Computability of Controllers for Discrete-Time Semicontinuous Systems

*Pieter Collins**

Abstract: In this paper we consider the computation of controllers for noisy nonlinear discrete-time systems described by upper-semicontinuous multivalued functions. We show that for the problem of controlling to a target set, if an open-loop solution exists, then a feedback controller with can be effectively computed in finite time from the problem data, and that the resulting system is robust with respect to perturbations. We extend the results for systems with partial observations and a dynamic output feedback law based on a finite automaton.

1 Introduction

In this paper we consider the problem of computing controllers for nonlinear discrete-time systems with disturbances. We consider the class of systems described by compact-valued upper-semicontinuous functions, and control problems of invariance and reachability of target sets. The aim is to prove for this very general case, that given natural information about the system, and natural solvability conditions for the control problem, then a robust, finitely presented feedback controller can be effectively computed.

In [1], the nonlinear control-to-target problem was studied and an explicit algorithm for control synthesis was given for linear systems with disturbances. The problem of state estimation in a similar setting was considered in [2]. Computability of control synthesis by constraint propagation was studied by[8]. More recently, reachability analysis for systems with disturbances has been considered in [6].

*P. Collins is affiliated with the Centrum voor Wiskunde en Informatica, Postbus 94079, 1090 GB Amsterdam, The Netherlands. E-mail: `Pieter.Collins`.

Much of this work is similar to the work on hybrid systems of Kohn and Nerode; see [5] and references therein.

One of the novel aspects of this paper is the use of the theory of computable analysis [9]. This gives a powerful framework for discussing effective computation (by Turing machines) on objects such as sets and functions in Euclidean space. We can often discuss the computability of complex operations in terms of elementary operations (intersection, image, preimage etc.) without needing to explicitly consider approximation errors, or resort to a particular geometric calculus. There is a strong relationship between computability and continuity properties, especially those studied in viability theory [3], and also with continuity issues studied in dynamic programming. An operator can only be (semi)computable if it is (semi)continuous, and continuity proofs can usually be easily converted to computability proofs.

2 Preliminaries

A multivalued function $F : X \rightrightarrows Y$ (equivalently, $F : X \to \mathcal{P}(X)$) is a function assigning to each $x \in X$, a set $F(x) \subset Y$. If $A \subset X$, define $F(A) := \bigcup\{F(x) \mid x \in X\}$, and if $B \subset Y$, define $F^{-1}(B) := \{x \in X \mid F(x) \cap B \neq \emptyset\}$ and $F^{\Leftarrow}(B) := \{x \in X \mid F(x) \subset B\}$. A multivalued function $F : X \rightrightarrows Y$ is said to be *upper-semicontinuous* if $F^{-1}(B)$ is closed whenever B is closed, *lower-semicontinuous* if $F^{-1}(B)$ is open whenever B is open, and *continuous* if it is both lower-semicontinuous and upper-semicontinuous. We say F is open, closed or compact valued if for all $x \in X$, $F(x)$ is open, closed or compact, respectively. If $f : X \times W \to Y$ is continuous and W is compact, then the function $F : X \rightrightarrows Y$ given by $F(x) := f(x, W) = \{f(x, w) \mid w \in W\}$ is a continuous compact-valued function.

The graph of $F : X \rightrightarrows Y$ is the set $\{(x, y) \in X \times Y \mid y \in F(x)\}$. The multivalued function F is open-valued lower-semicontinuous if, and only if, graph(F) is open, and is closed-valued upper-semicontinuous if, and only if, graph(F) is closed.

Given a topological space X, we denote the set of all subsets of X by $\mathcal{P}(X)$, and the spaces of open, closed and compact subsets of X by $\mathcal{O}(X)$, $\mathcal{A}(X)$ and $\mathcal{K}(X)$, respectively. The *lower topology* on $\mathcal{A}(X)$ is generated by sets of the form $\{A \in \mathcal{A}(X) \mid A \cap U \neq \emptyset\}$ for $U \in \mathcal{O}(X)$, and the *upper topology* on $\mathcal{K}(X)$ is generated by sets of the form $\{C \in \mathcal{K}(X) \mid C \subset U\}$ for $U \in \mathcal{O}(X)$.

3 Discrete-time control

In this paper, we consider control systems in discrete time defined by multivalued functions.

