Identifiability, recursive identification and spaces of linear dynamical systems part II

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CHAPTER 6
RIEMANNIAN GRADIENT ALGORITHMS FOR
RECURSIVE IDENTIFICATION

6.1. Introduction
In previous chapters we have seen that the set of stochastic linear systems of
fixed McMillan degree \( n \), which have no zeroes on the unit circle, forms a
differentiable manifold. In chapter 4 we have seen that it is not possible to
use only one parametrization.

There are several possibilities to handle this. One approach is to identify
the structural indices (Kronecker indices) first and then to identify the
parameters within the set of all stochastic systems with those structural
indices. (Even then the set of parameters is not a coordinate chart in
general, but only some open subset of Euclidean space). Another approach is to
use so-called overlapping parametrizations. (cf. [Glo-W1], [Owe-L], [Gui 81],
[Cla], i.a.).

Especially for recursive identification this is an important approach because
one does not have to decide a priori which structure to fix: one can switch
on-line from one parametrization to another. In this chapter we will present
an algorithm that uses overlapping parametrizations. It is desirable in such
an algorithm that its behaviour does not depend, or at least not very much, on
the actual choice of the parametrization at each time. Elsewhere we have
started to work out a version of the algorithm that is completely independent
of the choice of the parametrization at each time (cf. [Hnz 85b]).

In this chapter we will construct and analyze an algorithm that is (only)
asymptotically independent of the choice of parametrization, and apart from
that it does not depend very much on the choice of the parametrization at each
time, especially if the stepsize is small. We obtain this property by using a
so-called Riemannian gradient. As is well-known, on a differentiable manifold a
gradient is only defined with respect to a Riemannian metric. It is obtained
by premultiplying the gradient in terms of local coordinates (i.e. a chosen
parametrization) with the inverse of the Riemannian metric tensor. (see e.g.
[Ab-M]). However, for us this is not the basic argument to use a Riemannian
gradient. The basic argument is that it has the steepest ascent property: it
optimizes the increment of the objective function over all steps of fixed
(small) length over all possible directions. The length of such a step should be
measured in the model space, and not in the arbitrary parametrization at hand, because the parametrization is only an instrument to describe the model space, and nothing more. We will try to clarify this argument by way of an example.

Consider a (standard) gradient method

\[(6.1-1) \quad \hat{\theta}_{t+1} = \hat{\theta}_t + a_t \frac{\partial V}{\partial \theta} \]

for a stochastic linear model with parameters \( \theta \). The criterion function \( V \) depends on the parameters only in so far as \( V \) depends on the covariances \( \{\Gamma_k\} \) (otherwise the problem would not be identifiable). If a parametrization is such that at certain points, the \( \{\Gamma_k\} \) will be (rather) insensitive to certain parameter changes, one can expect in general, that \( V \) will also be rather insensitive to such parameter changes. This can be so even if \( V \) is not insensitive to changes of the \( \{\Gamma_k\} \) in the corresponding directions! As a simple example, consider the following scalar maximum likelihood problem:

\[(6.1-2) \quad \begin{cases} x_{t+1} = ax_t + bw_t, \\ y_t = x_t + dv_t, \end{cases} \]

\(v_t, w_t\) standard white noise. The log-likelihood of the observations \( \{y_0, y_1, \ldots, y_T\} \) is equal to

\[(6.1-3) \quad V = -\frac{1}{2} \ln |\Gamma(T)| - \frac{1}{2} (y_0, y_1, \ldots, y_T)^T \Gamma(T)^{-1} (y_0, y_1, \ldots, y_T)^T, \]

where

\[(6.1-4) \quad \Gamma(T) = \frac{b}{1-a^2} \left[ \begin{array}{cccc} 1 & a & a^2 & \cdots & a^{T-1} \\ a & 1 & a & \cdots & a^{T-2} \\ \vdots & & \ddots & \vdots & \vdots \\ a^{T-3} & \cdots & \ddots & 1 \end{array} \right] + d^2 I_{T+1}. \]

Then

\[(6.1-5) \quad \frac{\partial V}{\partial b} = -\frac{1}{2} \text{tr} [\Gamma(T)^{-1} \frac{\partial \Gamma(T)}{\partial b}] + \frac{1}{2} (y_0, y_1, \ldots, y_T)^T \Gamma(T)^{-1} \frac{\partial \Gamma(T)}{\partial b} \Gamma(T)^{-1} (y_0, y_1, \ldots, y_T)^T. \]
The partial derivative $\frac{\partial V}{\partial d}$ is given by a similar formula. Now consider these formulas at $a = 0$: then

\[(6.1-6) \quad \frac{\partial \Gamma(T)}{\partial b} = 2b\Gamma_{t+1} \quad \text{and} \quad \frac{\partial \Gamma(T)}{\partial d} = 2d\Gamma_{t+1}.\]

Therefore the partial derivative of $V$ in the direction $(a, b, d) = (0, d, -b)$ at $a = 0$, is

\[(6.1-7) \quad \frac{\partial^2 V}{\partial b^2} - \frac{\partial^2 V}{\partial d^2} = 0.\]

It follows that in a neighbourhood of a point $(a_0, b_0, d_0)$, $V$ will be rather insensitive to parameter changes in the direction $(a, b, d) = (0, d, -b)$. Therefore the gradient method described above is not likely to go in such a direction even if $V$ increases substantially in such a direction as a function of the covariances. A solution to these problems is obtained by using a Riemannian gradient.

Our basic objective in this chapter is to show that one can construct a recursive identification algorithm on a manifold of stochastic linear systems, which has asymptotic properties similar to those of corresponding algorithms on an open (or at least with non-empty interior) Euclidean parameter space (i.e. $\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}$). Or to put it concisely our basic objective is to show that 'one can do system identification on a manifold'. As our point of departure we have chosen for Ljung's prediction error algorithm for recursive identification. For a description see e.g. [Lj 81], [Lj-Söd]. We have chosen for the simplest version. Once it is understood how such an algorithm can be transformed to a Riemannian gradient algorithm on a manifold, it should not be very difficult to generalize to less simple versions. To be sure, the simplest version is already quite complicated.

One assumption that we make is rather crucial, especially for the convergence analysis of the algorithm, namely that our manifold is compact (and without boundary in the sense of manifold theory, cf. [Bool], p.11 and around p. 250). For manifolds this assumption is often made, and there are many examples of such manifolds, e.g. a sphere (of any dimension). Although compactness appears to be rather crucial for the algorithm and its convergence properties, this appears not to be the case for the assumption that the manifold has no boundaries. If one has a compact manifold with boundaries, analogous results
are expected. The only problem is that the algorithm for such a case has to be constructed such that if the algorithm reaches the boundary of the manifold, then the next change in parameter is constrained and must not point outside the manifold with boundary. This gives rise to some technical problems that we do not want to go into here. Therefore we assume there are no such boundaries. Because of this, we do not need to have an analogue of the 'projection facility' that is needed in Ljung's prediction error algorithm, and which complicates the algorithm and its analysis (cf. [Lj-Söd]).

The major part of this chapter is taken up by the convergence analysis of the algorithm. One of the reasons for the length of the analysis is that it turned out not to be easily possible (at least for this author) to simply generalize the proofs that exist for the non-manifold case. Instead we had to come up with a new complete proof. It is based on the so-called o.d.e.-method (see e.g. [Lj 77], [Lj-Söd]). In this method of analysis the asymptotic properties of the algorithm are shown to be related to the properties of an ordinary differential equation (o.d.e.). To obtain the object that satisfies the o.d.e. we follow the method of [Ku-Cl], (esp. chapter II). Their main idea is to apply a well-known theorem from topological analysis known as the Arzelà-Ascoli theorem, to a set of interpolation curves of the parameterpoints produced by the algorithm. It is a limit point of this set of interpolation curves that satisfies the o.d.e. In [Ku-Cl], p. 19, it is stated that 'the basic idea is simply an extension of the compactness technique as used to construct solutions to ordinary differential equations (cf. [Co-Le], pp. 42-45')". Kushner and Clark treat some applications of their methods to system identification ([Ku-Cl], pp. 88-98). Once we have established the o.d.e. we can draw rather strong conclusions, thanks to the construction of the solution of the o.d.e. It turns out that our algorithm converges to a compact connected set of critical points of the objective function $V_p$. Of course the objective function is constant on such a set. This implies that if the critical points of $V_p$ are all isolated, then the algorithm converges to a critical point. In distinction to the theorems of [Lj 77] there are no assumptions needed about the actual behaviour of the algorithm to reach this conclusion. To be more specific: [Lj 77] requires the sequence of parameter points generated by the algorithm to return to a certain set infinitely often, and only under that assumption guaranteed convergence is obtained. We do not need such an assumption.
In section 6.2 the algorithm is presented. In section 6.3 we present a refinement of the cover of the manifold which is needed for the analysis. In section 6.4 the asymptotic behaviour of the stepsizes and the times of coordinate-change is analyzed. In section 6.5 some spaces of interpolation curves are presented, their topological and metrical structure are treated and the relation with the algorithm is explained. One of the main difficulties we found on our way was to deal with probability-one convergence, and the sets of exceptional events of measure zero. All this is discussed in section 6.6. One of the important properties of the algorithm and of the system-to-be-identified is the asymptotic stability of the dynamic matrices involved. Because of this, the behaviour of the algorithm at points of time that lie far apart tend to be almost independent. To make this precise we define certain so-called 'exponential decay' properties, and prove some theorems about them. We hope that these concepts will turn out to be useful for other analyses as well. Together with its implications for convergence of the algorithm, this is treated in section 6.7. In section 6.8 we finally arrive at the associated ordinary differential equation, and in section 6.9 we draw the conclusions that follow from the o.d.e. for the convergence properties of the algorithm. We end the chapter with some final remarks in section 6.10.

6.2. Description of the algorithm

6.2.1. The model set

From theorem (4.8-8) we know that the set of all stochastic systems (4.8-1), with fixed McMillan degree and a fixed number \( m \) of output components, which have an innovations representation with asymptotically stable inverse, forms a differentiable manifold, diffeomorphic to \( \mathcal{M}_{\tilde{m},\tilde{n},\tilde{m}}^{\tilde{m},\tilde{n},\tilde{m}} \times \text{Pos}(m) \). Fixing the diffeomorphism of theorem (4.8-8), we will identify a stochastic system with the corresponding element of \( \mathcal{M}_{\tilde{m},\tilde{n},\tilde{m}}^{\tilde{m},\tilde{n},\tilde{m}} \times \text{Pos}(m) \), if there is no danger of confusion. Our model set will be of the form \( M \times \text{Pos}(m) \), with \( M \) a compact submanifold of \( \mathcal{M}_{\tilde{m},\tilde{n},\tilde{m}}^{\tilde{m},\tilde{n},\tilde{m}} \). To avoid additional technical complications we assume \( M \) to be a manifold without boundary (in the sense of manifold theory, cf. e.g. [Boo]). An example of such a space is a sphere (but not a ball). The true model \( (\tilde{\theta},\tilde{n}) \) will be assumed to lie in the model set:

\[(6.2.1-1) \quad (\tilde{\theta},\tilde{n}) \in M \times \text{Pos}(m).\]
Our attention will focus on identifying \( \widehat{\theta} \in \mathcal{M} \). (In fact, once \( \widehat{\theta} \) is identified, one can estimate \( \tilde{\theta} \) by standard procedures, see (6.2.2-4)). Together with \( \mathcal{M} \) there exists the corresponding (induced) state bundle \( \mathcal{E} \) and a corresponding principal fibre bundle \( \mathcal{L} \). These are defined as follows:

\[(6.2.1-2) \quad \mathcal{E} := \{ \gamma \in M_{1,m,n,m}^{m,a,f} | \pi(\gamma) \in \mathcal{M} \}, \]

\[(6.2.1-3) \quad \mathcal{L} := \{(A,B,C) \in L_{m,n,m}^{m,a,f} | (A,B,C) \in \mathcal{M} \} \]

(cf. section 4.6, esp. remark 4.6-7).

The manifold \( \mathcal{M} \) is made into a Riemannian manifold by defining a Riemannian metric on its tangent bundle \( T\mathcal{M} \). This can be done as described in chapter 5, although the constructions and results in this chapter hold for an arbitrary Riemannian metric.

The problem to be considered is to construct and analyze a recursive identification algorithm to identify \( \widehat{\theta} \) (and \( \tilde{\theta} \)) in the model set \( \mathcal{M}(\times \text{Pos}(m)) \).

6.2.2. Prediction error algorithms.

The algorithm that will be constructed is a generalization of the well-known prediction error algorithm (cf. [Lj 81], [Lj-Söd 81], [Lj 78]). Before describing the generalization in the following subsections, let us briefly review the standard prediction error algorithm.

To be able to apply it, one has to choose, and therefore be able to specify a parameter set \( \Theta \subseteq \mathbb{R}^d \), and a smooth mapping

\[(6.2.2-1) \quad \Theta \rightarrow L_{m,n,m}^{m,a,f}, \quad \Theta \rightarrow (A(\Theta), B(\Theta), C(\Theta)) \]

with the following two properties:

(i) \( \Theta \) is an open subset of \( \mathbb{R}^d \), or has at least a non-empty interior. If \( \Theta \) has a non-empty boundary in \( \mathbb{R}^d \), the standard prediction error algorithm has built-in a so-called projection facility to see to it that the sequence of parameter estimates that is produced by the algorithm remains within \( \Theta \). However, because this will not be needed in our algorithm, we will not go into that here.

(ii) The composition of the mapping (6.2.2-1) with \( \Theta \), i.e. the mapping
\( \theta \rightarrow \{(A(\theta), B(\theta), C(\theta)) \} \in \mathbb{M}_{m,n,m'}^{p} \) is injective.

If \( \theta \in \Theta \) is believed to be the true parameter value, then the corresponding prediction \( \hat{y}_t(\theta) \) of \( y_t \) will be given by the filter

\[
\begin{align*}
&x_{t+1}(\theta) = (A(\theta) - B(\theta)C(\theta))x_t(\theta) + B(\theta)y_t, \\
&\hat{y}_t(\theta) = C(\theta)x_t(\theta).
\end{align*}
\]

The corresponding prediction error is

\[
\epsilon_t(\theta) = y_t - \hat{y}_t(\theta).
\]

Note that the covariance matrix \( \Sigma \) does not occur in these formulas. If \( \theta = \bar{\theta} \), the true parameter value, then \( \Sigma \) can be estimated consistently by the sample covariance matrix

\[
\hat{\Sigma} := \frac{1}{T} \sum_{t=1}^{T} \epsilon_t(\bar{\theta})\epsilon_t(\bar{\theta})^T.
\]

The idea behind the prediction error algorithm is to try to minimize with respect to \( \theta \) the expected sum of squares \( V(\theta) \) of the prediction errors:

\[
V(\theta) = \frac{1}{2} \mathbb{E}_\theta \{ \epsilon(\theta) \epsilon(\theta)^T \}.
\]

From the properties of the steady state Kalman filter it follows that \( V(\theta) \) has a unique global minimum at \( \theta = \bar{\theta} \) (cf. [An-M]). So if \( V(\theta) \) were known, a method to find the minimum would be the well-known gradient algorithm

\[
\theta_{k+1} = \theta_k - \frac{\partial V}{\partial \theta_k},
\]

or more generally

\[
\theta_{k+1} = \theta_k - R_k(\theta_k)^{-1} \frac{\partial V}{\partial \theta_k},
\]

where \( R_k(\theta_k) \) is some nonsingular (weighting) matrix, usually positive definite symmetric. The gradient is given by
(6.2.2-7) \( \frac{\partial V}{\partial \theta} = E \frac{\partial}{\partial \theta} e_t(\theta). \)

The expectation, both in (6.2.2-5) and (6.2.2-7), is taken with respect to the true probability measure, which depends on \( \hat{\theta} \) and \( \hat{u} \). They are unknown, so \( V(\theta) \) and \( \frac{\partial V}{\partial \theta} \) are unknown.

