A New World Underneath Standard Logic: Cylindric Algebra, Modality and Quantification

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1. WORKING AT INTERFACES

Cor Baayen's broad interests span at least mathematics, logic, computer science and linguistics. Our paths have crossed on many occasions, starting in the early seventies, when he invited me to talk at his lively mathematics colloquium at the Free University. Through the years, Cor has been a benevolent influential presence in the background, who often came to visit scientific events in our logic community at the University of Amsterdam. It was good to know that the Lord of that fabled Mathematical Centre, though far away in a mythical country, was on our side. We have worked together in various ways – and indeed, when our new research institute ILLC was created in 1991, Cor was the unanimous choice of our mathematicians, philosophers and computer scientists for a distinguished outside board member. It is a great pleasure to be able to express my gratitude for all this on this festive occasion. I would like to add that I have always admired Cor for his personality: deeply honest, compassionate, but penetrating and incisive when needed. People with his qualities are scarce.

But enough by way of fan-mail confessions! My real offering here is a short story about some current logical research at the very interfaces where Cor has been active. Moreover, this story has a direct link with his own early work in mathematics, viz. his spell of cylindric algebra at Berkeley with the Tarski School, which resulted in the papers Baayen 1960, 1962. What I want to show is how current interests in so-called 'dynamic semantics' of information flow for natural and formal languages motivate a reappraisal of 'standard' logical semantics. And some powerful mathematical tools for this analysis can be taken from cylindric algebra. What we discover in this way is a whole landscape of dynamic logics underneath classical predicate logic, some of them very well-behaved (and even decidable). But to see all this, we have to start with the Received View in logic, and see where it can be challenged.

2. DECONSTRUCTING TARSKI SEMANTICS

Tarski's well-known semantics for first-order predicate logic has the following key clause explaining the existential quantifier:

$$\mathbf{M}, \alpha \models \exists x\, \phi \quad \text{iff} \quad \textit{for some}\ d \in |\mathbf{M}|:\ \mathbf{M}, \alpha^x_d \models \phi\ .$$

Intuitively, this clause calls a verification procedure: "keep shifting the value of state α in the register x until some verifying instance is found for ϕ " . Put differently, an existential quantifier calls a procedure of random assignment to its designated variable. This is no mere curiosity. The currently emerging program of Dynamic Semantics analyzes any kind of linguistic expression via dynamic 'update conditions', rather than (just) static truth conditions. For natural language, this view is found, amongst others, in Kamp 1984, Barwise 1987, Groenendijk & Stokhof 1991, Van Benthem 1991, Veltman 1991. Its paradigmatic examples are such linguistic processes as anaphora (changing bindings for pronouns across discourse), movement of temporal reference points in narratives, changing presuppositions across texts, and many other aspects of linguistic information flow from speakers/authors to hearers/readers. (A broad survey may be found in Muskens, Van Benthem & Visser 1994.) Independently, and in even greater generality, such dynamic views have been proposed in computer science and cognitive science. For instance, the influential Gärdenfors 1988 explains propositions, not as static assertions, but as transformations of information states. Thus, 'updating' of beliefs includes learning via conditionalizing probability functions, and expansion or revision of data bases. (Both traditions meet in the volume Van Eyck & Visser 1994.) In this paper, we stick with the modest case of variable assignment in quantification.

The above dynamic move will make the semantics of linguistic sentences very much like that of computer programs, viewed in the familiar Hoare-Dijkstra style as binary transition relations between assignments. This semantic perspective is powerful and suggestive, but it has one paradoxical feature. Its complexity is at least as high as that of standard predicate logic – whereas part of the motivation for dynamic semantics is precisely the desire to get at simple computational mechanisms in human language use. Therefore, we should reflect further, and look at the bare bones of state transitions. What makes first-order predicate logic tick at a more abstract computational level? This policy is known from Propositional Dynamic Logic (cf. the new textbook Harel & Kozen 1994), which employs labeled transition systems (poly-modal Kripke models), also a favourite vehicle of mathematical theorizing at CWI concerning computation. Thus, let us see what is really involved in Tarski semantics. The answer is as follows. Much less is needed than the above concrete assignment scheme to give a compositional semantics for first-order quantification (usually taken to be its essential achievement).
The abstract core pattern which makes the semantic recursion work is this:

$$\mathbf{M}, \alpha \models \exists x\, \phi \quad \text{iff} \qquad \textit{for some } \beta : R_x\, \alpha\beta \textit{ and } \mathbf{M}, \beta \models \phi\,.$$

