Rambling along paths, trees, flows, curves, knots, and rails

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As Professor of Mathematics at the Free University, Cor Baayen was an inspiring teacher. His lectures were lucid and skilful, and his broad knowledge enabled him to exhibit the students unexpected vistas and panoramas through several areas in mathematics and theoretical computer science, with topology, set theory, discrete mathematics, logic, and computability as landmarks. As a student you learned that everything is related to everything.

Another characteristic of Cor Baayen's lectures was that he always was eager to present courses on 'modern' topics in mathematics — modern in the sense of not belonging to the standard student curriculum in mathematics (many still don't belong to it). Thus we learned about boolean algebras, graphs, modal logic, proof theory, recursion theory, computability, etc. At the same time there was a strong interest in the historical side of the results discussed.

The courses of Cor Baayen (and his oral examinations, which generally outgrowed to private lessons of at least three hours) being stimulating, he added a personal touch by inviting students from their first year at his home, for further metamathematical background. He has stimulated the enthusiasm of several students for mathematics and for doing research.

I think it appropriate not to restrict myself in this paper to one area, but rather to try to link some of the areas of Cor Baayen's interest, by a ramble through topology, discrete mathematics, and algorithmics, with due attention to the historical roots and to some connections with a few of the other interests of Cor Baayen.

1. Roots of topology. It seems that Leibniz was one of the first interested in topology, or what he called *geometria situs.* In 1679 he wrote in a letter to Christiaan Huygens:

> ... mais apres tous les progres que j'ay faits en ces matieres, je ne suis pas encor content de l'Algebre, en ce qu'elle ne donne ny les plus courtes voyes, ny les plus belles constructions de Geometrie. C'est pourquoy lorsqu'il s'agit de cela, je croy qu'il nous faut encor une autre analyse proprement geometrique ou lineaire qui nous exprime directement *situm*, comme l'Algebre exprime *magnitudinem*. Et je croy d'en voir le moyen et qu'on pourrait representer des figures et mesme des machines et mouve-

mens en caracteres, comme l'Algebre represente les nombres ou grandeurs: et je vous envoye un *essay* qui me paroist considerable.

According to Listing, in his *Vorstudien zur Topologie* of 1847 [37], this was the first idea of a scientific and 'calculatory' elaboration of the modal side of the geometry,

> ... in welchen von einer Art Algorithmus die Rede ist, womit man die Lage räumlicher Gebilde eben so der Analyse unterwerfen müsste, wie es hinsichtlich der Grösse mittelst der Algebra geschieht.

(The essay referred to by Leibniz is following Listing not of 'eigentlich modalen Inhalts'.)

Listing also mentions work by Euler and others on 'die bekannte Aufgabe des sogenannten Rösselsprungs', by Vandermonde on the route by which a thread should go in order to represent for instance a braid or a garter of the weave of a stocking, and by Clausen on the smallest number of penstrokes with which a given figure can be drawn.

Listing, a student of Gauss, says that except for this, the modal side of geometry has 'to expect its elaboration and development almost completely from the future'. As reasons for the fact that since Leibniz not much has been done on the topic, Listing mentions the complexity of discovering effective methods to reduce spatial intuition to concepts, and the inadequacy of language for describing scientifically these, often highly entangled, concepts.

Listing does not claim that he had performed this hard job, and therefore he calls his treatise *Vorstudien zur Topologie*, thereby coining the name *topology*:

> Es mag erlaubt sein, für diese Art Untersuchungen räumlicher Gebilde den Namen "Topologie" zu gebrauchen statt der von Leibniz vorgeschlagenen Benennung "geometria situs", welche an den Begriff des Masses, der hier ganz untergeordnet ist, erinnert, und mit dem bereits für eine andere Art geometrischer Betrachtungen gebräuchlich gewordenen Namen "géométrie de position" collidirt. Unter der *Topologie* soll also die Lehre von den modalen Verhältnissen räumlicher Gebilde verstanden werden, oder von den Gesetzen des Zusammenhangs, der gegenseitigen Lage und der Aufeinanderfolge von Punkten, Linien, Flächen, Körpern und ihren Theilen oder ihren Aggregaten im Raume, abgesehen von den Mass- und Grössenverhältnissen.

Listing discusses how several spatial configurations could be represented by a calculus. In particular he focuses on the orientation of objects, and on how one can use his observations when looking through the micro- or telescope, especially when also mirrors are involved. Moreover, he considers dextro- and laevorotation of screws, springs, ropes, spiral staircases, snail's shells, and stalks.

Listing finds that it is difficult to describe the orientation of objects by words, claiming the inadequacy of the description of dextro- and laevorotatory in Linnaeus' *Philosophia Botanica* (1751):

Den Ausdruck caulis volubilis nämlich erklärt Linné so: spiraliter adscendens per ramum alienum und zwar sinistrorsum (☾) secundum solem vulgo, e.g. Humulus, Lonicera cet.; dextrorsum (☽) contra motum solis vulgi e.g. Convolvulus, Phaseolus, cet. Bei der Intorsio wiederholt er diese Bestimmung und stellt sie mit den Windungstypen am Cirrhus, an der Corolla und anderen Organen zusammen. In einer Anmerkung hierzu gibt nun Linné seine Definition von sinistrorsum und dextrorsum, welche später — zum Theil aus Anlass des dabei vorgefallenen Druckfehlers — die verschiedensten Exegesen erfahren hat. Linné setzt fest: *s*inistrorsum hoc est, quod respicit *dex*tram, si ponas Te ipsum, in centro constitutum, meridiem adspicere; dextrorsum itaque contrarium, und erklärt damit, dass er die nach der *rechten* Seite eines im Centrum stehende Beobachters hervorragenden Blumenblätter als Kennzeichen einer *links* gewundenen Corolla angesehen wissen wolle, und vice versa. Das meridiem adspicere ist in der concreten Sprache Linné's nicht sowohl ein überflüssiger, als vielmehr ein prägnanter Ausdruck für die aufrechte Stellung des mitten in der Blume gedachten Beobachters, der das Gesicht nach einem bestimmten Punkte des Horizonts kehren soll — versteht sich, den Scheitel nach oben gerichtet. Freilich bleibt bei diesen Erklärungen in topologischer Hinsicht manches zu ergänzen, manches zu fragen übrig.

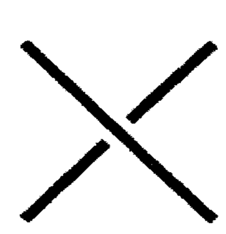


Figure 1

Studying orientation brings Listing to knots. (A *knot* is a simple closed curve in $\mathbb{R}^3$.) They were considered before by Gauss in computing inductance in a system of linked circular wires. Listing introduced a (now standard) planar representation of crossings, as in Figure 1.

Eine Kreuzung dieser Art, wobei sich nach angegebener Weise in der Projection oder Zeichnung der überliegende von dem untenliegenden Faden durch den blossen Anblick leicht unterscheiden lässt, nennen wir eine *Ueberkreuzung* im Gegensatz zur *Durchkreuzung*, wo ein wirklicher Durchschnittspunkt im Raume stattfindet, und die eben gedachte Entfernung beider Fäden bei K entweder Null ist, oder wenigstens als verschwindend betrachtet wird. Zwei Wege können demnach, wie beim gewöhnlichen Kreuzwege, einander durchkreuzen, oder aber, wie diess in manchen Städten und bei vielen Kreuzungen zwischen Eisenbahnen und anderen Fahrstrassen der Fall ist, einander überkreuzen.

He also introduces a calculus with λ (for laeotrop) and δ (for dexiotrop) indicating the corners at the crossing as in Figure 2, claiming that this signing will facilitate an algorithmic discussion ('wie sie ihres Ortes geführt werden muss') of the equivalence of knots.

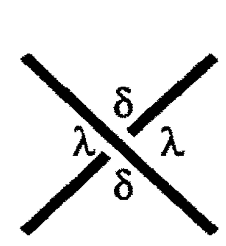


Figure 2

Without proof Listing states that the number of crossings in the trefoil knots (Figure 3) cannot be decreased, and that the two knots in the figure are not equivalent.

In particular, Listing was interested in knots in which each face of the projection is 'monotype' — that is, contains either only λ or only δ. Such knots are

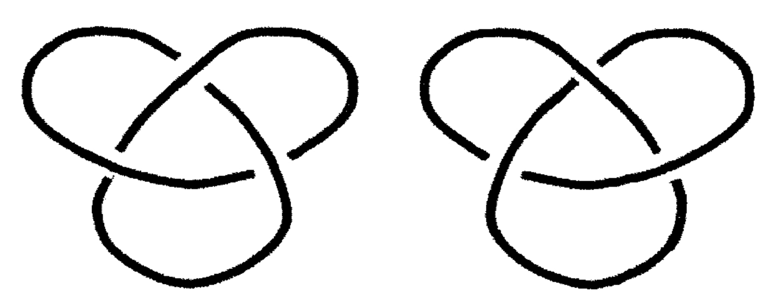


Figure 3

now known as *alternating* knots — indeed, when following the knot one goes alternatingly over and under. The *type-symbol* assigned to such knots is for instance $\delta^5 + 3\delta^3, \lambda^4 + 2\lambda^3 + 2\lambda^2$, indicating that there is 1 δ-face with 5 edges, 3 δ-faces with 3 edges each, 1 λ-face with 4 crossings, 2 λ-faces with 3 edges each, and 2 λ-faces with 2 edges each.

Clearly, the $\lambda\delta$ type-symbol is an invariant under the *trivial operations* on the diagram: rerouting an edge through the unbounded face, and mirroring the diagram, while interchanging 'up' and 'down' at each crossing. However, Listing realizes that the $\lambda\delta$ type-symbol does not give an invariant for alternating knots — he gives an example of two equivalent alternating knots (Figure 4) that have different $\lambda\delta$ type-symbols.

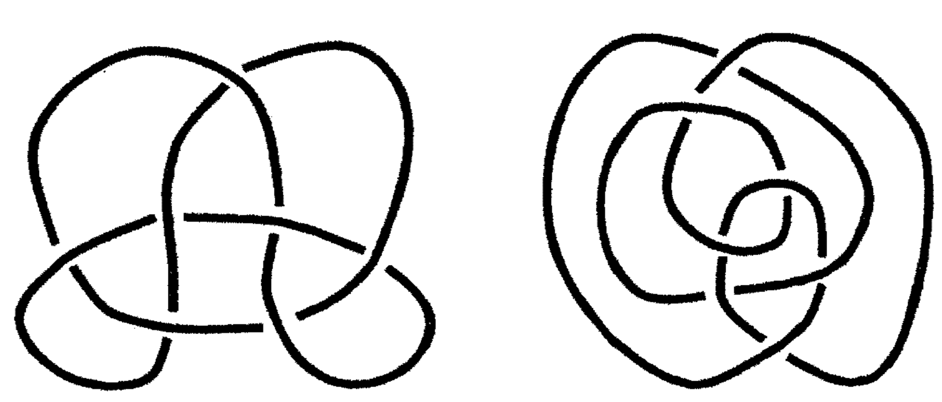


Figure 4

Interesting is that Listing mentions as one of the further applications of topology, beside natural sciences and art, also the area of industrial mechanics, for which Listing refers to the work of the computer pioneer Charles Babbage [4] on representing machine movements by symbols.

2. Tait and knots. Independently of Listing, P.G. Tait studied knots. He was interested in knots because of the 'vortex atom' model invented by his friend, the physicist W. Thomson (later Lord Kelvin), like Tait of Scottish origin.

Tait had a broad scientific interest in mathematics, physics and other disciplines, and published papers and notes on electrodynamics, magnetism, the molecular arrangement in crystals, determinants, quaternions, thermodynamics, the value of the Edinburgh Degree of M.A., the fecundity and fertility of women, earth rotation, comets, fluid dynamics, partial differential equations, spectral analysis, thermoelectricity, the retina, the pendulum motion, combinatorics, viscocity, integral calculus, sound and music, the double rainbow, thunderstorms, and the pace of a golf ball.

Studies of curves in the plane led him to investigating the four-colour problem, and he also applied them to knots. In a paper presented to the British Association in 1876, Tait [66] observed that the cells of a plane closed curve can be coloured black and white so that adjacent cells have different colours. He finishes by remarking:

> The development of this subject promises absolutely endless work — but work of a very interesting and useful kind — because it is intimately connected with the theory of knots, which (especially applied in Sir W. Thomson's Theory of *Vortex Atoms*) is likely soon to become an important branch of mathematics.

In the theory of 'vortex atoms' of Thomson [72], the internal coherence of atoms was assumed to be determined by a knot, or rather a link (a disjoint union of knots), connecting the different indivisible parts of the atom, the 'vortex tubes' (a theory soon abandoned by Thomson). By classifying knots, Tait hoped to shed light on the periodic table of elements.

In a note communicated to the Royal Society of Edinburgh on 18 December 1876, Tait [61] observed that any closed curve in the plane gives an alternating knot, just by going alternatingly over and under. He conjectures that if such an alternating knot is *reduced*, that is, cannot be decomposed as in Figure 5, then it has a minimum number of crossings among all knots equivalent to it; that is, 'cannot have the number of crossings reduced b.' any possible deformation.' As a motivation for considering alternating knots, Tait [65] mentioned that they occur on various sculptured stones and in woodcuts of Dürer.

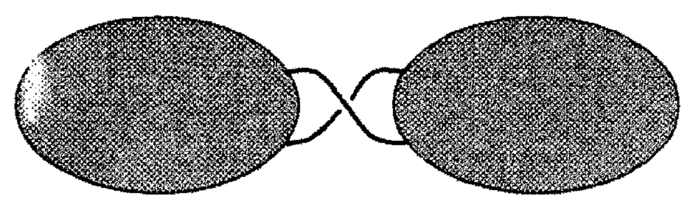
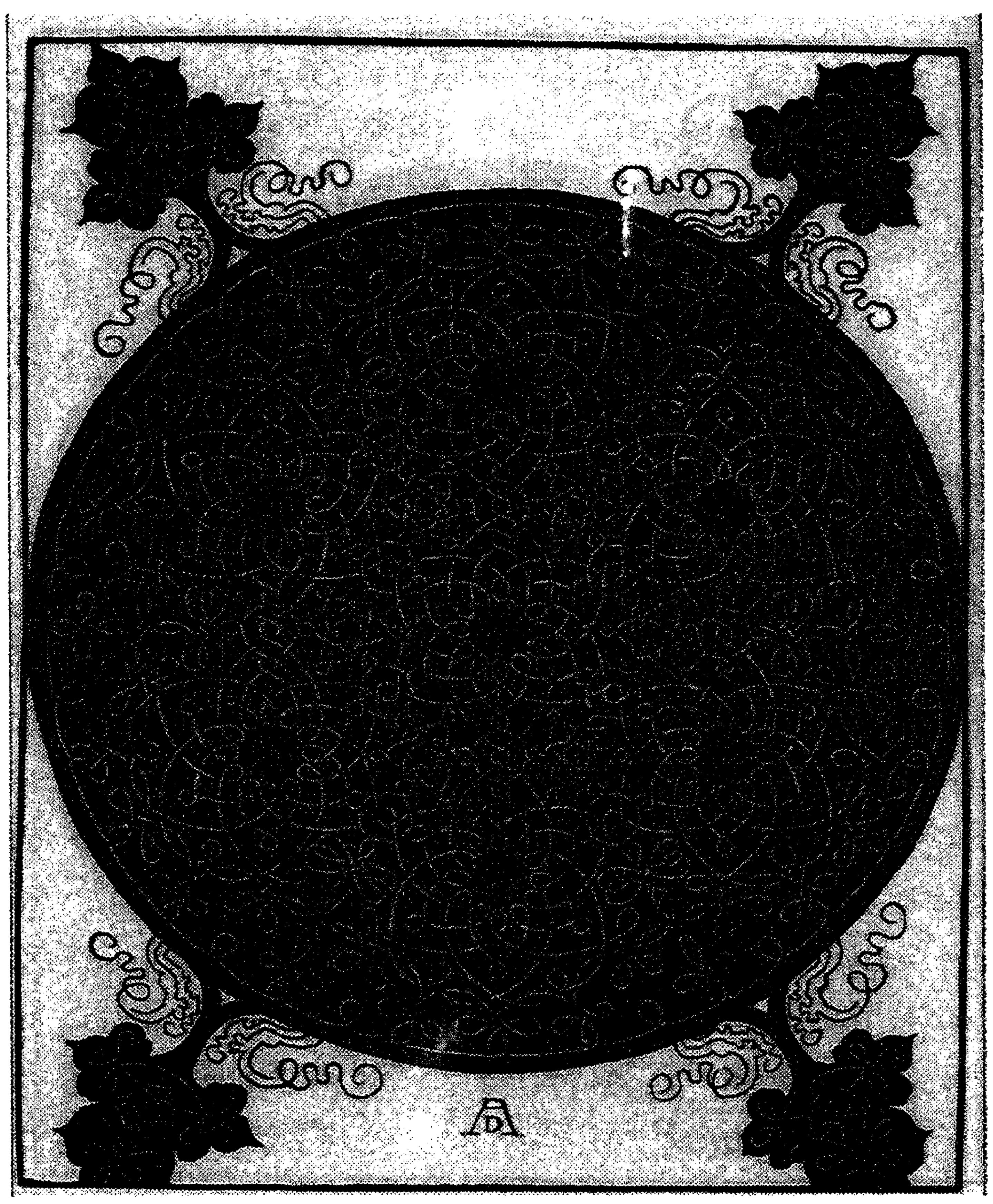


Figure 5



"I am indebted to Mr Dallas for a photograph of a remarkable engraving by Dürer, exhibiting a very complex but symmetrical linkage, in which this alternation is maintained throughout." (Tait [65])

After having presented his subsequent 'Note on the Measure of Beknottedness' (Tait [62]), Tait's attention was drawn by the physicist J.C. Maxwell (also Scottish) to Listing's *Vorstudien zur Topologie*, which Tait next studied with great enthusiasm, calling it an 'extremely valuable, but too brief, Essay'.