A technique to handle such a situation is the so-called stochastic approximation method (cf. e.g. [Ku-Cl], [Lj 81], [Lj-Söd], and the references given there). The idea of this method is to replace (6.2.2-6) by

(6.2.2-8) \( \theta_{k+1} = \theta_k + a_k R_k(\theta_k) \frac{1}{\theta_k} e_t(\theta_k), \)

where \( \{a_k\}_{k=k_0}^{\infty} \) is a sequence of positive numbers, tending to zero and adding up to infinity. One lets \( \{a_k\} \) converge to zero to help 'asymptotically cancel' the noise effects; having the sequence sum to infinity is usually necessary for convergence to the 'right' point or set. ([Ku-Cl], p.6). (The role of the parameter \( t \) in (6.2.2-8) is perhaps somewhat obscure but we will return to that shortly). From the filter (6.2.2-2), (6.2.2-3), for \( e_t(\theta) \) one can derive the filter equations for the \( d \times m \) matrix of partial derivatives

\( \frac{\partial e_t(\theta)}{\partial \theta}. \) Let \( \theta^i, i = 1, 2, \ldots, d, \) denote the components of the vector \( \theta. \) One has (compare (6.2.2-2), to keep the notation more transparent we drop the argument \( \theta \))

(6.2.2-9)

\[ \begin{align*}
\frac{3x_{t+1}}{3\theta} &= (A-BC)x_t + By_t, \\
\frac{3e_{t+1}}{3\theta} &= (A-BC)x_t + (A-BC)x_t + B\bar{y}_t, \\
\frac{3e_t}{3\theta} &= -C\bar{x}_t + y_t, \\
\frac{3\bar{x}_t}{3\theta} &= -C\bar{x}_t - C\bar{x}_t.
\end{align*} \]

This can be written in vector/matrix notation, as follows. Let

\[ e_t^{T} := (\frac{3x_t}{3\theta}, \frac{3x_t}{3\theta}, \ldots, \frac{3x_t}{3\theta}), \quad \text{the extended state vector;} \]

\[ \psi_t := \frac{3e_t}{3\theta}, \]

\( i = 1, 2, \ldots, d, \) then \( (e_t^{T}, \psi_t^{T}, \psi_{2t}^{T}, \ldots, \psi_{dt}^{T})^{T} \) is the (extended) output vector of
the filter, and \( \psi_t^T := [\psi_{1t} \psi_{2t} \ldots \psi_{dt}]^T \) \( \frac{\partial \mathcal{E}_t'(\theta)}{\partial \theta} \).

Furthermore let

\[
\begin{bmatrix}
A - BC & 0 & \ldots & 0 \\
\frac{3(A-BC)}{\partial \theta^1} & A - BC & \ldots & \\
\frac{3(A-BC)}{\partial \theta^2} & \ddots & \ddots & \\
\vdots & & \ddots & 0 \\
3(A - BC) & \frac{\partial \mathcal{E}_t'(\theta)}{\partial \theta^d} & 0 & \ldots & 0 & A - BC
\end{bmatrix}
\]

(6.2.2-10) \( F(\theta) := \)

\[
\begin{bmatrix}
B \\
\frac{\partial B}{\partial \theta^1} \\
\frac{\partial B}{\partial \theta^2} \\
\vdots \\
\frac{\partial B}{\partial \theta^d}
\end{bmatrix}
\]

(6.2.2-11) \( G(\theta) := \)

\[
\begin{bmatrix}
I \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

and \( K := \)

\[
\begin{bmatrix}
I \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

(6.2.2-12) \( H(\theta) := \)

\[
\begin{bmatrix}
-C & 0 & 0 & \ldots & 0 \\
-C & 0 & 0 & \ldots & 0 \\
\frac{3C}{\partial \theta^1} & -C & 0 & \ddots & \\
\frac{3C}{\partial \theta^2} & 0 & -C & \ddots & 0 \\
\frac{3C}{\partial \theta^d} & 0 & \ldots & 0 & -C
\end{bmatrix}
\]

Note that \( F(\theta) \) is asymptotically stable, because \( A - BC = A(\theta) - B(\theta)C(\theta) \) is asymptotically stable, for all \( \theta \in \Theta \). The filter (6.2.2-9) in vector/matrix notation is:

\[
\begin{bmatrix}
\xi_{t+1} = F(\theta)\xi_t + G(\theta)y_t \\
\xi_t \\
\end{bmatrix}
\]

(6.2.2-13) \( \psi_1 = H(\theta)\xi_t + Ky_t \),

\[
\begin{bmatrix}
\psi_{1t} \\
\psi_{2t} \\
\vdots \\
\psi_{dt}
\end{bmatrix}
\]
6.2.2-14. Remark. The state space dimension of this filter is \( n(d+1) \). In [Gu-Me] it is shown that the McMillan degree of this filter is in general smaller than or equal to \( n(m+1) \). So if \( d > m \) the representation (6.2.2-13) is not minimal, and could in principle be replaced by a minimal representation. We will not go further into this here.

Now if we take (6.2.2-8) as it stands, then the filter (6.2.2-13) has to be run over and over again, once for each \( \theta_k \), to compute \( \varepsilon_t(\theta_k) \) and \( \psi_t(\theta_k) \), for a fixed value of \( t \), and with different input sequences \( \{y_t\}_{t=t_0}^T \) (preferably independent) in each run. However, we want a recursive procedure, that adapts the parameter estimate on-line. Therefore one proceeds, more or less heuristically, in a 'diagonal' fashion by putting \( t = k \) in the formulas. In this way the following algorithm is obtained called the recursive prediction error algorithm

\[
\begin{align*}
\varepsilon_{t+1} &= F(\theta_t)\varepsilon_t + G(\theta_t)y_t, \\
\phi_t &= H(\theta_t)\varepsilon_t + Ky_t, \\
\vdots \\
\phi_{T_t} &= W(\theta_t)\varepsilon_t + Ky_t, \\
\bar{\theta}_{t+1} &= \bar{\theta}_t - a_t R_t(\bar{\theta}_t)^{-1} T_t^T e_t, \\
\tilde{\theta}_{t+1} &= \tilde{\theta}_t + a_t R_t(\tilde{\theta}_t)^{-1} T_t^T e_t,
\end{align*}
\]

with unspecified initial conditions on \( \varepsilon_{t_0} \) and \( \theta_{t_0} \). Note that we have replaced \( \varepsilon_t \) in (6.2.2.13) by \( \varepsilon_t \) in (6.2.2.5) to make absolutely clear that it is not a prediction error under the hypothesis of some parameter point, but just an auxiliary quantity in the algorithm.

6.2.2-16. Remarks. (i) Usually \( \{a_t\} \) is taken such that, besides the conditions mentioned before, it is square summable. The standard example for such a sequence is
\[ a_t = \frac{a}{(t-t_0) + \beta}, \quad \alpha > 0, \quad \beta > 0, \quad t \geq t_0. \]

If the contrary is not explicitly stated, we will assume that \( \{a_t\} \) satisfies this extra condition.

(iii) The coupling equation \( \theta_t = \hat{\theta}_t \) is introduced here explicitly for two reasons: (a) in the convergence analysis of the algorithm, it will be necessary to 'delete' the coupling equation and to investigate the resulting (data-dependent) map

\[ \{\theta_t\}_{t_0} \rightarrow \{\hat{\theta}_t\}_{t_0}. \]

(b) the coupling equation will be generalized (see section 6.2.9).

6.2.3. Differences between the new r.p.e. algorithm and the standard one

In this section we want to list the changes in the algorithm that will be made to obtain what we call a Riemannian gradient recursive prediction error algorithm that uses overlapping parametrizations, or, alternatively, a Riemannian gradient r.p.e. algorithm for manifolds of linear stochastic systems. In the introduction (section 6.2.1) we already mentioned some of the arguments and ideas that are used in the construction of this algorithm. The equations in the new algorithm deviate from the one presented in the previous section in several respects. Because our parameter space is a manifold, the parameter points will be described by local coordinates. In fact we need local coordinates of the state bundle, because we use the state space in our algorithm. Because of all this the following must be done.

(i) It must be described when and how the algorithm has to change from operating in one coordinate chart to another, and what the topological structure of the coordinate charts has to look like. This will be treated in section 6.2.4.

(ii) The algorithm equations have to be written down in local coordinates. This is treated in section 6.2.5.

(iii) Furthermore, because we work on a manifold, we have to use a more general definition of the concept 'gradient', namely the so-called Riemannian gradient. This is worked out in section 6.2.6.
(iv) If a coordinate change takes place, then a transformation of all relevant variables has to take place. The rules of transformation have to be described. This will be done in section 6.2.7.

(v) For several reasons we need to make sure that the stepsizes of the steps taken in the algorithm are bounded. In practice this will usually be the case, because of the 'physical bounds' of the problem under consideration. Therefore this is mainly a theoretical problem. On the other hand we want to stick to our assumptions and construct the algorithm such that the stepsizes are bounded indeed. We do not want to 'assume away' the problems by making alternative assumptions about the true model. Therefore we are led into an, alas (and hopefully only for the moment) somewhat uneven way to assure that the stepsizes are bounded. The reason for the way in which this is done is that this affects the probabilistic structure not too much. (Otherwise the proofs would become (even) more complicated). This will be treated in section 6.2.8. (The proof that the procedure presented indeed leads to a bounded stepsize will be given in section 6.4).

Having treated (i) - (v), we will be able to write down the complete set of equations of the new algorithm. This will be done in section 6.2.9.

6.2.4. The structure of the coordinate charts and the coordinate changes in the algorithm.

By definition, any manifold is covered by a set of coordinate charts. In our algorithm we shall make use of such a cover. However, we shall need a cover with a special structure. Making use of the compactness of the manifold \( M \) we will be able to show that there will always exist a cover with the required structure.

There are four conditions that we require for the cover \( \{ C_j \mid j \in J \} \) of the manifold \( M \). The first condition is that the cover is finite, i.e. \( |J| < \infty \). The second one is that it consists of coordinate charts of \( M \), i.e. there are smooth injective coordinate maps \( \phi_j : C_j \to \mathbb{R}^d \). The third one is that it consists of coordinate charts of the state bundle over \( M \), or, equivalently, of the corresponding principal bundle \( L \) over \( M \), cf. (6.2.1-3). This third condition means that for each chart \( C_j \) in the cover, there will exist a smooth, injective coordinate map \( \tilde{\phi}_j \) of the bundle \( L \).
\[ \tilde{\phi}_j : C_j \times G_{n_k}(R) \to L \]
\[ (\theta, T) \mapsto (\text{TA}(\theta, j)^{-1}, \text{TB}(\theta, j), C(\theta, j)^{-1}) \].

Taking \( T = I \), one obtains a smooth cross section
\[ C_j \to L \]
\[ \theta \mapsto (A(\theta, j), B(\theta, j), C(\theta, j)) \],

that will be useful. The fourth condition is somewhat more complicated. We first state a definition.

6.2.4-1. Definition. A set \( \{ (C_j^\prime, C_j) \mid j \in J \} \) will be called a nucleus-double-shell (n.d.s.) cover of the manifold \( M \) if

(i) \( C_j^\prime \) and \( C_j \) are open for all \( j \in J \),

(ii) \( \bigcup_{j \in J} C_j^\prime = M \), \( \bigcup_{j \in J} C_j = M \)

(iii) \( \overline{C_j^\prime} \subseteq C_j \subseteq \overline{C_j} \subseteq C_j \) for each \( j \in J \).

For each \( j \in J \), \( C_j^\prime \) is called the nucleus, \( C_j^\prime \setminus C_j \) is called the first or inner shell and \( C_j \setminus C_j^\prime \) is called the second or outer shell.

The fourth condition is that there exists an n.d.s.-cover \( \{ (C_j^\prime, C_j) \mid j \in J \} \). Note that because the cover \( \{ C_j \mid j \in J \} \) satisfies the previous three conditions, the same holds for the covers \( \{ C_j^\prime \mid j \in J \} \) and \( \{ C_j^\prime \mid j \in J \} \).

We will show that there always exists a cover which satisfies these four conditions. We need the following lemma. (This is standard topology and holds in fact for all paracompact manifolds).

6.2.4-2. Lemma. Let \( \{ C_j \mid j \in J \} \) be an open cover of coordinate charts of the compact manifold \( M \). Then there exists an open cover \( \{ C_j^\prime \mid j \in J \} \) of \( M \) with the property:

(6.2.4-3) \( \overline{C_j^\prime} \subseteq C_j \quad \forall j \in J \).
Proof. For each \( x \in M \) there exists a \( j \in J \) such that \( x \in C_j \) and there exists an open neighbourhood \( N(x) \) of \( x \), such that \( N(x) \subseteq C_j \). Of course \( \{N(x) | x \in M \} \) is an open cover of \( M \). Because \( M \) is compact there exists a finite subcover \( \{N(x_k) | k \in K \} \), \( |K| < \infty \). Let

\[
K_j = \{k | N(x_k) \subseteq C_j \},
\]

then

\[
K = \bigcup_{j \in J} K_j.
\]

Let

\[
C_j' = \bigcup_{k \in K_j} N(x_k), \text{ then } \bigcup_{j \in J} C_j' = \bigcup_{k \in K} N(x_k) = M.
\]

Because each \( K_j \) is finite, it follows that

\[
C_j' = \bigcup_{k \in K_j} N(x_k) \subseteq C_j.
\]

Q.E.D.

6.2.4-4. Corollary. Let \( \{C_j | j \in J \} \) be an open cover of coordinate charts of the compact manifold \( M \). Then there exists an n.d.s. cover \( \{(C_j, C_j', C_j') | j \in J \} \) of \( M \).

Proof. Apply the previous lemma twice.

6.2.4-5. Proposition. Let \( M \subseteq \mathbb{R}^m \) be a compact manifold. There exists a cover \( \{C_j | j \in J \} \) that satisfies the four condition mentioned before, namely: (i) \( |J| < \infty \); (ii) the \( C_j \) are coordinate charts of \( M \); (iii) the \( C_j \) are coordinate charts of the principal bundle \( L \) over \( M \); (iv) there exists an n.d.s. cover \( \{(C_j, C_j', C_j') | j \in J \} \).

Proof. Because \( M \) is a manifold there is an open cover of coordinate charts \( \{C_a \} \) of \( M \) and because \( L \) is a (principal) fibre bundle over \( M \), there exists an open cover of bundle-coordinate charts \( \{C_g \} \) of \( M \). Then the cover \( \{C_a \cap C_g \} \) of \( M \) clearly satisfies (ii) and (iii). Because \( M \) is compact there is a finite subcover which we denote by \( \{C_j | j \in J \}, |J| < \infty \). This cover satisfies (i), (ii),
and (iii). Apply corollary 6.2.4-4 to obtain a corresponding n.d.s.-cover 
\{(C_j',C_j,C_j')| j \in J\}. So (iv) is also satisfied.

Q.E.D.

6.2.4-6. Notation. (i) Let \( \phi_j : C_j \rightarrow \mathbb{R}^d \), \( j \in J \), denote coordinate mappings corresponding to the coordinate neighbourhoods. Because \( C_j'' \subseteq C_j' \subseteq C_j \), \( \phi_j \) is also defined on \( C_j'' \) and \( C_j' \).