Assignments α, β are now viewed as abstract states, and the concrete relation $\alpha =_x \beta$ (which holds between α and $\alpha^x{}_d$) has become just any binary update relation R_x . This greater freedom reflects current developments in Dynamic Semantics, where states can be much more diverse than just assignments (partial assignments, discourse stacks, or yet other data structures) and variable-value update transitions between them may vary accordingly. In this light, 'standard Tarski semantics' amounts to insisting (without explicit argumentation) on one particular set-theoretical implementation. States must be assignment functions in $|\mathbf{M}|^{VAR}$, all of which are to be present in our models, and 'variable update' must be the specific indifference relation $=_x$.

3. A MODAL PERSPECTIVE

The above pattern has a familiar mathematical form. It treats predicate logic as a modal logic, with existential quantifiers $\exists x$ as existential modalities $<x>$. This system has the usual possible worlds models $\mathbf{M} = (S, \{R_x\}_{x \in VAR}, I)$, with S a set of 'states', R_x a binary 'transition relation' for each variable x, and I a 'valuation' giving a truth value to atomic formulas Px, Rxy, ... in each state α. Henceforth, our language is the standard first-order one, with predicates and variables (but no function symbols). Some extensions will be considered at the end. Its modal truth definition is as follows:

$\mathbf{M}, \alpha \models Px$	iff	$I(\alpha, Px)$
$\mathbf{M}, \alpha \models \neg\phi$	iff	*not* $\mathbf{M}, \alpha \models \phi$
$\mathbf{M}, \alpha \models \phi \,\&\, \psi$	iff	$\mathbf{M}, \alpha \models \phi$ *and* $\mathbf{M}, \alpha \models \psi$
$\mathbf{M}, \alpha \models \exists x\, \phi$	iff	*for some* $\beta : R_x \alpha\beta$ *and* $\mathbf{M}, \beta \models \phi$.

The universal validities produced by this general semantics constitute the well-known *minimal modal logic*, whose principles are

(i) all classical Boolean propositional laws,
(ii) the axiom of Modal Distribution: $\exists x\, (\phi \vee \psi) \leftrightarrow \exists x\phi \vee \exists x\psi$,
(iii) the rule of Modal Necessitation: *if* $\vdash \phi$, *then* $\vdash \neg\exists x \neg\phi$,
(iv) the definition of $\forall x\phi$ as $\neg\exists x\neg\phi$.

A completeness theorem may be proved here using the standard Henkin construction. This poly-modal logic can be analyzed in a standard fashion (Andréka, van Benthem & Németi 1994 is a modern treatment), yielding the usual meta-properties such as the Craig Interpolation Theorem, and the Los-Tarski Preservation Theorem for submodels. Moreover, it is *decidable*, via any of the usual modal techniques (such as filtration). The model theory of this logic leads to interesting comparisons between 'bisimulations' for its models and 'partial isomorphism' in ordinary model theory (cf. de Rijke 1993). This modal perspective uncovers a whole *fine-structure* of predicate-logical validity. The minimal predicate logic consists of those laws which are 'very valid'. But we can analyze what other standard laws say, too, by the technique of *frame correspondence*. Recall that a modal formula ϕ defines a relational condition C on state frames if ϕ holds (for all states and interpretation functions) in just those frames where C obtains. Effective methods exist for finding such conditions, given suitable modal formulas (in particular, the following examples are well-behaved 'Sahlqvist forms'). Here are three illustrations involving key principles from cylindric algebra (cf. Baayen 1960):

- $\phi \,\&\, \exists x\phi \leftrightarrow \phi$ *expresses that* R_x *is reflexive*
- $\exists x\, (\phi \,\&\, \exists x\psi) \leftrightarrow \exists x\phi \,\&\, \exists x\psi$ *expresses that* R_x *is transitive and euclidean.*

These constraints make the R_x into equivalence relations, as with the modal logic S5. These universal conditions do not impose existence of any particular states in frames. By contrast, the following axiom is existential in nature:

- $\exists x\exists y\phi \leftrightarrow \exists y\exists x\phi$ *expresses that* $R_x;R_y = R_y;R_x$

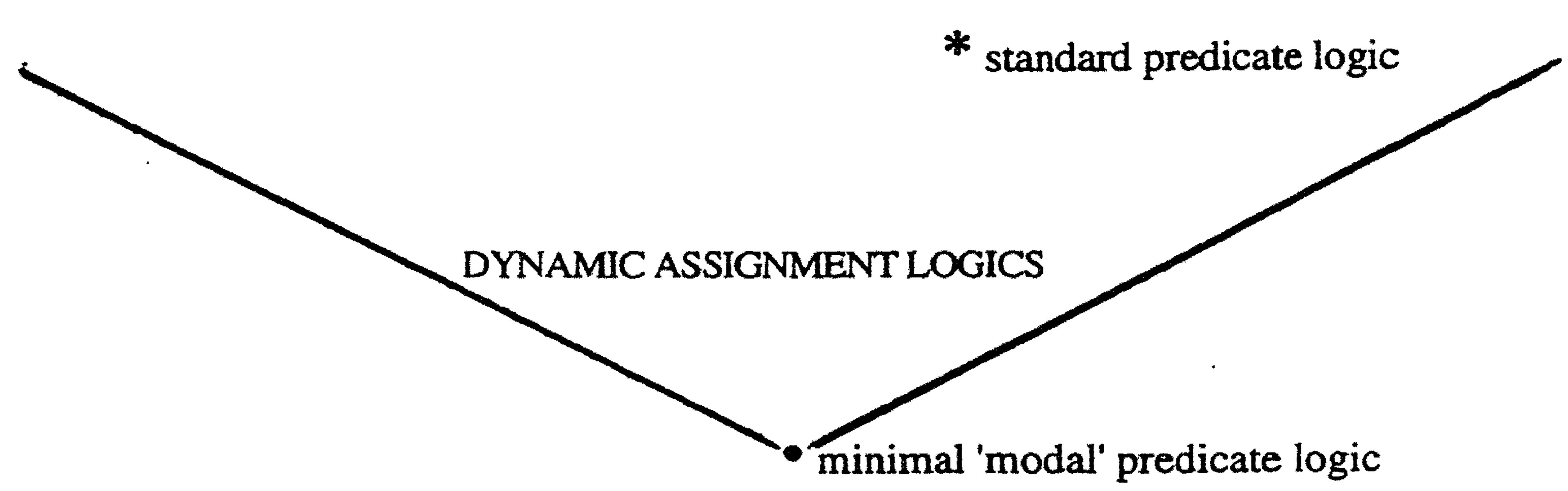
This says that sequences of state changes may be traversed in any order. Abstract state models need not have enough intermediate states to follow all these alternative routes. As a final example, consider another well-known valid quantifier shift:

- $\exists x \forall y \phi \rightarrow \forall y \exists x \phi$ *expresses Confluence of variable update: whenever* $\alpha R_x \beta R_y \gamma$, *there is a state* δ *with* $\alpha R_y \delta R_x \gamma$.

This is a natural Church-Rosser property of computational processes, whose adoption again has an existential price. Thus, the valid laws of predicate logic turn out to have quite different dynamic content, when analyzed in the light of this broader semantics.

4. THE LANDSCAPE OF DYNAMIC ASSIGNMENT LOGICS

Once again, we are now viewing first-order predicate logic as a dynamic logic for variable assignment, whose atomic computations shift values in registers x, y, z, ... This perspective yields a whole hierarchy of fine-structure underneath standard predicate logic. The latter system merely becomes the (undecidable) theory of one particular class of 'rich assignment models'. The result is a broad semantic landscape of options, rather than one canonical standard. (The same plurality is known in many other areas of logical analysis, witness the case of Modal Logic or Categorial Logic. For a principled defense of this phenomenon, cf. van Benthem 1991.) We have already found a minimal system at the bottom, and standard logic at the top, while intermediate systems arise by imposing varying requirements on assignments and updates R_x :



In this landscape, we want to find expressive logics that share important properties with predicate logic (Interpolation, Effective Axiomatizability) and that even *improve* on this, preferably by being decidable. The minimal predicate logic satisfies these demands – but what about more powerful candidates? Here Cylindric Algebra becomes important. Equational theories in the latter field correspond with modal logics in our landscape, via a well-known representation (cf. Venema 1991, Marx 1994). (Subdirectly irreducible algebras play a key role here. Cf. Baayen 1960, Blok 1977, van Benthem 1985.) Natural intermediate systems have been identified in this way (cf. Henkin-Monk-Tarski 1985, Németi 1991, 1993), by a method of 'relativization' from the algebraic literature.