It made Tait aware of the fact that there exist al ternating knots that are e- quivalent but cannot be obtained from each other by trivial operations, as they have different $\lambda\delta$ type-symbols. In fact, in [63] he states that the sole point of Listing's paper which (as far as knots are concerned) was thoroughly new to Tait — 'though not unexpected' — was an operation that Tait extracted from Listing's assertion that the knots in Figure 4 are equivalent.

The operation transforms one alternating knot into another. To apply it, one needs to decompose the knot into two blocks as in the first picture in Figure

6. Then one of the blocks is rotated 180°, as indicated in the second picture of Figure 6. Later, Tait called this operation *flyping*. Note that also the trivial operations can be obtained as the result of a series of flypings.

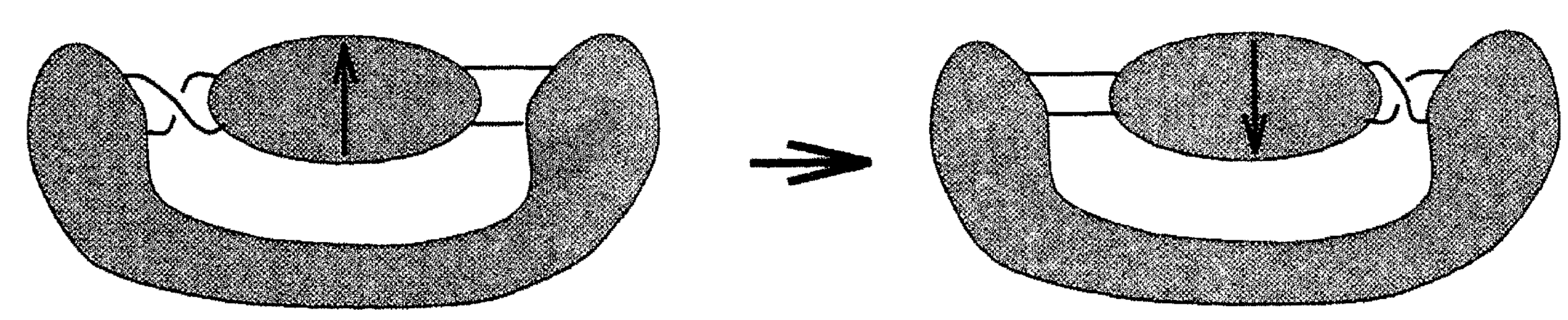


Figure 6

The new operation made Tait conclude that the classification of knots is much more difficult than Tait initially thought,

> and it is so because the number of really distinct species of each order is very much *less* than I was prepared to find it.

It made him plan to give up the whole area of knots, as the note ends with:

> And here I am glad to leave it, for at this stage it is entirely out of my usual sphere of work, and it has already occupied too much of my time.

But saying farewell to knots is not that easy, and Tait's abstinence was of very short duration. In the same 'Session 1876-77' of the Royal Society of Edinburgh he published five more notes on knots and links, including one on 'Sevenfold Knottiness' [64]. In this paper, the reduced alternating knots with seven crossings are classified. This may be considered as the root of 'Tait's flyping conjecture' (although in [64] the term 'flyping' is not used yet).

In his classification, the equivalence of knots is derived by applying only flyping (including the trivial operations). On the other hand, Tait seemed to have only intuitive means of showing that certain knots are *non*equivalent — at least, he does not describe in his paper why certain knots are nonequivalent. So Tait assumed without proof that equivalence of alternating knots is completely determined by flyping. Therefore one may say that Tait conjectured:

Tait's flyping conjecture. *Two reduced alternating knots are equivalent if and only if they can be obtained from each other by a series of flypings.*

Tait was aware of the fact that he did not yet have a way of proving nonequivalence of knots, as in [68] he wrote:

> ... and thus, though I have grouped together many widely different but equivalent forms, I cannot be *absolutely* certain that all those groups are essentially different one from another.

Tait's big article 'On knots' [65] seems the first in which he uses the term *flyping*:

> The deformation process is, in fact, one of *flyping*, an excellent word, very inadequately represented by the nearest equivalent English phrase "turning outside in".

Although it seems that he restricted the term for turning a knot completely upside down, earlier in the paper the operation of Figure 6 was mentioned:

> ... this process ... gets rid of a crossing at one place only by introducing it at another. It will be seen later that this process may in certain cases be employed *to change the scheme* of a knot, ...

Moreover, in a later paper, Tait [67] speaks of 'flyping of individual parts' of a knot, thereby indicating that the general operation described above indeed should be called *flyping*.

The word 'flype' is old Scottish and means according to *The Concise Scots Dictionary*: 'fold back; turn wholly or partially inside out; tear off (the skin) in strips, peel'. *A Dictionary of the Older Scottish Tongue, from the Twelfth Century to the End of the Seventeenth* has as lemma:

> **Flyp(e**, *v.* [e.m.E. and ME. *flype* (c. 1400), of obscure origin; current in later Sc. and northern Eng. dialects.] *tr.* To fold back; to turn outwards. Thare laithlie lyning furthwart flypit; LYND. *Syde Taillis* 97. Ane pair of wyd slevis of arming flypand bakward; 1561 *Inv. Wardrobe* 128. Sum flyrand, thair phisnomeis thai flyp [*v.r.* flipe]; MONTG. *Flyt.* 510 (T). I used often to flype up the lids of my eyes; Row 452.

The Scottish National Dictionary, designed partly on regional lines and partly on historical principles, and containing all the Scottish words known to be in use or to have been in use since c. 1700 gives among other the following usage:

> Sc. 1896 Stevenson *W. of Hermiston* vi.:
> "Miss Christina, if you please, Mr. Weir!" says I, and just flyped up my skirt tails.
> ...
> Sc. 1721 J. Kelly *Proverbs* 218:
> I will sooner see you fleip-ey'd, like a French Cat. A disdainful rejecting of an unworthy Proposal; spoken by bold Maids to the vile offers of young Fellows.

In a discussion of Listing's *Vorstudien*, Tait [67] describes flyping as follows:

> When we *flype* a glove (as in taking it very wet, or as we skin a hare), we perform an operation which (not describable in English by any shorter phrase than "*turning outside in*") changes it character from a right-hand glove to a left. A pair of trousers or a so-called *reversible* waterproof coat is, after this operation has been transformed, still a pair of trousers or a coat, but the legs or arms are interchanged; unless the garments, like those of "Paddius à Corko", are buttoned behind.

The processes described by (Peter) Tait and the vocabulary introduced by him inspired the physicist (Jack) Maxwell to the following poem:

(CATS) CRADLE SONG
By a Babe in Knots.

Peter the Repeater
Platted round a platter
Slips of silvered paper
Basting them with batter.

Flype 'em, slit 'em, twist 'em,
Lop-looped laps of paper;
Setting out the system
By the bones of Neper.

Clear your coil of kinkings
Into perfect plaiting,
Locking loops and linkings
Interpenetrating.

Why should a man benighted,
Beduped, befooled, besotted,
Call knotful knittings plighted,
Not knotty but beknotted?

It's monstruous, horrid, shocking,
Beyond the power of thinking,
Not to know, interlocking
Is no mere form of linking.

But little Jacky Horner,
Will teach you what is proper,
So pitch him, in his corner,
Your silver and your copper.

Tait [65] also introduced a convenient auxiliary graphical representation of knot and link diagrams (more generally, sets of closed curves) in the plane. Colour the faces of a link diagram K black and white, so that adjacent faces have different colours, and so that the unbounded face has colour white. Now put a point in each of the black faces. If any two black faces f, f' are adja cent to a common crossing, draw a line connecting the points in f and f' — cf. Figure 7. In this way we obtain a plane graph H_K, that uniquely determines the projection of the link diagram K, at least combinatorially. If the link diagram is alternating, we can reconstruct it from H_K (after adopting a convention on whether each black face corresponds to a dexiotrop or a laeotrop face of the link). We thus obtain an equivalence of combinatorial questions on alternating knots and on plane graphs.

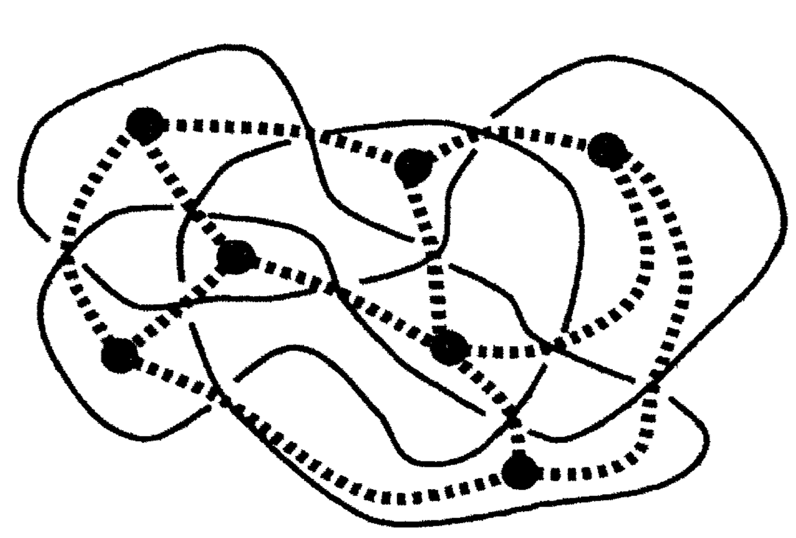


Figure 7

3. Work on Tait's conjectures. Since the work of Listing and Tait, the study of knots has come to great flourishing. Work on distinguishing knots by polynomial invariants (including the well-known Jones polynomial), the connections to mathematical physics, and the applications for instance to DNA have contributed to that. Especially, the work on polynomials has made it possible to prove the nonequivalence of several pairs of knots.

In this ramble I just want to restrict myself to some of the work done on Tait's conjectures. Using the Jones polynomial, Kauffman [27], Murasugi [43], and Thistlethwaite [69] were able to show Tait's conjecture that a reduced alternating link diagram attains a minimum number of crossings, taken over all (not necessarily alternating) links equivalent to it. In particular, any two equivalent reduced alternating links have the same number of crossings.

A special case of Tait's flyping conjecture was considered in [57]. Call a link diagram K *well-connected* if it does not have a nontrivial cut that crosses the diagram in at most four curves only. That is, for any decomposition of the diagram as in Figure 8, one of the blocks should contain at most one crossing.

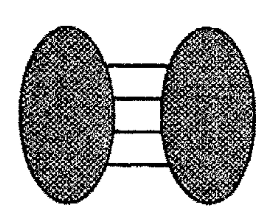


Figure 8

For a well-connected alternating link diagram, flyping clearly loses most of its lustre. For well-connected links Tait's flyping conjecture reduces to:

Theorem 1. *Let K and K' be links with well-connected alternating diagrams. Then K and K' are equivalent if and only if the diagrams arise from each other by trivial operations.*

Meantime, Menasco and Thistlethwaite [39] have announced a proof of Tait's flyping conjecture in full generality.

We sketch some elements of the proofs. Let K and K' be two links, with reduced alternating diagrams. We must show that if K and K' are equivalent, then their diagrams arise from each other by a series of flypings. In both proofs, surfaces are introduced to trace the movements when transforming K' to K.

Let K be an alternating link, with link diagram having a dextrotrop unbounded face. Then the compact bordered surface Σ_K is 'the' surface with boundary K and with projection equal to the closure of the union of the laeotrop faces. A pictorial impression is given in Figure 9.

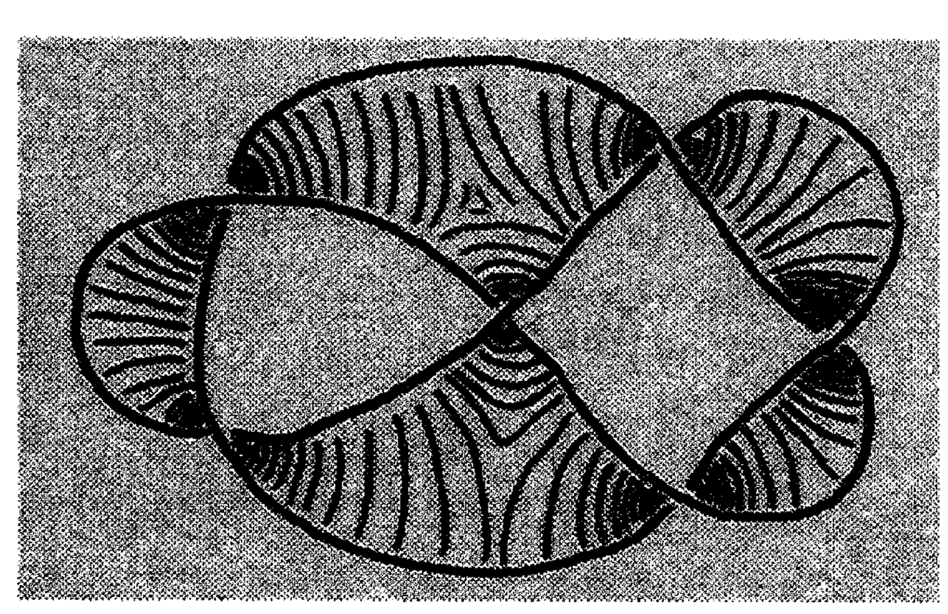


Figure 9

Now note that if we move link K' to link K, there will be two surfaces with boundary K: first the surface Σ_K associated with K; second the transformed surface $\tau(\Sigma_{K'})$, where $\tau : S^3 \longrightarrow S^3$ describes the isotopy bringing K' to K. Thus the surface $\tau(\Sigma_{K'})$ in a way bears the 'history' of moving K' to K.