(ii) For each point \( \theta \in M \), let

\[
(6.2.4-7) \quad J''(\theta) := \{ j \in J | \theta \in C_j'' \}
\]

and

\[
(6.2.4-8) \quad J'(\theta) := \{ j \in J | \theta \in C_j' \}.
\]

Clearly

\[
J''(\theta) \subseteq J'(\theta) \subseteq J.
\]

6.2.4-9. Prescription. Let \( \theta(r), r \in [a,b) \) be a continuous curve of \( M \). We assign a coordinate chart \( C_j' \) with index \( j = j(r) \) to each \( r \in [a,b) \). We prescribe \( j(r) \) to be piecewise constant and left continuous, and \( j(a) \in J'(\theta(a)) \). A change of coordinates takes place at \( r \in [a,b) \) if and only if \( \theta(r) \in \partial C_j \). If so, then \( j(r^+) := \lim_{\epsilon \to 0^+} j(r+\epsilon) \), has to be an element of \( J''(\theta(r)) \).

The prescription is such that a certain 'inner shell' has to be crossed completely between any two coordinate changes. It is clear that this implies that at least a certain fixed positive distance has to be covered between any two coordinate changes. One could call this a form of hysteresis. The procedure can be considered as a generalization of the procedure of [Cla]. For our results it is immaterial which nucleus \( C_j'' \) is chosen at a change of coordinates, provided \( \theta \in C_j'' \) holds. To finish this subsection we give a related proposition for later reference.

6.2.4-10. Proposition. There exists a finite cover \( \{U_j\} \) of \( M \) with the following property. Let \( \theta(r), r \in (a,b) \subseteq R \), be a continuous curve in \( M \).

Suppose we assign a coordinate chart \( C_j' \) with index \( j = j(r) \in J(\theta(r)) \) to each
r, such that the prescription (6.2.4-9) is satisfied. (I.e., j(r) is piecewise constant and changes at r₀ only if \( \partial(r) \in \partial C_j(r) \) and changes to
\[ i \in J'(\partial(r) \cap (a,b)). \]
Then, if \((c,d) \subseteq (a,b)\) is such that for some i,
\[ \{ \partial(r) | c < r < d \} \subseteq U_i, \]
i.e., \( \partial(r) \) remains within \( U_i \) for \( r \in (c,d) \), then at most one coordinate change occurs on \((c,d)\).

Proof. Let \( \partial \in M \). Consider the compact set
\[ \bigcup_{\theta \in \partial C_j} U \bigcup_{\theta \notin \partial C_j} U. \]
Clearly, this set does not contain \( \partial \), and therefore there exists an open, (connected) neighborhood \( U \) of \( \partial \) in the complement of this set. It follows that if, for some \( j \in J \), \( U \cap C_j'' \neq \emptyset \) then (using the connectedness of \( U \)) \( U \subseteq C_j'' \cup \theta \in \partial C_j'' \), so \( \theta \in C_j'' \). This implies \( \theta \notin \partial C_j' \) and so \( U \cap \partial C_j' = \emptyset \). The conclusion is that \( U \) has the following property
\[ (6.2.4-11) \quad \forall j \in J: (U \cap C_j'' \neq \emptyset \Rightarrow U \cap \partial C_j' = \emptyset). \]
(Note that \( \partial \) does not occur in this implication). The sets \( U = U(\partial), \partial \in M \) form a cover of \( M \). Because \( M \) is compact there exists a finite subcover that will be denoted by \( \{ U_i \}_{i \in I} \). Each \( U_i \) has the property (6.2.4-11) and therefore if \( \exists i \in I: \forall r \in (c,d): \partial(r) \in U_i \), then at most one change of coordinates can occur. Q.E.D.

6.2.5. On the use of local coordinates in the algorithm

First we have to introduce some notation. As described in (6.2.4-6), \( \phi_j: C_j \rightarrow \mathbb{R}^d, j \in J \), denotes the coordinate mapping of the coordinate neighbourhood \( C_j \). If the value of \( j \) is clear from the context, we will drop the lower index \( j \) and write \( \phi \). With some abuse of notation,
\[ \phi = (\phi_1, \phi_2, \ldots, \phi_d)^T \]
will not only denote the mapping, but also the local coordinates themselves. Furthermore, using a similar abuse of notation, we will denote the smooth section defined just before (6.2.4-1), in local coordinates by
\[ (6.2.5-1) \]
\[ \begin{cases} 
\phi_j(C_j) \subseteq \mathbb{R}^d + L, \\
\phi \mapsto (A(\phi, j), B(\phi, j), C(\phi, j)).
\end{cases} \]
Using this, one can take derivatives with respect to $\phi^1, \phi^2, \ldots, \phi^d$ and define $F(\phi, j), G(\phi, j), H(\phi, j)$ (and $K$) in complete analogy with the definitions (6.2.2-10,11,12). The only changes are that $\theta$ has to be replaced by $\phi(e^d)$ and that the index $j$ has to be added to the notation, to replace $F(\theta)$ by $F(\phi, j), A(\theta)$ by $A(\phi, j)$ etc. Corresponding to this, the algorithm state vector $\xi_t$ is replaced by its local coordinates version $\xi(t, j)$ (and $\nu_t$ by $\nu(t, j)$ and $e(t)$ by $e(t, j)$). In fact, $\xi(t, j)$ is the representation in local coordinates of an element of the tangent bundle $TE$ of the state bundle $E$ (in which, as is standard (cf. e.g. [Ko-N]) the state-vector space is identified with its corresponding part of the tangent space $TE$). This element will be denoted by $\xi_t \in TE$. This will play a role in the transformation formulas of a coordinate change, that will be treated in section 6.2.7.

Now consider the equations (6.2.2-15). To generalize them to the manifold case, we have to replace the parameter update equation, because the parameter update equation of (6.2.2-15) makes use of addition, which is possible because of the vector space structure of the parameter space there. Two possible solutions to this problem present themselves. (a) One is using the geodesics structure of the manifold (once the Riemannian metric is defined). This is worked out, along with other things, in [Hnz 85b]. The disadvantage of this approach is that it requires propagation of a differential equation in most cases, instead of a simple addition. So it is more complex and it may be computationally burdensome. (b) The other solution is simply to do the addition in the local coordinates. In that case the parameter update equation will be of the form

$$(6.2.5-2) \quad \hat{\phi}_j(\theta_{t+1}) = \hat{\phi}_j(\theta_t) \text{ - "proxy for the gradient"},$$

at least if the right-hand side is an element of $\phi_j(\tilde{E}_j)$. This equation will be worked out further in the next subsections.

6.2.5-3. Remark. In fact, by generalizing the coupling equation $\theta_t = \hat{\theta}_t$, to one of the form

$$(6.2.5-4) \quad \theta_t \in \mathbb{R}(\hat{\theta}_t, \delta_t),$$

for some sequence of positive numbers $\{\delta_t\}$, (which has to satisfy certain
conditions that will be specified in section 6.2.9), not only the possibility (b) will be captured by the algorithm, but - as it appears - also possibility (a), although this will not be shown here.

6.2.6. The Riemannian gradient

Because the parameter space in our new set-up is a manifold and not a Euclidean space, a more general definition of the concept of the gradient of a function will be used. For the gradient of a real valued differentiable function to be well-defined, one needs a Riemannian metric on the manifold (cf. e.g. [Ab-M], pp. 127-128, especially Def. 2.5.14). In chapter 5 several Riemannian metrics on $M^m$, were presented. By restriction to $\mathcal{M}^m$, (which is assumed to be a smooth embedding) one obtains a Riemannian metric on $M$. Which (smooth) Riemannian metric on $M$ is chosen is immaterial for the construction of the algorithm, as presented here, and for the theorems about the convergence behaviour of the algorithm, that will be presented in sections 6.3 - 6.9. With respect to the local coordinates of chart $C_j$, $j \in J$, the Riemannian metric tensor at a point $\theta \in C_j$ is a positive definite matrix that will be denoted by $R(\theta, j)$. The Riemannian gradient of a differentiable real-valued function $V$ on $M$ is an element of the tangent bundle of $M$, that is given in local coordinates by

\[(6.2.6-1) \quad R(\theta, j)^{-1} \frac{\partial V}{\partial \phi_j},\]

where $\frac{\partial V}{\partial \phi_j} = \left(\frac{\partial V}{\partial \phi_j^1}, \ldots, \frac{\partial V}{\partial \phi_j^d}\right)^T$ denotes the vector of partial derivatives of $V$ considered as a function of the local coordinates $(\phi_j^1, \ldots, \phi_j^d)^T \in \phi_j(C_j) \subset \mathbb{R}^d$.

It is remarkable that in the standard parameter update formula (see (6.2.2-15)), one already finds the expression $R^{-1} \times 'proxy for vector of partial derivatives of $V'. However, the meaning of the matrix $R$ in the standard case is not completely clear. In the present algorithm $R$ will be taken equal to the Riemannian metric matrix $R(\theta, j)$. In fact, one (and probably more) of the standard choices for $R$ in the literature can be interpreted as a Riemannian metric tensor asymptotically (cf. [Hnz 85a]).

6.2.7. The transformation rules for a coordinate change

A change of coordinates in fact means a change of coordinates of the state
bundle, or equivalently, of the corresponding principal fibre bundle. Therefore such a coordinate change involves the following:

(a) A change in the local coordinates representation of the parameter point \( \theta \in M \). In section 6.2.4 it is described which changes of local coordinates are allowed in the algorithm. Suppose the change that takes place is from \( \mathcal{C}_j^i \) (so \( \theta \in \mathcal{AC}_j^i \)) to \( \mathcal{C}_j^{n_i} \) (so \( \theta \in \mathcal{AC}_j^{n_i} \)), \( j,i \in J \). Then, if \( x = \psi_j(\theta) \in \mathbb{R}^d \) denotes the old local coordinate vector, then

\[
(6.2.7-1) \quad y = \psi_i \circ \psi_j^{-1}(x)
\]

is the new one, representing \( \theta \) in the new local coordinates.

To give an impression of how \( \psi_i \circ \psi_j^{-1} \) may look, consider the space \( W_{m,m}^{n,n} \). Of course \( M \neq W_{m,m}^{n,n} \), because \( W_{m,m}^{n,n} \) is not compact. This is only meant as an impression, nothing more. In sections 4.4 and 4.5 local coordinates are constructed for this space, using a set of local, continuous canonical forms \( \{ c \} \) a nice choice. For a coordinate change corresponding to a change from nice selection \( \alpha \) to nice selection \( \beta \) the equivalent of the mapping in (6.2.7-1) is the composition of mappings

\[
(6.2.7-2)
\]

\[
z \in V_{a}^{m} \rightarrow (A(z)B(z)C(z)) \in W_{a}^{m} \rightarrow (Q^{-1}A(z)0,0^{-1}B(z),C(z)0) \in W_{b}^{m} \rightarrow
\]

\[
\left[ (RQ^{-1}A(z)Q,0^{-1}B(z)) \right]_{s(b,j)} \rightarrow \left[ C_{(z)0} \right]_{s(b,j)} \rightarrow V_{b}^{m}.
\]

where \( Q = R(A(z),B(z)) \) is invertible, because \( z \in \mathcal{V}_{\beta}^{m} \) (if not, this coordinate change would of course be impossible). How (6.2.7-1) will look like in detail in our case of the (compact) manifold \( M \) will depend on the specific choice of \( M \) and on the choice of the coordinate neighbourhoods.

(b) A change of the local section of the principal fibre bundle \( L \) of the state bundle \( \pi: E \rightarrow M \). In other words: instead of the mapping

\[
(6.2.7-3) \quad \begin{cases} C_{j} \rightarrow L, \\ \phi \rightarrow (A(\phi,j),B(\phi,j),C(\phi,j)) \end{cases}
\]
the mapping

\[(6.2.7-4) \left\{ \begin{array}{l}
C_1 \rightarrow L,
\phi_j \mapsto (A(\phi,1), B(\phi,1), C(\phi,1))
\end{array} \right.\]

is going to be used.

From (a) we can see that (a) and (b) will be intimately related. In fact, often the local coordinates \(\phi_j\) for the parameter point \(\theta \in \phi_j(C_1)\) are certain specified entries of the matrix triple \((A(\theta,j), B(\theta,j), C(\theta,j))\). This is the case for the local coordinates of \(M_{n,m}^m\) as specified in chapter 4 (and in (a)). However, this is not necessarily so, and therefore we make this clear distinction.

The effect of the change from \((6.2.7-3)\) to \((6.2.7-4)\) can be described by a nonsingular matrix \(T(\theta)\), which describes the state space basis change involved at \(\theta\), as follows

\[(6.2.7-4)\]

\[(A(\theta,1), B(\theta,1), C(\theta,1)) = (T(\theta)A(\theta,j)T(\theta)^{-1}, T(\theta)B(\theta,j), C(\theta,j)T(\theta)^{-1}).\]

To derive the transformation rule for the local coordinates representation of \(\xi \in T_E\) we proceed as follows. Consider a smooth local section

\[N(\theta_0) \subseteq M \times E, \theta_0 \mapsto x(\theta).\] Its derivative with respect to \(\theta\) at \(\theta_0\) is an element of \(T_E\). On the other hand each element of \(T_E\) can be represented by such a section, in a coordinate independent way. In local coordinates (of the state bundle) \(x(\theta)\) can be represented by \(x(\phi_j, j) \in \mathbb{R}^m, \ j \in J(\theta)\), (formally together with \(\phi_j\) itself). Its derivative with respect to \(\phi_j = (\phi_j^1, \ldots, \phi_j^d)^T\) can be computed and \(x(\phi_j, j)\)

\[\frac{\partial x(\phi_j, j)}{\partial \phi_j^a}\]

together represent an element in \(T_E\). The same element of \(T_E\) is represented with respect to the coordinates corresponding to \(C_1\)

by \(\phi_1 x(\phi_1, 1)\) and \(\frac{\partial x(\phi_1, 1)}{\partial \phi_1^a}\), where

\[(6.2.7-5) \ x(\phi_1, 1) = T(\theta)x(\phi_j, j),\] and so
\[
\begin{align*}
(6.2.7-6) \quad \frac{3x(\phi_i,1)}{\partial \phi_i} &= \frac{3}{k} [T(\theta)] x(\phi_j,j) + T(\theta) \cdot \frac{3x(\phi_i,j)}{\partial \phi_i} \\
&= \frac{2}{k} [T(\theta)] x(\phi_j,j) + \frac{3x(\phi_i,j)}{\partial \phi_i} \frac{3x(\phi_i,j)}{\partial \phi_i}, \quad k = 1, 2, 3, \ldots, d.
\end{align*}
\]

Because \( \phi_i, \phi_j \) and \( T(\theta) \) are known, the Jacobian \( \frac{3x}{\partial \phi_i} \) and the derivatives \( \frac{3x(\phi_i,j)}{\partial \phi_i} \), \( k = 1, \ldots, d \), can be computed. As an alternative to direct computations, one can also compute \( \frac{3T(\theta)}{\partial \phi_i} \), \( k = 1, \ldots, d \), using Lyapunov equations and the Jacobian. This is treated in appendix 6A. There also a method is given to compute the Jacobian, in case one does not know the transformation mapping \( \phi_i \circ \phi_j^{-1} \) explicitly as a function of the parameter.