One attractive candidate is **CRS**, consisting of all predicate-logical validities in the state frames satisfying all *universal frame conditions* true in standard assignment models. These are the general logical properties of assignments, that do not make existential demands on their supply. (The latter would be more 'mathematical' or 'set-theoretic'.) CRS is known to be decidable, though non-finitely axiomatizable. Moreover, its frame definition needs only universal *Horn* clauses, from which Craig Interpolation follows (van Benthem 1994). Another way of describing CRS may have independent appeal. Consider state frames where S is a family of ordinary assignments (but not necessarily the full function space D^{VAR}), and the R_x are the standard relations $=_x$. Such frames admit 'assignment gaps', which model 'dependencies' between variables: i.e., changes in value for one variable x may induce, or be correlated with changes in value for another variable y (van Lambalgen 1991, Fine 1985 give natural illustrations). This phenomenon cannot be modeled in standard Tarskian semantics, which changes values for variables completely independently. The latter is the 'degenerate case' where all interesting dependencies between variables have been suppressed. From CRS, one can move upward, by considering only families of assignments that satisfy natural closure conditions. For instance, assignment sets might be closed under local shifts in values to variables, or under reassignment of values for one variable to another. Such further structure tends to support the introduction of further operators into the language (e.g., permutation or substitution operators, as well as a predicate for identity). For the resulting logics, cf. Venema 1991, Németi 1993, Marx 1994.

5. EXPLORING THE RICHER SEMANTICS

The landscape of dynamic assignment logics invites obvious geographical research. What are its natural landmarks? Current research by algebraic logicians is bringing to light various interesting mathematical phenomena here. For instance, intermediate logics may have better properties than standard logic. (E.g., the strong Interpolation Theorem for CRS in van Benthem 1994 fails for predicate logic.) Next, generalized assignment semantics throws new light on old questions in standard model theory. (E.g., it improves the poor behaviour of 'finite-variable fragments' of predicate logic that are currently used in defining complexity classes semantically via query languages: cf. Andréka, Németi & van Benthem 1994.) There are also challenging issues of mathematical representation for abstract state frames (some sample results are found in Henkin-Monk-Tarski 1985, Venema 1991, van Benthem 1994). This is an area where modal logicians and algebraists have made common cause by now.

Perhaps the most striking consequence of the new perspective, however, concerns the *language* of predicate logic. A generalized semantics, with its weaker logics, often invites re-design of the original formal language. Distinctions become visible which were suppressed or overlooked in the 'standard semantics'. This general point is well-known from earlier work on, e.g., intuitionistic logic, relevant logic or linear logic. (For instance, classical 'conjunction' splits into two relevant or linear versions, and some connectives in these weaker logics have no classical counterparts at all.) Again, the algebraic tradition has been aware of this issue. Weaker cylindric equational logics may support expanded languages with desirable items like 'discriminator terms', which allow one to pass from algebraic quasi-equations to ordinary equations (Németi 1991). Likewise, modal semantics supports an infinite hierarchy of ever more expressive formalisms (cf. de Rijke 1993). When analyzing predicate logic, two striking examples occur of such expressive enrichment. First, there is a case for adding *substitutions*. Consider the central first-order axiom of 'Existential Generalization': $[t/x]\phi \rightarrow \exists x\phi$. Its computational content is this: 'definite assignment implies random assignment'. To express this intuition, one treats the substitution operator $[t/x]$ as a new modality (metabletically, its very notation made this historically inevitable ...). The earlier state frames must then be expanded with matching update relations $A_{x,t}$ saying that the target state has its x-value replaced by the t-value of the source state. This move brings definite assignment as such into our models. The previous modal analysis still applies. Notably, standard substitution laws show dynamic content via frame correspondence. For instance, $[t/x](\phi \vee \psi) \leftrightarrow [t/x]\phi \vee [t/x]\psi$ is universally valid in the minimal logic, whereas $[t/x]\neg \phi \leftrightarrow \neg [t/x]\phi$ expresses that the relation $A_{x,t}$ must be a total *function.* (Van Benthem 1994 also considers backward 'temporal' versions of substitution.)