There are some parameters of compact bordered surfaces that remain invariant under isotopy. First, the Euler characteristic is an invariant. A second parameter invariant under isotopy is the *twisting number*, which is about the number of twists one makes when driving on the surface, close to the boundary, like on a roller coaster (added up over all boundaries).

Now one can show that if K is a link with well-connected alternating diagram and if Σ is any compact bordered surface with boundary K and with the same Euler characteristic and twisting number as Σ_K, then there is an isotopy bringing Σ to Σ_K.

This directly gives, for any two equivalent links K and K' with well-connected alternating diagrams, that there is an isotopy bringing $\Sigma_{K'}$ to Σ_K. Indeed, for this it suffices to show that Σ_K and $\Sigma_{K'}$ have the same Euler characteristic and the same twisting number. This follows directly from earlier results on the invariance of the number of black faces and of the 'writhe' of a link (Murasugi [44], Thistlethwaite [70], [71]).

Finally, to finish the proof of Theorem 1, one has for links K and K' with well-connected alternating diagrams: if there is an isotopy bringing $\Sigma_{K'}$ to Σ_K, then the diagrams arise from each other by trivial operations. This fact is proved by showing that if Σ_K and $\Sigma_{K'}$ are isotopic, then the cycle spaces of H_K and $H_{K'}$ form isomorphic matroids. This is shown by comparing the twisting numbers of circuits in Σ_K and $\Sigma_{K'}$.

Hence, by a theorem of Whitney [76], H_K and $H_{K'}$ are the same up to trivial operations (note that these plane graphs are 3-connected by the well-connectedness of the diagrams). This gives that the diagrams are the same up to trivial operations, and thus we have Tait's flyping conjecture for well-connected links.

The proof of the full Tait flyping conjecture as announced by Menasco and Thistlethwaite [39] makes a more extensive use of invariants, including polynomial invariants, and applies them simultaneaously to the surface Σ_K and to the surface Σ'_K obtained similarly as Σ_K but with respect to the dextrotrop faces (assuming the link diagram being on the 2-sphere).

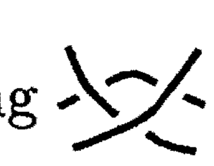
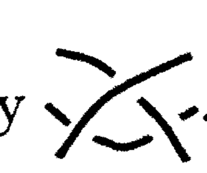
4. Reidemeister moves. A basis of representing a knot by its diagram is that never more than two points of a knot project to the same point in the plane, and if two points have the same projection, it is a crossing. By this one does not lose generality.

Reidemeister [48] observed that this principle can be extended. If one considers the isotopic move of a knot, one has a fourth dimension, the time. Then one may assume that the move is so that at any fixed moment not more than three points of the knot project to the same point in the plane, and if three points have the same projection, they pairwise cross.

Further analysis led Reidemeister to showing that if two links are equivalent, then their diagrams can be moved to each other by a series of simple operations, called *Reidemeister moves*:

(1) *type I:* replacing  by , and conversely;

type II: replacing  by , and conversely;

type III: replacing  by .

(In Reidemeister's book *Knotentheorie* [49], these operations are called $\Omega.1$, $\Omega.2$, and $\Omega.3$.)

It enables to study knot equivalence just by diagrams, and it reduces knot equivalence to a combinatorial question. Most of the knot polynomials have been shown to be invariant by showing that they are invariant under the Reidemeister moves.

On the other hand, Reidemeister moves do not imply a finite algorithm to test if two given knots are equivalent. There is no upper bound known (expressed in the number of crossings of the knots) for the number of Reidemeister moves to be made to transform one knot to another, equivalent, knot. Equivalently, there is no upper bound known for the maximum number of crossings at intermediate diagrams when transforming two equivalent knots to each other by Reidemeister moves.

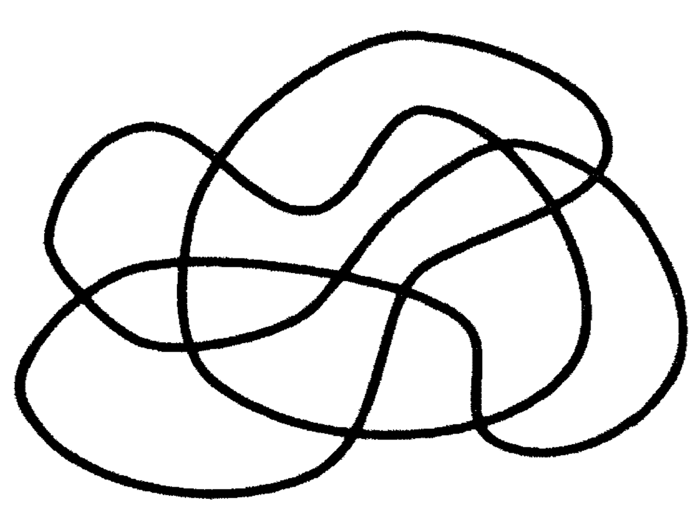


Figure 10

Consider next a closed curve in the plane, like in Figure 10, assuming that there are only a finite number of double points, each being a crossing of two curve parts. It is quite trivial to show that it can be unwrapped to a *simple* closed curve by a series of the following operations — which are also called Reidemeister moves:

(2) *type I:* replacing by , and conversely;

type II: replacing by , and conversely;

type III: replacing by .

Next it is an easy exercise to show something stronger: in transforming a plane closed curve to a simple curve we can restrict the Reidemeister moves to those *not increasing the number of crossings.* That is, the Reidemeister moves of types I and II are only applied from left to right in (2). A similar statement holds when transforming a system of plane closed curves to a system of pairwise disjoint simple closed curves, except that we should add a Reidemeister move of type 0:

(3) *type 0:* replacing by .

(Using the analogy between a system K of plane closed curves and the plane graph H_K as introduced by Tait (see Figure 7), one can derive from this the result of Grünbaum [23] that each plane graph can be obtained from the empty graph by a series of the following operations: (i) adding a new vertex, possibly connected by a new edge to an existing vertex; (ii) adding a new edge parallel to an existing edge; (iii) adding a new vertex in the 'midst' of an existing edge; (iv) 'YΔ', that is, replacing a vertex v of degree 3, and the three edges incident with v, by a triangle connecting the three vertices adjacent to v; (v) 'ΔY', that is, the operation reverse to (iv).)

If we have a closed curve C on a compact surface S it is clear that in general one cannot make it simple by Reidemeister moves. The best one may hope for is to reduce the number of crossings to the minimum number of crossings taken over all closed curves freely homotopic to C.

That is, define

(4) $$\operatorname{mincr}(C) := \min\{\operatorname{cr}(C') | C' \text{ freely homotopic to } C\}.$$

Here $\mathrm{cr}(C')$ denotes the number of selfcrossings of C', counting multiplicities. Two closed curves $C, C' : S^1 \longrightarrow S$ are *freely homotopic*, in notation $C \sim C'$, if there exists a continuous function $\Phi : S^1 \times [0,1] \longrightarrow S$ such that $\Phi(x,0) = C(x)$ and $\Phi(x,1) = C'(x)$ for each $x \in S^1$.

Call C *minimally crossing* if $\mathrm{cr}(C) = \mathrm{mincr}(C)$. Then it is shown in [22] that each closed curve C can be transformed to a minimally crossing closed curve by Reidemeister moves, *without increasing the number of crossings* throughout the moves.

This holds more generally for systems of closed curves. To this end define for closed curves C and D on S:

$$(5) \qquad \mathrm{mincr}(C,D) := \min\{\mathrm{cr}(C',D') | C' \sim C, D' \sim D\}.$$

Here $\mathrm{cr}(C',D')$ is the number of crossings of C' and D', counting multiplicities. A system $C_1,\ldots,C_k$ of closed curves on S is called *minimally crossing* if each C_i is minimally crossing and if $\mathrm{cr}(C_i,C_j) = \mathrm{mincr}(C_i,C_j)$ for all $i \neq j$.

Then the following is proved in [22]:

Theorem 2. *Any system of closed curves on a surface can be transformed to a minimally crossing system by a series of Reidemeister moves, without increasing the number of crossings during the moves.*

(To be precise, one should add some tameness assumptions: the surface should be triangulizable, and the system of closed curves should have only a finite number of double points, each being a crossing.)

It is important to note that the main content of Theorem 2 is that one does not need to apply any of the operations (2) in the reverse direction — otherwise the result would follow quite straightforwardly with the techniques of simplicial approximation.

The idea of the proof is as follows (for one nontrivial closed curve C). First it is shown that one may assume that S is 'hyperbolic', that is, has a hyperbolic distance on it. Then C is freely homotopic to a unique shortest closed curve C' on S. Consider the following operation. Choose a closed disk Δ on S, convex with respect to the hyperbolic distance. Straighten out the intersections of C with Δ; that is, replace each intersection I by the shortest curve that has the same end points as I. Due to an extension of a theorem of Ringel [50], this can be done by applying Reidemeister moves to Δ.

Now one may show that by choosing a finite number of closed disks Δ, one can move C arbitrarily close to C'. Then making C minimally crossing essentially is reduced to making a closed curve on the annulus or the Möbius strip minimally crossing (depending on whether C is orientation preserving or not). This last turns out to boil down to the following auxiliary results on permutations.

Let π be a permutation of $\{1,\ldots,n\}$. A *crossing pair of* π is a pair $\{i,j\}$ with $(i-j)(\pi(i)-\pi(j)) < 0$. The *crossing number* (or *length* (cf. Bourbaki [7]))

$\mathrm{cr}(\pi)$ of π is the number of crossing pairs of π.

Let $\mathrm{mincr}(\pi)$ denote the minimum of $\mathrm{cr}(\pi')$ taken over all conjugates π' of π. So $\mathrm{mincr}(\pi)$ only depends on the sizes of the orbits of π. A permutation is *minimally crossing* if $\mathrm{cr}(\pi) = \mathrm{mincr}(\pi)$. Similarly, *maximally crossing* is defined.

A *transposition* is any permutation $(k, k+1)$ for some $k \in \{1, \ldots, n-1\}$. Since each permutation σ is a product of transpositions, it is trivial to say that each permutation π can be transformed to a minimally crossing permutation by a series of operations

$$(6) \qquad \pi \longrightarrow \tau\pi\tau,$$

where τ is a transposition. Similarly for maximally crossing.

What however can be proved more strongly is:

Lemma. *Each permutation π of $\{1, \ldots, n\}$ can be transformed to a minimally crossing permutation by a series of operations* (6), *while never increasing the number of crossing pairs. A similar statement holds for maximally crossing.*

Geck and Pfeiffer [21] proved the first part of the Lemma more generally for any Weyl group (instead of just a permutation group). It is not known if also the 'maximally crossing' part also holds for Weyl groups.

5. Curves and circulations on surfaces. One motivation for studying Reidemeister moves on surfaces was to derive a homotopic circulation theorem for graphs embedded on a surface. Once one has Theorem 2, such a circulation theorem can be derived by a number of straightforward arguments based on two kinds of duality: duality of graphs on surfaces and linear programming duality (Farkas' lemma).

Again, let S be a surface, and let $G = (V, E)$ be an undirected graph embedded on S. For any closed curve D on S, let $\mathrm{cr}(G, D)$ denote the number of intersections of G and D (counting multiplicities). Moreover, $\mathrm{mincr}(G, D)$ denotes the minimum of $\mathrm{cr}(G, D')$ where D' ranges over all closed curves freely homotopic to D and *not intersecting* V.

We first derive the following theorem from Theorem 2, which was proved for the projective plane by Lins [36]:

Theorem 3. *Let $G = (V, E)$ be an Eulerian graph embedded on a surface S. Then the edges of G can be decomposed into closed curves $C_1, \ldots, C_k$ such that for each closed curve D on S:*

$$(7) \qquad \mathrm{mincr}(G, D) = \sum_{i=1}^{k} \mathrm{mincr}(C_i, D).$$

Here a graph is *Eulerian* if each vertex has even degree. (Connectedness of the graph is not assumed.) Moreover, *decomposing* the edges into $C_1, \ldots, C_k$ means that each edge of G is traversed by exactly one of the C_i.

Note that the inequality $\geq$ in (7) trivially holds, for *any* decomposition of the edges into closed curves $C_1, \ldots, C_k$. The content of the theorem is that there exists a decomposition attaining equality for each D.

The idea of the proof is as follows. First, by an easy construction we may assume that each vertex v of G has degree at most four. Next, we define the *straight decomposition* of G as the system of closed curves that decomposes the edges of G in such a way that in each vertex of G, opposite edges are traversed consecutively. So each vertex of G of degree four represents a (self-)crossing of $C_1, \ldots, C_k$.

Up to some trivial operations, such a decomposition is unique, and conversely, it uniquely describes G. So any Reidemeister move applied to $C_1, \ldots, C_k$ carries over a modification of G. Hence we can speak of Reidemeister moves applied to G.

The following is easy to see:

(8) if G' arises from G by one Reidemeister move of type III, then $\mathrm{mincr}(G', D) = \mathrm{mincr}(G, D)$ for each closed curve D.

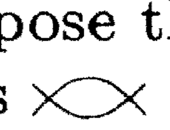
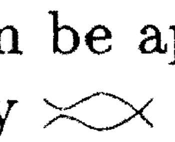
Let us call any graph $G = (V, E)$ that is a counterexample to the theorem with each vertex having degree at most four and with a minimal number of faces, a *minimal counterexample.*

From (8) it directly follows that:

(9) if G' arises from a minimal counterexample G by one Reidemeister move of type III, then G' is a minimal counterexample again.

Moreover one has:

(10) if G is a minimal counterexample, then no Reidemeister move of type 0, I or II can be applied to G without increasing the number of vertices of G.

For suppose that a Reidemeister move of type II can be applied to G. Then G contains  as subconfiguration. Replacing this by  would give a smaller counterexample (since the function $\mathrm{mincr}(G, D)$ does not change by this operation), contradicting the minimality of G.

One similarly sees that no Reidemeister move of type 0 or I can be applied.

The proof is finished by showing the contradictory statement that the straight decomposition $C_1, \ldots, C_k$ of any minimal counterexample G satisfies (7).

Choose a closed curve D. By Theorem 2 we can apply Reidemeister moves to the system $D, C_1, \ldots, C_k$ so as to obtain a minimally crossing system $D', C'_1, \ldots, C'_k$.

By (10) we did not apply Reidemeister moves of type 0, I or II to $C_1, \ldots, C_k$. Hence by (8) for the graph G' obtained from the final $C'_1, \ldots, C'_k$ we have $\mathrm{mincr}(G', D) = \mathrm{mincr}(G, D)$. So

$$\text{(11)} \qquad \operatorname{mincr}(G,D) = \operatorname{mincr}(G',D) \le \operatorname{cr}(G',D') = \sum_{i=1}^{k} \operatorname{cr}(C_i',D')$$
$$= \sum_{i=1}^{k} \operatorname{mincr}(C_i',D') = \sum_{i=1}^{k} \operatorname{mincr}(C_i,D).$$

This proves Theorem 3.

Using surface duality one directly obtains from Theorem 3 the next theorem. If G is a graph embedded on a surface S and C is a closed curve in G, then $\operatorname{minlength}_G(C)$ denotes the minimum length of any closed curve $C' \sim C$ in G. (The *length* of C' is the number of edges traversed by C', counting multiplicities.)