3.1 Control systems

Definition 1. *A* discrete-time control system *with state space X, input space U and output space Y is a pair (F, H) where $F : X \times U \rightrightarrows V$ and $H : X \rightrightarrows Y$ are multivalued functions. The evolution of the system satisfies*

$$\forall n \in N : \quad x_{n+1} \in F(x_n, u_n) \text{ and } y_n \in H(x_n).$$

If $Y = X$ *and* $H = \mathrm{id}$ *then we have* complete observations*, and the system is defined by* F *alone.*

In this paper, we shall restrict to control systems for which F and H are upper-semicontinuous with compact values. Upper-semicontinuity is natural, since it means that if we have a good estimate of x and u, then we can compute a good bound for $F(x, u)$ and $H(x)$.

Remark 2. An common description of a discrete-time control system is as a pair of continuous functions $f : X \times U \times V \to X$ and $h : X \times W \to Y$ where V and W are compact sets giving the *state* and *output* disturbances, respectively. We can recover the description in terms of multivalued functions by taking $F(x, u) := f(x, u, V)$ and $H(x) := h(x, W)$.

3.2 Control laws

There are many different classes of controller used in the literature, each suitable for a different purpose. A very general definition is given below:

Definition 3. *Let* (F, H) *be a control system with input space* U *and output space* X*. A* (nondeterministic) control law *is defined by a multivalued function* $K : Y^{*+1} \times U^{*} \to \mathcal{P}(U)$ *with nonempty values. The input is given by*

$$u_n \in K(y_0, \ldots, y_n, u_0, \ldots, u_{n-1}).$$

An execution *of the* closed-loop system (F, H, K) *is a triple of sequences* $(\vec{x}, \vec{y}, \vec{u}) \in X^{\omega} \times Y^{\omega} \times U^{\omega}$ *such that*

$$\forall n \in \mathbb{N} : \ x_{n+1} \in F(x_n, u_n), \ y_n \in H(x_n) \ \textit{and} \ u_n \in K(y_0, \ldots, y_n, u_0, \ldots, u_{n-1}).$$

A deterministic control law is defined by a function $k : Y^{*} \to U$*. The input is given by* $u_n = k(y_0, \ldots, y_n)$*.*

The definition of control law given above is based on open-loop control and does not impose any regularity conditions on the controller. Ideally, we would like to have a control law defined as a feedback controller with a finite description. For systems with complete observations, we would like to be able to construct a state feedback controller, whereas for systems with partial observations, we would like to be able to construct a dynamic output feedback controller based on an observer.

It is well-known that even for discrete-time with complete observations, simple control problems need not have a continuous state feedback controller [7]. Hence we need to look for controllers which are discontinuous in general. Since arbitrary discontinuous single-valued functions are typically hard to work with, in this paper we will describe our control laws using semicontinuous multivalued functions.

Definition 4. *A* state feedback controller *is a defined by a function* $G : X \to \mathcal{P}(U)$*. The input is given by* $u_n \in G(x_n)$*.*

A set-based dynamic output feedback controller *is defined by a function* $G : \mathcal{P}(X) \to \mathcal{P}(U)$*. The operation of the controller is given by taking* $\widehat{X}_0 = X_0$*,* $\widehat{X}_{n+1} = F(\widehat{X}_n, u_n) \cap H^{-1}(y_{n+1})$ *and input* $u_n \in G(\widehat{X}_n)$*.*

We will typically use lower-semicontinuous open-valued functions, since these have an open graph, and can hence be used as an intermediate For example, given an open-valued lower-semicontinuous function $G : X \rightrightarrows U$, we can effectively compute a piecewise-constant, piecewise-affine or piecewise-polynomial selection $g : X \to U$ which can be used as the implementation.

The set-based dynamic output feedback controller is a special case of a general dynamic feedback controller.

Definition 5. *A* dynamic output feedback controller *is defined by a tuple* $G = (Q, q_0, \delta, \gamma)$ *where* Q *is a topological space,* $q_0 \in Q$*,* $\delta : Q \times Y \rightrightarrows Q$ *and* $\gamma : Q \rightrightarrows U$*. The* closed-loop system *is the dynamic system with executions* $(\vec{x}, \vec{y}, \vec{u}, \vec{q})$ *such that*

$$x_{n+1} \in F(x_n, u_n), \quad y_n \in H(x_n), \quad q_{n+1} \in \delta(q_n, y_{n+1}), \quad u_n \in \gamma(q_n)$$

