Symbolic Dynamics for a Piecewise-Affine System with Hysteresis

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Abstract

In this paper we present the computation of symbolic dynamics of a one dimensional return map of a piecewise-affine hybrid system. The system arises as a simple electrical circuit with hysteresis switching, and exhibits chaotic dynamics. Our method allows us to rigorously obtain a qualitative description of the discrete behaviour of the system. We show how the discrete dynamics changes as a parameter is varied, and we compute bounds for the topological entropy to provide a measure of the complexity of the system.

Key words: Hybrid systems, symbolic dynamics, hysteresis, return map, chaotic dynamics.

1 Introduction

Hysteresis systems are a very common class of hybrid system which model many phenomena and devices in engineering and sciences. Even in two dimensions the behaviour of such systems can present very complicated dynamics, including chaotic behaviour [3]. One way of obtaining insight into the system behaviour is to study the pure discrete part of the evolution as given by the *symbolic dynamics*.

In previous work [5] we developed and implemented algorithms to approximate the symbolic dynamics of one dimensional return maps, and applied these algorithms to the study of of the hysteretic system introduced in [3]. In this work we give a detailed study of the symbolic dynamics of an electrical circuit with hysteresis switching introduced in [4]. This system is chaotic, and we find infinitely many distinct discrete behaviours for the return map of the system.

The extended abstract is organized as follows. In Section 2 we present some basic notions about symbolic dynamics and we mention the algorithms of approximation of symbolic dynamics of one dimensional map. In Section 3 we describe the hysteresis system and the related return map and we present the approximation of the the symbolic dynamics of the map.

2 Symbolic Dynamics

2.1 Shifts Spaces

In this subsection, we introduce some basic notation and terminology regarding symbolic dynamics. For more information, see [2] and [1]

Definition 2.1 *(Shifts).* Let A be a finite alphabet. The *shift map* σ on sequences A^ω is defined by $(\sigma\vec{s})_i = s_{i+1}$ for $i \in \mathbb{N}$. A *shift space* on A is a compact subset Σ of A^ω (in the topology induced by the metric $d(\vec{s},\vec{t}) = 2^{-min\{n\in\mathbb{N}|s_n\neq t_n\}}$) which is invariant under σ. A *shift* is the restriction of the shift map σ to a shift space Σ. A shift $\sigma|_\Sigma$ is a *subshift* of $\sigma|_{\widehat{\Sigma}}$ if $\Sigma \subset \widehat{\Sigma}$.

Definition 2.2 *(Sofic and finite type shifts).* A shift is called *sofic* if there exists a graph $G(N,E)$ and a labelling function $L : N \longrightarrow A$ such that $S = \{s \in A^\omega \mid \forall n \in \mathbb{N} : L^{-1}(s_n)L^{-1}(s_{n+1}) \subseteq E\}$. If L is injective the shift is of *finite type*.

Definition 2.3 *(Topological partition).* A *topological partition* of an interval $I \subset \mathbb{R}$ is a finite collection $\mathcal{P} = \{P_1, P_2, ..., P_n\}$ of mutually disjoint open intervals such that $I = \bigcup_{i=1}^n \overline{P}_i$. The *boundary points* of $\mathcal{P}$ are elements of $\partial\mathcal{P} := \bigcup_{P\in\mathcal{P}} \partial P$.

Given topological partitions $\mathcal{P}$ and $\mathcal{Q}$, we say that $\mathcal{P}$ is a *refinement* of $\mathcal{Q}$ if for all $P \in \mathcal{P}$, there exists $Q \in \mathcal{Q}$ such that $P \subset Q$. The *join* of $\mathcal{P}$ and $\mathcal{Q}$ is defined by $\mathcal{P} \vee \mathcal{Q} = \{P \cap Q \mid P \in \mathcal{P},\ Q \in \mathcal{Q} \text{ and } P \cap Q \neq \emptyset\}$.