Theorem 4. *Let $G = (V,E)$ be a bipartite graph cellularly embedded on a compact surface S. Then there exist closed curves $D_1, \ldots, D_t$ on $S \setminus V$ such that each edge of G is crossed by exactly one D_j and by this D_j only once and such that for each closed curve C:*

$$\text{(12)} \qquad \operatorname{minlength}_G(C) = \sum_{j=1}^{t} \operatorname{mincr}(C,D_j).$$

Now with linear programming duality (Farkas' lemma) one derives from Theorem 4 the following 'homotopic circulation theorem' — a fractional packing theorem for cycles of given homotopies in a graph on a compact surface.

Let $G = (V,E)$ be a graph embedded on a compact surface S. For any closed curve C on G and any edge e of G let $\operatorname{tr}_C(e)$ denote the number of times C traverses e. So $\operatorname{tr}_C \in \mathbb{R}^E$.

Call a function $f : E \longrightarrow \mathbb{R}$ a *circulation* (of value 1) if f is a convex combination of functions tr_C. We say that f is *freely homotopic to* a closed curve C_0 if we can take each C freely homotopic to C_0.

Theorem 5 (homotopic circulation theorem). *Let $G = (V,E)$ be an undirected graph embedded on a compact surface S and let $C_1, \ldots, C_k$ be closed curves on S. Then there exist circulations $f_1, \ldots, f_k$ such that f_i is freely homotopic to C_i $(i = 1, \ldots, k)$ and such that $\sum_{i=1}^{k} f_i(e) \le 1$ for each edge e, if and only if for each closed curve D on $S \setminus V$ one has*

$$\text{(13)} \qquad \operatorname{cr}(G,D) \ge \sum_{i=1}^{k} \operatorname{mincr}(C_i,D).$$

We sketch the proof if G is cellularly embedded. Necessity of the condition is direct. To show sufficiency, by Farkas' lemma (cf. [54]) it suffices to show that if $d \in \mathbb{Q}^k$ and $l \in \mathbb{Q}_+^E$ such that $\sum_{e \in E} \operatorname{tr}_C(e) \ge d_i$ for each i and each closed curve $C \sim C_i$ in G, then $\sum_{e \in E} l(e) \ge \sum_{i=1}^{k} d_i$.

Then one can show that it may be assumed that each d_i and each $l(e)$ is an even integer, and that $l(e) > 0$ for each e. Replacing each edge e by a path of length $l(e)$ makes G into a bipartite graph G'. Applying (13) to each of the D_j of Theorem 4 gives the required inequality.

6. Disjoint curves in graphs on surfaces. In the homotopic circulation theorem one may wonder when there exists an integer-valued circulation. This would correspond to a system of pairwise edge-disjoint cycles $C'_1, \ldots, C'_k$ in G with C'_i freely homotopic to C_i. However, the conditions given in the theorem are not sufficient to get an integer-valued circulation; and no additional conditions are known to ensure the existence of an integer-valued circulation.

If we want to have *vertex*-disjoint circuits, such conditions have been given in [55], proving a conjecture of L. Lovász and P.D. Seymour:

Theorem 6. *Let G be an undirected graph embedded on a compact surface S and let $C_1, \ldots, C_k$ be pairwise disjoint simple closed curves on S. Then there exist pairwise disjoint simple circuits $C'_1, \ldots, C'_k$ in G where C'_i is freely homotopic to C_i for $i = 1, \ldots, k$, if and only if*

$$\text{(14)} \qquad \mathrm{cr}(G, D) \geq \sum_{i=1}^{k} \mathrm{mincr}(C_i, D)$$

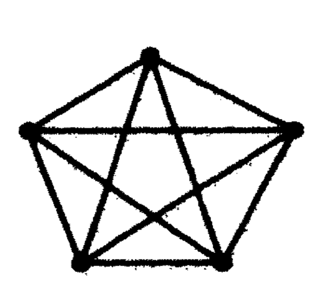
for each closed curve D on S, with strict inequality if D is doubly odd.

Here a closed curve D is *doubly odd* if D is the concatenation of two closed curves D_1 and D_2, with a common beginning (= end) point, which is not on G, in such a way that $\mathrm{cr}(G, D_j) + \sum_{i=1}^{k} \mathrm{mincr}(C_i, D_j)$ is odd for $j = 1, 2$. It is not difficult to see that the condition given in the theorem is necessary.

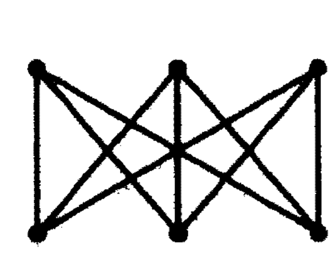
The problem solved in Theorem 6 arose during the graph minors project of N. Robertson and P.D. Seymour. Principal result of this deep project is a proof ([53]) of Wagner's conjecture: in any infinite class of graphs there are graphs G and H such that H is a minor of G. (H is a *minor* of G if H arises from G by a series of deletions and contractions of edges.)

Equivalent to Robertson and Seymour's theorem is that if $\mathcal{G}$ is a class of graphs closed under taking minors, then there is a *finite* collection $\mathcal{H}$ of graphs with the property that a graph G belongs to $\mathcal{G}$ if and only if G does not have a minor H with $H \in \mathcal{H}$.

We may assume that $\mathcal{H}$ does not contain two graphs H, H' such that H' is a minor of H. Then $\mathcal{H}$ is called the set of *forbidden minors* of $\mathcal{G}$.



K_5



$K_{3,3}$

The well-known theorem of Kuratowski [34] (or rather, its equivalent formulation by Wagner [74]) states that if $\mathcal{G}$ is the class of planar graphs, then $\{K_5, K_{3,3}\}$ is the set of forbidden minors.

A consequence of Robertson and Seymour's theorem is that for *any* surface S there is a finite class of forbidden minors for the class of graphs embeddable on S. This was shown before by Archdeacon [2] for the projective plane and by Archdeacon and Huneke [3] for compact nonorientable surfaces.

Very roughly speaking, the proof of Robertson and Seymour of Wagner's conjecture is as follows. It can be shown that for any graph G there is a finite collection of surfaces such that each graph not containing G as a minor can be expressed as a tree-structure of 'pieces' such that each piece can 'almost' be drawn on a surface in the collection. Part of the proof next is that any graph H embedded on a surface S is a minor of each graph that is embedded densely enough on S ('enough' depending on H).

Related to this last statement is the question under which conditions for two given graphs G and H embedded on S, H is a minor of G *on* S. That is, when can we delete and contract edges of G, while keeping the embedding, so as to obtain H (possibly after a homotopic shift of H over S). The case where H consists of disjoint loops only is solved in Theorem 6.

The more general case of this question where H is an arbitrary graph is not solved completely, but can be approached slightly similarly as follows. Let G and H be graphs embedded on S. For each edge f of H choose an edge e_f of G. Now we wish to complete these edges to a minor of G isomorphic to H. By this it is meant that one should find for each vertex v of H a tree T_v in G such that the T_v are mutually disjoint and such that for each edge f of H, e_f is incident with T_v if and only if f is incident with v. Thus contracting each tree T_v to one vertex, the edges e_f would give a minor isomorphic to H.

Now an extension of Theorem 6 (cf. [56]) characterizes under which conditions such trees exist, given the homotopy of the trees. It amounts to finding disjoint trees $T_1, \ldots, T_k$ such that each T_i connects a given set V_i of vertices. If each V_i just consists of two vertices, it reduces to a *disjoint paths problem.*

7. Menger and König. Disjoint paths problems belong to the heart of classical graph theory. They go back to 1927, when the topologist Karl Menger [40] published an article called *Zur allgemeinen Kurventheorie* in which he showed a result that now is one of the most fundamental results in graph theory:

> Satz β. *Ist K ein kompakter regulär eindimensionaler Raum, welcher zwischen den beiden endlichen Mengen P und Q n-punktig zusammenhängend ist, dann enthält K n paarweise fremde Bögen, von denen jeder einen Punkt von P und einen Punkt von Q verbindet.*

The result can be formulated as a maximum-minimum theorem in terms of graphs, as follows:

Menger's theorem. *Let $G = (V, E)$ be an undirected graph and let $P, Q \subseteq V$. Then the maximum number of pairwise disjoint $P - Q$ paths is equal to the minimum cardinality n of any set of vertices that intersects each $P - Q$ path.*

Here a $P - Q$ *path* is a path starting in P and ending in Q. Two paths are *disjoint* if they do not have any vertex or edge in common. The result became also known as the *n-chain theorem* or the *n-arc theorem.* Knaster [28] observed that (by an easy construction) Menger's theorem is equivalent to:

Menger's theorem (variant). *Let $G = (V, E)$ be an undirected graph and let $s, t \in V$ with $st \notin E$. Then the maximum number of pairwise internally disjoint $s - t$ paths is equal to the minimum cardinality of any subset of $V \setminus \{s, t\}$ that intersects each $s - t$ path.*

Here an $s - t$ *path* is a path starting in s and ending in t. Two paths are *internally disjoint* if they do not have a vertex or edge in common, except for the end vertices.

Why was Menger interested in this question? In his article he investigates a certain class of topological spaces called 'Kurven': a *curve* is a connected compact topological space X with the property that for each $x \in X$ and each neighbourhood N of x there exists a neighbourhood $N' \subseteq N$ of x such that $\mathrm{bd}(N')$ is totally disconnected. Here bd stands for 'boundary'; a space is *totally disconnected* if each point forms an open set. Notice that each graph, considered as a topological space, is a curve in Menger's terminology.

In particular, Menger was motivated by characterizing a certain furcation number of curves. To this end, a curve X is called *regular* if for each $x \in X$ and each neighbourhood N of x there exists a neighbourhood $N' \subseteq N$ of x such that $|\mathrm{bd}(N')|$ is finite. The *order* of a point $x \in X$ is equal to the minimum natural number n such that for each neighbourhood N of x there exists a neighbourhood $N' \subseteq N$ of x satisfying $|\mathrm{bd}(N')| \leq n$.

According to Menger:

> Eines der wichtigsten Probleme der Kurventheorie ist die Frage nach die Beziehungen zwischen der Ordnungszahl eines Punktes der regulären Kurve K und der Anzahl der im betreffenden Punkt zusammenstossenden und sonst fremden Teilbögen von K.

In fact, Menger used 'Satz β' to show that if a point in a regular curve K has order n, then there exists a topological *n-leg* with p as top; that is, K contains n arcs $P_1, \ldots, P_n$ such that $P_i \cap P_j = \{p\}$ for all i, j with $i \neq j$.

The proof idea is as follows. There exists a series $N_1 \supset N_2 \supset \cdots$ of open neighbourhoods of p such that $N_1 \cap N_2 \cap \cdots = \{p\}$ and $|\mathrm{bd}(N_i)| = n$ for all $i = 1, 2, \ldots$, and such that

(15) $$|\mathrm{bd}(N)| \geq n \text{ for each neighbourhood } N \subseteq N_1.$$

This follows quite directly from the definition of order.

Now Menger showed that we may assume that the space $G_i := \overline{N_i} \setminus N_{i+1}$ is a (topological) graph. For each i, let $Q_i := \mathrm{bd}(N_i)$. Then (15) gives with Menger's theorem that there exist n pairwise disjoint paths $P_{i,1}, \ldots, P_{i,n}$ in G

such that each $P_{i,j}$ runs from Q_i to Q_{i+1}. Properly connecting these paths for $i = 1, 2, \ldots$ we obtain n arcs forming the required n-leg.

It was however noticed by Kőnig [30] that Menger gave a lacunary proof of 'Satz β'. Menger applies induction on $|E|$, where E is the edge set of the graph G. Menger first claims that one easily shows that $|E| \geq n$, and that if $|E| = n$ then G consists of n disjoint arcs connecting P and Q. He states that if $|E| > n$ then there is a vertex $s \notin P \cup Q$, or in his words (where the 'Grad' denotes $|E|$):

> Wir nehmen also an, der irreduzibel n-punktig zusammenhängende Raum K' besitze den Grad g ($> n$). Offenbar enthält dann K' ein punktförmiges Stück s, welches in der Menge $P + Q$ nicht enthalten ist.

Indeed, as Menger shows, if such a vertex s exists one is done: If s is not contained in any set W intersecting each $P - Q$ path such that $|W| = n$, then we can delete s and the edges incident with s without decreasing the minimum in the theorem. If s is contained in some set W intersecting each $P - Q$ path such that $|W| = n$, then we can split G into two subgraphs G_1 and G_2 that intersect in W in such a way that $P \subseteq G_1$ and $Q \subseteq G_2$. By the induction hypothesis, there exist n pairwise disjoint $P - W$ paths in G_1 and n pairwise disjoint $W - Q$ paths in G_2. By pairwise sticking these paths together at W we obtain paths as required.

However, such a vertex s need not exist. It might be that V is the disjoint union of P and Q in such a way that each edge connects P and Q. In that case, G is a bipartite graph, and what should be shown is that G contains a matching (= set of disjoint edges) of size n. This is a nontrivial basis of the proof.

It is unclear when Menger became aware of the hole. In his reminiscences on the origin of the n-arc theorem, Menger [42] wrote in 1981:

> In the spring of 1930, I came through Budapest and met there a galaxy of Hungarian mathematicians. In particular, I enjoyed making the acquaintance of Dénes Kőnig, for I greatly admired the work on set theory of his father, the late Julius Kőnig—to this day one of the most significant contributions to the continuum problem—and I had read with interest some of Dénes papers. Kőnig told me that he was about to finish a book that would include all that was known about graphs. I assured him that such a book would fill a great need; and I brought up my n-Arc Theorem which, having been published as a lemma in a curve-theoretical paper, had not yet come to his attention. Kőnig was greatly interested, but did not believe that the theorem was correct. "This evening," he said to me in parting, "I won't go to sleep before having constructed a counterexample." When we met the next day he greeted me with the words, "A sleepless night!" and asked me to sketch my proof for him. He then said that he would add to his book a final section devoted to my theorem. This he did; and it is largely thanks to Kőnig's valuable book that the n-Arc Theorem has become widely known among graph theorists.

Dénes Kőnig was a pioneer in graph theory and in applying graphs to other areas like set theory, matrix theory, and topology. He had published in the

1910s theorems on perfect matchings and on factorizations of regular bipartite graphs in relation to the study of determinants by Frobenius.

At the meeting of 26 March 1931 of the Eötvös Loránd Matematikai és Fizikai Társulat (Loránd Eötvös Mathematical and Physical Society) in Budapest, Kőnig [29] presented a result that formed in fact the induction basis for Menger's theorem:

> Páros körüljárású graphban az éleket kimerítő szögpontok minimális száma megegyezik a páronként közös végpontot nem tartalmazó élek maximális számával.

In other words:

Kőnig's theorem. *In a bipartite graph $G = (V, E)$, the maximum size of a matching is equal to the minimum number of vertices needed to cover all edges.*

Kőnig did not mention in his paper that this result provided the missing induction basis in Menger's proof, although he finishes with:

> Megemlítjük végül, hogy eredményeink szorosan összefüggnek FROBENIUSnak determinánsokra és MENGERnek graphokra vonatkozó némely vizsgálatával. E kapcsolatokra másutt fogunk kiterjeszkedni.