From (6.2.7-5) and (6.2.7-6) it follows that the vector
\[
\left( x(\phi_i,1)^T, \frac{3x(\phi_i,1)}{\partial \phi_i}, \ldots, \frac{3x(\phi_i,d)}{\partial \phi_i} \right)^T \quad \text{at} \quad \phi_i = \phi_i(\theta),
\]

is a linear transformation of the vector
\[
\left( x(\phi_j,j)^T, \frac{3x(\phi_j,j)}{\partial \phi_j}, \ldots, \frac{3x(\phi_j,d)}{\partial \phi_j} \right)^T \quad \text{and} \quad \phi(\theta).
\]

Let the matrix of this linear transformation be denoted by \( S(\theta;i,j) \). This matrix is completely specified by (6.2.7-5) and (6.2.7-6). Then the transformation rule for \( \xi \in \text{TE} \) is
\[
(6.2.7-7) \quad \xi(t,i) = S(\theta;i,j)\xi(t,j).
\]

6.2.8. Bounding the stepsize

As motivated in section 6.2.3 (v) we will construct the algorithm such that its stepsize in the parameter space are uniformly bounded. The boundedness of the stepsize will be crucial in our convergence proof for the algorithm. The uniform boundedness is obtained in a perhaps somewhat unelegant way, but it is done such that it does not complicate the probabilistic structure of our algorithm too much.
There are two parts to this so-called boundary provision.

(a) Let $\theta \in \Theta$. For each $j$ such that $\theta \in C_j$, i.e., $j \in J(\theta)$, consider the spectrum of $A(\theta,j)-B(\theta,j)C(\theta,j)$. It is obvious to show that this spectrum is the same for all $j \in J(\theta)$. Indeed this follows directly from the state-space basis change transformation rule (6.2.7-4). Let us denote it by $\sigma(\theta)$, and let $\lambda_M(\theta)$ be an element of $\sigma(\theta)$ with the maximum modulus.

$$\lambda_M(\theta) = \max_{\lambda \in \sigma(\theta)} |\lambda| < 1.$$  

Consider $|\lambda_M(\theta)|$ as a function of $\theta \in \Theta$.

This is a continuous function of $\theta$. This can be shown in three steps.

(i) The coefficients of the characteristic polynomial depend continuously on $\theta$.

(ii) the set $\sigma(\theta)$ of roots of the polynomial depends continuously on its coefficients (cf. [Mar], p.4) and

(iii) $|\lambda_M(\theta)|$ depends continuously on the set $\sigma(\theta)$.

(The details are left to the reader).

Because $\Theta$ is compact, one can define

$$\lambda_0 := \max_{\theta \in \Theta} |\lambda_M(\theta)| \in (0,1).$$

Choose $\lambda_1 \in (\lambda_0,1)$, and define recursively the nonnegative variables $v_t$ as follows

$$\begin{align*}
v_{t-1} &:= 0, \\
v_t &= \lambda_1 v_{t-1} + \|y_t\|, \quad t = t_0, t_0 + 1, \ldots
\end{align*}$$

Note that $v_t$ depends only on the observations $y_t$ and on the choice of $\lambda_1$; it does not depend on any quantities computed in the algorithm.

Next choose a ('large') constant $K' > 0$ and define the function

$$\alpha : [0,\infty) \to [0,1], \alpha(v_t) = (v_t 

\begin{cases}
1 & \text{if } v_t \leq K', \\
0 & \text{if } v_t > K'.
\end{cases}$$

\begin{align*}
\alpha &:= 1_{v_t \leq K'}, \\
\alpha &:= 0_{v_t > K'}
\end{align*}$$
The idea behind this is to measure with \( g_1 \) whether there are outliers in the observations. If \( v_t > K' \) then the update equation is 'turned off', and the parameter estimate is kept constant in the algorithm, because otherwise these outliers could destabilize the algorithm. This is comparable to e.g. the method in [Ku-C1], p. 94 (2.6.8), (2.6.9). They have a simpler scheme which we found, however, harder to analyze than ours, because of the dependency on the current parameter estimate.

(b) Let \( \{K_t\}_t \) be a sequence of nonnegative numbers such that

(i) \( \lim_{t \to +\infty} K_t = 0 \) and

(ii) \( \lim_{t \to +\infty} a_t = 0. \)

\( \{a_t\} \) as in (6.2.2-8) and (6.2.2-16)).

Such a sequence \( \{K_t\}_{t=0}^\infty \) certainly exists, because \( \lim_{t \to +\infty} a_t = 0. \) For example, one can take \( K_t := a_t^{-1} \) if \( a_t > 0 \) and \( K_t := t \) if \( a_t = 0. \)

If the algorithm at time \( t \) operates in the \( j \)-th coordinate chart, the parameter update equation will be of the following form, if the right-hand side is an element of \( \phi_j(c_j) \):

\[
(6.2.8-5) \quad \phi_j(\hat{\theta}_{t+1}) = \phi_j(\hat{\theta}_t) + a_t g_1(v_t)g_2(t)R^{-1}h,
\]

with \( R = R(\theta_{t+1}) \) and \( h = \Psi(t,j)^T e(t,j) \). The function \( g_2(t) \) will be defined as follows:

\[
(6.2.8-6) \quad g_2(t) = \begin{cases} 
1 & \text{if } \|\Psi(t,j)^T e(t,j)\|_R = g_1(v_t)h^T R^{-1} h \leq K_t, \\
0 & \text{if } \|\Psi(t,j)^T e(t,j)\|_R = g_1(v_t)h^T R^{-1} h > K_t.
\end{cases}
\]

Here \( \|x\|_R := x^T R x \) denotes the Riemannian length of \( x \), considered as an element of the tangent space \( T\theta \) at \( \theta \).

Because \( K_t \to 0 \) for \( t \to +\infty \), the effect of \( g_2 \) is vanishing asymptotically.

In fact it is only needed to prevent that the algorithm is destabilized if the \( a_t \) are too big compared to the sizes of the coordinate charts. Because \( \lim_{t \to +\infty} a_t = 0 \), \( g_2 \) plays a role only during a finite time. For more details we
refer to the proof of the uniform boundedness of the stepsize in section 6.4.

6.2.9. The complete set of update equations of the algorithm

Two more things have to be settled before we can write down the complete set of update equations of the algorithm.

(a) If the right-hand side of the parameter update equation (6.2.8-5) lies outside of \( \phi_j(C_j^t) \), then \( \hat{\theta}_{t+1} \) will be defined to be the point where 'the boundary of \( \phi_j(C_j^t) \) is hit' using linear interpolation in the coordinate chart. Define \( \lambda_t \in (0,1) \) by

\[
(6.2.9-1) \quad \lambda_t := \min\{1, \phi_j(\hat{\theta}_t) + \lambda_a_1 g_1(v_t)g_2(t)R^{-1}h\phi_j(3C_j^t)\},
\]

and let \( \hat{\theta}_{t+1} \) be defined by

\[
(6.2.9-2) \quad \phi_j(\hat{\theta}_{t+1}) = \phi_j(\theta_t) + \lambda_t a_1 g_1(v_t)g_2(t)R^{-1}h.
\]

Then \( \hat{\theta}_{t+1} \) is well-defined in all cases. Of course if \( \lambda_t < 1 \) then \( \hat{\theta}_{t+1} \in 3C_j^t \).

If \( \hat{\theta}_{t+1} \in 3C_j^t \), then a change of coordinates will take place. Note that \( \lambda_t \neq 0 \)
because \( \phi_j(\hat{\theta}_t) \in \phi_j(3C_j^t) \) in (6.2.9-1). The reason is that if

\( \phi_j(\theta_t) \in \phi_j(3C_j^t) \), then a coordinate change will take place immediately,

before the parameter update equation is formed.

(b) In (6.2.2-15) the so-called coupling equation \( \theta_t = \hat{\theta}_t \) was introduced explicitly. This equation will be generalized as follows. Let \( \{\delta_t\}_{t=0}^{\infty} \)

be a sequence of nonnegative numbers that converges to zero and let

\[
(6.2.9-3) \quad \delta_t := a_1 \delta_{t-1}, \quad t = t_0, t_0+1, t_0+2, \ldots.
\]

Instead of requiring \( \theta_t \) to be equal to \( \hat{\theta}_t \), we allow \( \theta_t \) to be chosen arbitrarily from the (nonempty) intersection of

(i) a closed ball \( B(\hat{\theta}_t, \delta_t) \) with centre \( \hat{\theta}_t \) and radius \( \delta_t \) and

(ii) the set \( C_j^t \), if \( \hat{\theta}_t \in C_j^t \), and \( C_j^t \) is the coordinate chart in which the algorithm operates at time \( t \). One needs \( \hat{\theta}_t \in C_j^t \), because one must be able to represent \( \theta_t \) in the local coordinates \( \phi_j = \phi_j(\theta_t) \). So the coupling equation is:
(6.2.9-4) \( \theta_t \in \overline{B}(\hat{\theta}_t, \delta_t) \cap C_j \).

Note that, because \( \hat{\theta}_t \in C_j' \), \( C_j' \) compact and \( C_j \) open, \( C_j' \subseteq C_j \), it follows that for \( \delta_t \) small enough

(6.2.9-5) \( \overline{B}(\hat{\theta}_t, \delta_t) \subseteq C_j \).

Because \( \delta_t \) converges to zero, it follows that for the asymptotic analysis one can assume without loss of generality that the coupling equation is

(6.2.9-6) \( \theta_t \in \overline{B}(\hat{\theta}_t, \delta_t) \).

For the metric that goes into the definition of \( \mathcal{H}(\hat{\theta}_t, \delta_t) \) one can choose (i) the inner metric of \( M \), (ii) the Euclidean metric in the local coordinate chart in which the algorithm operates at time \( t \). Those two metrics are, of course locally equivalent around \( \hat{\theta}_t \). This will be shown in section 6.3.4. It can be shown that if option (ii) is chosen, there exists a data-independent sequence \( \{ \delta'_t \} \), (which satisfies the conditions), such that \( \{ \delta'_t, \hat{\theta}_t \} \) satisfies (6.2.9-4) with option (i), too. Therefore without loss of generality in the analysis we will work with option (i), unless otherwise is stated.

Note that one is allowed to take \( \delta'_t = 0 \) for all \( t \geq t_o \). Then the coupling equation reduces to \( \hat{\theta}_t = \hat{\theta}_t \). As an example of how the more general coupling 'equation' can be used, one can take the following parameter update scheme:

(6.2.9-7) \[
\theta_{t+1} = \begin{cases} 
\theta_t & \text{if } \theta_t \in \overline{B}(\hat{\theta}_{t+1}, \delta_{t+1}) \cap C'_j \\
\theta_{t+1} & \text{if } \theta_t \notin \overline{B}(\hat{\theta}_{t+1}, \delta_{t+1}) \cap C'_j
\end{cases}
\]

where \( j \) now indicates the coordinate chart in which the algorithm operates at time \( t+1 \). This has the advantage that if the parameter estimate \( \hat{\theta}_{t+1} \) is close enough to \( \hat{\theta}_t \), then the parameter \( \theta \) in the computations of the algorithm can be kept constant.

Let us now summarize the update equations of the algorithm

(6.2.9-8) \( \xi(t+1, j) = F(\phi, j)\xi(t, j) + G(\phi, j)y_t \);
(6.2.9-9) \[
\begin{bmatrix}
e(t,j) \\
\gamma_1(t,j) \\
\vdots \\
\gamma_d(t,j)
\end{bmatrix} = H(\phi, j) \xi(t,j) + K y_t,
\]

where

(6.2.9-10) \( \phi = \phi^t(0) \); 

(6.2.9-11) \( \phi^t(0)^{t+1} = \phi^t(0) + \lambda^t a_t r_1 g_2^{t-1} h, \)

with \( \lambda_t \) as in (6.2.9-1), \( r_1 \) as in (6.2.8-4), \( g_2 \) as in (6.2.8-6), and \( R \) and \( h \) as in (6.2.8-5);

(6.2.9-12) \( \theta_{t+1} \in B(\theta_{t+1}, \delta_{t+1}), t \geq t_0 - 1, (\text{cf. } (6.2.9-4)). \)

If \( \theta_{t+1} \in S^t \), then a coordinate change has to take place to a chart \( i \in J \) with the property

(6.2.9-13) \( \hat{\theta}_{t+1} \in C^i_t \), i.e. \( i \in J(\hat{\theta}_{t+1}) \);

\( \phi^t(0)^{t+1} \) has to be replaced by

(6.2.9-14) \( \phi_i(\theta_{t+1}) = \phi_i \circ \phi^t_j(\theta_{t+1}), \)

and \( \xi(t+1,j) \) has to be replaced by

(6.2.9-15) \( \xi(t+1,i) = S(\theta_{t+1}; i, j) \xi(t+1,j) \).

As soon as \( y_t \) is known, the calculation (6.2.9-8) - (6.2.9-11) can be made, and the choice (6.2.9-12) can be made. If necessary, a change of coordinates can be made. If all this is done, the algorithm can wait till \( y_{t+1} \) becomes available and do all this again, but now with \( t \) replaced by \( t+1 \) (and if a coordinate change from \( j \) to \( i \) has taken place, with \( j \) replaced by \( i \)). This specifies the algorithm except for the initial conditions. Any choice of \( j \in J, \theta_{t_0} \in C^j, \theta_{t_0} \in B(\theta_{t_0}, \delta_{t_0}) \cap C^1, \xi(t_0, j) \in \mathbb{R}^{(1+d)} \) will do in principle. As a standard choice for \( \xi(t_0, j) \) one can take \( \xi(t_0, j) = 0 \), and this
is the case that will be analyzed. If $\xi(t, j) \neq 0$, then the definition of $v_t$ has to be changed somewhat. We will not go into this.

This finishes the description of the algorithm, or better, the class of algorithms, because many choices can be made within the class of algorithms: the manifold $M$, the Riemannian metric, the precise decision rule for a coordinate change (as long as it meets our requirements) and the precise coupling equation (as long as it meets our requirements) etc.

6.3. A refinement of the cover of the manifold

6.3.1. Introduction

'I would rather discover one proof, then to earn the throne of Persia' - Democritos

In the following sections, 6.3-6.10, the asymptotic behaviour of the algorithm presented in section 6.1 and section 6.2 will be analyzed. In the analysis we will make use of a refinement of the n.d.s. cover $\{(C_j, C'_j, C_j')\}_{j \in J}$ of $M$. The refinement is introduced only for the sake of the analysis, the algorithm will not be changed. In section 6.3.2 the structure of the refinement, that is needed for our purposes, is described. By choosing a refinement we can make sure that certain properties hold within each coordinate chart. This is applied in section 6.3.3 and section 6.3.4. In section 6.3.3 it is applied to establish asymptotic stability of arbitrary products of dynamic $F$-matrices occurring in the coordinate neighbourhood involved. In section 6.3.4 it is applied to establish equivalence in each coordinate neighbourhood between the inner metric of the Riemannian manifold and the metric defined by using the Euclidean metric of the local coordinates.
6.3.2. The structure of a refinement of the n.d.s. cover of the manifold

The refinements we will consider will again have a "nucleus-double-shell" (n.d.s.) structure just as the cover \( \{(C_j', C_j') | j \in J \} \). Consider a finite open cover \( \{E_i | i \in I \} \) of \( M \), \( |I| < \infty \).

Let

\[
(6.3.2-1) \quad E_{ij} := E_i \cap C_j \text{ for all } i \in I, j \in J.
\]

Then \( \{E_{ij} \} \) forms a finite open cover of \( M \) and for fixed \( j \), \( \{E_{ij} | i \in I \} \) forms a finite open cover of \( C_j \), and therefore \( C_j' \subseteq \bigcup_{i \in I} E_{ij} \).