Definition 2.4 *(Itinerary).* Let $\mathcal{P} = \{P_j \mid j \in J\}$ be a topological partition of I, and $\vec{x} = (x_i)_{i\in\mathbb{N}}$ a sequence in I. A sequence $\vec{s}$ is a $\mathcal{P}$*-itinerary* of $\vec{x}$ if $x_i \in P_{s_i}$ for all $i \in \mathbb{N}$.

Given a topological partition, we can define the symbolic dynamics of a piecewise-continuous function f.

Definition 2.5 *(Symbolic dynamics).* Let $\mathcal{P} = \{P_j \mid j \in J\}$ be a topological partition of I, $f : I \to I$ a function continuous over $P_j : j \in J$. The *symbolic dynamics* $\Sigma(f)$ is the closure of the set of all $\mathcal{P}$-itineraries of orbits of f.

2.2 Algorithms

The following algorithm can be used to compute the symbolic dynamics of a one-dimensional piecewise-monotone map. The algorithm can be implemented either with exact arithmetic, or with interval arithmetic to control numerical errors.

Algorithm 2.6. *Let $\mathcal{Q}$ be a partition of I and $\mathcal{E}$ a topological partition of I, f be a piecewise-continuous function with nondegenerate critical points.*

1. *Fix a numerical precision ϵ and a maximum number of steps n.*
2. *Compute an approximate topological partition $\mathcal{C}$ refining $\mathcal{Q}$ and $\mathcal{E}$ such that f is continuous on each piece of $\mathcal{C}$*
3. *Refine the partition $\mathcal{C}$ to obtain an approximate topological partition $\mathcal{M}$ such that f is monotone on each partition element.*
4. *Refine the partition $\mathcal{M}$ by repeating the one of the following partitioning strategies at most n times to obtain a partition $\mathcal{R}$.*
   - *Forward refinement: Refine a partition $\mathcal{P}$ by introducing new partition boundary points at $f(p)$ for boundary points $p \in \partial\mathcal{P}$.*
   - *Backward refinement: Refine a a partition $\mathcal{P}$ by introducing new partition boundary points at $f^{-1}(p)$ for $p \in \partial\mathcal{P}$*

   In either strategy, do not introduce any new points $\lfloor y \rceil$ which overlap existing boundary points $\lfloor p \rceil$.
5. *Build directed graphs $G_1 = (V, E_1)$, and $G_2 = (V, E_2)$ as follows. The common vertex set V is labelled by $\{1, \dots, N\}$ where $N = |\mathcal{R}|$. The edge sets $E_1, E_2 \subset V \times V$ are defined by $E_1 = \{(i, j) \mid f(R_i) \cap R_j \neq \emptyset\}$ and $E_2 = \{(i, j) \mid f(R_i) \supseteq R_j\}$.*

6. *Build lower-approximation $\Lambda = \Lambda_{\epsilon,n}$ the finite type shift associated to G_1 and over-approximation $\Upsilon = \Upsilon_{\epsilon,n}$ the finite type shift associated to G_2.*

The outputs of the algorithm are shift space Λ and Υ which have the following property: $\Lambda \subseteq \Sigma(f) \subseteq \Upsilon$. Both Λ and Υ converge to $\Sigma(f)$ for n tending to infinity and ϵ tending to zero. Notice that we can alternatively consider two sofic shift if we choose as labelling function $\tilde{L} : N \longrightarrow \mathcal{C} : \tilde{L}(n) = C_i : L(n) \subseteq C_i$ and they would express each itinerary in terms of the partition $\mathcal{C}$ computed at the end of the initialization, which is the simplest partition which generates sofic shifts and the closest to the in ital partition.