One common way of implementing a feedback controller is to discretise the output in Y into finitely many values, and choose an input from some countable subset of U. The discretisation map cannot be continuous (unless it is constant on components of Y), which leads us to consider semicontinuous multivalued discretisations. Here, the nondeterminism corresponds to small errors in the discretiser. A discretiser is lower-semincontinuous if it is constant on open sets, and upper-semicontinuous if it is constant on closed sets. Ideally, we want the level sets of the discretisation to have a finite description, such as a finite union of rational boxes in Euclidean space. Likewise, the input values should have a finite description, such as rational points in Euclidean space. We say such sets and points are *definable.*

Definition 6. *A* digital state feedback controller *controller is a state feedback controller* $G : X \rightrightarrows U$ *such that* $G(X)$ *is a finite set of definable points and* $G^{-1}(u)$ *is an definable set.*

A digital output feedback controller *is a dynamic output feedback controller* (Q, q_0, δ, γ) *where* Q *is finite,* $\gamma(Q)$ *consists of definable points, and for all* $q, q' \in Q$*,* $\{y \in Y \mid \delta(q, y) = q'\}$ *is a definable set.*

3.3 Control problems

In this paper we consider the *control-to-target* (a kind of liveness problem) and the *safety* problem.

Problem 7. *The* control-to-target problem *for initial set* $X_0 \subset X$ *and target set* $T \subset X$ *is satisfied if all sequences* $(\vec{u}, \vec{y})$*, there exists* $n \in \mathbb{N}$ *such that for any* $\vec{x}$ *such that* $(\vec{x}, \vec{u}, \vec{y})$ *satisfies the closed-loop system, we have* $x_n \in T$*.*

Note that for a solution of the control-to-target problem, we require that at the time n we know from the input-output behaviour that $x_n \in T$. This is a stronger condition than the *control-through-target* problem, for which we only require that for all solutions $(\vec{x}, \vec{u}, \vec{y})$ of the closed-loop system, there exists $n \in \mathbb{N}$ such that $x_n \in T$. In other words, we know that the state passes through the target set, without necessarily knowing when this occurs.

Problem 8. *The* safety problem *for initial set $X_0 \subset X$ and safe set $S \subset X$ is satisfied if all solutions $(\vec{x}, \vec{u}, \vec{y})$ of the closed-loop system with $x_0 \in X_0$ satisfy $x_n \in S$ for all n.*

Other control problems, such as the "safe-control-to target" problem, or language inclusion problems, can be reformulated in terms of the safety problem and the control-to-target problem.

Definition 9. *We say that a control problem is* solvable *for the control system (F, H) if there exists a deterministic control law $k : Y^* \to U$ such that the resulting closed-loop system satisfies the control objective.*

Note that we place no regularity conditions on the control law used. In particular, the control law is a potentially wild discontinuous function which cannot be effectively implemented. If F and H are given in terms of approximations, or we wish to use numerical methods to solve the problem, then we only expect to be able to find an implementable solution if the problem is sufficiently *robust*.

Definition 10. *We say that a control problem is* robustly solvable *if there is a control system $(\widetilde{F}, \widetilde{H})$ such that the closures of the graphs of F and H are subsets of the interiors of the graphs of respectively $\widetilde{F}$ and $\widetilde{H}$, and the problem is solvable for $(\widetilde{F}, \widetilde{H})$.*

We are interested in conditions under which a solvable problem can be solved by a control law which is regular enough to be implementable, and which can be effectively computed from a description of F and H.

4 Computable analysis

Let X be a locally-compact Hausdorff space and β a countable base for the topology of X consisting of open sets with compact closures. In Euclidean space $\mathbb{R}^n$, we may take β to be a list of all open rational cuboids. We say $I \in \beta$ is a *basic set*. A finite union of basic sets, $\bigcup_{j=0}^{k} I_j$, is a *denotable set*. For computability purposes, we implicitly assume an enumeration $\nu : \mathbb{N} \to \beta$, and identify a sequence of basic sets $(I_0, I_1, \ldots) \in \beta^\omega$ with an element of $\mathbb{N}^{\mathbb{N}}$. A *name* of a point, set or function is a specification of that object in terms of the basic open sets. A *representation* is a naming system for a class of objects. An operator is *computable* if there is a Turing machine which transforms names of the inputs to a name of the output. See [9] for more details on computable analysis. We will use the following representations.