6.3.2-2. Proposition. Let \( \{E_{ij} | i \in I, j \in J \} \) be any finite open cover of \( M \) with the property \( \bigcup_{i \in I} E_{ij} \subseteq C_j' \). Then there exists an n.d.s. - cover

\[
(6.3.2-3) \quad \bigcup_{i \in I} E''_{ij} = C_j', \quad \forall j \in J.
\]

6.3.2-3. Remark. This is also standard topology; note that some of the \( E''_{ij} \) and \( E^*_{ij} \) may be empty sets. Let

\[
(6.3.2-4) \quad I(j) := \{i \in I | E''_{ij} \neq \emptyset \}.
\]

Then \( \{(E''_{ij}, E^*_{ij}) | i \in I(j), j \in J \} \) will again be an n.d.s. cover of \( M \!'

Proof of proposition (6.3.2-2) (sketch).

This is analogous to the proofs of lemma 6.2.4-2 and corollary 6.2.4-4. First one shows that if \( \{E_{ij} | i \in I \} \) covers \( C_j' \) in the sense that

\[
(6.3.2-5) \quad C_j' \subseteq \bigcup_{i \in I} E_{ij}
\]

then there exists a nucleus-shell (n.s.) cover \( \{(E^*_{ij}, E''_{ij}) | i \in I \} \) of \( C_j' \), analogous to lemma (6.2.4-2) and its proof, using the fact that \( C_j' \) is compact.

This means that \( E_{ij} \subseteq E^*_{ij}, \forall i \in I \) and that

\[
(6.3.2-6) \quad C_j' \subseteq \bigcup_{i \in I} E_{ij}.
\]
Then by applying the same argument to \( \{E_{1j}'\} \), (compare corollary (6.2.4-4)) one concludes that for each \( j \in J \) there exists also an open n.d.s. cover 
\[ \{ (E_{1j}'', E_{1j}', E_{1j}) \mid i \in I \} \]

such that

\[(6.3.2-7) \quad C_j'' \subseteq \bigcup_{i \in I} E_{1j}'.\]

Now let \( E_{1j}'' := E_{1j}'' \cap C_j'. \) Then \( E_{1j}'' \subseteq E_{1j}' \) clearly. It follows furthermore that

\[(6.3.2-8) \quad \bigcup_{i \in I} E_{1j}'' = C_j'.\]

and

\[(6.3.2-9) \quad \bigcup_{j \in J} \bigcup_{i \in I} E_{1j}'' = M.\]

and so \( \{ (E_{1j}'', E_{1j}', E_{1j}) \mid i \in I(\cdot), j \in J \} \) is indeed an open n.d.s. cover of \( M \) with the required property. Q.E.D.

Now let for all \( i \in I(\cdot), \) all \( j \in J \)

\[(6.3.2-10) \quad D_{1j}'' = E_{1j}'', D_{1j}' = E_{1j}' \cap C_j', D_{1j} = E_{1j} \cap C_j'.\]

6.3.2-11. **Proposition.** (a) \( \{ (D_{1j}'', D_{1j}', D_{1j}) \mid i \in I(\cdot), j \in J \} \) has the following properties

1. \( D_{1j}'', D_{1j}', D_{1j} \) are open sets for each \( i, j, \)
2. \( D_{1j}'' \subseteq D_{1j}' \subseteq D_{1j} \) holds for all \( i, j, \)
3. \( \bigcup_{i \in I(\cdot)} D_{1j}' = C_j' \) for all \( j \in J \) and
4. for each \( i \in I(\cdot), j \in J \) there exist open sets \( N_{1j}', N_{1j} \subseteq C_j', \) such that

\[ D_{1j}'' \subseteq N_{1j}' \quad \text{and} \quad N_{1j}' \cap C_j' \subseteq D_{1j}' \]

and

\[ D_{1j}'' \subseteq N_{1j} \quad \text{and} \quad N_{1j} \cap C_j' \subseteq D_{1j}.\]

(6.3.2-12)
(b) If \( \{D^n_{i,j}, D'_{i,j}, D''_{i,j}\}_{i \in I(j), j \in J} \) has the properties (i), (ii), (iii), (iv) mentioned in (a), then there exists an n.d.s. cover
\( \{E^n_{i,j}, E'_i, E''_{i,j}\}_{i \in I(j), j \in J} \) of \( M \), such that for all \( i \in I(j), j \in J \):

1. \( E_{ij} \subseteq C_j \),
2. \( E''_{ij} = D''_{ij}; E'_i = D'_i \cap C_i; E_{ij} = D_{ij} \cap C_j \).

Before we go to the proof of this proposition we give a definition and some remarks.

6.3.2-13. Definition. \( \{D^n_{i,j}, D'_{i,j}, D''_{i,j}\}_{i \in I(j), j \in J} \) having properties (i), (ii), (iii), (iv) of the previous proposition, will be called a second order n.d.s. cover.

6.3.2-14. Remarks (i) It is called a second order n.d.s. cover because it is a refinement of the n.d.s. cover \( \{C_i, C'_i, C_j\}_{j \in J} \) with respect to the n.d.s. cover \( \{E^n_{i,j}, E'_i, E''_{i,j}\}_{i \in I(j), j \in J} \).
(ii) \( \{D^n_{i,j}, D'_{i,j}, D''_{i,j}\}_{i \in I(j), j \in J} \) is itself not an n.d.s. cover because \( \bar{D}''_{ij} \subseteq D'_{ij} \) etc. does not necessarily hold. On the other hand for all \( i \in I(j), j \in J \):

\( \bar{D}''_{ij} \cap C_j \subseteq D'_{ij}; \bar{D}''_{ij} \cap C_j \subseteq D_{ij} \).

However, these inclusion are not sufficient for a second order n.d.s. cover.

The reason behind it is that (6.3.2-15) does not exclude shells with 'thickness' vanishing at certain points, while definition (6.3.2-13) does exclude this, as will be shown after the proof of the proposition.

Proof of proposition 6.3.2-11.

(a) (i), (ii), (iii) are trivial; (iv) can be shown simply by taking \( N'_{ij} := E'_{ij} \) and \( N''_{ij} := E''_{ij} \), for all \( i \in I(j), j \in J \).

(b) Let \( C'_j \) be an open set such that \( \bar{C}''_j \subseteq \bar{C}'_j \subseteq C'_{ij} \subseteq C_j \). Let \( N'_{ij} \) and \( N''_{ij} \) be as in (6.3.2-12). Because \( \bar{D}''_{ij} \subseteq N''_{ij} \), there exists an open set \( N''_{ij} \) such that
\[ D_{ij}^* \subseteq N_{ij} \subseteq D_{ij} \subseteq N_{ij}^*. \] Define (for all \( i \in \mathbf{I}(j), \ j \in \mathbf{J} \))

\[ E_{ij}^* := D_{ij}, \]
\[ E_{ij} := D_{ij} \cup D_{ij} \]
\[ E_{ij} := D_{ij} \cup D_{ij} \cup D_{ij} \]

Then

\[ \overline{E}_{ij} = \overline{D}_{ij} \subseteq N_{ij} \subseteq E_{ij} \]
\[ \overline{E}_{ij} = \overline{D}_{ij} \cup D_{ij} \subseteq N_{ij} \cup N_{ij} \subseteq E_{ij} \]

Furthermore, one has:

(6.3.2-16) \( E_{ij}^* \cap C_j^* = D_{ij} \)

because \( E_{ij} = D_{ij} \subseteq C_j \);  

(6.3.2-17) \( E_{ij}^* \cap C_j^* = D_{ij} \)

because \( E_{ij} = D_{ij} \cup D_{ij} \) and \( N_{ij} \cap C_j^* \subseteq N_{ij} \cap C_j^* \); 

(6.3.2-18) \( E_{ij} \cap C_j^* = D_{ij} \)

because \( E_{ij} = N_{ij} \cup D_{ij} \cup D_{ij} \) and \( N_{ij} \cap C_j^* \subseteq D_{ij} \subseteq N_{ij} \).

\[ N_{ij} \cap C_j \subseteq D_{ij} \subseteq C_j. \]

Q.E.D.

For a cover with this structure we have the following important property.

6.3.2-19. **Proposition.** There exists a constant \( c > 0 \) such that for all \( j \in \mathbf{J} \) and for all \( i \in \mathbf{I}(j) \), one has

(6.3.2-20) \( d(D_{ij}^*, \overline{D}_{ij} \cap C_j^*) \geq c, \)
where the distance \(d\) between sets \(A\) and \(B\) is defined as
\[
d(A,B) = \inf_{a \in A, b \in B} d(a, b) = \infty \quad \text{if} \quad A = \emptyset \quad \text{or} \quad B = \emptyset.
\]

Proof. Because enlarging the sets cannot increase the distance, one has
\[
d'_{ij} \geq d(\overline{E''_{ij}}, \overline{E'_{ij}}).
\]
(the \((E''_{ij}, E'_{ij}, \overline{E_{ij}})\) are as in proposition 6.3.2-11(b)). Here it is used that
\[
D''_{ij} = E''_{ij} \quad \text{and} \quad D'_{ij} = C'_{ij} \cap E'_{ij}
\]
which implies \(\overline{D''_{ij}} \subseteq \overline{C'_{ij}} \cup \overline{E'_{ij}}\), and so
\[
\overline{D''_{ij}} \setminus \overline{C'_{ij}} \subseteq \overline{E''_{ij}} \setminus \overline{E'_{ij}} \subseteq \overline{E'_{ij}}.
\]
Because \(E''_{ij} \subseteq E'_{ij}, \overline{E'_{ij}}\) open, it follows that \(E''_{ij} \cap \overline{E'_{ij}} = \emptyset\). Furthermore \(E''_{ij}\)
and \(\overline{E'_{ij}}\) are closed and therefore compact, so \(d(\overline{E''_{ij}}, \overline{E'_{ij}})\) is positive, say:
\[
d(\overline{E''_{ij}}, \overline{E'_{ij}}) = d'_{ij} > 0. \quad \text{Now let} \quad c = \min_{i \in I, j \in J} d'_{ij} > 0. \quad \text{Q.E.D.}
\]

For the sake of the analysis of the algorithm we will associate not only a coordinate chart \(C_{ij}\), but also a subchart \(D_{ij} \subseteq C_{ij}\), with each stage of the algorithm. The rules of changing from \(D_{ij}\) to \(D_{kl}\) are a generalization of those of changing from \(C_{ij}\) to \(C_{kl}\). Furthermore, for the sake of the analysis, we will generalize the prescription somewhat, such that curves that enter \(D_{ij}\) but do not leave \(D_{ij}\) are allowed to go without a change of coordinates. One could generalize the prescription (6.2.4-9) likewise, in which case the algorithm itself would be generalized. However, for the sake of definiteness of the rule for changing coordinate charts in the algorithm, we choose not to do so.

6.3.2-21. Prescription. (Compare (6.2.4-9))
Let \(\theta(t), r \in [a, b]\) be a continuous curve of \(M\). We assign a coordinate chart \(n'_{ij}\) with index pair \((i,j) = (i(r), j(r))\), to each \(r \in [a, b]\). We prescribe \((i(r), j(r))\) to be piecewise constant and left continuous and \((i(r), j(r))\) has to be such that
∀r ∈ (a, b): θ(r) ∈ D_i(r), j(r).

A change of coordinates is allowed to take place at r ∈ (a, b) only if

θ(r) ∈ \mathcal{A}^{D_i(r), j(r)}.

If it takes place then (i(r^+), j(r^+)) := \lim_{ε \to 0}(i(r+ε), j(r+ε)) has to be such that

\begin{align*}
\begin{cases}
\text{if } θ(r) ∈ \mathcal{C}^{D_i(r), j(r)} \text{ then } j(r^+) \text{ such that } &θ(r) ∈ C_j^{D_i(r), j(r^+)}, \\
i(r^+) &j(r^+),
\end{cases}
\end{align*}

\begin{align*}
\begin{cases}
\text{if } θ(r) ∈ \mathcal{D}_i(r), j(r) \setminus \mathcal{C}^{D_i(r), j(r)} \text{ then } j(r^+) = j(r) \text{ and } i(r^+) &\text{ such that } \\
θ(r) ∈ D_i^{D_i(r), j(r^+)} &i(r^+), j(r^+).
\end{cases}
\end{align*}

It is very important that for j(r) this prescription is a generalization of the one given before in (6.2.4-9). (Note that \mathcal{C}_j \subseteq \bigcup_{i ∈ I(j)} \mathcal{D}_{i,j}). To conclude this subsection we state the analog of (6.2.4-10).

6.3.2-22. Proposition. There exists a finite cover \{U_n\} of M with the following property. Let θ(t), t ∈ (a, b) be a continuous curve in M. If an interval (c, d) ⊆ (a, b) is such that for some value of k and for all t ∈ (c, d), θ(t) ∈ U_k, then at most two coordinate changes occur on the interval (c, d). I.e., (i(r), j(r)) takes on at most three values for r ∈ (c, d).

Proof. Let \{(E_{i,j}^{D}, F_{i,j}^{D}, E_{i,j})\} i ∈ I(j), j ∈ J be an n.d.s. cover of M such that

\begin{align*}
D_{i,j} = E_{i,j} \cap C_j; & D_{i,j} = E_{i,j} \cap C_j \text{ and } D_{i,j} = E_{i,j}, \text{ as before. Let } θ ∈ M. \text{ Consider the compact set}
\end{align*}

\begin{align*}
\left( \bigcup \mathcal{C}_j \right) &\cup \left( \bigcup \mathcal{C}_j \right) \cup \left( \bigcup \mathcal{C}_j \right) \cup \left( \bigcup \mathcal{C}_j \right) \cup \left( \bigcup \mathcal{C}_j \right),
\end{align*}

\begin{align*}
θ \notin \mathcal{C}_j &\theta \notin \mathcal{C}_j \theta \notin \mathcal{C}_j \theta \notin \mathcal{C}_j \theta \notin \mathcal{C}_j.
\end{align*}

Clearly this set does not contain θ, and therefore there exists an open connected neighbourhood U = U(θ) of θ in the complement of this set. Just as in the proof of (6.2.4-10) U has property (6.2.4-11):
(6.3.2-23) \( \forall j \in J: U \cap C'_j \neq \emptyset \Rightarrow U \cap \partial C'_j = \emptyset \)

and similarly

(6.3.2-24) \( \forall i \in I(j), j \in J: U \cap E'_{ij} \neq \emptyset \Rightarrow U \cap \partial E'_{ij} = \emptyset \).

We claim that if \( \theta(r), r \in [a, b) \) is a continuous curve which takes its values in \( U \), then at most two coordinate changes (in the sense of the prescription (6.3.2-21)) can take place. According to the prescription, after a first coordinate change at \( r = r_1 \) (say) we have \( \theta(r_1^+) \in D_{ij} \) with \( i = i(r_1^+) \), \( j = j(r_1^+) \). Because \( \theta(r_1^+) \in U \) and \( D_{ij} = u_{ij} \), (6.3.2-24) implies that \( U \cap \partial E'_{ij} = \emptyset \). A second coordinate change can only take place at \( u_{ij} \cap U \).

We know that \( u_{ij} \subseteq \partial C'_j \cap \partial E'_{ij} \). It follows that

\[ u_{ij} \cap U \subseteq \partial C'_j \cap \partial E'_{ij} . \]

Therefore a second coordinate change at \( r_2 \) can only take place if \( \theta(r_2) \in \partial C'_j \). In that case \( j \) will be changed. According to the prescription we will have

\[ \theta(r_2) \in D_{kl} \cap C'_k \text{ with } k = k(r_2^+) , l = j(r_2^+) . \]

It then follows from (6.3.2-23), (6.3.2-24) that

\[ U \cap \partial C'_j = \emptyset \text{ and } U \cap \partial E'_{ij} = \emptyset . \]

So \( U \cap u_{ij} \subseteq U \cap (\partial C'_j \cup \partial E'_{ij}) = \emptyset \).