3 Analysis of a Circuit with Hysteresis Switching

3.1 System model

In [4] there is analyzed a piecewise affine hysteresis circuit whose dynamics is described by the following equation:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \delta & 1 \\ -1 & \delta \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} p \\ q \end{bmatrix} h(x) \right)$$

$$h(x) = \begin{cases} 1 & \text{for } x \geq -1, \\ -1 & \text{for } x \leq 1, \end{cases}$$

The authors compute an analytic expression for the return map f of the system from the set $L = \{(x, y) \in \mathbb{R}^2 \mid x = p, h(x) = 1\}$ to itself, obtaining

$$f(x) = \begin{cases} f_1(x), & \text{for } x \geq D, \\ f_3 \circ f_2(x), & \text{for } x < D; \end{cases}$$

with

$$\begin{aligned} f_1(x) &= -e^{-\delta\pi}(x - q) + q; \\ f_2^{-1}(x) &= -\sqrt{(x-q)^2 + (p+1)^2} e^{-\delta\tau_1(x)} + q; \\ f_3(x) &= \begin{cases} -\sqrt{(x+q)^2 + (p-1)^2} e^{\delta\tau_2(x)} + q & \text{for } p < 1, \\ -x & \text{for } p = 1, \\ -\sqrt{(x+q)^2 + (p-1)^2} e^{-\delta\tau_3(x)} + q & \text{for } p > 1; \end{cases} \end{aligned}$$

the functions τ_i are given by

$$\tau_1(x) = \frac{\pi}{2} - \arctan\frac{q-x}{p+1}, \quad \tau_2(x) = \frac{\pi}{2} - \arctan\frac{q+x}{1-p}, \quad \tau_3(x) = \frac{\pi}{2} - \arctan\frac{q+x}{p-1};$$

and

$$D = f_2^{-1}(q + \delta(p+1)).$$

All parameters of the equation can be expressed as functions of a parameter b which characterises the original electronic circuit.

$$\delta = \frac{5b-1}{2\omega}, \quad p = \frac{4b}{1-b}, \quad q = \frac{1}{2\omega}\left(8 - \frac{(5b-1)4b}{1-b}\right), \quad \omega = \sqrt{5(1-b) - \frac{1}{4}(5b-1)}.$$

In Figure 1 we sketch some trajectories of the systems. The two dashed vertical lines are the switching thresholds, while the dotted vertical line is the set L.

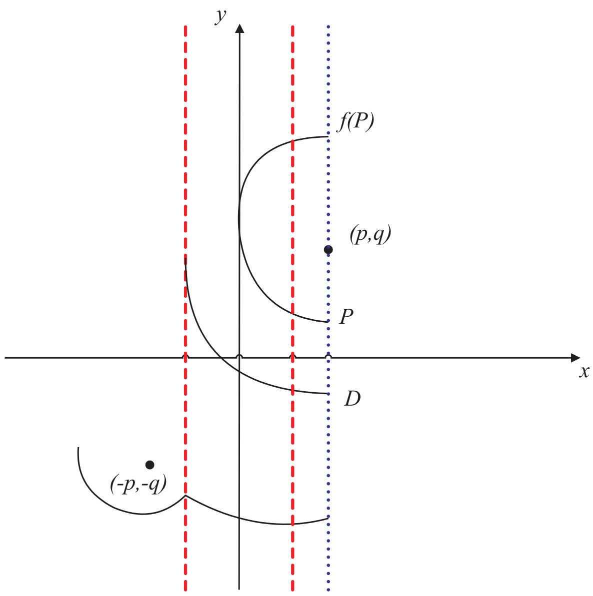

Figure 1: Trajectories of the hysteresis system.

3.2 Computation symbolic dynamics and entropy

In this section we present some results on computation of symbolic dynamics and entropy for $b = 0.24$ and $b = 0.22$.

Figure 2 shows the graph of the return map for $b = 0.24$, together with points s_i forming the boundaries of elements of a topological partition. The graph of the return map for $b = 0.22$ is qualitatively the same.

There is a point of discontinuity at s_{5a} which corresponds to point D in Figure 1, when the trajectory is tangent to the threshold of the hysteresis. We refer to the system with fix point in $(-p, -q)$ as left mode and the one with fixed point in (p, q) as right mode. We consider symbolic dynamics on the intervals $I_0 = [s_0, s_{5a}]$ and $I_1 = [s_{5a}, s_8]$. Points in I_0 correspond to orbits which switch to the right mode without makig any whole turn around (p, q) while points in I_1 correspond to orbits which make one or more turns before switching. The initial partition into monotone pieces consists of intervals, $[s_0, s_1]$, $[s_1, s_{5a}]$ and $[s_{5a}, s_8]$. Since the interval $[s_1, s_{7a}]$ is positively invariant under the return map, we can restrict to considering the dynamics in this interval, which has monotone pieces $[s_1, s_{5a}]$ and $[s_{5a}, s_7]$.

