Finite graphs in which the point neighbourhoods are the maximal independent sets

A.E. Brouwer

We determine all graphs as in the title.

In [vdH] certain graphs \( L_k \) occur. Noticing that they have the property mentioned in the title, I wondered whether they are the only such graphs. This note shows that, essentially, this is indeed the case.

For \( k \leq 1 \), let \( L_k \) be the graph with vertex set \( \mathbb{Z}_{3k-1} \) (the integers mod \( 3k-1 \)) and adjacencies \( x \sim y \) iff \( y - x \in \{1, 4, 7, \ldots, 3k - 2\} \). (Thus, \( L_1 \) is the complete graph on two vertices, and \( L_2 \) is the pentagon.) The \emph{neighbourhood} of a vertex \( x \) is the set \( N(x) = \{y \mid y \sim x\} \). A graph \( G \) is called \emph{reduced} when distinct vertices have distinct neighbourhoods.

**Theorem 0.1.** The finite reduced triangle-free graphs in which each independent set is contained in a point neighbourhood are precisely the graphs \( L_k \) (\( k \geq 1 \)).

**Proof:** First we show that the graphs \( L_k \) have the stated property. That they are finite, reduced and triangle-free is clear. Now it suffices to show that if \( S \) is an independent set contained in \( N(x) \), and \( S \cup \{y\} \) is independent for some \( y, y \neq x \), then \( S \cup \{y\} \subseteq N(z) \) for some \( z \). But if \( y = x + 3i - 1 \) or \( y = x + 3i \) for some \( i \) (\( 1 \leq i \leq k - 1 \)), and we can take \( z = x + 3i \) or \( z = x + 3i - 1 \), respectively.

Conversely, let the graph \( G \) have the stated property. We show that \( G \simeq L_k \) for some \( k \leq 1 \). Since \( \emptyset \) is independent, \( G \) has a vertex, and since a singleton is independent, each vertex has a neighbour, and since two nonadjacent vertices have a common neighbour, \( G \) has diameter at most 2. Clearly, if \( G \) is complete, then \( G \simeq L_1 \), so we may assume that \( G \) has diameter 2.

Step 1. Given two nonadjacent vertices \( x, y \), there is a unique vertex \( z = \sigma(x; y) \) such that \( y \sim z \) and \( N(x) \cap N(z) = N(x) \setminus (N(x) \cap N(y)) \).
PROOF: The set \( \{y\} \cup N(x) \setminus (N(x) \cap N(y)) \) is independent and hence contained in \( N(z) \) for some \( z \). If it is also contained in \( N(z') \), then, since \( G \) is reduced, the vertices \( z \) and \( z' \) have distinct neighbourhoods, and we may assume that \( z \sim u, z' \not\sim u \) for some vertex \( u \). But now \( \{x, u, z'\} \) is independent and not contained in a point neighbourhood. Contradiction.

Step 2. \( G \) is regular of valency \( k \), say. If \( k > 1 \), then there is a pair of nonadjacent vertices with \( k-1 \) common neighbours.

PROOF: Let \( x, y \) be nonadjacent. If \( |N(y) \setminus N(x)| > 1 \), then choose \( u \in N(y) \setminus N(x), u \neq \sigma(x; y) \). By the uniqueness part of the previous step, there is a vertex \( v \in N(x) \setminus (N(y) \cup N(u)) \), so that also \( |N(x) \setminus N(y)| > 1 \). Now \( (N(x) \cap N(y)) \cup \{u, v\} \) is independent, and hence contained in \( N(z) \) for some \( z \). By downward induction on \( |N(x) \cap N(y)| \) it follows that \( |N(x)| = |N(y)| \) (since we have either \( |N(x)| = |N(y)| + 1 = |N(y)| \)), or, by induction, \( |N(x)| = |N(z)| = |N(y)| \). Now regularity of \( G \) follows since its complementary graph \( \overline{G} \) is connected.

Step 3. \( G \cong L_k \).