- A $\theta_<$ name of open $U \subset X$ encodes a list of all $I \in \beta_X$ such that $\overline{I} \subset U$
- A $\psi_>$ name of closed $A \subset X$ encodes a list of all $I \in \beta_X$ such that $A \cap \overline{I} = \emptyset$.
- A $\kappa_>$ name of compact $C \subset X$ encodes a list of all $(J_1, \ldots, J_k) \in \beta_X^*$ such that $C \subset \bigcup_{i=1}^k J_i$.
- A $\mu_<^{\mathcal{O}}$ name of an open-valued lower-semicontinuous function $F : X \rightrightarrows Y$ encodes a list of all $(I, J) \in \beta_X \times \beta_Y$ such that $F(x) \supset J$ for all $x \in \overline{I}$.
- A $\mu_>^{\mathcal{K}}$ name of a compact-valued upper-semicontinuous function $F : X \rightrightarrows Y$ encodes a list of all $(I, J_1, \ldots, J_k) \in \beta_X \times \beta_Y^*$ such that $F(\overline{I}) \subset \bigcup_{i=1}^k J_i$.

Note that an object has many names, corresponding to different orderings of the basic sets.

The main strength of the computable analysis setting is that we can discuss effective computation of various operators directly, without having to go into the details of ϵ-δ proofs.

Theorem 11. *The following operators are computable [4].*

1. *Union $(U_1, U_2) \mapsto U_1 \cup U_2$ is $(\theta_<, \theta_<; \theta_<)$-computable for open sets U_1, U_2.*
2. *Intersection $(C, A) \mapsto C \cap A$ is $(\kappa_>, \psi_>; \kappa_>)$-computable for compact C and closed A.*
3. *Image $(F, C) \mapsto F(C)$ is $(\mu_>^{\mathcal{K}}, \kappa_>; \kappa_>)$-computable for compact-valued upper-semicontinuous F and compact C.*
4. *Weak preimage $(F, A) \mapsto F^{-1}(A)$ is $(\mu_>^{\mathcal{K}}, \psi_>; \psi_>)$-computable for compact-valued upper-semicontinuous F and closed A.*
5. *Strong preimage $(F, U) \mapsto F^{\Leftarrow}(U)$ is $(\mu_>^{\mathcal{K}}, \theta_<; \theta_<)$-computable for compact-valued upper-semicontinuous F and open U.*

As well as describing open and compact sets approximately, we will sometimes want to describe sets exactly in terms of a finite union of basic sets. Recall that a set A is *regular* if $\overline{A}^\circ = A^\circ$ and $\overline{A^\circ} = \overline{A}$. We say that a set A is *denotable* if it can be written as the finite union of basic sets. More precisely, a regular set A is denotable if there exists $(I_1, \ldots, I_k) \in \beta^*$ such that $\bigcup_{j=1}^k I_j = A^\circ$ and $\bigcup_{j=1}^k \overline{I}_j = \overline{A}$.

Theorem 12.

1. *If C is compact and U is open, then $C \subset U$ can be effectively verified.*
2. *If C is compact and U is open, then it is possible to construct a regular denotable set A such that $C \subset A^\circ$ and $\overline{A} \subset U$ given a $\theta_<$-name of U and a $\kappa_>$-name of C.*
3. *If $A = \bigcup_{j=1}^k I_j$ is a regular denotable set, then we can effectively compute a $\theta_<$-name of A° and a $\kappa_>$-name of $\overline{A}$.*

Note that the construction of a regular denotable set A with $C \subset A^\circ$ and $\overline{A} \subset U$ is not canonical; it depends on the names used to describe C and U.

5 Control with complete observations

We now consider control problems for a control system with complete observations, as described by a compact-valued upper-semicontinuous system $F : X \times U \rightrightarrows X$. We will make frequent use of the following key lemma:

Lemma 13. *Let $F : X \times U \rightrightarrows X$ be compact-valued upper-semicomputable, and $W \subset X$ be open. Define $\mathrm{Pre}(F,W) = \{x \in X \mid \exists u \in U \text{ s.t. } F(x,u) \in W\}$, and $\mathrm{Ctl}(F,W) : X \times U \rightrightarrows X$ by $\mathrm{Ctl}(F,W)(x) = \{u \in U \mid F(x,u) \in W\}$. Then*

1. *the operator $(F,W) \mapsto \mathrm{Pre}(F,W)$ is $(\mu_>,\theta_<;\theta_<)$-computable, and*
2. *the operator $(F,W) \mapsto \mathrm{Ctl}(F,W)$ is $(\mu_>,\theta_<;\mu_<)$-computable.*

In other words, given a $\mu_>$-name of F and a $\theta_<$-name of W, we can effectively compute a $\theta_<$-name of the set of points which can be controlled into W, and a $\mu_<$-name of the minimally-restrictive one-step feedback law.

5.1 Control-to-target

In this section, we show that if the control-to-target problem with complete observations is solvable, then a solution can be effectively computed.