Secondly, generalized assignment models suggest a natural distinction between singular quantifiers and *polyadic quantifiers* (cf. Keenan & Westerståhl 1994 for extensive linguistic motivation of the latter). One can interpret a polyadic existential formula like $\exists xy \bullet \phi$ as saying that there exists some state satisfying ϕ with possibly different x– and y–values from the current one. In general, no intermediate states need exist allowing the stepwise singular decompositions $\exists x \exists y \bullet \phi$ or $\exists y \exists x \bullet \phi$ that would be equivalent in standard logic. In state frames, direct interpretation of polyadic quantifiers involves simultaneous updates R_X for sets or sequences X of individual variables. A similar move will be needed to model simultaneous substitutions $[t_1/x_1, ..., t_k/x_k]$, which are known to be irreducible to iterations of singular substitutions. Another view of these linguistic extensions is as follows. From the earlier poly-modal logic with only atomic assignment programs, we are now passing on to a full dynamic logic with operators forming complex programs. In particular, an iterated singular quantifier $\exists x \exists y \bullet$ involves a *sequential* composition of update relations $R_x ; R_y$, whereas the polyadic quantifier $\exists xy \bullet$ involves a form of *parallel* composition. Evidently, these are just first steps on a longer road.

6. CONCLUSIONS

The above re-analysis of what is arguably the basic tool of modern logic may be seen as an instance of a more general philosophical enterprise. What we are trying to do is locate the 'computational core' of a phenomenon – in this case the dynamics of variable-value assignment – and detach it from its 'mathematical wrappings', i.e., more negotiable aspects of its accidental mathematical presentation. We are after the former: the rest is imported complexity. Such a philosophical program may have great practical repercussions. In particular, the hallowed 'undecidability of predicate logic' might merely reflect an infelicity of its traditional Tarskian modeling: namely, the import of extraneous set-theoretic facts about full function spaces D^{VAR} – rather than the core facts about quantification and variable assignment. Thus, adopting 'dynamic semantics' and thinking it through might lead to decreased logical complexity – once we have the courage of our convictions. This provocative statement needs to be backed up, of course, by concrete analysis of predicate-logical reasoning found in applications. Which universal validities are really used (that is, under appropriate formalizations)?

I am not quite sure that Cor Baayen will be overjoyed by this radical departure from the tenets of our Founding Fathers. But he will certainly appreciate the following points. At least, our case study demonstrates a commonality in key interests between such apparently diverse disciplines as logic, computer science and linguistics. In particular, it demonstrates that genuine 'application' is not a one-way process, but an interaction. Standard logic has inspired an illuminating analysis of computational processes via 'dynamic logics' and their ilk. But what happens now is that, conversely, dynamic viewpoints may 'turn around' and start challenging received views of what standard logic is all about. This move does not invalidate the achievements of previous periods. On the contrary, as we have seen, it is driven by insights from cylindric algebra, an enterprise squarely within mathematical logic - and it will no doubt inspire that area too. I conclude that Cor Baayen's scientific interests, outlined at the beginning of this paper, have proved fruitful and topical: both generally, and in their technical bent.

7. REFERENCES

Andréka, H., I. Németi & J. van Benthem, 1994, 'Back and Forth Between Modal Logic and Classical Logic', Mathematical Institute, Hungarian Academy of Sciences, Budapest / Institute for Logic, Language and Computation, University of Amsterdam.

Baayen, C., 1960, 'Subdirect Oplosbare Cylinder-Algebra's', Rapport ZW 1960-006, Stichting Mathematisch Centrum, Amsterdam.

Baayen, C., 1962, 'Cylinder-Algebra's', Rapport ZW 1962-004, Stichting Mathematisch Centrum, Amsterdam.

Barwise, J. , 1987, 'Noun Phrases, Generalized Quantifiers and Anaphora', in P. Gärdenfors, ed., *Generalized Quantifiers. Logical and Linguistic Approaches*, Reidel, Dordrecht, 1-29.