PROOF: Let \( x_0 \neq y_0 \) and \( |N(x_0) \cap N(y_0)| = k-1 \). Define vertices \( x_i, y_i \) \( (i \in \mathbb{Z}) \) by \( y_{i+1} = \sigma(x_i; y_i) \) and \( x_i = \sigma(y_i; x_{i-1}) \). Then \( |N(x_i) \cap N(y_i)| = k-1 \) and \( N(x_i) \cap N(y_{i+1}) = \{x_{i+1}\} = \{y_{i+2}\} \) for all \( i \). By induction on \( j \) \( (1 \leq j \leq k-1) \) we see that \( |N(x_0) \cap N(x_{3j})| = k-j \), and that \( x_0 \sim x_1, x_4, \ldots, x_{3j-2} \) and \( x_{3j} \sim x_2, x_5, \ldots, x_{3j-1} \). Indeed, for \( j = 1 \) this is clear, since \( x_0 = y_3 \). But \( x_{3j} \) and \( x_{3j+3} \) have the same neighbours except for \( x_{3j+1}, x_{3j+2} \), and \( x_0 \) and \( x_{3j} \) have the same neighbours except for the vertices \( x_{3j+1}, x_{3j+2} (0 \leq i \leq j-1) \), so \( x_0 \sim x_{3j+1} \) and similarly \( x_2 \sim x_{3j+3} \). As long as \( x_0 \) and \( x_{3j} \) have common neighbours, it follows that \( x_0 \neq x_{3j+1} \). However, \( x_0 \) and \( x_{3k-1} \) have the same neighbours, so \( x_0 = x_{3k-1} \). If there is a vertex \( z \) distinct from all \( x_i \), then \( z \) is adjacent to either all or none of the \( x_i \), contradiction, since \( G \) is triangle-free and connected. \( \square \)

This theorem can be generalized by deleting the hypothesis that \( G \) is reduced. Now the conclusion becomes that \( G \) is a clique extension of one of the \( L_k \). (In particular, if \( G \) is regular, that \( G \) is a lexicographic product \( L_k[K_m] \).)

Finally the finiteness hypothesis can be dropped as well, but the conclusion becomes more complicated, and I have not investigated this further.

The reason that the graphs \( L_k,m \) occur in \( [vdH] \) is that (for \( m \geq 3 \)) they have the maximal possible toughness \( t = n/k - 1 \) for triangle-free regular graphs. (The toughness \( t(G) \) of a connected non-complete graph \( G \) with vertex set \( V \) is by definition \( \min |V \setminus X| / \omega(X) \) taken over all subsets \( X \) of \( V \) such that the number of connected components \( \omega(X) \) of \( X \) is at least two. Clearly, \( t(G) \leq \left( |V| - 2 \right) / 2 \).)

**Lemma 0.2** Let \( G \) be a connected non-complete graph. The toughness of the lexicographic product \( G[K_m] \) equals \( \min |V \setminus X| / \omega(X) \), where \( \omega(X) \) is the number of singleton components of \( X \) plus \( 1/m \)-th of the number of other components of \( X \), and \( X \) runs through the subsets of \( V \) with \( \omega(X) > 1 \). \( \square \)
Proposition 0.3. The toughness of $L_{k,m}$ equals $\min\left(2 - \frac{1}{k}, 2 - \frac{2}{m(k-1)+1}\right)$ ($k \geq 1, m \geq 1$).

Proof: By the above lemma, we only have to investigate $G = L_k$. Taking $X = N(0)$ shows that $t(G) \leq (3k-1-k)/k = 2 - 1/k$. Taking $X = N(0) \cup \{2\}$ shows that $t(G) \leq ((3k-1) - (k-1))/(k - 1 + 1/m) = 2 - 2/(m(k-1) + 1)$. Conversely, if $\{x, y\}$ is an edge of $G$, then $V \setminus (N(x) \cup N(y))$ is complete bipartite or a coclique. Thus, if some subgraph $X$ of $G$ has at least two non-singleton components, then $w(X) = 2/m$ and $|V \setminus X|/w(X) \geq 4/(2/m) = 2m \geq 2$ so that $X$ does not determine the toughness. If $X$ has precisely one non-singleton component, say containing the edge $\{0, 3t+1\}$, then the set $S$ of all vertices $s$ such that $\{s\}$ is a component of $X$ is contained in one part of the bipartition on the vertices nonadjacent to both 0 and $3t+1$; say, $S \subseteq \{3t+3, \ldots, 3k-3\}$. Now $|V \setminus X|/w(X) \geq |N(S)|/(|S| + 1/m)$. But when $|S|$ is given, $|N(S)|$ is minimal when $S$ is 'consecutive': $S = \{3a, 3a + 3, \ldots, 3a + 3r\}$, and then $|N(S)|/(|S| + 1/m) = (k + r)/(r + 1 + 1/m)$. This again is minimal when $|S|$ is maximal, i.e., for $t = 0$ and $r = k - 2$, and then $|N(S)|/(|S| + 1/m) = 2 - 2/(m(k-1) + 1)$. Finally, if $X$ has only singleton components, a similar but easier argument again shows that we get the smallest quotient by taking $X$ a maximal coclique, and then this quotient equals $2 - 1/k$. \qed

References