Theorem 14. *Suppose the control-to-target problem for F with compact initial set X_0 and open target set T has a solution. Then there is a digital state feedback controller G which is computable from a $(\mu_>,\kappa_>,\theta_<)$-name of (F,X_0,T).*

Proof. [Sketch of proof] Recursively define $W_0 = T$ and $W_{n+1} = \mathrm{Pre}(F,W_n)$. Then W_n is the set of points which can be controlled into T in time n. Since the control problem is solvable, $X_0 \subset \bigcup_{n=0}^{\infty} W_n$, and since X_0 is compact and the W_i are open, there exists $k \in \mathbb{N}$ such that $X_0 \subset \bigcup_{n=0}^{k} W_n$. From Lemma 13.1, we can compute $\theta_<$-names of the W_n, from Theorem 11.1 we can compute $\bigcup_{n=0}^{k} W_n$ and from Theorem 12.1 we can prove that $X_0 \subset \bigcup_{n=0}^{k} W_n$.

To compute a controller for the orginal control problem, it is not sufficient to compute a controller G_i for each step W_i to W_{i-1}. For if W_j and W_i overlap, then the controller G_j may take a point in $W_i \cap W_j$ to a point not in W_{i-1}, and we do not obtain a suitable controller. We therefore need to restrict the sets W_i.

We first construct a sequence of sets A_j as follows, starting with A_k and ending with A_0. We choose A_j such that $\overline{A}_j \subset W_j$, $\mathrm{Pre}(F,A_j) \supset W_{j+1}$ if $j < k$, and $A_j \supset X_0 \setminus \big(\bigcup_{i=0}^{j-1} W_i \cup \bigcup_{i=j+1}^{k} A_i\big)$. The set A_j can be effectively constructed so that $\mathrm{Pre}(F,A_j) \supset W_{j+1}$ and $\overline{A}_j \subset W_j$ since $\mathrm{Pre}(F,W_j) \supset W_{j+1}$ and $\mathrm{Pre}(F,A_j) \nearrow \mathrm{Pre}(F,W_j)$ as $A_j \nearrow W_j$.

We then compute sets $B_i = \mathrm{Pre}(F,A_{i-1}) \setminus \bigcup_{j=1}^{i-1} \overline{A}_j$. We let K_i be the minimally-restrictive controller taking for B_i into A_{i-1}, so $K_i(x) = \{u \in U \mid F(x,u) \in A_{i-1}\}$ for $x \in B_i$, and $K_i(x) = \emptyset$ otherwise. We take the control law $K(x) = \bigcup_{i=1}^{k} K_i(x)$ solves the control problem.

We now show that the constructed control law K solves the control-to-target problem. We first show that $\bigcup_{i=1}^{j} B_j = \bigcup_{i=0}^{j-1} \mathrm{Pre}(F, A_i)$ by induction. For $B_1 = \mathrm{Pre}(F, A_0)$ by definition, and assuming the inductive hypothesis, we have $\bigcup_{i=1}^{j} B_j = \big(\mathrm{Pre}(F, A_{j-1}) \setminus \bigcup_{i=1}^{j-1} \overline{A}_i\big) \cup \bigcup_{i=1}^{j-1} B_i \supset \big(\mathrm{Pre}(F, A_{j-1}) \setminus \bigcup_{i=1}^{j-2} \mathrm{Pre}(F, A_i)\big) \cup \bigcup_{i=1}^{j-1} B_i = \mathrm{Pre}(F, A_{j-1}) \setminus \bigcup_{i=1}^{j-1} B_i$.

If $x \in B_j$, then $F(x, B_j(x)) \subset A_{j-1}$. If $x \in A_j$ for some j, then $x \notin B_i$ for any $i > j$, so $F(x, K(x)) \subset \bigcup_{i=1}^{j} F(x, K_i(x)) \subset \bigcup_{i=0}^{j-1} A_{j-1}$. Hence if $j(x) = \max\{i \in 0, \ldots, k \mid x \in B_i\}$, then the control law K forces x into T in at most $j(x)$ steps.

To compute a digital state feedback controller, we take a selection G of K. □

It is easy to see from the construction of the control law that the same controller also works for any sufficiently small perturbation of F. We therefore have the following corollary.