'Másutt' became Kőnig [30], where a full proof of Menger's theorem is given, with the following footnote:

> Der Beweis von MENGER enthält eine Lücke, da es vorausgesetzt wird (S. 102, Zeile 3–4) daß "K' ein punktförmiges Stück s enthält, welches in der Menge $P + Q$ nicht enthalten ist", während es recht wohl möglich ist, daß — mit der hier gewählten Bezeichnungsweise ausgedrückt — jeder Knotenpunkt von G zu $H_1 + H_2$ gehört. Dieser — keineswegs einfacher — Fall wurde in unserer Darstellung durch den Beweis des Satzes 13 erledigt. Die weiteren — hier folgenden — Überlegungen, die uns zum Mengerschen Satz führen werden, stimmen in Wesentlichen mit dem — sehr kurz gefaßten — Beweis von MENGER überein. In Anbetracht der Allgemeinheit und Wichtigheit des Mengerschen Satzes wird im Folgenden auch dieser Teil ganz ausführlich und den Forderungen der *reinkombinatorischen* Graphentheorie entsprechend dargestellt.
>
> [Zusatz bei der Korrektur, 10.V.1933] Herr MENGER hat die Freundlichkeit gehabt — nachdem ich ihm die Korrektur meiner vorliegenden Arbeit zugeschickt habe — mir mitzuteilen, daß ihm die oben beanstandete Lücke seines Beweises schon bekannt war, daß jedoch sein vor Kurzem erschienenes Buch *Kurventheorie* (Leipzig, 1932) einen vollkommen lückenlosen und rein kombinatorischen Beweis des Mengerschen Satzes (des "n-Kettensatzes") enthält. Mir blieb dieser Beweis bis jetzt unbekannt.

This book of Menger [41] was published in 1932, and contains a complete proof of Menger's theorem. Menger did not refer to any hole in his proof, but remarked:

> Über den n-Kettensatz für Graphen und die im vorangehenden zum Beweise verwendete Methode vgl. Menger (Fund. Math. 10, 1927, S. 101 f.). Die obige detaillierte Ausarbeitung und Darstellung stammt von Nöbeling.

In his book *Theorie der endlichen und unendlichen Graphen*, published in 1936, Kőnig [31] calls his theorem *ein wichtiger Satz*, and he emphasizes the chronological order of the proofs of Menger's theorem and of Kőnig's theorem (which is implied by Menger's theorem):

> Ich habe diesen Satz 1931 ausgesprochen und bewiesen, s. Kőnig [9 und 11]. 1932 erschien dann der erste lückenlose Beweis des Mengerschen Graphensatzes, von dem in §4 die Rede sein wird und welcher als eine Verallgemeinerung dieses Satzes 13 (falls dieser *nur für endliche* Graphen formuliert wird) angesehen werden kann.

8. Disjoint paths and trees. Menger's theorem addresses the problem of finding a set of paths with one common beginning vertex and one common end vertex. A more general problem is the following *disjoint paths problem*:

(16) given: a graph $G = (V, E)$ and k pairs of vertices $s_1, t_1, \ldots, s_k, t_k$;
find: pairwise disjoint paths $P_1, \ldots, P_k$ where P_i runs from s_i to t_i $(i = 1, \ldots, k)$.

This covers four variants of the problem: the graph can be directed or undirected, and 'disjoint' can mean: vertex-disjoint or edge-disjoint.

In 1974, D.E. Knuth (see [26]) showed that the edge-disjoint undirected variant, and hence also each of the other variants, is NP-complete — and this is even so if we restrict ourselves to planar graphs (Lynch [38]). This destroys (for those believing NP$\neq$co-NP or NP$\neq$P) the hope for nice theorems (like Menger's theorem) and for fast algorithms for solving this problem.

On the other hand, Robertson and Seymour [52], as another important result of their graph minors project, proved that for each *fixed* k, there exists a polynomial-time algorithm for the disjoint paths problem for undirected graphs. Their algorithm has running time bounded by $c_k|V|^3$, for some constant c_k heavily depending on k. (It implies that for each fixed graph H there exists a polynomial time algorithm to test if a given graph G contains H as a minor.)

For *directed* graphs, the situation seems different. In 1980, Fortune, Hopcroft, and Wyllie [20] showed the NP-completeness of the vertex-disjoint paths problem for directed graphs, even when restricted to the case $k = 2$.

For *planar* directed graphs however there is a positive result ([58]):

Theorem 7. *For each fixed k there is a polynomial-time algorithm for the k vertex-disjoint paths problem for directed planar graphs.*

This is a result only of interest from the point of view of theoretical complexity: the degree of the polynomial bounding the running time of the algorithm is quadratic in k.

The proof of Theorem 7 is based on representing disjoint paths as 'flows' over a free group. Indeed, let a directed planar graph $D = (V, A)$ and $s_1, t_1, \ldots, s_k, t_k \in V$ be given. Let G_k be the free group with k generators $g_1, \ldots, g_k$. If $\Pi = (P_1, \ldots, P_k)$ is a solution to the disjoint paths problem, let $\phi_\Pi : A \longrightarrow G_k$ be defined by, for $a \in A$: $\phi_\Pi(a) := g_i$ if P_i traverses a ($i = 1, \ldots, k$), and $:= 1$ if no P_i traverses a.

Let F be the set of faces of D. Call two functions $\phi, \psi : A \longrightarrow G_k$ *homologous* if there exists a function $p : F \longrightarrow G_k$ such that for each arc a of D one has:

$$\psi(a) = p(f)^{-1}\phi(a)p(f'), \tag{17}$$

where f and f' are the faces at the left hand side and the right hand side of a respectively (with respect to the orientation of the plane and of the arc a).

This defines an equivalence relation on functions $A \longrightarrow G_k$. We now enumerate representatives of homology classes of functions $A \longrightarrow G_k$. Generally there are infinitely many homology classes, but one can find in polynomial time a collection of $O(|V|^{2k^2+3})$ homology classes of which one can be sure that it covers all functions ϕ_Π with Π a solution to the vertex-disjoint paths problem (without having these functions explicitly).

For the representative ψ of each of these classes one should test if there is a path packing function ϕ_Π homologous to ψ. This can be done in polynomial time, by reducing it to the following dual problem.

Given any directed graph $D = (V, A)$ (not necessarily planar) and any group G, call two functions $\phi, \psi : A \longrightarrow G$ *cohomologous* if there exists a function $p : V \longrightarrow G$ such that for each arc $a = (u, w)$ of D one has:

$$\psi(a) = p(u)^{-1}\phi(a)p(w). \tag{18}$$

Again this is an equivalence relation.

Consider the following *cohomology feasibility problem:*

(19) given: a directed graph $D = (V, A)$ and functions $\phi : A \longrightarrow G$ and $H : A \longrightarrow \mathcal{P}(G)$;

find: a function ψ cohomologous to ϕ with $\psi(a) \in H(a)$ for each $a \in A$.

This is in its general form an NP-complete problem: when $G = C_3$ (the group with three elements) and $\phi(a) = 1$ and $H(a) = C_3 \setminus \{1\}$ for each arc a, the problem amounts to the 3-colourability of the vertices of D. However:

Theorem 8. *If G is the free group and each $H(a)$ is hereditary, then the cohomology feasibility problem is solvable in polynomial time.*

Here a subset H of the free group is *hereditary* if for each (reduced) word $w'ww''$ in H, also the word w belongs to H.

Now the problem of finding a path packing function ϕ_Π homologous to a given function ψ, can be reduced to the cohomology feasibility problem on an extension of the dual graph of D, where each $H(a)$ is equal to $\{1, g_1, \ldots, g_k\}$ or to $\{1, g_1, g_1^{-1}, \ldots, g_k, g_k^{-1}\}$. This finishes the outline of the proof of Theorem 7.

Theorem 7 can be generalized to disjoint *trees* connecting given sets of vertices, and Theorem 8 can be generalized to *free partially commutative groups* — see [59]. Moreover, necessary and sufficient conditions for the existence of a solution can be described in terms of cycles in the graaf D.

9. VLSI-routing. The approach described above for the vertex-disjoint paths problem in directed planar graphs is analogous to a method developed for the *VLSI-routing problem.* This problem asks for the routes that wires should make on a chip so as to connect certain pairs of pins and so that wires connecting different pairs of pins are disjoint.

As the routes that the wires potentially can make form a graph, the problem to be solved can be modeled as a disjoint paths problem. Consider an example of such a problem as in Figure 11 — relatively simple, since generally the number of pins to be connected is of the order of several thousands. The grey areas are 'modules' on which the pins are located. Points with the same label should be connected.

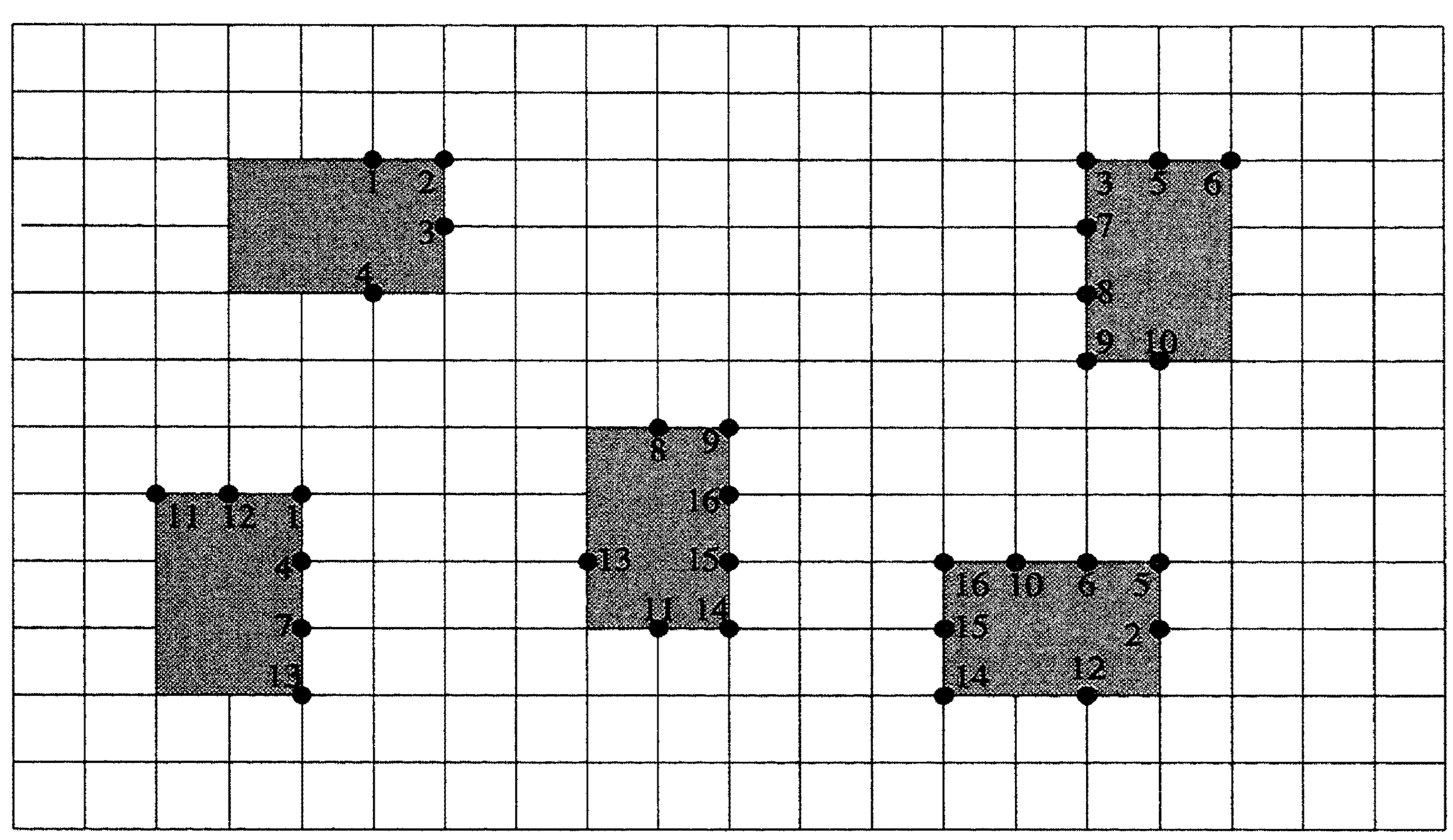


Figure 11

In the example, the graph is a 'grid graph', which is typical in VLSI-design since it facilitates the manufacturing of the chip and it ensures a certain minimum distance between disjoint wires. But even for such graphs the disjoint paths problem is NP-complete.

Now the following two-step approach was proposed by Pinter [46]. First choose the homotopies of the wires; for instance like in Figure 12. That is, for each i one chooses a curve C_i in the plane connecting the two vertices i, in such a way that they are pairwise disjoint, and such that the modules are not traversed.

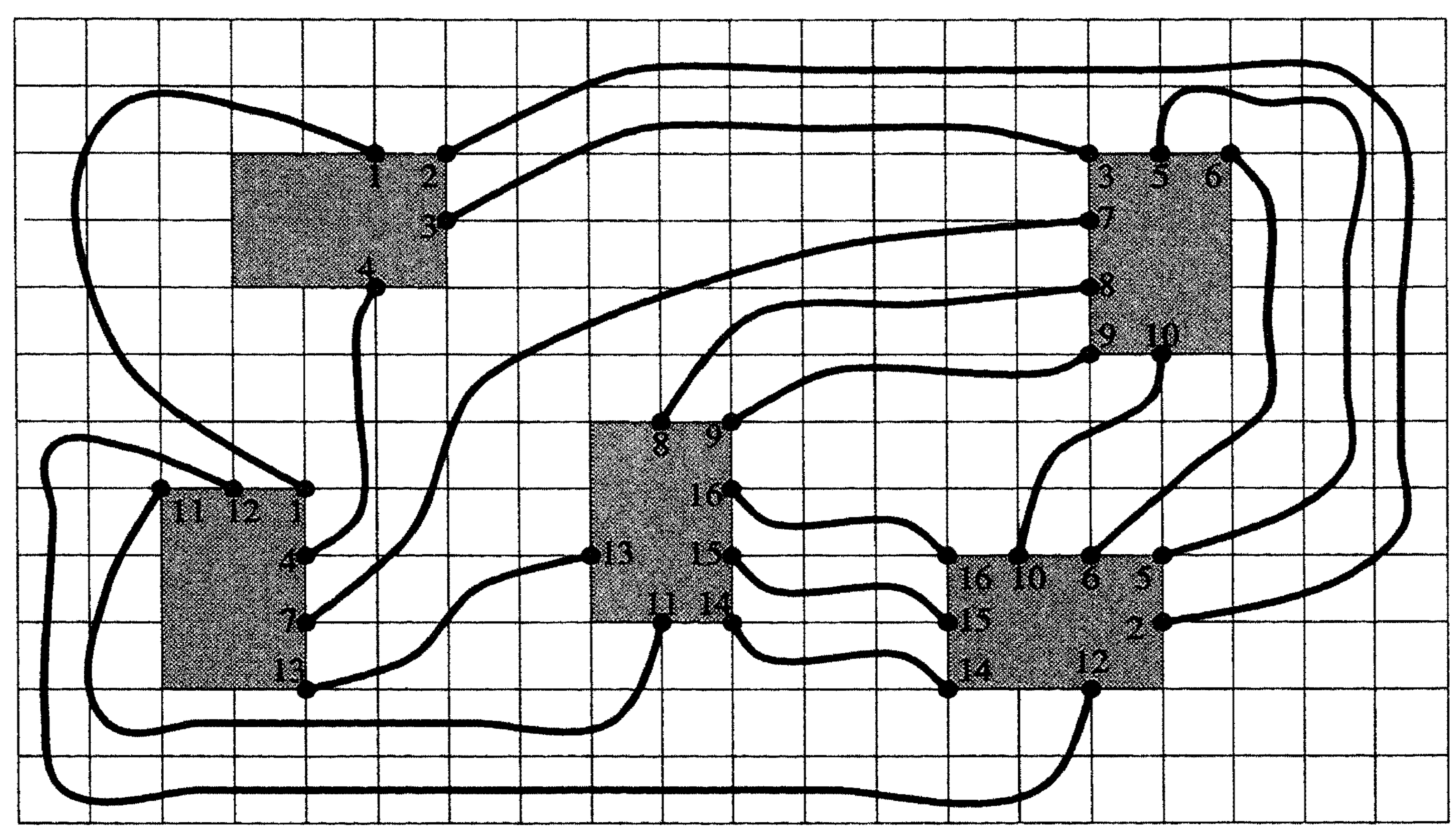


Figure 12

Second, try to find disjoint paths $P_1, \ldots, P_k$ in the graph such that P_i is homotopic to C_i, in the space obtained from the plane by taking out the rectangles forming the modules. In Figure 13 such a solution is given.