Therefore after a second coordinate change no more coordinate changes will take place for \( \theta(r) \).

It is clear that \( \{U(\theta) \mid \theta \in M\} \) is an open cover of \( M \). Because \( M \) is compact there is a finite subcover \( \{U_k\} \) of \( M \), and each \( U_k \) has the required property.

Q.E.D.

6.3.2-25. Remark. According to the proof \( \{U_k\} \) can be chosen such that if two coordinate changes take place then the first one will leave \( j \) constant, while the second one changes \( j \). I.e. the first one will be within \( C'_j \), while the second one will be a coordinate change from \( C'_j \) to \( C'_k \).
6.3.3. Application to the question of asymptotic stability of products of asymptotically stable matrices

Let us consider the problem of possible instability of a product of asymptotically stable matrices. If $A_1$ and $A_2$ are asymptotically stable matrices, then the product does not have to be asymptotically stable. For example, let

$$ (6.3.3-1) \quad A_1 = \begin{bmatrix} 0.5 & 0 \\ 10 & 0.5 \end{bmatrix}, \quad A_2 = A_1^T. $$

Then

$$ (6.3.3-2) \quad A_1 A_2 = \begin{bmatrix} 0.25 & 5 \\ 5 & 100.25 \end{bmatrix} $$

which is unstable.

This implies that a time varying linear system

$$ (6.3.3-3) \quad \begin{cases} x_{t+1} = A_t x_t + B_t u_t \\ y_t = C_t x_t \end{cases} $$

with $A_t$ an asymptotically stable matrix for each $t$ (i.e., $\forall t: \sigma(A_t) \subseteq D(0,1)$), can be unstable. An example is obtained by taking $A_{2t+1} := A_1$ and $A_{2t} := A_2$, for all $t$, with $A_1$ and $A_2$ as above.

In our algorithms the dynamic matrix is $F_t$, which is asymptotically stable (for each $t$) but time-varying. So the question arises whether the resulting time-varying system is stable. To treat this problem we will investigate the relationship between asymptotic stability and asymptotically stable norms.

6.3.3-4. Definition (cf. [Gan]).

The right norm of a matrix $A$ is $AA^*$ and the left norm $A^*A$.

It is well-known that $AA^*$ and $A^*A$ have the same nonzero eigenvalues (they are the squares of the nonzero singular values of $A$); and we will say that $A$ has asymptotically stable norm(s) if $AA^*$ and $A^*A$ are asymptotically stable, i.e., if their eigenvalues are smaller than one, or, equivalently, if all the singular values of $A$ are smaller than one. Clearly a has asymptotically stable norm if and only if $\|A\|_S < 1$ ($\|\cdot\|_S$ denotes again the spectral norm).
6.3.3-5. Proposition. Let \( A \) be square and let \( \lambda_M \) denote the eigenvalue of \( A \) with maximum modulus:

\[
|\lambda_M| = \max_{\lambda_i \in \sigma(A)} |\lambda_i|.
\]

Then

(a) \( |\lambda_M| \leq |A|_S \).

(b) For each \( \varepsilon > 0 \) there exists a nonsingular matrix \( T \) and an open neighbourhood \( N \) of \( A \) such that for all \( \tilde{A} \in N \)

\[
|TAT^{-1}|_S < |\lambda_M| + \varepsilon.
\]

Before going to the proof let us state a corollary.

6.3.3-7. Corollary. Let \( A \) be square.

(a) If \( A \) has asymptotically stable norm then \( A \) is asymptotically stable.

(b) If \( A \) is asymptotically stable then there exists a nonsingular matrix \( T \) and an open neighbourhood \( N \) of \( A \) such that for all \( \tilde{A} \in N \), \( TAT^{-1} \) has asymptotically stable norm. (and especially \( TAT^{-1} \) has asymptotically stable norm).

Proof of proposition (6.3.3-5). (a) Let \( Ax = \lambda x, \|x_0\| = 1, x_0 \in \mathbb{C}^n \) then it follows that \( |A|_S = \max_{\|x\|=1} \|Ax\| = |\lambda_M| \).

(b) Because \( |TAT^{-1}|_S \) is continuous in the entries of \( A \), it suffices to show that for each \( \varepsilon > 0 \) there exists a nonsingular matrix \( T \) such that

\[
|TAT^{-1}|_S < |\lambda_M| + \varepsilon.
\]

Let \( S \) be an invertible matrix such that

\[
J_1 := SAS^{-1}
\]

is in Jordan normal form. Then \( J_1 \) is block-diagonal with \( n_i \times n_i \) blocks \( J^{(i)}_1, i = 1, 2, \ldots, l_0 \) (say), of the form

\[
J^{(i)}_1 = \begin{bmatrix}
\lambda_i & 1 & 0 \\
& \ddots & \ddots \\
& & \lambda_i
\end{bmatrix}.
\]
Let $J_\mu$, with $\mu > 0$, be defined by

$$(6.3.3-10) \quad J_\mu := A_\mu^{-1} J_1 A_\mu,$$

where $A_\mu := \text{diag}(1, \mu, \mu^2, \ldots, \mu^{n-1})$. Then $J_\mu$ has the same block structure as $J_1$, with $\mu$th block

$$J^{(1)}_\mu = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ \mu & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_1 \end{bmatrix}.$$ 

Clearly $\lim_{\mu \to 0} J^{(1)}_\mu = \text{diag}(\lambda_1, \lambda_1, \ldots, \lambda_1)$, and $J_0 := \lim_{\mu \to 0} J_\mu$ is a diagonal matrix with the eigenvalues of $A$ on its main diagonal. Therefore $\|J_0\|_S = |\lambda M|_1$.

Because the spectral norm depends continuously on the entries of the matrix, it follows that for each $\varepsilon > 0$ there exists a number $\mu_0 > 0$ such that

$$\|J - J_0\|_S < |\lambda M|_1 + \varepsilon. \text{ So let } T = A_\mu^{-1} S \text{ and (b) follows.}$$

Q.E.D.

This will now be applied, as follows. Let $\theta \in M$ and $j$ such that $\theta \in C_j$.

Consider $F(\phi_j(\theta), j)$ as defined in section 6.2. Its eigenvalues are the same as those of $A(\theta, j) - B(\theta, j)C(\theta, j)$ (compare (6.2.2-10)), but with higher multiplicity. Therefore its eigenvalue with maximum modulus is $\lambda M(\theta)$ (cf. (6.2.8-1)), and $\forall \theta \in M$: $|\lambda M(\theta)| < \lambda_0 < 1$ (cf. (6.2.8-2)). Let $\varepsilon \in (0, 1 - \lambda_0)$ be fixed. According to the previous proposition there exists a neighbourhood $N \subseteq C_j$ of $\theta$ and a (constant, nonsingular) matrix $T$ such that

$$\forall \tilde{\theta} \in N: T F(\phi_j(\tilde{\theta}), j) T^{-1} S < \lambda_0 + \varepsilon (< 1).$$

The neighbourhoods cover $M$. Because each $C_j$ is compact and $|J| < \omega$, there is a finite subcover $\{E_{a_j}\}$ with the property $C_j a_j \subseteq \cup \tilde{a}_j$. At the end of the next subsection this cover will be used to obtain a second order n.d.s. cover $\{(D_{ij}, D_1, D_{ij}) \mid i \in I(j), j \in J\}$ corresponding to $\{(C_{ij}, C_{ij}, C_{ij}) \mid j \in J\}$, such that for each set $D_{ij}$ there exists a nonsingular matrix $T_{ij}$ such that for all $\theta \in D_{ij}$
\( (6.3.3-11) \quad F(\psi^j_{i}(t); i, j) := T^{i}_{ij}F(\psi^j_{i}(0), j)T^{-1}_{ij} \)

has spectral norm smaller than \( \lambda^0 \). In accordance with this we will define (compare (6.2.9-8) and (6.2.9-9))

\( (6.3.3-12) \quad G(\psi^j_{i}(t); i, j) = T^{i}_{ij}G(\psi^j_{i}(0), j) \)

and

\( (6.3.3-13) \quad H(\psi^j_{i}(t); i, j) = H(\psi^j_{i}(0); i, j)T^{-1}_{ij} \)

and

\( (6.3.3-14) \quad \xi(t; i, j) = T^{i}_{ij}\xi(t, j). \)

Equations (6.2.9-8) and (6.2.9-9) can then be rewritten as

\( (6.3.3-15) \quad \xi(t+1; i, j) = F(\psi; i, j)\xi(t; i, j) + G(\psi; i, j)\gamma_t, \)

\[
\begin{bmatrix}
\varepsilon(t, j) \\
\psi_1(t, j) \\
\vdots \\
\psi_d(t, j)
\end{bmatrix} = H(\psi; i, j)\xi(t; i, j) + Ky_t,
\]

if \( \psi = \psi^j_{i}(t) \), where \( \theta \in D_{ij} \).

6.3.4. **Local equivalence of the coordinate chart metrics with the inner metric**

(This is of course standard differentiable geometry, included for completeness sake). Within a coordinate chart \( (C^j, \psi^j_{i}) \) one can make use of the Euclidean metric \( d_j \) of the coordinates. To be precise, \( d_j \) is given by:

\( (6.3.4-1) \quad d_j(\theta^j_0, \theta^j_1) := \| \psi^j_{i}(\theta^j_0) - \psi^j_{i}(\theta^j_1) \| \) for all \( \theta^j_0, \theta^j_1 \in C^j. \)

The length of a differentiable curve \( \gamma: [0,1] \rightarrow C^j, \) in this metric, is given by the formula
(6.3.4-2) \[ I_j(\gamma) = \frac{1}{l} \int_0^l \frac{d\phi_j(\gamma(t))}{dt} \, dt. \]

Let \( \Gamma(\theta_0, \theta_1, U) \) denote the set of all differentiable curves \( \gamma: [0, 1] \to U \) with \( \gamma(0) = \theta_0, \gamma(1) = \theta_1 \). Suppose \( \phi_j(U) \) contains the straight line segment between \( \phi_j(\theta_0) \) and \( \phi_j(\theta_1) \). Then clearly

(6.3.4-3) \[ d_j(\theta_0, \theta_1) = \min_{\gamma \in \Gamma(\theta_0, \theta_1, U)} I_j(\gamma). \]

The length of a curve \( \gamma \in \Gamma(\theta_0, \theta_1, U) \) with respect to the Riemannian metric with Riemannian metric tensor \( R(\theta, j) \) is given by the formula

(6.3.4-4) \[ I(\gamma) = \frac{1}{2} \int_0^1 \frac{d\phi_j(\gamma(t))}{dt} \frac{d\phi_j(\gamma(t))}{dt} R(\gamma(t), j) \, dt. \]

For each pair of points \( \theta_0, \theta_1 \in C_j \) the inner metric \( d \) (with respect to the Riemannian metric) is given by

(6.3.4-5) \[ d(\theta_0, \theta_1) = \inf_{\gamma \in \Gamma(\theta_0, \theta_1, M)} I(\gamma). \]

Recall the well-known definition of equivalence of metrics.

6.3.4-6. Definition. Two metrics \( d', d'' \) on a space \( S \) are called equivalent, if there exists a constant \( K_\epsilon \in \mathbb{R}_+ \) such that

(6.3.4-7) \[ \forall x, y \in S: d'(x, y) \leq K_\epsilon d''(x, y) \text{ and } c''(x, y) \leq K_\epsilon d'(x, y). \]

6.3.4-8. Proposition. Let \( \theta \in C_j^\dagger \). There exists an open neighbourhood \( N = N_\theta, j \subseteq C_j \), \( \theta \in N \), such that \( d_j \) and \( d \) are equivalent on \( N \).

Proof. Let \( C_j^\dagger \) be an open set such that

(6.3.4-9) \[ \overline{C_j} \subseteq C_j^\dagger \subseteq C_j \subseteq C_j^\dagger. \]

Because \( R(\theta, j) \) is a positive definite matrix depending smoothly and hence continuously on \( \theta \) in \( C_j \), it follows that on the compact set \( \overline{C_j}^\dagger \) this matrix has a maximum eigenvalue \( \lambda^*_N > 0 \) and a minimum eigenvalue \( \lambda_m > 0 \). It follows that

if \( \gamma \) lies entirely in \( \overline{C_j}^\dagger \) then
(6.3.4-10) $\lambda'_m I_j(\gamma) \leq I(\gamma) \leq \lambda_m I_j(\gamma)$.

Consider a fixed $\theta \in \mathcal{C}^i_j$ and an open neighbourhood $N_1$ of $\theta$, $N_1 \subset \mathcal{C}^i_j$, such that $\phi_j(N_1) \subset \mathbb{R}^d$ is convex. It follows that

(6.3.4-11) $\forall \theta_0, \theta_1 \in N_1: d_{j}(\theta_0, \theta_1) = \min_{\gamma \in \Gamma(\theta_0, \theta_1, N_1)} I_j(\gamma)$.

According to a theorem from Riemannian geometry there exists an open neighbourhood $N \subset N_1$, $\theta \in N$ and a number $\varepsilon > 0$, such that any two points in $N$ can be joined by a unique geodesic of length $< \varepsilon$. This geodesic lies entirely in $N_1$ (cf. [Boo], chapter VII, theorem (6.9) pp. 336-337). It follows that

(6.3.4-12) $\forall \theta_0, \theta_1 \in N: d(\theta_0, \theta_1) = \min_{\gamma \in \Gamma(\theta_0, \theta_1, N_1)} I(\gamma)$.

It follows that for all $\theta_0, \theta_1 \in N$:

(6.3.4-13) $d_{j}(\theta_0, \theta_1) \leq \lambda_{\theta}^{-1} d(\theta_0, \theta_1)$

and

(6.3.4-14) $d(\theta_0, \theta_1) \leq \lambda_m d_{j}(\theta_0, \theta_1)$.

Taking $K_{\varepsilon} := \max(\lambda_{\theta}^{-1}, \lambda_m)$ gives the desired result.

Q.E.D.

