Finite graphs in which the point neighbourhoods are the maximal independent sets

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We determine all graphs as in the title.

In [vdH] certain graphs L_k occur. Noticing that they have the property mentioned in the title, I wondered whether they are the only such graphs. This note shows that, essentially, this is indeed the case.

For $k \leq 1$, let L_k be the graph with vertex set $\mathbf{Z}_{3k-1}$ (the integers mod $3k-1$) and adjacencies $x \sim y$ iff $y - x \in \{1, 4, 7, ..., 3k-2\}$. (Thus, L_1 is the complete graph on two vertices, and L_2 is the pentagon.) The *neighbourhood* of a vertex x is the set $N(x) = \{y | y \sim x\}$. A graph G is called *reduced* when distinct vertices have distinct neighbourhoods.

THEOREM 0.1 *The finite reduced triangle-free graphs in which each independent set is contained in a point neighbourhood are precisely the graphs L_k $(k \geq 1)$.*

PROOF: First we show that the graphs L_k have the stated property. That they are finite, reduced and triangle-free is clear. Now it suffices to show that if S is an independent set contained in $N(x)$, and $S \cup \{y\}$ is independent for some y, $y \not\sim x$, then $S \cup \{y\} \subseteq N(z)$ for some z. But $y = x + 3i - 1$ or $y = x + 3i$ for some i $(1 \leq i \leq k-1)$, and we can take $z = x + 3i$ or $z = x + 3i - 1$, respectively.

Conversely, let the graph G have the stated property. We show that $G \simeq L_k$ for some $k \leq 1$. Since $\emptyset$ is independent, G has a vertex, and since a singleton is independent, each vertex has a neighbour, and since two nonadjacent vertices have a common neighbour, G has diameter at most 2. Clearly, if G is complete, then $G \simeq L_1$, so we may assume that G has diameter 2.

Step 1. *Given two nonadjacent vertices x, y, there is a unique vertex $z = \sigma(x; y)$ such that $y \sim z$ and $N(x) \cap N(z) = N(x) \setminus (N(x) \cap N(y))$.*

PROOF: The set $\{y\} \cup N(x) \setminus (N(x) \cap N(y))$ is independent and hence contained in $N(z)$ for some z. If it is also contained in $N(z')$, then, since G is reduced, the vertices z and z' have distinct neighbourhoods, and we may assume that $z \sim u$, $z' \not\sim u$ for some vertex u. But now $\{x, u, z'\}$ is independent and not contained in a point neighbourhood. Contradiction.

Step 2. *G is regular of valency k, say. If $k > 1$, then there is a pair of nonadjacent vertices with $k - 1$ common neighbours.*
PROOF: Let x, y be nonadjacent. If $|N(y) \setminus N(x)| > 1$, then choose $u \in N(y) \setminus N(x)$, $u \neq \sigma(x; y)$. By the uniqueness part of the previous step, there is a vertex $v \in N(x) \setminus (N(y) \cup N(u))$, so that also $|N(x) \setminus N(y)| > 1$. Now $(N(x) \cap N(y)) \cup \{u, v\}$ is independent, and hence contained in $N(z)$ for some z. By downward induction on $|N(x) \cap N(y)|$ it follows that $|N(x)| = |N(y)|$ (since we have either $|N(x)| = |N(x) \cap N(y)| + 1 = |N(y)|$, or, by induction, $|N(x)| = |N(z)| = |N(y)|$). Now regularity of G follows since its complementary graph $\overline{G}$ is connected.