Corollary 15. *If the control-to-target problem is solvable, then it is robustly solvable.*

5.2 Safety

Theorem 16. *Suppose F is an upper-semicontinuous system, that X_0 is a compact set of initial states, and S an open set of safe states with compact closure. Then the safety control problem for (F, X_0, S) is robustly solvable if and only if there exists a regular denotable set A such that*

$$X_0 \subset A^\circ,\ \ \overline{A} \subset S \ \text{ and } \mathrm{Pre}(F, A^\circ) \supset \overline{A}.$$

Proof. Suppose the safety problem is robustly solvable, and let $\widetilde{F}$ be an open-valued lower-semicontinuous system for which a solution exists. Let $\widetilde{W}_0 = S$ and $\widetilde{W}_{n+1} = \widetilde{W}_n \cap \mathrm{Pre}(\widetilde{F}, \widetilde{W}_n)$, and let $\widetilde{W}_\infty = \bigcap_{n=0}^{\infty} \widetilde{W}_n$. Then $\widetilde{W}_n$ is the set of points which can be controlled within S for times up to n, and since the safety control problem is solvable for $\widetilde{F}$, we have $X_0 \subset \widetilde{W}_\infty$.

Let $\widehat{F}$ be a compact-valued upper-semicontinuous system such that graph$(\widehat{F})$ is a subset of graph$(\widetilde{F})$ and a neighbourhood of graph$(\widehat{F})$. Let $\widehat{S}$ be an open set containing X_0 whose closure lies in S and such that $\mathrm{Pre}(\widetilde{F}, S) \subset \mathrm{Pre}(\widehat{F}, \widehat{S})$. Define $\widehat{W}_0 = \widehat{S}_0$ and $\widehat{W}_{n+1} = \widehat{W}_n \cap \mathrm{Pre}(\widehat{F}, \widehat{W}_n)$. Since for any set W, $\mathrm{Pre}(\widehat{F}, W) \supset \mathrm{cl}(\mathrm{Pre}(\widetilde{F}, W))$, we have $\widehat{W}_\infty \supset \mathrm{cl}(\widetilde{W}_\infty) \cap \widehat{S}$.

We therefore have $\widehat{X}_0 \subset \widehat{W}_\infty$ for a compact neighbourhood of $\widehat{X}_0$ X_0. Further, since $\widehat{W}_\infty = \mathrm{Pre}(\widehat{F}, \widehat{W}_\infty)$, we have $\mathrm{cl}(\widehat{W}_\infty) \subset \mathrm{Pre}(F, \widehat{W}_\infty)$. Then there exists a regular set A approximating $\widehat{W}_\infty$ such that $\overline{A} \subset \mathrm{Pre}(F, A^\circ)$, $X_0 \subset A^\circ$ and $\overline{A} \subset S$.

The converse is trivial. □

Theorem 17. *Suppose F is an upper-semicontinuous system, that X_0 is a compact set of initial states, and S an open set of safe states. Then if the safety control* *problem is robustly solvable, there is a digital state feedback controller which can be computed from a* $(\mu_>, \kappa_>, \theta_<)$*-name of* (F, X_0, S).

Proof. Let A be a regular denotable open set as given by Theorem 16. Note that W can be found by brute-force search over all regular denotable open sets, since the conditions $X_0 \subset A$, $\overline{A} \subset S$ and $\overline{A} \subset F^{-1}(A)$ are all verifyable. A lower-semicontinuous open-valued controller for the problem is the function $G : X \rightrightarrows U$ given by

$$G(x) = \{u \in U \mid F(x,u) \in A\} \text{ if } x \in A; \quad G(x) = U \text{ otherwise.}$$

and a $\mu_<$-name of G can be computed from a $(\mu_>, \theta_<)$-name of (F, A). Finally, we restrict G to a digital state feedback controller. □

6 Control with partial observations

We now consider the control-to-target and safety problems with partial observations.

6.1 Control-to-target

Theorem 18. *Let* (F, H) *be a control system where* $F : X \times U \rightrightarrows X$ *and* $H : X \rightrightarrows Y$ *are compact-valued upper-semicontinuous,* $X_0 \subset X$ *be compact and* $T \subset X$ *open. Suppose that the control-to-target problem is solvable for* (F, H, X_0, T)*. Then there exists a automaton-based feedback controller which can be computed from a* $(\mu_>, \mu_>, \kappa_>, \theta_<)$*-name of* (F, H, X_0, T).