In Figure 3 we show the lower and upper approximations of the lower symbolic dynamics computed by refining the initial partition by forward refinement. The lower approximation includes only solid edges while the upper includes both solid and dotted edges. Each node is labelled with a number k which represent the interval $[s_k, s_j]$ where j is $k + 1$ or k labelled with the next letter of the alphabet. The dark nodes in any graph represent partition elements subset of I_0, while light nodes represent intervals subset of I_1. After 2 step of forward refinement we obtain the end points s_{3a}, s_4, s_6, s_{7a}. After one more iteration we obtain points s_{3b}, s_{5b}, s_{7_b}, s_{7c}. We can see that from iteration 2 to iteration 3 of the forward refinement the upper approximation loses a number of loops, the upper shift shrinks. For instance the loop $5-7-6$ or the loop $4-3$ disappear. The upper entropy for 2 steps of the forward refinement is 0.908740 while for 3 steps is 0.730193, in both cases the lower entropy is zero. With 2 and 3 steps of forward refinement we can not prove the existence of any periodic cycle including both symbol 1 and 0 of the initial partition.

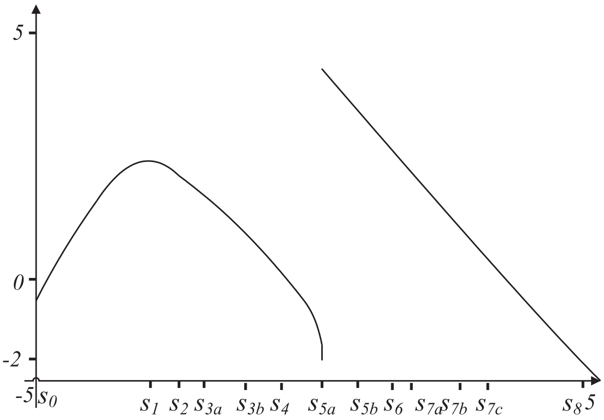

Figure 2: The return map for the hysteresis system

Figure 4 (left) shows the maximally connected components of the approximation of the symbolic dynamics of the return map with $b = 0.24$ after 6 steps of forward refinement, which is the minimal number of steps of the forward refinement strategy which produces a partition whose lower shift approximation has positive entropy. We can deduce the existence of a period-3 orbit through region $3b$ $5a$ $7h$ which has symbols $0, 1, 1$ in terms of itinerary of I_0 and I_1.

For a value of b in the interval $[0.22, 0.24]$ there is a first period doubling bifurcation which transforms a double screw chaotic attractor to a single screw chaotic attractor [4].

Figure 4 (right) shows the maximally connected components of the approximation of the symbolic dynamics of the return map in figure with $b = 0.22$ after 11 steps, which is the minimal number of steps to have positive lower entropy. The labelling follows the same rules for the previous shift.

The dynamics described by graph in Figure 4 (left) shows that there is loop of 0 (dark nodes) in terms of symbol of original partition which means there are trajectories which never make a whole turn around (p, q). There are not loops of 1 (light nodes) which means there are not trajectories which never switch. The entropy of the graphs is 0.343990 while the entropy of the map is in the interval $[0.461165 : 0.479751]$.