The set $(N_j, \mathcal{E}_j \subset \mathcal{C}^i_j)$ forms an open cover of $\mathcal{C}^i_j$, and so there is a finite subcover, because $\mathcal{C}^i_j$ is compact. This holds for each $j \in J$. Taking the union of the finite subcovers for the different values of $j \in J$, one obtains a finite cover of $\mathcal{C}$, which we will denote by $\{E_{Bj}\}$. Of course

(6.3.4-15) $\mathcal{C}^i_j \subseteq \bigcup_{B \in B_j} E_{Bj}$

and for each $B$ and $j$ there exists a positive constant $K_{Bj}$ such that

(6.3.4-16) $\forall \theta_0, \theta_1 \in E_{Bj}: d(\theta_0, \theta_1) \leq K_{Bj} d_{j}(\theta_0, \theta_1)$ and $d_{j}(\theta_0, \theta_1) \leq K_{Bj} d(\theta_0, \theta_1)$.
Take
\[ K_e := \max_{\beta, j} K_{\beta j} > 0. \]

It follows that

\[ (6.3.4-17) \quad \forall \beta, \forall j, \forall \theta_0, \theta_1 \in E_{\beta j} : d(\theta_0, \theta_1) \leq K_{\beta j}(\theta_0, \theta_1) \text{ and } c_j(\theta_0, \theta_1) \leq K_{\beta}(\theta_0, \theta_1). \]

In the previous subsection a finite cover \( \{E_{\alpha j}\} \) was found with the property

\[ (6.3.4-18) \quad E_{\alpha j}^r \subseteq \bigcup_{\alpha} E_{\alpha j}. \]

Now let us consider the finite cover \( \{E_{\alpha j} \cap E_{\beta j}\} \) of \( M \), which we shall denote by \( \{E_{ij} \mid i \in I(j), j \in J\} \),

\[ (6.3.4-19) \quad E_{ij}^r \subseteq \bigcup_{i \in I(j)} E_{ij}. \]

According to proposition (6.3.2-2) there exists an n.d.s. cover

\[ \{(E_{ij}, E_{ij}^r, E_{ij}^l) \mid i \in I(j), j \in J\} \]

of \( M \) with the property

\[ (6.3.4-20) \quad \bigcup_{i \in I(j)} E_{ij}^r = C_j. \]

Now form \( \{(D_{ij}, D_{ij}^r, D_{ij}^l) \mid i \in I(j), j \in J\} \) as in (6.3.2-10). According to proposition (6.3.2-11) this is a second order n.d.s. cover. Summarizing this section 6.3, we have found the following result

6.3.4-21. Theorem. Let \( \{(C_j^r, C_j, C_j^l) \mid j \in J\} \) be an n.d.s. cover of \( M \) consisting of coordinate charts. Let \( \varepsilon \in (0, 1-\lambda_0) \) be fixed. Then there exists an n.d.s. cover \( \{(E_{ij}^r, E_{ij}, E_{ij}^l) \mid i \in I(j), j \in J\} \) of \( M \) with the properties

\[ (a) \quad E_{ij} \subseteq C_j \text{ for all } i \in I(j), j \in J, \]

\[ (b) \quad \bigcup_{i \in I(j)} E_{ij} = C_j \text{ for all } j \in J, \]

\[ (c) \quad \text{for each } i \in I(j), j \in J \text{ there exists a nonsingular matrix } T_{ij} \text{ such that} \]


has spectral norm smaller than $\lambda_0 + \varepsilon(<1)$,
(d) there exists a positive number $K_0$ such that for all $i \in I(j)$, $j \in J$,

\[(6.3.4-22) \ \forall \theta \in E_{ij} : \ F(\phi_j(\theta);i,j) \equiv \mathcal{T}_{ij} F(\phi_j(\theta),i,j)^{-1}\]

There exists a corresponding second order n.d.s. cover
\[(6.3.4-23) \ \forall \theta_0, \theta_1 \in E_{ij} : \ d_1(\theta_0, \theta_1) \leq K_0 \ d(\theta_0, \theta_1) \text{ and } d(\theta_0, \theta_1) \leq K_1 \ d_1(\theta_0, \theta_1) .
\]

6.4. On the asymptotic behaviour of the stepsize and the coordinate change times

One of the main problems in analyzing algorithms like the one under investigation is the data-dependence of the parameter sequence $\{\theta_t\}$. If one fixes the sequence of parameters $\{\theta_t\}_{t=t_0}$ and a corresponding sequence of coordinate chart indices $\{(i_t, j_t)\}_{t=t_0}$, $i_t \in I(j_t)$, $j_t \in J$, then the state equation of the algorithm (cf. (6.2.9-8)) becomes linear (time-varying), which makes it more tractable. From the algorithm state vector $\xi(t,j)$ one can compute $e(t,j)$ and $v(t,j)$ as in (6.2.9-9) and one can compute $a_t b_1 b_2 R^{-1} h$ (compare (6.2.9-11)). However, in general, the equation (6.2.9-11) no longer makes sense, because the right-hand side may be outside the image $\phi_j(C_t) \subseteq \mathbb{R}^d$, and then $\phi_j^{t+1}$ is no longer well-defined by (6.2.9-11). Also, of course, the so-called coupling equation (6.2.9-12) and the rules for coordinate change that follow (6.2.9-12) no longer make sense because the parameters $\{\theta_t\}$ and the coordinate charts are fixed a priori. The set of equations that compute $\xi(t,j)$, $e(t,j)$, $v(t,j)$ and $a_t b_1 b_2 R^{-1} h$ in the case of a fixed sequence $\{(i_t, j_t)\}_{t=t_0}$ with $\theta_t \in \mathcal{D}(i_t, j_t)$, $i_t \in I(j_t)$ and $j_t \in J$, will be called the decoupled algorithm. If equation (6.2.9-11) happens to define a sequence $\{\theta_t\}$, i.e. if the right-hand side of (6.2.9-11) happens to be
in \( \phi_{j_t}(C_{j_t}) \) for each \( t \geq t_o \), then and only then, the equation (6.2.9-11) will be considered as part of the decoupled algorithm.

Our approach will be to derive results for the decoupled algorithm, in which special types of parameter/index sequences \( \{\theta_t, i_t, j_t\}_{t=t_o}^{t} \) are used, and to show what the implications are for the 'coupled' algorithm.

In this subsection we want to analyze the stepsizes and the number of steps minimally taken in each coordinate chart. Because the asymptotic behaviour is investigated, the estimates on the stepsizes etc. do not have to be sharp, they only have to be sufficient for our purposes.

The different types of parameter sequences will be described in the form of numbered properties.

6.4.1. Property 0. There exists a sequence \( \{a_t\}_{t=t_0}^{t} \) with the properties

(a) for all \( t < t_o \), \( a_t = a_{t_o} \);

(b) \( \{a_t\} \) is monotonically non-increasing;

(c) \( \forall t: a_t > 0 \) and

\[
\sum_{t=t_0}^{t} a_t = \infty
\]

such that the parameter sequence \( \{\theta^+_t\}_{t=t_0}^{t} \) with \( \theta^+_t = (\theta_t, i_t, j_t) \in \Theta^+ \) for each \( t \), satisfies the following:

(i) \( \forall t < t_o: \theta_t^+ = \theta_t^+ \),

(ii) \( \forall t \geq t_o: d(\theta_{t+1}, \theta_t^+) \leq a_t \),

(iii) a coordinate change can take place only if \( \theta_t \) is close enough to the boundary of \( D_{i_t, j_t} \), to be precise:

\[
(i_{t+1}, j_{t+1}) \neq (i_t, j_t) \Rightarrow d(\tilde{\theta}_t, D_{i_t, j_t}) \leq a_t
\]

(iv) a coordinate change has to take place before \( \theta_t \) is too far from \( D_{i_t, j_t} \), to be precise:

\[
\forall t: d(\theta_t, \bar{D}_{i_t, j_t}) \leq a_t
\]

(v) if a coordinate change takes place then \( \theta_{t+1} \) has to be close enough to \( \tilde{D}_{i_{t+1}, j_{t+1}} \), to be precise:
\[(i_{t+1}, j_{t+1}) \neq (i_t, j_t) = d(\theta_{t+1}, D_{t+1}^{-1}j_{t+1}) \leq a_t.\]

6.4-2. Notation. \(\lim_{t \to \infty} a_t' =: a'_\infty.\)

6.4-3. Property 1. Property 0 holds and the sequence \(\{a_t'\}_{t=t_0}\) has the extra property \(a'_\infty = 0.\)

6.4-4. Notation.
Before we proceed let us first introduce a useful notation. The matrices that we encounter, like \(G, H, K, R^{-1}\) etc. are all smooth functions of the local coordinates and therefore their spectral norm takes on a maximum on each \(D_{i,j}^{-1}.\)
Because the number of indices \(i,j\) is finite it follows that for each of the matrices there is an over-all upper bound for the norm. Without loss of generality this upper bound can and will be taken greater than or equal to one. Let it be denoted by \(\overline{R}(G), \overline{R}(H),\) etc.

6.4-5. Theorem. The sequence \(\{\theta_t'\}_{t \geq t_0}\) that is produced by the (coupled) algorithm satisfies property 1.

Proof. This is mainly a consequence of the use of \(g_2\) in the parameter update equation in the algorithm (cf. (6.2.9-1) and (6.2.8-6)). It can easily be shown that

\[
(6.4-6) \quad \|a_t g_1 g_2^{-1} h_1 \| \leq a_t \overline{K}^{-1}(R^{-1}).
\]

Consider the curve \(\gamma\) given in local coordinates of \(G_j\) by

\[\phi_j(\gamma(\lambda)) = \phi_j(\theta_t') + \lambda a_t g_1 (v_t) g_2 (t) R^{-1} h, \quad \lambda \in [0, \lambda_t].\]

Clearly, by definition of \(\lambda_t\) (cf. (6.2.9-1)), \(\gamma(\lambda_t) = \theta_{t+1}'\), and \(\forall \lambda \in [0, \lambda_t], \gamma(\lambda) \in C_j'.\) It follows that the Riemannian length of \(\gamma\) is bounded from above by \(a_t \overline{K}^{-1}(R^{-1}) \overline{R}(R).\) A fortiori one has

\[d(\theta_{t+1}, \theta_t') \leq a_t \overline{K}^{-1}(R^{-1}) \overline{R}(R),\]

Using the coupling equation (6.2.9-12), one finds
\[ d(\theta_{t+1}, \theta_t) \leq d(\hat{\theta}_{t+1}, \hat{\theta}_t) + d(\hat{\theta}_t, \theta_t) \leq \delta_{t+1} + a_t K_t \mathbb{K}(R^{-1}) \mathbb{K}(R) + \delta_t. \]

Recall that \( \lim_{t \to \infty} \delta_t = 0 \) and \( \lim_{t \to \infty} a_t K_t = 0 \). Therefore one can take
\[
a_t := \max_{s \geq t} (\delta_{s+1} + a_s K_s \mathbb{K}(R^{-1}) \mathbb{K}(R) + \delta_s),
\]
to obtain a monotonically non-increasing sequence with
\[
a_t' = \lim_{t \to \infty} a_t = 0, \quad a_t > 0 \quad \text{for all } t \quad \text{and} \quad t_{a_t} = \infty.
\]
Using this sequence \( \{a_t'\} \) it follows that the parameter sequence \( \{\theta_t'\}_{t=t_0}^\infty \)
from the (coupled) algorithm satisfies \( d(\theta_{t+1}, \theta_t') \leq a_t' \). Using the fact that
\[
a_t' \geq \delta_t \quad \text{and} \quad a_t' \geq \delta_{t+1},
\]

it can easily be checked that also the other conditions of property 1 are satisfied. (This is left to the reader).

Q.E.D.

The idea is now to show results for the decoupled algorithm with parameter sequences satisfying property 0 with \( a_m' > 0 \) 'small enough' or property 1. Property 1, i.e. \( a_m' = 0 \) will be the most important case but the results will be needed also for \( a_m' > 0 \) now and then. Using the previous theorem, one can then draw conclusions for the coupled algorithm.

Of course one of the main problems is to estimate the effects of the coordinate changes. It has to be demonstrated that these effects do not destabilize the algorithms. One of the main results of this subsection will be that if the sequence \( \{\hat{\theta}_t\} \) is well-defined by the decoupled algorithm, then the stepsize \( d(\hat{\theta}_{t+1}, \hat{\theta}_t) \) is bounded by \( a_t \mathbb{K} \) for some positive constant \( \mathbb{K} \) which is independent of \( t \) and the data.

In the algorithm at each time instance \( t \) there is a coordinate chart \( \mathcal{D}_{ij} \) in which the algorithm operates. Let the indices \( i,j \) at time \( t \) be denoted by \( i_t \) and \( j_t \). Consider the following indicator functions of coordinate changes

\[
X_2(t) = \begin{cases} 1 & \text{if } j_{t+1} \neq j_t, \\ 0 & \text{elsewhere, and} \end{cases}
\]

and

\[
X_1(t) = \begin{cases} 1 & \text{if } j_{t+1} = j_t \text{ and } i_{t+1} = i_t, \\ 0 & \text{elsewhere.} \end{cases}
\]

(6.4-7)
As a first step we present a lemma which gives a sequence \( \{ \overline{v}_t \}_{t_0}^{\infty} \) of upper bounds for the stepsizes.

6.4-8. Lemma. Let \( \lambda_2 \in (\lambda_0, 1) \) be fixed. Let \( \{(D_i^{(1)}, D_i^{(2)}, D_i^{(3)}) | i \in I(j), j \in J\} \) be as in theorem (6.3.4-21) with \( 0 < \epsilon < \lambda_2 - \lambda_0 \). There exist positive data-independent constants \( c_0, c_1, c_2 \) such that if a sequence \( \{\overline{v}_t\}_{t_0}^{\infty} \) is defined by

\[
\begin{align*}
(1) & \quad \overline{v}_{t_0-1} = 0, \\
(6.4-9) & \quad \overline{v}_{t+1} = \lambda_2 \left[ 1 + c_1 x_1(t) + c_2 x_2(t) \right] \overline{v}_t + c_0 y_t + 1, \quad \forall t \geq t_0 - 1,
\end{align*}
\]

then

\[
\begin{align*}
(6.4-10) & \quad \forall t \geq t_0 : |d(\hat{\theta}_{t+1}, \hat{\theta}_t) | \leq \overline{v}_t^2, \quad \text{if } \{\hat{\theta}_t\} \text{ is well-defined}, \\
(6.4-11) & \quad \forall t \geq t_0 : \| E_1(v_t) R^{-1} h_t R \| \leq \overline{v}_t, \quad \text{and} \\
(6.4-12) & \quad \forall t \geq t_0 : \| E_t \| \leq \overline{v}_t.
\end{align*}
\]

Here the quantities are quantities of the decoupled algorithm.

6.4-13. Remarks. (i) Recall that \( \xi(t_0, i, j) = 0 \) is assumed, this is of importance here.

(ii) It is not necessary to assume in this lemma that the parameter sequence \( \{\hat{\theta}_t\} \) in the decoupled algorithm satisfies property 0 or property 1.

Proof of the lemma. The two-vector \((x_1(t), x_2(t))\) can take three values, namely \((0, 0), (1, 0)\) and \((0, 1)\). Each of these cases will be treated separately.

(a) \((x_1(t), x_2(t)) = (0, 0)\).

Consider the algorithm equations (cf. section 6.2.9 and section 6.3.3)

\[\xi(t+1,i,j) = F(\varphi; i,j) \xi(t;i,j) + G(\varphi; i,j) y_t,\]
\[ z(t, j) := \begin{bmatrix} e(t, j) \\ \psi_1(t, j) \\ \vdots \\ \psi_d(t, j) \end{bmatrix} = H(\phi; i, j)\xi(t; i, j) + Ky(t, j), \]

with \( \phi = \phi_j(\hat{\theta}_{t+1}) \), and if \( \{\hat{\theta}_t\} \) is well-defined,

\[ \phi_j(\hat{\theta}_{t+1}) = \phi_j(\hat{\theta}_t) + \lambda_t \hat{\theta}_t R^{-1} \mathbf{h}, \]

with \( \lambda_t = 1 \), as no coordinate change takes place in the case considered here.