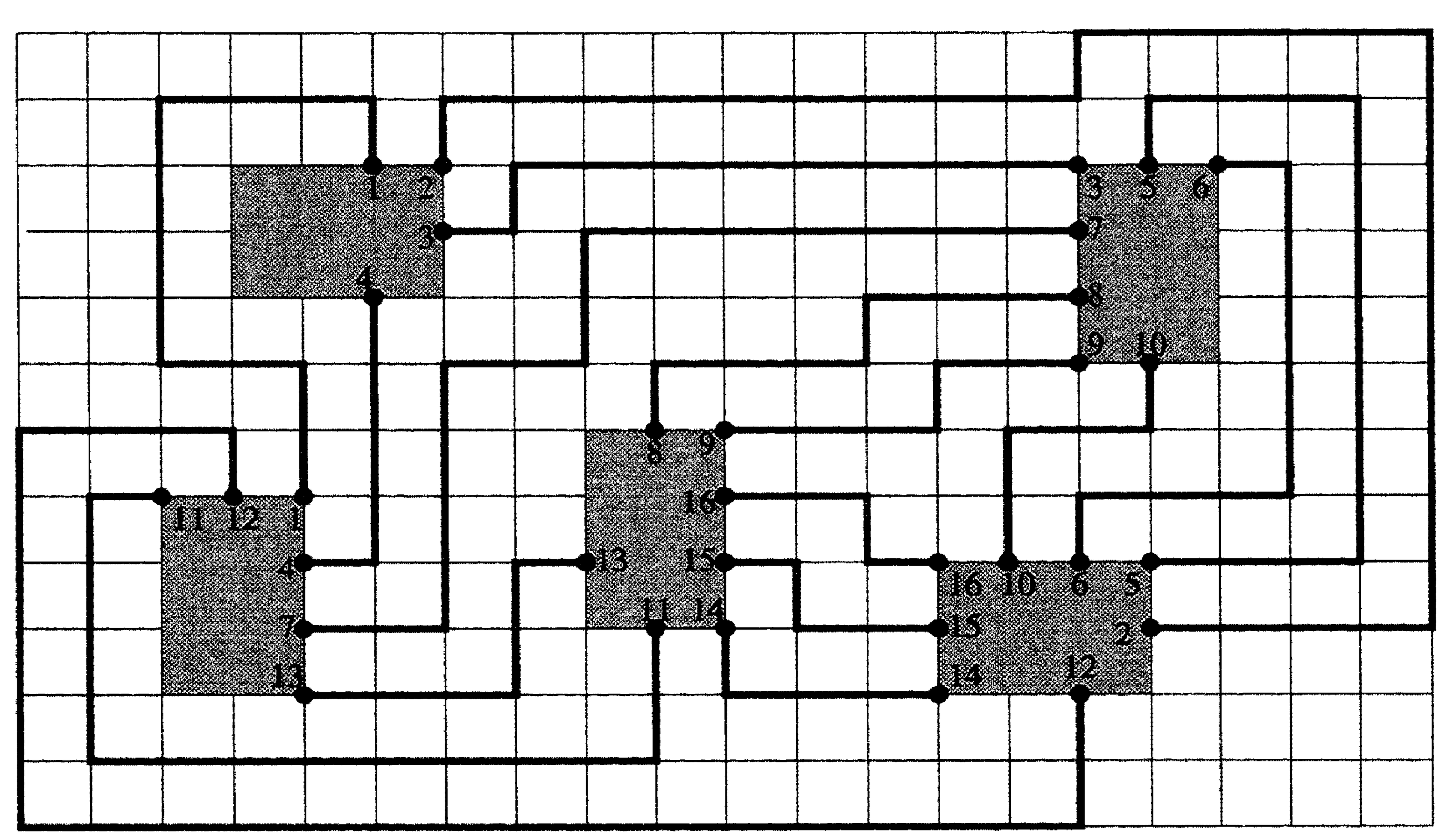


Figure 13

It was shown by Leiserson and Maley [35] that this second step can be performed in polynomial time. So the hard part of the problem is the first step: finding the right topology of the layout.

Cole and Siegel [8] proved a Menger-type cut theorem characterizing the existence of a solution in the second step. That is, if there is no solution for the disjoint paths problem given the homotopies, there is an 'oversaturated' cut: a curve D connecting two holes in the plane and intersecting the graph less than the number of times D necessarily crosses the curves C_i.

This can be used in a heuristic practical algorithm for the VLSI-routing problem: first guess the homotopies of the solution; second try to find disjoint paths of the guessed homotopies; if you find them you can stop; if you don't find them, the oversaturated cut will indicate a bottleneck in the chosen homotopies; amend the bottleneck and repeat.

Similar results hold if one wants to pack trees instead of paths (which is generally the case at VLSI-design), and the result can be extended to any planar graph [56]. As a theoretical consequence one has (by an enumeration argument similar to the one used for Theorem 7):

Theorem 9. *For each fixed number of modules, the planar VLSI-routing problem can be solved in polynomial time.*

10. Railway timetabling. The cohomology feasibility problem also shows up in the problem of making the timetable for Nederlandse Spoorwegen (Dutch Railways), a project currently performed for NS by CWI (Adri Steenbeek and me). The Dutch railway system belongs to the busiest in the world, with several short distance trajectories, while many connections are offered, with short transfer time.

Task is to provide algorithmic means to decide if a given set of conditions on the timetable can be satisfied. In particular, the hourly pattern of the timetable is considered. The basis of the NS-timetable is a periodic cycle of one hour, so that on each line there is a train at least once an hour.

How can this problem be modeled? First of all, each departure time to be determined is represented by a variable v_t. Here t is a train leg that should go every hour once. So v_t represents a variable in the cyclic group $C_{60} = \mathbb{Z}/60\mathbb{Z}$. Similarly, the arrival time is represented by a variable a_t in C_{60}.

In the problem considered by us, a fixed running time was assumed for each leg. This implies that if train leg t has a running time of 11 minutes, then $a_t - v_t = 11$. The waiting period of a train in a station is prescribed by an interval. E.g., if t and t' are two consecutive train legs of one hourly train, and if it is required that the train stops at the intermediate station for a period of at least 2 and at most 5 minutes, then one poses the condition $v_{t'} - a_t \in [2, 5]$ (as interval of C_{60}).

This gives relations between train legs of one hourly train. To make connections, one has to consider train legs in two different trains. So if one wants to make a connection from leg t, arriving in Utrecht say, of one train, to a leg t' departing from Utrecht of another train, so that the transfer time is at least 3 and at most 7 minutes, then one gets the condition $v_{t'} - a_t \in [3, 7]$.

Finally, there is the condition that for safety each two trains on the same trajectory should have a timetable distance of at least 3 minutes. That is, if train leg t of one train and train leg t' of another train run on the same railway section, then one should pose the condition $v_{t'} - v_t \in [3, 57]$.

By representing each variable by a vertex, the problem can be modeled as follows. Let $D = (V, A)$ be a directed graph, and for each $a \in A$, let $H(a)$ be an interval on C_{60}. Find a function $p : V \longrightarrow C_{60}$ such that $p(w) - p(u) \in H(a)$ for each arc $a = (u, w)$ of D.

This is a special case of the cohomology feasibility problem. Note that (as C_{60} is abelian) one may equivalently find a 'length' function $l : A \longrightarrow C_{60}$ such that $l(a) \in H(a)$ for each $a \in A$ and such that each undirected circuit in D has length 0. (For arcs a in the circuit traversed backward one takes $-l(a)$ for its length.)

It is not difficult to formulate this problem as an integer linear programming problem. Indeed, if for any arc $a = (u, w)$, $H(a)$ is equal to the interval $[l_a, u_a]$, we can put:

$$(20) \qquad l_a \leq x_w - x_u + 60y_a \leq u_a,$$

where y_a is required to be an integer. Thus we get a system of $|A|$ linear inequalities with $|V|$ real variables x_v and $|A|$ integer variables y_a. In fact, if there is a solution, there is also one with the x_v being integer as well (as the x variables make a network matrix).

Now in solving (20), one may choose a spanning tree T in D, and assume that $y_a = 0$ for each arc a in T (cf. Serafini and Ukovich [60]). Alternatively, one may consider the problem as follows.

A *circulation* is a function $f : A \longrightarrow \mathbb{R}$ such that the 'flow conservation law':

$$(21) \qquad \sum_{a \in \delta^-(v)} f(a) = \sum_{a \in \delta^+(v)} f(a)$$

holds for each vertex v of D. Here $\delta^-(v)$ and $\delta^+(v)$ denote the sets of arcs entering v and leaving v, respectively.

Let L be the lattice of all integer-valued circulations. Now one can describe the problem as one of finding a linear function $\Phi : L \longrightarrow \mathbb{Z}$ such that there exist z_a (for $a \in A$) with the properties that $l_a \leq z_a \leq u_a$ for each arc A and $z^T f = 60\Phi(f)$ for each $f \in L$.

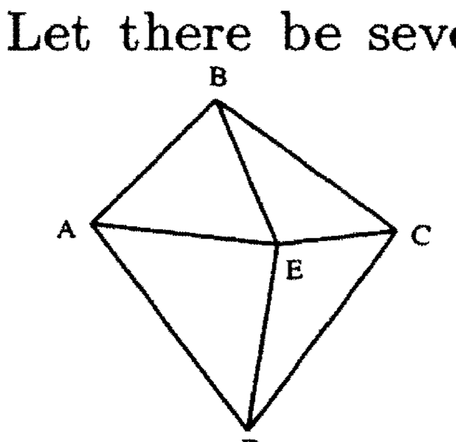
The existence of such z_a can be checked in polynomial time, given the values of Φ on a basis of L. Hence, in a searching for a feasible timetable one can branch on values of Φ on an appropriate basis of L. Given Φ, if there exist z_a, one can optimize the z_a under any linear (or convex piecewise linear) objective function (for instance, passenger waiting time).

Typically, the problems coming from NS have about 3000 variables with about 10,000 constraints. In a straightforward way they can be reduced to about 200 variables with about 600 constraints. The above observations turn out to require a too heavy framework in order to solve the problem fast in practice (although they are of help in optimizing a given solution).

The package CADANS (Combinatorisch-Algebraïsch Dienstregeling-Algoritme voor de Nederlandse Spoorwegen) that CWI is developing for NS for solving the problem above, is based on a fast constraint propagation technique and fast branching heuristics designed by Adri Steenbeek. It gives, within time of the order of 1-10 minutes either a solution (i.e., a feasible timetable), or an inclusionwise minimal set of constraints that is infeasible. If CADANS gives the latter answer, the user should drop, or relax, at least one of the constraints in the minimal set in order to make the constraints feasible. Thus CADANS can be used interactively to support the planner. Alternatively, it can uncover bottlenecks in the infrastructure, and indicate where extra infrastructure (viaducts, flyovers, four-tracks) should be built in order to make a given set of conditions feasible.

11. Transportation and flow problems. Railway transportation forms a classical source of problems studied in operations research. In 1939, Kantorovich [25] published in Leningrad a monograph called *Mathematical Methods of Organizing and Planning Production*, in which he outlined a new method to maximize a linear function under given linear inequality constraints, thus laying the fundaments for *linear programming.* He gave the following application:

> Let there be several points A, B, C, D, E which are connected to one another by a railroad network. It is possible to make the shipments from B to D by the shortest route BED, but it is also possible to use other routes as well: namely BCD, BAD. Let there also be given a schedule of freight shipments; that is, it is necessary to ship from A to B a certain number of carloads, from D to C a certain number, and so on. The problem consists of the following. There is given a maximum capacity for each route under the given conditions (it can of course change under new methods of operation in transportation). It is necessary to distribute the freight flows among the different routes in such a way as to complete the necessary shipments with a minimum expenditure of fuel, under the condition of minimizing the empty runs of freight cars and taking account of the maximum capacities of the routes. As was already shown, this problem can also be solved by our methods.



In 1941, Hitchcock [24] formulated another variant of a transportation problem. Independently, during the Second World War, Koopmans was on the staff of the Combined Shipping Adjustment Board (an agency formed by the Allied to coordinate the use of their merchant fleets). Influenced by his teacher Tinbergen (cf. [73]) he was interested in the topic of ship freights and capacities. His task at the Board was the planning of assigning ships to convoys so as to accomplish prescribed deliveries, while minimizing empty voyages (cf. [12]). Koopmans found in 1943 a method for the *transshipment problem*, but due to wartime restrictions he published it only after the war [32].

Koopmans and Reiter [33] investigated the economic implications of the method:

> For the sake of definiteness we shall speak in terms of the transportation of cargoes on ocean-going ships. In considering only shipping we do not lose generality of application since ships may be "translated" into trucks, aircraft, or, in first approximation, trains, and ports into the various sorts of terminals. Such translation is possible because all the above examples involve particular types of movable transportation equipment.
>
> The cultural lag of economic thought in the application of mathematical methods is strikingly illustrated by the fact that linear graphs are making their entrance into transportation theory just about a century after they were first studied in relation to electrical networks, although organized transportation systems are much older than the study of electricity.

The breakthrough in linear programming came around 1950 when Dantzig [10] published the *simplex method* for the linear programming problem. The success of the method was caused by a very simple tableau-form and pivoting rule and by the large efficiency in practice. Dantzig also described a direct implementation of the simplex method to the transportation problem ([9]).

In the beginning of the 1950s, T.E. Harris at the RAND Corporation (the think tank of the U.S. Air Force in Santa Monica, California) called attention for the following special case of the problem considered by Kantorovich:

> Consider a rail network connecting two cities by way of a number of intermediate cities, where each link of the network has a number assigned to it representing its capacity. Assuming a steady state condition, find a maximal flow from one given city to the other.

This question raised a stream of research at RAND. The problem can be formalized as follows.

Let be given a directed graph $D = (V, A)$, with two special vertices, a 'source' s and a 'sink' or 'terminal' t. Then an $s - t$ *flow* is a function $f : A \longrightarrow \mathbb{R}_+$ such that for each vertex $v \neq s, t$ the flow conservation law (21) holds. The *value* of f is equal to the net flow leaving s; that is:

(22) $$\text{value}(f) := \sum_{a \in \delta^-(s)} f(a) - \sum_{a \in \delta^+(s)} f(a).$$

It is not difficult to prove that this value is equal to the net flow entering t.

If moreover a 'capacity' function $c : A \longrightarrow \mathbb{R}_+$ is given, one says that f is *subject to* c if $f(a) \leq c(a)$ for each arc a.

Now the *maximum flow problem* can be formulated:

(23) given: a directed graph $D = (V, A)$, vertices $s, t \in V$, and a 'capacity' function $c : A \longrightarrow \mathbb{R}_+$;

find: a flow f subject to c maximizing value(f).

In their basic paper "Maximal flow through a network" (published as a RAND Report of 19 November 1954), Ford and Fulkerson [17] observed that this is just a linear programming problem, and hence can be solved with Dantzig's simplex method.

Main result of Ford and Fulkerson's paper is the famous *max-flow min-cut theorem*. To this end, the concept of a *cut* is defined. Let U is any set with $s \in U$ and $t \notin U$. Then $\delta^+(U)$ (the set of all arcs leaving U) is an $s-t$ *cut*. The *capacity* of the cut is the sum of all $c(a)$ for $a \in \delta^+(U)$.