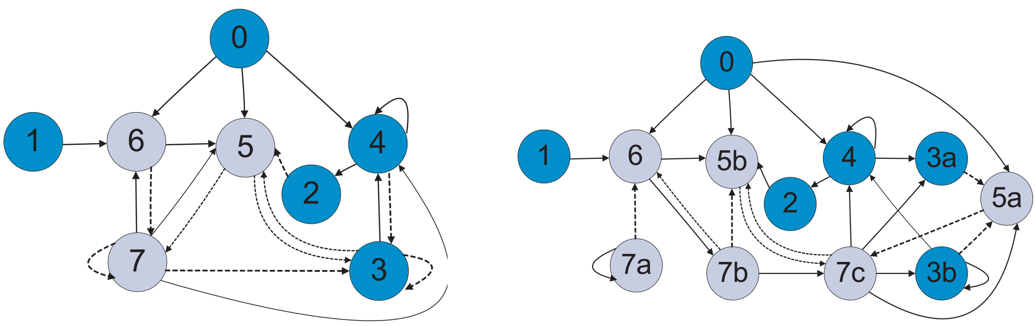

Figure 3: Symbolic dynamics for the hysteresis system with parameter $b = 0.24$ computed using (left) two steps and (right) three steps of forward refinement. The dark shaded vertices represent partition elements with symbolic code 0, and the light shaded vertices represent partition elements with symbolic code 1.

The dynamics described by graph in Figure 4(right) shows that there are not loops of only 0 in terms of symbols of the original partition as instead there are for $b = 0.24$.

This is a relatively tight approximation to the full dynamics because the entropy of this

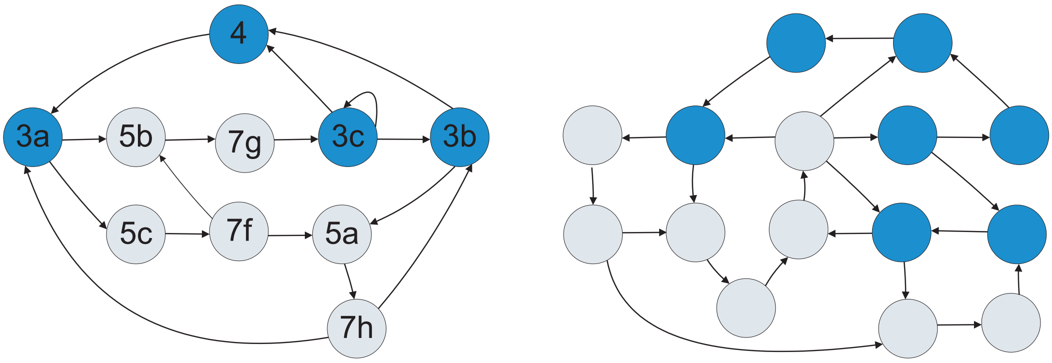

Figure 4: Maximally connected component of the lower symbolic dynamics for (left) $b = 0.24$ computed using 6 steps of forward refinement, and (right) $b = 0.22$ computed using 11 steps of forward refinement.

graph is 0.452698 and the entropy of the map is not higher than 0.476055. We can see that with a relatively small number of iteration we can already see a qualitatively different behavior of return maps of the 2 different parameters of b.

4 Conclusions and future work

In this work we have considered the return map of an hysteretic systems, a piecewise affine flow. We have applied an algorithm for the computation of approximation of symbolic dynamics of a one dimensional return map. The results of this algorithm detects qualitative different behaviour of the return map for parameter values on either side of a bifurcation. The algorithm provides arbitrarily accurate approximation. We are presently developing algorithms for study return maps of two and more dimensions which require more sophisticated techniques than in one dimension, such as Conley index theory.

References

[1] Anatole Katok and Boris Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge University Press, Cambridge, 1995.

[2] Douglas Lind and Brian Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge, 1995.

[3] Robert W. Newcomb and Nevine El-Leithy, *Chaos generation using binary hysteresis*, Circuits Systems Signal Process. **5** (1986), no. 3, 321–341. MR MR893937 (88g:58119)

[4] T. Saito and S. Nakagawa, *Chaos from hysteresis and switched circuit*, Phil. Trans. R. Soc. Lond. **353** (1995), 47–57.

[5] L. Sella and P. Collins, *Discrete dynamics of two-dimensional nonlinear hybrid automata*, HSCC 2008, 2008.