Benthem, J. van, 1985, *Modal Logic and Classical Logic*, Bibliopolis, Napoli.

Benthem, J. van, 1991, *Language in Action. Categories, Lambdas and Dynamic Logic*, North-Holland, Amsterdam.

Benthem, J. van, 1994, 'Modal Foundations for Predicate Logic', CSLI Research Report, Center for the Study of Language and Information, Stanford University.

Benthem, J. van & J. Bergstra, 1993, 'Logic of Transition Systems', Report CT-93-03, Institute for Logic, Language and Computation, University of Amsterdam. (To appear in *Journal of Logic, Language and Information*.)

Blok, W., 1977, *Varieties of Interior Algebras*, Dissertation, Mathematical Institute, University of Amsterdam.

Eyck, J. van & A. Visser, eds., 1994, *Dynamic Logic and Information Flow*, The MIT Press, Cambridge (Mass.).

Fine, K., 1985, *Reasoning With Arbitrary Objects*, Blackwell, Oxford.

Gärdenfors, P., 1988, *Knowledge in Flux. Modelling the Dynamics of Epistemic States*, The MIT Press, Cambridge (Mass.).

Goldblatt, R., 1987, *Logics of Time and Computation*, CSLI Lecture Notes, Chicago University Press.

Groenendijk, J. & M. Stokhof, 1991, 'Dynamic Predicate Logic', *Linguistics and Philosophy* 14, 39- 100.

Harel, D. & D. Kozen, 1994, *Dynamic Logic*, Department of Computer Science, Technion, Haifa / Department of Computer Science, Cornell University.

Henkin, L., D. Monk & A. Tarski, 1985, *Cylindric Algebra*, part II, North-Holland, Amsterdam.

Jaspars, J., 1994, *Calculi for Constructive Communication*, ILLC Dissertation Series 1994-1, Institute for Logic, Language and Computation, University of Amsterdam / Institute for Language and Knowledge Technology, University of Tilburg

Kamp, H., 1984, 'A Theory of Truth and Semantic Representation', in J. Groenendijk et al., eds., *Truth, Interpretation and Information*, Foris, Dordrecht, 1-41.
Keenan, E. & D. Westerståhl, 1994, 'Quantifiers', a chapter in J. van Benthem & A. ter Meulen, eds., *Handbook of Logic and Language*, Elsevier Science Publishers, Amsterdam.
Lambalgen, M. van, 1991, 'Natural Deduction for Generalized Quantifiers'. In J. van der Does & J. van Eyck, eds., *Generalized Quantifiers: Theory and Applications*, Dutch Ph. D. Network for Logic, Language and Information, Amsterdam, 143-154. To appear with Cambridge University Press.
Marx, M., 1994, *Arrow Logic and Relativized Algebras of Relations*, Dissertation, CCSOM, Faculty of Social Sciences / Institute for Logic, Language and Computation, University of Amsterdam.
Muskens R., J. van Benthem & A. Visser, 1994, 'Dynamics', a chapter in J. van Benthem & A. ter Meulen, eds., *Handbook of Logic and Language*, Elsevier Science Publishers, Amsterdam.
Németi, I., 1991, 'Algebraizations of Quantifier Logics: An Introductory Overview', Mathematical Institute, Hungarian Academy of Sciences, Budapest.
Németi, I., 1993, 'Decidability of Weakened Versions of First-Order Logic', Lecture Notes Workshop on Algebraization of Logics, Fifth European Summer School in Logic, Language and Information, Lisbon.
Rijke, M. de, 1993, *Extending Modal Logic*, Dissertation Series 1993-4, Institute for Logic, Language and Computation, University of Amsterdam.
Veltman, F., 1991, 'Defaults in Update Semantics', Report LP-91-02, Institute for Logic, Language and Computation, University of Amsterdam. To appear in the *Journal of Philosophical Logic*.
Venema, Y., 1991, *Many-Dimensional Modal Logic*, Dissertation, Institute for Logic, Language and Computation, University of Amsterdam.
Vermeulen, K., 1994, *Exploring the Dynamic Environment*, Dissertation, Onderzoeksinstituut voor Taal en Spraak, University of Utrecht.