It is clear that the capacity of any cut is an upper bound on the maximal value of $s-t$ flows. What Ford and Fulkerson [17] showed is:

Max-flow min-cut theorem. *The maximal value of the $s-t$ flows is equal to the minimal capacity of the $s-t$ cuts.*

Since (as follows from an observation of Dantzig [9]) there is an integer-valued maximum flow if all capacities are integer, an arc-disjoint version of Menger's theorem follows from the max-flow min-cut theorem.

Alternative proofs of the max-flow min-cut theorem were given by Robacker [51] and by Elias, Feinstein, and Shannon [14]. In this last paper it is claimed that the result was known by workers in communication theory:

> This theorem may appear almost obvious on physical grounds and appears to have been accepted without proof for some time by workers in communication theory. However, while the fact that this flow cannot be exceeded is indeed almost trivial, the fact that it can actually be achieved is by no means obvious. We understand that proofs of the theorem have been given by Ford and Fulkerson and Fulkerson and Dantzig. The following proof is relatively simple, and we believe different in principle.

The max-flow min-cut theorem being also a combinatorial result, one was interested in obtaining combinatorial methods for finding maximum flows. First, Ford and Fulkerson [17] gave a simple algorithm for the maximal flow problem in case the graph, added with an extra edge connecting s and t, is planar.

Next, a heuristic method, the *flooding technique*, was presented by Boldyreff [6] on 3 June 1955 at the New York meeting of the Operations Research Society of America (RAND Report of 5 August 1955). The method was intuitive, and the author did not claim generality:

> It has been previously assumed that a highly complex railway transportation system, too complicated to be amenable to analysis, can be represented by a much simpler model. This was accomplished by representing each complete railway operating division by a point, and by joining pairs of such points by arcs (lines) with traffic carrying capacities equal to the maximum possible volume of traffic (expressed in some convenient unit, such as trains per day) between the corresponding operating divisions.
>
> In this fashion, a network is obtained consisting of three sets of points — points of origin, intermediate or junction points, and the terminal points (or points of destination) — and a set of arcs of specified traffic carrying capacities, joining these points to each other.

Boldyreff's arguments for designing a heuristic procedure are formulated as:

> In the process of searching for the methods of solving this problem the following objectives were used as a guide:
>
> **1.** That the solution could be obtained quickly, even for complex networks.
>
> **2.** That the method could be explained easily to personnel without specialized technical training and used by them effectively.
>
> **3.** That the validity of the solution be subject to easy, direct verification.
>
> **4.** That the method would not depend on the use of high-speed computing or other specialized equipment.

Boldyreff's 'flooding technique' pushes a maximum amount of flow greedily through the network. If at some vertex a 'bottleneck' arises (i.e., there are more trains arriving than can be pushed further through the network), it is eliminated by returning the excess trains to the origin. It is empirical, not using backtracking, and not leading to an optimum solution in all cases:

> Whenever arbitrary decisions have to be made, ordinary common sense is used as a guide. At each step the guiding principle is to move forward the maximum possible number of trains, and to maintain the greatest flexibility for the remaining network.

Boldyreff speculates that 'in dealing with the usual railway networks a single flooding, followed by removal of bottlenecks, should lead to a maximal flow.' He gives as an example of a complex network, a railway transportation system with 41 vertices and 85 arcs, for which 'the total time of solving the problem is less than thirty minutes.'

Soon after, Ford and Fulkerson presented in a RAND Report of 29 December 1955 [18] their 'very simple algorithm' for the maximum flow problem, based on finding 'augmenting paths'. The algorithm finds in a finite number of steps a maximum flow, if all capacities have rational values. After mentioning the maximum flow problem, they remark:

> This is of course a linear programming problem, and hence may be solved by Dantzig's simplex algorithm. In fact, the simplex computation for a problem of this kind is particularly efficient, since it can be shown that the sets of equations one solves in the process are always triangular. However, for the flow problem, we shall describe what appears to be a considerably more efficient algorithm; it is, moreover, readily learned by a person with no special training, and may easily be mechanized for handling large networks. We believe that problems involving more than 500 nodes and 4,000 arcs are within reach of present computing machines.

Ford and Fulkerson's algorithm for the maximum-flow problem formed a breakthrough. It has implementations that require only polynomially bounded running time, as was shown by Dinits [11] and Edmonds and Karp [13]. In the latter paper, also a polynomial-time algorithm is given for the *minimum-cost* flow problem. It implies a polynomial-time algorithm for the minimum-cost circulation problem.

12. Routing of railway stock. The work on the minimum-cost circulation problem can be applied to minimizing the railway stock needed to run a schedule. NS (Nederlandse Spoorwegen) runs an hourly train service on its route Amsterdam - Schiphol Airport - Leyden - The Hague - Rotterdam - Dordrecht - Roosendaal - Middelburg - Vlissingen *vice versa*, with timetable as in Table 1.

train number		2123	2127	2131	2135	2139	2143	2147	2151	2155
Amsterdam	V		6.48	7.55	8.56	9.56	10.56	11.56	12.56	13.56
Rotterdam	A		7.55	8.58	9.58	10.58	11.58	12.58	13.58	14.58
Rotterdam	V	7.00	8.01	9.02	10.03	11.02	12.03	13.02	14.02	15.02
Roosendaal	A	7.40	8.41	9.41	10.43	11.41	12.41	13.41	14.41	15.41
Roosendaal	V	7.43	8.43	9.43	10.45	11.43	12.43	13.43	14.43	15.43
Vlissingen	A	8.38	9.38	10.38	11.38	12.38	13.38	14.38	15.38	16.38

train number		2159	2163	2167	2171	2175	2179	2183	2187	2191
Amsterdam	V	14.56	15.56	16.56	17.56	18.56	19.56	20.56	21.56	22.56
Rotterdam	A	15.58	16.58	17.58	18.58	19.58	20.58	21.58	22.58	23.58
Rotterdam	V	16.00	17.01	18.01	19.02	20.02	21.02	22.02	23.02	
Roosendaal	A	16.43	17.43	18.42	19.41	20.41	21.41	22.41	23.54	
Roosendaal	V	16.45	17.45	18.44	19.43	20.43	21.43			
Vlissingen	A	17.40	18.40	19.39	20.38	21.38	22.38			

train number		2108	2112	2116	2120	2124	2128	2132	2136	2140
Vlissingen	V			5.30	6.54	7.56	8.56	9.56	10.56	11.56
Roosendaal	A			6.35	7.48	8.50	9.50	10.50	11.50	12.50
Roosendaal	V		5.29	6.43	7.52	8.53	9.53	10.53	11.53	12.53
Rotterdam	A		6.28	7.26	8.32	9.32	10.32	11.32	12.32	13.32
Rotterdam	V	5.31	6.29	7.32	8.35	9.34	10.34	11.34	12.34	13.35
Amsterdam	A	6.39	7.38	8.38	9.40	10.38	11.38	12.38	13.38	14.38

train number		2144	2148	2152	2156	2160	2164	2168	2172	2176
Vlissingen	V	12.56	13.56	14.56	15.56	16.56	17.56	18.56	19.55	
Roosendaal	A	13.50	14.50	15.50	16.50	17.50	18.50	19.50	20.49	
Roosendaal	V	13.53	14.53	15.53	16.53	17.53	18.53	19.53	20.52	21.53
Rotterdam	A	14.32	15.32	16.32	17.33	18.32	19.32	20.32	21.30	22.32
Rotterdam	V	14.35	15.34	16.34	17.35	18.34	19.34	20.35	21.32	22.34
Amsterdam	A	15.38	16.40	17.38	18.38	19.38	20.38	21.38	22.38	23.38

Table 1: Timetable Amsterdam-Vlissingen vice versa

The trains have more stops, but for our purposes only those given in the table are of interest.

For each of the legs of any scheduled train, Nederlandse Spoorwegen has determined an expected number of (second-class) passengers, given in Table 2. The problem to be solved is: What is the minimum amount of train stock necessary to perform the service in such a way that at each leg there are enough seats?

train number	2123	2127	2131	2135	2139	2143	2147	2151	2155
Amsterdam-Rotterdam		340	616	407	336	282	287	297	292
Rotterdam-Roosendaal	58	272	396	364	240	221	252	267	287
Roosendaal-Vlissingen	328	181	270	237	208	188	180	195	290
train number	2159	2163	2167	2171	2175	2179	2183	2187	2191
Amsterdam-Rotterdam	378	527	616	563	320	184	161	190	123
Rotterdam-Roosendaal	497	749	594	395	254	165	130	77	
Roosendaal-Vlissingen	388	504	381	276	187	136			

train number	2108	2112	2116	2120	2124	2128	2132	2136	2140
Vlissingen-Roosendaal			138	448	449	436	224	177	184
Roosendaal-Rotterdam		167	449	628	397	521	281	214	218
Rotterdam-Amsterdam	61	230	586	545	427	512	344	303	283
train number	2144	2148	2152	2156	2160	2164	2168	2172	2176
Vlissingen-Roosendaal	181	165	225	332	309	164	142	121	
Roosendaal-Rotterdam	174	206	298	422	313	156	155	130	64
Rotterdam-Amsterdam	330	338	518	606	327	169	157	154	143

Table 2: Numbers of required seats

In order to answer this question, one should know a number of further characteristics and constraints. In a first variant of the problem considered, the train stock consists of one type of two-way train-units, each consisting of three carriages, and each having 163 seats. Each unit has at both ends an engineer's cabin, and units can be coupled together, up to 15 carriages, that is, 5 train-units.

The train length can be changed, by coupling or decoupling units, at the terminal stations of the line, that is at Amsterdam and Vlissingen, and *en route* at two intermediate stations: Rotterdam and Roosendaal. Any train-unit decoupled from a train arriving at place X at time t can be linked up to any other train departing from X at any time later than t. (The Amsterdam-Vlissingen schedule is such that in practice this gives enough time to make the necessary switchings.)

A last condition is that, for logistic reasons, for each place $X \in$ {Amsterdam, Rotterdam, Roosendaal, Vlissingen}, the number of train-units staying overnight at X should be constant during the week (but may vary for different places).

Given these problem data and characteristics, one may ask for the minimum number of train-units that should be available to perform the daily cycle of train rides required.

If only one type of railway stock is used, the classical min-cost circulation method can be applied (Bartlett [5], cf. [15], [16], [45], [47], [75]). To this end, a directed graph $D = (V, A)$ is constructed as follows. For each place $X \in$ {Amsterdam, Rotterdam, Roosendaal, Vlissingen} and for each time t at which any train leaves or arrives at X, we make a vertex (X, t). So the vertices of D correspond to all 198 time entries in the timetable (Table 1).

For any leg of any train, leaving place X at time t and arriving at place Y at time t', we make a directed arc from (X, t) to (Y, t'). For instance, there is an arc from (Roosendaal, 7.43) to (Vlissingen, 8.38).

Moreover, for any place X and any two successive times t, t' at which any train leaves or arrives at X, we make an arc from (X, t) to (X, t'). Thus in our example there will be arcs, e.g., from (Rotterdam, 8.01) to (Rotterdam, 8.32), from (Rotterdam, 8.32) to (Rotterdam, 8.35), from (Vlissingen, 8.38) to (Vlissingen,8.56), and from (Vlissingen, 8.56) to (Vlissingen, 9.38).

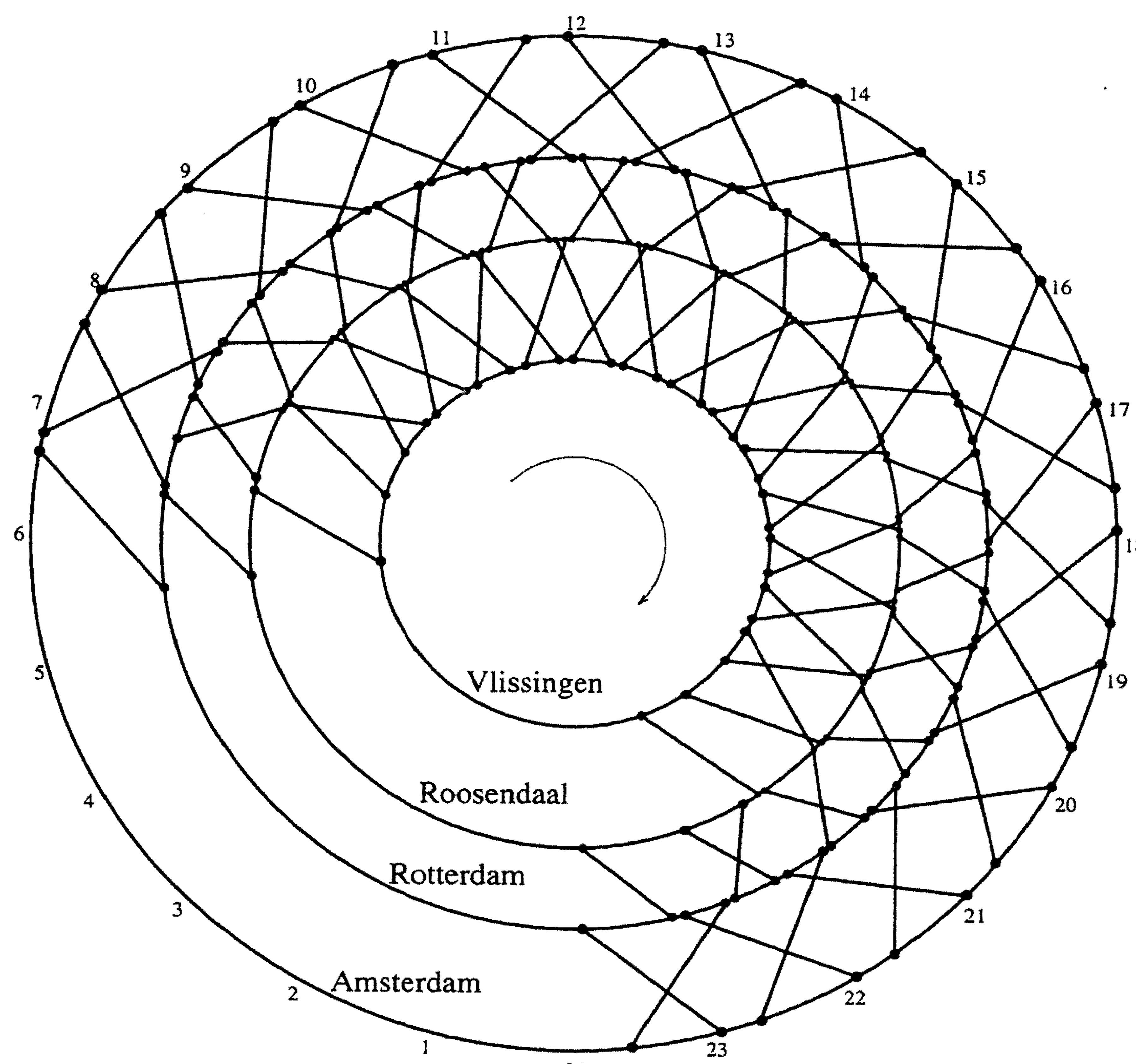


Figure 14: The graph D. All arcs are oriented clockwise

Finally, for each place X there will be an arc from (X, t) to (X, t'), where t is the last time of the day at which any train leaves or arrives at X and where t' is the first time of the day at which any train leaves or arrives at X. So there is an arc from (Roosendaal, 23.54) to (Roosendaal, 5.29).

We can now describe any possible routing of train stock as a function $f : A \longrightarrow \mathbb{Z}_+$, where $f(a)$ denotes the following. If a corresponds to a leg, then $f(a)$ is the number of units deployed for that leg. If a corresponds to an arc from (X, t) to (X, t'), then $f(a)$ is equal to the number of units present at place X in the time period t–t' (possibly overnight).

First of all, this function is a *circulation*, that is, the *flow conservation law* (21) holds. Moreover, in order to satisfy the demand and capacity constraints, $f(a)$ should satisfy $d(a) \leq f(a) \leq 5$, where $d(a)$ is the minimum number of train-units necessary for leg a, based on the lower bound on seats for leg a.

