Actions on the Hilbert cube

To Cor Baayen, at the occasion of his retirement.

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We provide a negative answer to Problem 933 in the “Open Problems in Topology Book”.

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1 Introduction

Let $Q$ denote the Hilbert cube $\prod_{i=1}^{\infty} [-1,1]$. In the “Open Problems in Topology Book”, West [2] asks the following (Problem #933):

Let the compact Lie group $G$ act semifreely on $Q$ in two ways such that their fixed point sets are identical. If the orbit spaces are ANR’s, are the actions conjugate?

The aim of this note is to present a counterexample to this problem. For all undefined notions we refer to [1].

2 The Example

Let $G$ be a group and let $\pi: G \times X \to X$ be an action from $G$ on $X$. Define $\text{Fix}(G) = \{x \in X : (\forall g \in G)(\pi(g, x) = x)\}$. It is clear that $\text{Fix}(G)$ is a closed subset of $X$: it is called the fixed-point set of $G$. The action $\pi$ is called semifree if it is free off $\text{Fix}(G)$, i.e., if $x \in X \setminus \text{Fix}(G)$ and $\pi(g, x) = x$ for some $g \in G$ then $g$ is the identity element of $G$. The space of orbits of the action $\pi$ will be denoted by $X/G$. Let $I$ denote the interval $[0,1]$. 

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Let $G$ denote the compact Lie group $T \times Z_2$, where $T$ denotes the circle group. We identify $Z_2$ and the subgroup $\{-1, 1\}$ of $T$. In addition, $D$ denotes $\{z \in C : |z| \leq 1\}$. We let $G$ act on $D \times D$ in the obvious way:

$$((g, \epsilon), (x, y)) \mapsto (g \cdot x, \epsilon \cdot y) \quad (g \in T, \epsilon \in \{-1, 1\}, x, y \in D),$$

where $\cdot$ means complex multiplication. Observe that this action is semifree, and that its fixed-point set contains the point $(0, 0)$ only. Also, observe that $(D \times D)/G \cong I \times D$.

**Lemma 2.1** Let $H$ denote either $G$ or $T$. There is a semifree action of $H$ on $Q \times I$ having $Q \times \{0\}$ as its fixed-point set. Moreover, $(Q \times I)/G$ and $Q$ are homeomorphic.

**Proof.** We will only prove the lemma for $G$ since the proof for $T$ is entirely similar. We first let $G$ act on $X = D \times D \times Q$ as follows:

$$((g, \epsilon), (x, y, z)) \mapsto (g \cdot x, \epsilon \cdot y, z) \quad (g \in T, \epsilon \in \{-1, 1\}, x, y \in D, z \in Q).$$

This action is semifree and its fixed-point set is equal to $\{(0, 0)\} \times Q$. Also observe that $X/G \cong I \times D \times Q$.

We now let $G$ act coordinatewise on the infinite product $X^\infty$. This action is again semifree, having the diagonal $\Delta$ of $\{(0, 0)\} \times Q$ in $X^\infty$ as its fixed-point set. Also, $X^\infty/G$ is homeomorphic to $(I \times D \times Q)^\infty \cong Q$. Since $\Delta$ projects onto a proper subset of $X$ in every coordinate direction of $X^\infty$, it is a $Z$-set. Since $X^\infty \cong Q$ there consequently is a homeomorphism of pairs $(X^\infty, \Delta) \rightarrow (Q \times I, Q \times \{0\})$. We are done.

We will now describe two actions of $G$ on $Q \times [-1, 1]$. By Lemma 2.1 there is a semifree action $\alpha_r : T \times Q \times I \rightarrow Q \times I$ having $Q \times \{0\}$ as its fixed-point set, while moreover $Q \times I/G \cong Q$. We let $T$ act on $Q \times [-1, 0]$ as follows:

$$(z, (q, t)) \mapsto (\tilde{q}, s) \quad \text{iff} \quad \alpha_r(z, (q, -t)) = (\tilde{q}, -s).$$

We will denote this action by $\alpha_l$. So $\alpha = \alpha_l \cup \alpha_r$ is an action of $T$ onto $Q \times [-1, 1]$, having $Q \times \{0\}$ as its fixed-point set. Now define $\tilde{\alpha} : G \times (Q \times [-1, 1]) \rightarrow Q \times [-1, 1]$ as follows:

$$\tilde{\alpha}((z, \epsilon), (q, t)) = \begin{cases} \alpha(z, (q, t)), & (\epsilon = 1), \\ \alpha(z, (q, -t)), & (\epsilon = -1). \end{cases}$$

Then $\tilde{\alpha}$ is a semifree action of $G$ onto $Q \times [-1, 1]$ having $Q \times \{0\}$ as its fixed-point set, while moreover $(Q \times [-1, 1])/\tilde{\alpha} \cong Q$. Observe the following triviality.

**Lemma 2.2** If $A \subseteq Q \times [-1, 1]$ is $\tilde{\alpha}$-invariant such that $A$ is not contained in $Q \times \{0\}$, then $A$ intersects $Q \times (0, 1]$ as well as $Q \times [-1, 0)$. 

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We will now describe the second action on \( Q \times [-1,1] \). By Lemma 2.1 there is a semifree action \( \beta_r: G \times Q \times I \to Q \times I \) having \( Q \times \{0\} \) as its fixed point set, while moreover \( Q \times I/G \approx Q \). Construct \( \beta_l \) from \( \beta_r \) in the same way we constructed \( \alpha_l \) from \( \alpha_r \). Then \( \beta = \beta_l \cup \beta_r \) is a semifree action from \( G \) onto \( Q \times [-1,1] \) having \( Q \times \{0\} \) as its fixed-point set. Moreover, \( (Q \times I)/\beta \) is the union of two Hilbert cubes, meeting in a third Hilbert cube, hence is an AR. (It can be shown that \( (Q \times I)/\beta \approx Q_s \)).

Now assume that the two axioms \( \alpha \) and \( \beta \) are conjugate. Let \( \tau: Q \times [-1,1] \to Q \times [-1,1] \) be a homeomorphism such that for every \( g \in G \), \( \beta(g) = \tau^{-1} \circ \alpha(g) \circ \tau \). Then \( \tau(Q \times \{0,1\}) \) is a connected \( \alpha \)-invariant subset of \( Q \times [-1,1] \) which misses \( Q \times \{0\} \). This contradicts Lemma 2.2.

References