Proof. Define collections of compact sets $\mathcal{C}_n$ by $\mathcal{C}_0 = \{C \in \mathcal{K}(X) \mid C \subset T\}$ and

$$\mathcal{C}_{n+1} = \mathcal{C}_n \cup \{C \in \mathcal{K}(X) \mid \exists u \in U,\ \forall y \in Y,\ \exists B \in \mathcal{C}_n \colon\ F(C,u) \cap H^{-1}(y) \subset B\}.$$

Hence $\mathcal{C}_n$ is the set of compact sets C such that if we know $x \in C$, then we can control the state to the target in at most n steps. Since the control-to-target problem is solvable, there exists $N \in \mathbb{N}$ such that $X_0 \in \mathcal{C}_N$. The reason we need only consider compact sets is that at any time, the best state estimate is $\widehat{X}_n = F(\widehat{X}_{n-1}, u_{n-1}) \cap H^{-1}(y_n)$, so is compact.

We now show that we can find $\mathcal{W}_n \subset \mathcal{O}(X)$ such that

$$\mathcal{C}_n = \{C \in \mathcal{K}(X) \mid \exists W \in \mathcal{W}_n \text{ s.t. } C \subset W\}.$$

Clearly, we can take $\mathcal{W}_0 = T$. Suppose $C \in \mathcal{C}_n \setminus \mathcal{C}_{n-1}$ Then $\exists u \in U$, $\forall y \in Y$, $\exists w \in \mathcal{W}_{n-1}$ such that $F(C,u) \cap H^{-1}(y) \subset W$. Then there is an open cover $Z_1, \ldots, Z_k$ of $H(C)$ and sets $W_1, \ldots, W_k \in \mathcal{W}_{n-1}$ such that $C \cap H^{-1}(\overline{Z}_i) \subset W_i$ for all i. Therefore we can construct an open neighbourhood V of C such that $H(\overline{V}) \subset \bigcup_{i=1}^k Z_i$ and $\overline{V} \cap H^{-1}(\overline{Z}_i) \subset W_i$. Then $H(B) \subset \bigcup_{i=1}^k Z_i$ and $B \cap H^{-1}(\overline{Z}_i) \subset W_i$ for all compact $B \subset V$. We take $\mathcal{W}_{n+1}$ to be the collection of all such V.

Suppose $U_i \subset U$ and $Y_i \subset Y$ are compact sets, and define sets $\widehat{X}_i$ by $\widehat{X}_0 = X_0 \cap H^{-1}(Y_i)$ and $\widehat{X}_{i+1} = F(\widehat{X}_i, U_i) \cap H^{-1}(Y_i)$. Suppose $\widehat{X}_n \subset T$. Construct open W_n such that $\widehat{X}_n \subset W_n$ and $\overline{W}_n \subset T$. Then $W_n \supset F(\widehat{X}_{n-1}, U_{n-1}) \cap H^{-1}(Y_n)$, and since H is upper-semicontinuous and $F(\widehat{X}_{n-1}, U_{n-1})$ is compact, we can contstruct open $Z_n \supset Y_n$ such that $W_n \supset F(\widehat{X}_{n-1}, U_{n-1}) \cap H^{-1}(\overline{Z}_n)$. Since F is upper-semicontinuous, there exists open $W_{n-1} \supset \widehat{X}_{n-1}$ and $V_{n-1} \supset U_{n-1}$ such that $W_n \supset F(\overline{W}_{n-1}, \overline{V}_{n-1}) \cap H^{-1}(\overline{Z}_n)$. A recursive procedure constructs open sets $W_i \subset X$, $V_i \subset U$ and $Z_i \subset Y$ for $i = 0, \ldots, n$ such that $\widehat{X}_i \subset W_i$, $U_i \subset V_i$, $Y_i \subset Z_i$ and $W_i \supset F(\overline{W}_{i-1}, \overline{V}_{i-1}) \cap H^{-1}(\overline{Z}_i)$ for $i = 1, \ldots, n$. In other words, we can enlarge the sets U_i and Y_i and still prove that the state at time n lies in T.

Combining the above constructions, we can find finite collections $\mathcal{A}_n$ of denotable regular sets, and a locally-finite open cover $\mathcal{Z}$ of Y such that

$$\forall A_{n+1} \in \mathcal{A}_{n+1},\ \exists I \subset U,\ \forall Z \in \mathcal{Z},\ \exists A_n \in \mathcal{A}_n \text{ s.t. } F(\overline{A}_{n+1}, \overline{I}) \cap H^{-1}(\overline{Z}) \subset A_n^\circ,$$

and that $\mathcal{A}_0 = \{A_0\}$ with $\overline{A}_0 \subset T$, and $\exists A_N \in \mathcal{A}_N$ with $X_0 \subset A_N^\circ$. We can use the disjoint union of the $\mathcal{A}_n$ as the states of an digital output feedback controller. The update law is $B \in \delta(A, y)$ and $u \in \gamma(A, y)$ if $A \in \mathcal{A}_{n+1}$, $B \in \mathcal{A}_n$, $y \in Z$ and and $F(\overline{A}, u) \cap H^{-1}(\overline{Z}) \subset B$. □

6.2 Safety

We now consider the safety problem. We first give a checkable condition for a system to be robustly stable.