Now observe that the total number of units needed, is equal to the total flow on the 'overnight' arcs. So if we wish to minimize the total number of units deployed, we could restrict ourselves to minimizing $\sum_{a \in A^\circ} f(a)$, where A° denotes the set of overnight arcs. (So $|A^\circ| = 4$ in the Amsterdam - Vlissingen example.)

It is easy to see that this fully models the problem. Hence determining the minimum number of train-units amounts to solving a minimum-cost circulation problem, where the cost function is quite trivial: we have $\text{cost}(a) = 1$ if a is an overnight arc, and $\text{cost}(a) = 0$ for all other arcs.

Having this model, we can apply standard min-cost circulation algorithms, based on min-cost augmenting paths and cycles (cf. Ford and Fulkerson [19] and Ahuja, Magnanti, and Orlin [1]). Implementation gives solutions of the problem (for the above data) in about 0.05 CPUseconds on an SGI R4400.

Alternatively, the problem can be solved easily with any linear programming package, since by the integrality of the input data and by the total unimodularity of the underlying matrix the optimum basic solution will have integer values only. With the linear programming package CPLEX (version 2.1) the optimum solution given in Table 3 was obtained again in about 0.05 CPUseconds (on an SGI R4400):

train number	2123	2127	2131	2135	2139	2143	2147	2151	2155
Amsterdam-Rotterdam		3	4	3	3	2	2	2	2
Rotterdam-Roosendaal	1	2	3	3	2	2	2	2	2
Roosendaal-Vlissingen	3	2	2	2	2	2	2	2	2
train number	**2159**	**2163**	**2167**	**2171**	**2175**	**2179**	**2183**	**2187**	**2191**
Amsterdam-Rotterdam	5	5	4	4	2	2	1	2	1
Rotterdam-Roosendaal	4	5	4	3	2	2	1	1	
Roosendaal-Vlissingen	3	4	3	2	2	1			

train number	2108	2112	2116	2120	2124	2128	2132	2136	2140
Vlissingen-Roosendaal			1	3	3	3	2	2	2
Roosendaal-Rotterdam		2	4	4	3	4	2	2	2
Rotterdam-Amsterdam	1	2	4	4	3	4	3	2	2
train number	**2144**	**2148**	**2152**	**2156**	**2160**	**2164**	**2168**	**2172**	**2176**
Vlissingen-Roosendaal	2	2	2	3	2	2	1	4	
Roosendaal-Rotterdam	2	3	2	4	3	1	1	1	1
Rotterdam-Amsterdam	3	3	4	4	3	2	1	1	1

Table 3: Minimum circulation with one type of stock

Required are 22 units, divided during the night over Amsterdam: 4, Rotterdam: 2, Roosendaal: 8, and Vlissingen: 8.

It is quite direct to modify and extend the model. Instead of minimizing the number of train-units one can minimize the amount of carriage-kilometers that should be made every day, or any linear combination of both quantities. In addition, one can put an upper bound on the number of units that can be stored at any of the stations.

Instead of considering one line only, one can more generally consider *networks* of lines that share the same railway stock, including trains that are scheduled to be split or combined. (Nederlandse Spoorwegen has trains from The Hague and Rotterdam to Leeuwarden and Groningen that are combined to one train on the common trajectory between Utrecht and Zwolle.)

If only one type of unit is employed for that part of the network, each unit having the same capacity, the problem can be solved fast even for large networks.

13. Two types of stock. The problem becomes harder if there are several types of trains that can be deployed for the train service. Clearly, if for each scheduled train we would prescribe which type of unit should be deployed, the problem could be decomposed into separate problems of the type above. But if we do not make such a prescription, and if some of the types can be coupled together to form a train of mixed composition, we should extend the model to a 'multi-commodity circulation' model.

Let us restrict ourselves to the case Amsterdam-Vlissingen again, where now we can deploy two types of two-way train-units, that can be coupled together. The two types are type IC3, each unit of which consists of 3 carriages and has 163 seats, and type IC4, each unit of which consists of 4 carriages and has 218 seats.

Again, the demands of the train legs are given in Table 2. The maximum number of carriages that can be in any train again is 15. This means that if a train consists of x units of type IC3 and y units of type IC4 then $3x + 4y \leq 15$ should hold.

It is quite easy to extend the model above to the present case. Again we consider the directed graph $D = (V, A)$ as above. At each arc a let $f(a)$ be the number of units of type IC3 on the leg corresponding to a and let $g(a)$ similarly represent type IC4. So both $f : A \longrightarrow \mathbb{Z}_+$ and $g : A \longrightarrow \mathbb{Z}_+$ are circulations, that is, satisfy the flow circulation law:

$$(24)\qquad \sum_{a\in\delta^-(v)} f(a) = \sum_{a\in\delta^+(v)} f(a),$$
$$\sum_{a\in\delta^-(v)} g(a) = \sum_{a\in\delta^+(v)} g(a),$$

for each vertex v. The capacity constraint now is:

$$(25)\qquad 3f(a) + 4g(a) \leq 15$$

for each arc a representing a leg. The demand constraint can be formulated as follows:

(26) $$163f(a) + 218g(a) \geq p(a),$$

for each arc a representing a leg, where $p(a)$ denotes the number of seats required (Table 2). Note that in contrary to the case of one type of unit, now we cannot speak of a minimum number of units required: now there are two dimensions, so that minimum train compositions need not be unique.

If cost_{IC3} and cost_{IC4} represent the cost of purchasing one unit of type IC3 and of type IC4, respectively, then the problem is to find f and g so as to

(27) $$\text{minimize} \sum_{a \in A^\circ} (\text{cost}_{\text{IC3}} f(a) + \text{cost}_{\text{IC4}} g(a)).$$

The classical min-cost circulation algorithms do not apply now. Moreover, when solving the problem as a linear programming problem, we lose the pleasant phenomenon observed above that we automatically would obtain an optimum solution $f, g : A \longrightarrow \mathbb{R}$ with *integer* values only.

So the problem is an integer linear programming problem, with 198 integer variables. Solving the problem in this form with the integer programming package CPLEX (version 2.1) would give (for the Amsterdam-Vlissingen example) a running time of several hours, which is too long, for instance when one wishes to compare several problem data.

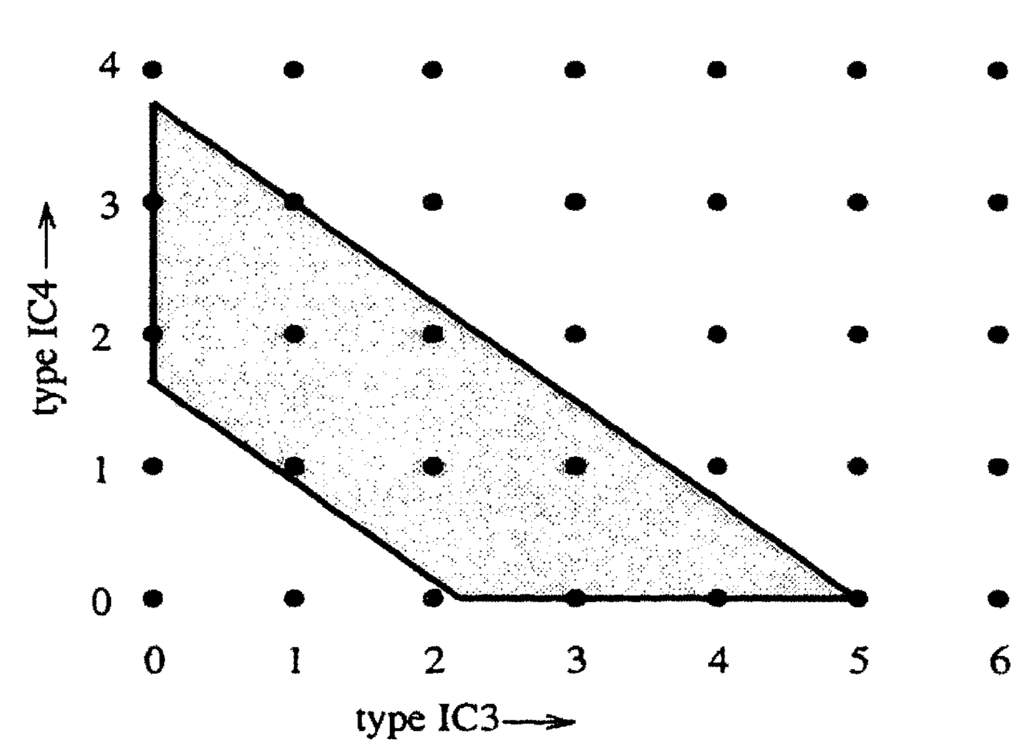
However, there are ways of speeding up the process, by sharpening the constraints and by exploiting more facilities offered by CPLEX. The conditions (25) and (26) can be sharpened in the following way. For each arc a representing a leg, the two-dimensional vector $(f(a), g(a))$ should be an integer vector in the polygon

(28) $$P_a := \{(x, y) | x \geq 0, y \geq 0, 163x + 218y \geq p(a), 3x + 4y \leq 15\}.$$

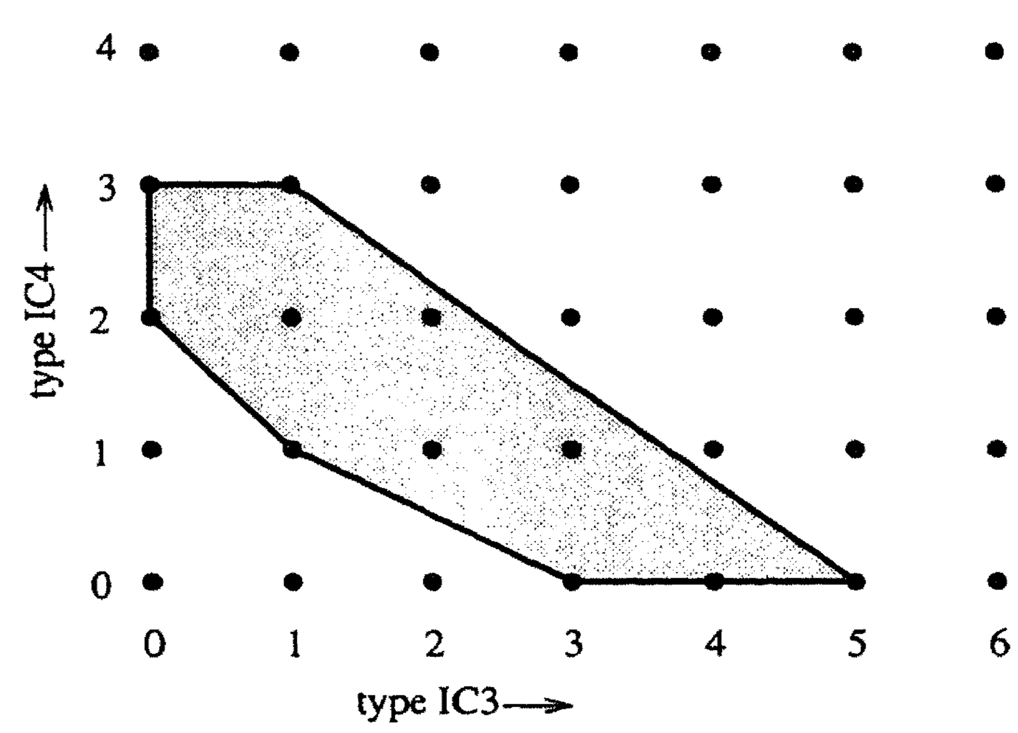
For instance, the trajectory Rotterdam-Amsterdam of train 2132 gives the polygon

(29) $$P_a = \{(x, y) | x \geq 0, y \geq 0, 163x + 218y \geq 344, 3x + 4y \leq 15\}.$$

In a picture:



In a sense, the inequalities are too wide. The constraints given in (29) could be tightened so as to describe exactly the convex hull of the integer vectors in the polygon P_a (the 'integer hull'), as in:



Thus for segment Rotterdam-Amsterdam of train 2132 the constraints in (29) can be sharpened to:

(30) $$x \geq 0, y \geq 0, x + y \geq 2, x + 2y \geq 3, y \leq 3, 3x + 4y \leq 15.$$

Doing this for each of the 99 polygons representing a leg gives a sharper set of inequalities, which helps to obtain more easily an integer optimum solution from a fractional solution. (This is a weak form of application of the technique of *polyhedral combinatorics*.) Finding all these sharpened inequalities can be done in a pre-processing phase, and takes about 0.04 CPUseconds.

Implementation of these techniques makes that CPLEX gives a solution to the Amsterdam-Vlissingen problem in 1.58 CPUseconds — see Table 4.

train number	2123	2127	2131	2135	2139	2143	2147	2151	2155
Amsterdam-Rotterdam		0+2	0+3	4+0	0+2	0+2	1+2	0+2	1+1
Rotterdam-Roosendaal	0+1	0+2	0+2	4+0	0+2	0+2	1+3	0+3	1+1
Roosendaal-Vlissingen	0+2	0+2	0+2	2+0	0+1	0+1	0+2	0+2	2+0
train number	2159	2163	2167	2171	2175	2179	2183	2187	2191
Amsterdam-Rotterdam	0+3	2+1	0+3	1+2	0+2	0+1	1+2	0+1	0+1
Rotterdam-Roosendaal	0+3	2+2	0+3	0+2	1+1	2+0	1+3	1+0	
Roosendaal-Vlissingen	0+2	2+1	0+2	0+2	2+0	0+1			

train number	2108	2112	2116	2120	2124	2128	2132	2136	2140
Vlissingen-Roosendaal			1+0	0+3	1+2	0+2	0+2	0+1	1+1
Roosendaal-Rotterdam		1+2	3+0	0+3	0+2	1+2	0+2	2+1	1+3
Rotterdam-Amsterdam	0+1	0+2	4+0	0+3	0+3	1+2	0+2	2+0	0+2
train number	2144	2148	2152	2156	2160	2164	2168	2172	2176
Vlissingen-Roosendaal	1+1	0+1	0+2	0+2	2+0	0+2	2+0	0+1	
Roosendaal-Rotterdam	0+1	0+3	1+3	0+3	1+1	0+1	2+2	0+1	1+0
Rotterdam-Amsterdam	1+1	0+3	1+2	0+3	1+1	0+1	0+2	0+1	0+1

Table 4: Minimum circulation with two types of stock

In this table $x + y$ means: x units of type IC3 and y units of type IC4. In total, one needs 7 units of type IC3 and 12 units of type IC4, divided during the night as in Table 5.

	number of units of type IC3	number of units of type IC4	total number of units	total number of carriages
Amsterdam	0	2	2	8
Rotterdam	0	2	2	8
Roosendaal	3	3	6	21
Vlissingen	2	5	7	26
Total	5	12	17	63

Table 5: Required stock (two types)

So compared with the solution for one type only, the possibility of having two types gives both a decrease in the number of train-units (17 instead of 22) and in the number of carriages (63 instead of 66).

Our research for NS in fact has focused on more extended problems that require more complicated models and techniques. One requirement is that in any train ride Amsterdam-Vlissingen there should be at least one unit that makes the whole trip. Moreover, it is required that, at any of the four stations given (Amsterdam, Rotterdam, Roosendaal, Vlissingen) one may either couple units to or decouple units from a train, but not both simultaneously. Moreover, one may couple fresh units only to the front of the train, and decouple laid off units only from the rear. (One may check that these conditions are not met by all trains in the solution given in Table 4.)

This all causes that the order of the different units in a train does matter, and that conditions have a more global impact: the order of the units in a certain morning train can still influence the order in some evening train. This does not fit directly in the circulation model described above, and requires a combinatorial extension.

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