{\mathtt{rn}}\, (B \to C)) \to (B \to C)$. □

The lemma in combination with the characterization theorem yields

10.4. Proposition. $\mathbf{HA}^* + \mathrm{ECT}_0$ *is conservative over* $\mathbf{HA}^*$ *w.r.t. formulas in* $\mathrm{CC}(\underline{\mathtt{rn}})$*:*

$$(\mathbf{HA}^* + \mathrm{ECT}_0) \cap \mathrm{CC}(\underline{\mathtt{rn}}) = \mathbf{HA}^* \cap \mathrm{CC}(\underline{\mathtt{rn}}).$$

The following proposition follows from $\underline{\mathtt{rnt}}$-realizability.

10.5. Proposition. *(Derived rules) In* $\mathbf{HA}^*$

(i) For sentences $\vdash A \vee B \Rightarrow \vdash A$ *or* $\vdash B$ *(Disjunction property* DP*),*

(ii) For sentences $\vdash \exists x A \Rightarrow \vdash A[x/\bar{n}]$ *for some numeral* $\bar{n}$ *(Explicit Definability for Numbers* EDN*),*

(iii) Extended Church's Rule: for $\exists$*-free* A

$$\text{ECR}\ \vdash \forall x(A \to \exists y B x y) \ \Rightarrow\ \vdash \exists z \forall x(A \to z{\bullet}x{\downarrow} \wedge B(x, z{\bullet}x)).$$

PROOF. (i) follows from (ii) (actually, (i) and (ii) are equivalent for systems containing a minimum of arithmetic, see Friedman (1975)). As to (ii), let $\vdash \exists x A$, then, by the strong soundness for $\underline{\mathtt{rnt}}$-realizability, $\vdash \bar{m}\,\underline{\mathtt{rnt}}\,\exists x A$ for some numeral $\bar{m}$, so $\vdash \mathbf{p}_1\bar{m}\,\underline{\mathtt{rnt}}\,A[x/\mathbf{p}_0\bar{m}]$, and hence $\vdash A[x/\mathbf{p}_0\bar{m}]$.

(iii) Assume $\vdash \forall x(A \to \exists y B x y)$, then for a suitable $t \vdash t\,\underline{\mathtt{rnt}}\,\forall x(A \to \exists y B x y)$, i.e.

$$\vdash \forall x \forall z(z\,\underline{\mathtt{rnt}}\,A \to \mathbf{p}_1(t{\bullet}x{\bullet}z)\,\underline{\mathtt{rnt}}\,B(x, \mathbf{p}_0(t{\bullet}x{\bullet}z)).$$

Since $t_A\,\underline{\mathtt{rnt}}\,A$,

$$\vdash \forall x(A \to \mathbf{p}_1(t{\bullet}x{\bullet}t_A)\,\underline{\mathtt{rnt}}\,B(x, \mathbf{p}_0(t{\bullet}x{\bullet}t_A))),$$

and therefore $\vdash \forall x(A \to B(x, \mathbf{p}_0(t{\bullet}x{\bullet}t_A))$. So we can take $z = \Lambda x.\mathbf{p}_0(t{\bullet}x{\bullet}t_A)$. □

10.6. REMARK. The DP cannot be formalized in any consistent extension of **HA** itself (Myhill (1973), Friedman (1977)). We sketch Myhill's argument (the result of Friedman is even stronger). Assume that there is a provably recursive function f satisfying

$$\vdash \mathrm{Prf}(x, \ulcorner A \vee B \urcorner) \to ((fx = 0 \wedge \mathrm{Pr}(\ulcorner A \urcorner)) \vee ((fx = 1 \wedge \mathrm{Pr}(\ulcorner B \urcorner))).$$

where $\mathrm{Pr}(x) := \exists y \mathrm{Prf}(y, x)$. So $f = \{\bar{p}\}$, and $\vdash \forall x \exists y T \bar{p} x y$. Let F enumerate all primitive recursive functions, i.e. $\lambda n.F(i, n)$ is the i-th primitive recursive function. Put

$$D(n) := \bar{p}{\bullet}F(n, n) \neq 0,$$

then $\vdash \forall n(Dn \vee \neg Dn)$ (i.e. $\mathrm{Prf}(\bar{k}, \ulcorner \forall n(Dn \vee \neg Dn) \urcorner)$ for a specific $\bar{k}$), from which we can find a particular primitive recursive $\lambda n.F(\bar{m}, n)$ such that $\vdash \mathrm{Prf}(F(\bar{m}, \bar{n}), \ulcorner D\bar{n} \vee \neg D\bar{n} \urcorner)$. Then $D\bar{m} \to \bar{p}{\bullet}F(\bar{m}, \bar{m}) \neq 0 \to \mathrm{Prf}(F(\bar{m}, \bar{m}), \ulcorner D\bar{m} \vee \neg D\bar{m} \urcorner) \wedge \mathrm{Pr}(\ulcorner \neg D\bar{m} \urcorner)$, hence $\neg D\bar{m}$ follows, since $\mathbf{HA}^*$ is consistent. If we start assuming $\neg D\bar{m}$, we similarly obtain a contradiction.

From this we see that DP cannot be proved in $\mathbf{HA}^*$ itself; for if DP were provable in $\mathbf{HA}^*$, then a function f as above would be given by

$f(x) := \mathbf{p}_0$(the least y s.t.(x does not prove a closed disjunction and $y = 0$)
or (for some closed $\ulcorner A \vee B \urcorner$, $\mathrm{Prf}(x, \ulcorner A \vee B \urcorner) \wedge \mathbf{p}_0 y = 0 \wedge \mathrm{Prf}(\mathbf{p}_1 y, \ulcorner A \urcorner)$
or (for some closed $\ulcorner A \vee B \urcorner$, $\mathrm{Prf}(x, \ulcorner A \vee B \urcorner) \wedge \mathbf{p}_1 y = 1 \wedge \mathrm{Prf}(\mathbf{p}_1 y, \ulcorner B \urcorner)$)).

This in turn implies that the strong soundness theorem is not formalizable in $\mathbf{HA}^*$, since strong soundness for $\underline{\mathtt{rn}}$-realizability immediately implies EDN for $\mathbf{HA}^* + \mathrm{ECT}_0$.

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