Theorem 19. *Let (F, H) be an upper-semicontinuous control system, $X_0 \subset X$ compact and $S \subset X$ open. The the safety control problem to remain in S is robustly solvable if, and only if, there exist finitely many regular sets $\mathcal{A}$, and a locally-finite open cover $\mathcal{Z}$ of Y such that*

$$\forall A \in \mathcal{A}, Z \in \mathcal{Z},\ \exists B \in \mathcal{A},\ \text{open } I \subset U \text{ s.t. } F(\overline{A}, \overline{I}) \cap H^{-1}(\overline{Z}) \subset B^\circ,$$

and $\forall A \in \mathcal{A},\ \overline{A} \subset S$, *and* $\forall Z \in \mathcal{Z},\ \exists B \in \mathcal{A}$ *s.t.* $X_0 \cap H^{-1}(\overline{Z}) \subset B^\circ$.

Proof. Suppose the collections of regular sets $\mathcal{A}$ and $\mathcal{Z}$ exist. Then we can construct a feedback controller with state space $\mathcal{A}$, nondeterministic update rule $\delta(A_n, y_n) = A_{n+1}$ and $\gamma(A_n, y_n) \in I_n$ if $y_n \in Z_n$ and $F(\overline{A}_n, \overline{I}_n) \cap H^{-1}(\overline{Z}_n) \subset A_{n+1}^\circ$.

Suppose that the safety control problem is robustly solvable. Let $(\widetilde{F}, \widetilde{H})$ be over-approximations of (F, H) for which the problem is still solvable. Then we can construct $(\widehat{F}, \widehat{H})$ which are restrictions of $(\widetilde{F}, \widetilde{H})$ such that the possible sets $\mathcal{A} = \{\widehat{F}(x, u) \cap \widehat{H}^{-1}(y) \mid x \in X,\ u \in U,\ y \in Y\}$, and the sets $\mathcal{Z} = \{\widehat{H}(x) \mid x \in X\}$ are finite. The result follows. □

From the collection of subset $\mathcal{A}$, we can easily construct a controller.

Theorem 20. *Let (F, H) be a control system where F and H are compact-valued upper-semicontinuous, $X_0 \subset X$ compact and $S \subset X$ open. Suppose the safety control problem for (F, H, X_0, S) is robustly solvable. Then there is a dynamic feedback controller which can be effectively computed from a $(\mu_>, \mu_>, \kappa_>, \theta_<)$-name of (F, H, X_0, S).*

Proof. By a brute-force search over all finite collections of regular sets $\mathcal{A}$ and $\mathcal{Z}$, we can verify the conditions of Theorem 19. The states of the digital output feedback controller are then $\mathcal{A}$, and the discretised outputs are $\mathcal{Z}$. The controller is given by $B \in \delta(A, y)$ and $u \in \gamma(A, y)$ if $y \in Z$ and $F(\overline{A}, u) \cap H^{-1}(\overline{Z}) \subset B^{\circ}$. □

7 Conclusions

In this paper we have considered effective control synthesis for discrete-time systems defined by compact-valued upper-semicontinuous functions. We have seen that for control-to-target problems that if the problem is solvable, then we can compute a controller, and for safety problems, that if the problem is robustly solvable, then we can also compute a controller. For systems with complete observations, we first construct an open-valued lower-semicontinuous state feedback map, which we may then restrict to an implementable controller, such as a digital state feedback controller. For systems with partial observations, we construct a digital output feedback controller by using an approximate set-based observer.

The information used to define the system is given by the standard representations of compact-valued upper-semicontinuous maps, as defined in computable analysis. This facilitates some of the constructions, since we can consider computation of fundamental operators on sets and functions abstractly.

The approach to partial output feedback in this paper is to work directly with the state estimates $\widehat{X}_n$ as much as possible. However, it may be possible to obtain simpler derivations of the results by considering the input/output history $(y_0, \ldots, y_n, u_0, \ldots, u_{n-1})$ directly instead. An alternative approach is to prove robustness results, and then to work directly with discretized approximations of the system.

In future work, we plan to investigate these alternative approaches, as well as to extend the results to controllers for continuous-time and hybrid-time systems.

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