Step 3. $G \simeq L_k$.
PROOF: Let $x_0 \not\sim y_0$ and $|N(x_0) \cap N(y_0)| = k - 1$. Define vertices x_i, y_i ($i \in \mathbf{Z}$) by $y_{i+1} = \sigma(x_i; y_i)$ and $x_i = \sigma(y_i; x_{i-1})$. Then $|N(x_i) \cap N(y_i)| = k - 1$ and $N(x_i) \cap N(y_{i+1}) = \{x_{i-1}\} = \{y_{i+2}\}$ for all i. By induction on j ($1 \leq j \leq k-1$) we see that $|N(x_0) \cap N(x_{3j})| = k - j$, and that $x_0 \sim x_1, x_4, \ldots, x_{3j-2}$ and $x_{3j} \sim x_2, x_5, \ldots, x_{3j-1}$. Indeed, for $j = 1$ this is clear, since $x_0 = y_3$. But x_{3j} and x_{3j+3} have the same neighbours except for x_{3j+1}, x_{3j+2}, and x_0 and x_{3j} have the same neighbours except for the vertices x_{3i+1}, x_{3i+2} $(0 \leq i \leq j-1)$, so $x_0 \sim x_{3j+1}$ and similarly $x_2 \sim x_{3j+3}$. As long as x_0 and x_{3j} have common neighbours, it follows that $x_0 \neq x_{3j \pm 1}$. However, x_0 and x_{3k-1} have the same neighbours, so $x_0 = x_{3k-1}$. If there is a vertex z distinct from all x_i, then z is adjacent to either all or none of the x_i, contradiction, since G is triangle-free and connected. □

This theorem can be generalized by deleting the hypothesis that G is reduced. Now the conclusion becomes that G is a coclique extension of one of the L_k. (In particular, if G is regular, that G is a lexicographic product $L_{k,m} := L_k[\overline{K_m}]$.) Probably the finiteness hypothesis can be dropped as well, but the conclusion becomes more complicated, and I have not investigated this further.

The reason that the graphs $L_{k,m}$ occur in [vdH] is that (for $m \geq 3$) they have the maximal possible toughness $t = n/k - 1$ for triangle-free regular graphs. (The *toughness* $t(G)$ of a connected non-complete graph G with vertex set V is by definition $\min |V \setminus X| / \omega(X)$ taken over all subsets X of V such that the number of connected components $\omega(X)$ of X is at least two. Clearly, $t(G) \leq (|V| - 2)/2$.)

LEMMA 0.2 *Let G be a connected non-complete graph. The toughness of the lexicographic product $G[\overline{K_m}]$ equals $\min |V \setminus X| / w(X)$, where $w(X)$ is the number of singleton components of X plus $1/m$-th of the number of other components of X, and X runs through the subsets of V with $\omega(X) > 1$.* □

PROPOSITION 0.3 *The toughness of $L_{k,m}$ equals* $\min(2-\frac{1}{k}, 2-\frac{2}{m(k-1)+1})$ $(k \geq 1,\ m \geq 1)$.

PROOF: By the above lemma, we only have to investigate $G = L_k$. Taking $X = N(0)$ shows that $t(G) \leq (3k-1-k)/k = 2-1/k$. Taking $X = N(0) \cup \{2\}$ shows that $t(G) \leq ((3k-1)-(k-1))/(k-1+1/m) = 2-2/(m(k-1)+1)$. Conversely, if $\{x,y\}$ is an edge of G, then $V \setminus (N(x) \cup N(y))$ is complete bipartite or a coclique. Thus, if some subgraph X of G has at least two non-singleton components, then $w(X) = 2/m$ and $|V \setminus X|/w(X) \geq 4/(2/m) = 2m \geq 2$ so that X does not determine the toughness. If X has precisely one non-singleton component, say containing the edge $\{0, 3t+1\}$, then the set S of all vertices s such that $\{s\}$ is a component of X is contained in one part of the bipartition on the vertices nonadjacent to both 0 and $3t+1$; say, $S \subseteq \{3t+3, ..., 3k-3\}$. Now $|V \setminus X|/w(X) \geq |N(S)|/(|S|+1/m)$. But when $|S|$ is given, $|N(S)|$ is minimal when S is 'consecutive': $S = \{3a, 3a+3, ..., 3a+3r\}$, and then $|N(S)|/(|S|+1/m) = (k+r)/(r+1+1/m)$. This again is minimal when $|S|$ is maximal, i.e., for $t = 0$ and $r = k-2$, and then $|N(S)|/(|S|+1/m) = 2-2/(m(k-1)+1)$. Finally, if X has only singleton components, a similar but easier argument again shows that we get the smallest quotient by taking X a maximal coclique, and then this quotient equals $2-1/k$. □

REFERENCES

[vdH] Jan van den Heuvel, *Degree and Toughness Conditions for Cycles in Graphs*, Ph.D. thesis, Techn. Univ. Twente, 1993.