Taking norms, one obtains

\[ \|z(t; i, j)\| \leq \|H(\phi; i, j)\| \|\xi(t; i, j)\| + \|G(\phi; i, j)\| \|y(t, j)\|, \]

\[ \|z(t, j)\| \leq \|H(\phi; i, j)\| \|\xi(t; i, j)\| + \|K\| \|y(t, j)\|, \]

and if \( \{\hat{\theta}_t\} \) is well-defined,

\[ \|\phi_j(\hat{\theta}_{t+1}) - \phi_j(\hat{\theta}_t)\| \leq a_t \|R^{-1}\| \|h(t, j)\|. \]

From theorem (6.3.4-21) it follows that the spectral norm of \( F \) is bounded from above by \( \lambda_0 + \varepsilon < \lambda_2 \) and furthermore that \( d(\hat{\theta}_{t+1}, \hat{\theta}_t) \leq \varepsilon d(\hat{\theta}_t, \hat{\theta}_t) \) for all \( \theta, \hat{\theta}_t \in \mathbb{F}_{i, j} \). Applying this to (6.4-14,15,16) one obtains

\[ \|z(t; i, j)\| \leq (\lambda_0 + \varepsilon) \|\xi(t; i, j)\| + \|G\| \|y(t, j)\|, \]

\[ \|z(t, j)\| \leq \|H\| \|\xi(t; i, j)\| + \|K\| \|y(t, j)\|, \]

and if \( \{\hat{\theta}_t\} \) is well-defined,

\[ d(\hat{\theta}_{t+1}, \hat{\theta}_t) \leq k e^{-a_t} \|R^{-1}\| \|h(t, j)\|. \]

Because \( h = \psi(t, j) e(t, j) \) and \( z(t, j)^T = [e(t, j)^T, \psi_1(t, j)^T, \ldots, \psi_d(t, j)^T] \) it follows that

\[ \|h\| \leq \sum_{i=1}^{d} \|\psi_i(t, j)^T e(t, j)\| \leq \|H\| \|\xi(t, j)\| \|e(t, j)\| \leq \|H\| \|\xi(t, j)\| \|z(t, j)\|, \]

\[ \|h\| \leq \sum_{i=1}^{d} \|\xi(t, j)\| \|z(t, j)\| = d \|z(t, j)\|^2. \]
(6.4-21) \[ d(\hat{\theta}_{t+1}, \hat{\theta}_t) \leq a_t \hat{R}^{-1} d(t, j) \] \[ \| z(t, j) \|_2^2 = a_t \hat{R}^{-1} d(t, j) \] \[ \| z(t, j) \|_2^2. \]

If \( \{ \hat{\theta}_t \} \) is well-defined.

Also,

(6.4-22) \[ R_L^{-1} h_i R \leq \hat{R}^{-1} h_i \leq \hat{R}^{-1} d(t, j) \| z(t, j) \|_2. \]

(b) \((x_1(t), x_2(t)) = (1, 0).\)

In this case \( j(t+1) = j(t) \) and \( i(t+1) \neq i(t). \) Let \( i' = i(t+1), \ i = i(t). \) Then \( \xi \) is premultiplied by a matrix \( T_{i', i} \) (representing the change of basis of the state space of the algorithm). Because there are only a finite number of indices \( i, i', \) i.e., \( | \bigcup I(\{ j \}) | < \infty, \) there exists a finite upper bound for the spectral norms of all the matrices \( T_{i', i}. \) Let \( c_1' > 0 \) be such that \( 1 + c_1' \) is such an upper bound. Then

(6.4-23) \[ \| I(c_1') \| \leq (1 + c_1') \| I(c_1') \| \leq (1 + c_1') \| I(c_1') \| \leq \text{upper bound}. \]

(c) \((x_1(t), x_2(t)) = (0, 1).\)

In this case \( j_L \) is changed into \( j_{L+1}. \) Let \( j = j_L \) and \( j' = j_{L+1}, \ i = i_L \) and \( i' = i_L. \) In general \( \xi \) will now be premultiplied by three matrices, to obtain \( (t+1; i_L, j_L) \):

1) By a matrix \( T_L(j) \) to transform back to the \( C_j \)-state space basis of the algorithm.

2) By a matrix \( S(\hat{\theta}; j', j), \) cf. (6.2.7-7).

3) By a matrix \( T_{i', j'}^{-1} \) to transform to the \( D(i', j') \)-state-space basis of the algorithm.

Of course, 1 and 3 can be treated analogously to (b). Because \( S(\hat{\theta}; j', j) \) is continuous, a constant \( c_2'' > 0 \) can be found such that

(6.4-24) \[ \| S(\hat{\theta}'; j'; j) \|_2 \leq 1 + c_2'', \ \forall \hat{\theta} \in C''_{j}, \ \exists \hat{\theta}', \ \forall j, j' \in J. \]
Combining (1), (2) and (3) it can be concludec that a constant \( c_2' > 0 \) exists such that

\[
(6.4-25) \quad \| \xi(t+1; t_{t+1}^i; t_{t+1}^i) \| \leq (1+c_2') \| \xi(t+1; t_{t}^i; t_{t}^i) \|. 
\]

Let \( \xi_t := \xi(t; t_{t}^i; t_{t}^i) \). Combining (a), (b) and (c) one obtains

\[
(6.4-26) \quad \| \xi_{t+1} \| \leq (\lambda_0 + c)(1+c_1' X_1(t) + c_2' X_2(t)) \| \xi_t \| + (1+c_1')(1+c_2') \| K(G) \| \| \gamma_t \|. 
\]

First, let \( u_{t_o} > 0 \) satisfy

\[
(6.4-27) \quad u_{t_o} = 0
\]

and

\[
(6.4-28) \quad u_{t+1} = (\lambda_0 + c)(1+c_1' X_1(t) + c_2' X_2(t)) u_{t} + (1+c_1')(1+c_2') \| K(G) \| \| \gamma_t \|. 
\]

Then clearly \( u_{t} \geq \| \xi_t \| \) for all \( t \geq t_o \) (recall \( \xi_{t_o} = 0 \)).

Next consider

\[
(6.4-29) \quad \begin{cases} 
\tilde{u}_t := K_{e}^{-\frac{1}{2}}(R^{-1})^{-\frac{1}{2}} d^{\frac{1}{2}} (\tilde{K}(H) u_{t} + \tilde{K}(K) \| \gamma_t \|), \forall t \geq t_o, \\
\tilde{u}_{t-o} := 0.
\end{cases}
\]

Then

\[
(6.4-30) \quad \tilde{u}_t \geq K_{e}^{-\frac{1}{2}}(R^{-1})^{-\frac{1}{2}} d^{\frac{1}{2}} (\tilde{K}(H) \| \xi_t \| + \tilde{K}(K) \| \gamma_t \|) \geq K_{e}^{-\frac{1}{2}}(R^{-1})^{-\frac{1}{2}} d^{\frac{1}{2}} \| \xi(t, j) \|, \quad \forall t \geq t_o,
\]

so

\[
(6.4-31) \quad d(\hat{\theta}_{t+1}, \hat{\theta}_t) \leq a_{t} \tilde{u}_t^2, \forall t \geq t_o, \text{ if } \{ \hat{\theta}_t \} \text{ is well-defined, and}
\]

\[
(6.4-32) \quad \| \theta_1 - R^{-1} h_R \| \leq \tilde{u}_t^2 K_{e}^{-\frac{1}{2}}(R^{-1})^{-\frac{1}{2}} \leq \tilde{u}_t^2.
\]
Now $\{\tilde{u}_t\}_{t=t_0-1}^\infty$ can be considered as the output of a linear system with inputs

$\{(\tilde{y}_{t+1})\}_{t=t_0-1}^\infty$, and zero initial conditions, as follows:

\[
\begin{bmatrix}
u_{t+1} \\
\tilde{y}_{t+1}^1
\end{bmatrix} = \begin{bmatrix}
(\lambda_0 + \varepsilon)(1+c_1^1 x_1(t)+c_2^1 x_2(t)) \\
(1+c_1^1)(1+c_2^1)\bar{K}(G)
\end{bmatrix} \times
\begin{bmatrix}
u_t \\
\tilde{y}_{t}^1
\end{bmatrix} + (0)\tilde{y}_{t+1}^1, \quad \forall t \geq t_0 - 1,
\]

with $u_{t_0 - 1} := 0$, $y_{t_0 - 1} := 0$ and $x_1(t_0 - 1) = 0$, $x_2(t_0 - 1) = 0$ by convention. The output equation is (6.4-29). Let $T$ be a nonsingular matrix such that

\[
T \begin{bmatrix}
\lambda_0 + \varepsilon \\
0
\end{bmatrix} = \begin{bmatrix}
(1+c_1^1)(1+c_2^1)\bar{K}(G)
0
\end{bmatrix}^{-1}
\]

has spectral norm smaller than or equal to $\lambda_2$ (recall that $\lambda_2 > \lambda_0 + \varepsilon$; such a $T$ exists according to proposition (6.3.3-5)). Let $c_1 > 0$, $c_2 > 0$ be such that the spectral norm of

\[
T \begin{bmatrix}
(\lambda_0 + \varepsilon)(1+c_1^1 x_1(t)+c_2^1 x_2(t)) \\
0
\end{bmatrix} = \begin{bmatrix}
(1+c_1^1)(1+c_2^1)\bar{K}(G)
0
\end{bmatrix}^{-1}
\]

is bounded from above by $\lambda_2 (1+c_1^1 x_1(t)+c_2^1 x_2(t))$ for all $t \geq t_0$.

Let

\[
(6.4-34) \quad \tilde{u}_t := T(\tilde{y}_{t}^1)^1, \quad \forall t \geq t_0 - 1,
\]

then

\[
(6.4-35) \quad \tilde{u}_{t+1} \leq \lambda_2 (1+c_1^1 x_1(t)+c_2^1 x_2(t))\tilde{u}_t + T\tilde{y}_{t+1}^1, \quad \forall t \geq t_0 - 1
\]

and from (6.4-29) it follows that

\[
(6.4-36) \quad \tilde{u}_t \leq \frac{1}{e}\bar{K}(\bar{K}^{-1})^{1/2} d^1 \| \bar{K}(H)\bar{K}(K) \| T^{-1} \tilde{u}_t, \quad \forall t \geq t_0.
\]

Now let
(6.4-37) \[ c_0 := \frac{1}{t} \int_{\mathbb{R}(H), \mathbb{R}(K)} \mathbb{K}_t \mathbb{K}^{-1} d \mathbb{K}(H), \mathbb{K}(K) t^{-1} \mathbb{K}_t \]

and \[ \int_{\mathbb{V}_t} v_{t+1} = 0 \text{ and } \]

\[ \int_{\mathbb{V}_t} \lambda_2 (1+c_1 x_1(t) + c_2 x_2(t)) v_t + c_3 y_{t+1}, \quad \forall t \geq t_0 - 1, \]

then \( \bar{v}_t \leq \bar{v}_t \), \( \forall t \geq t_0 \) and so from (6.4-31) one has

(6.4-38) \[ d(\hat{\theta}_{t+1}, \hat{\theta}_t) \leq a t^{-1/2}, \quad \text{if } \{ \hat{\theta}_t \} \text{ is well-defined}, \]

and

(6.4-39) \[ \| R_1(v_t) - h v_t \| \leq \| v_t \|. \]

Furthermore, combining the inequalities: \( \forall t \geq t_0 : \| v_t \| \leq u_t, \quad u_t \leq \bar{u}_t \)

(this follows from (6.4-29)) and \( \bar{u}_t \leq \bar{v}_t \), it follows that

(6.4-40) \[ \| v_t \| \leq \bar{v}_t, \quad \forall t \geq t_0. \]

Q.E.D.

6.4-41. Lemma. Consider the decoupled algorithm and assume its parameter sequence \( \{ \theta_t \} \) satisfies property 0 with \( a^2 > 0 \) sufficiently small or property 1. Then

(a) there exists a data-independent positive constant \( c_3 \) such that

\[ \forall t \geq t_0 : \bar{v}_t \leq c_3 v_t, \]

and

(b) \( \forall c_1, c_2 \geq 0, \forall t > 0, \exists n_1 \) such that

\[ \forall t \geq t_0, k \geq n_1 : \| (1+c_1 x_1(t) + c_2 x_2(t)) < (1+\delta)^k. \]

Proof. The idea of the proof is the following. Due to property 0 the number of times that a coordinate change takes place in some given phase of the algorithm is bounded.

The proof of (a) will be given in seven steps, of which we first give an
overview.

1. Because the parameter sequence \( \{a_t\} \) satisfies property 0 with \( a_t > 0 \) small enough or property 0, the step-length \( d(\theta_t, \theta_{t+1}) \) has a small nonnegative limes superior for \( t \rightarrow \infty \). Define \( \{k_t\} \), \( k_t + k = \in R \cup \{\infty\} \), such that on the time interval \([t, t+k_t]\), \( x_1 = 1 \) occurs at most once and \( x_2 = 1 \) occurs at most once.

2. For \( t \) large enough, \( t \geq t_1 \), the product \( \prod_{t=t}^{t+k_t-1} (1+c_i x_i + c_j x_j) \) is majorized by the expression \( (1+c_i)(1+c_j) \), for a certain \( i \).

3. On \([t_1, t_1+1] \cap Z \) one has \( \bar{v}_t < \bar{v}_t \times \text{constant} \), where \( \bar{v}_t = c_0 (1+c_1)(1+c_2) \bar{v}_t \).

4. For a certain \( m \), it can be shown that on \([t_1+m, t_1+m+1] \cap Z \) the inequality \( \bar{v}_t \geq \bar{v}_t \) holds.

5. If \( \bar{v}_t \geq \bar{v}_t \) then \( \bar{v}_{t+1} \geq \bar{v}_{t+1} \).

6. Combination of 4 and 5 gives \( \forall t \geq t_1+m: \bar{v}_t \geq \bar{v}_t \).

7. It can be concluded that for some \( c_3 > 0 \), \( \bar{v}_t < c_3 v_t \), for all \( t \geq t_0 \).

ad 1. From proposition (6.3.2-19) it follows that at least a distance \( c > 0 \) has to be covered between any two occurrences of \( x_1 = 1 \). A similar result holds for \( x_2 \), as can be shown easily. For notational simplicity let us denote the minimum of the two distances (again) by \( c \).

Let

(6.4-42) \( k_t := \max\{n \in N: n < (c/a_t^3)^{-2} \} \cup \{0\} \).

Then \( \{k_t\} \) is monotonically nondecreasing and \( \lim_{t \rightarrow \infty} k_t = k_0 \in R \cup \{\infty\} \). It follows from property 0 that on the time interval \([t, t+k_t]\), \( x_1 = 1 \) can occur at most once and \( x_2 = 1 \) can occur at most once.

ad 2. Fix \( \lambda_2 \in (\lambda_0, \lambda_1) \) (cf. (6.2.8-2,3)) and \( \lambda_3 \in (\lambda_2, \lambda_1) \). Define the natural number \( k \) by

(6.4-43) \( k := \min\{\lambda_3^{\lambda_3^4} > \max\{(1+c_1)^4, (1+c_2)^4\}\} \).
Because $\lambda_2^*/\lambda_2 > 1$, $\ell$ is well-defined. By taking $s^*_m > 0$ small enough

$k_m > \ell$ will hold and if $s^*_m = 0$, then $k_m = + > \ell$.

Let

\[(6.4.44) \quad t_1 := \min\{t : k_\ell \geq \ell\}.\] This will be well-defined.

The following inequality will be shown,

\[(6.4.45) \quad \forall t \geq t_1, \forall k \in \mathbb{N}: \prod_{\tau = t}^{t+k-1} \lambda_2(1+c_1\chi_1(\tau)+c_2\chi_2(\tau)) < (1+c_1)^{1-k/\ell}(1+c_2)^{-k/\ell} \cdot \lambda_3^k.\]

Note that the right-hand side of (6.4.45) is always smaller than or equal to $(1+c_1)(1+c_2)^k$ and that if $k \geq \ell$, then the right-hand side is smaller than or equal to $\lambda_3^k$.

The proof of (6.4.45) is straightforward. It is mainly a matter of counting how often $\chi_1(\tau) = 1$ resp. $\chi_2(\tau) = 1$ can occur on $[t, t+k-1] \cap \mathbb{Z}$. This is no more than $1 + \left\lfloor k/k_\ell \right\rfloor$ times, where $[k/k_\ell]$ denotes the enter of $k/k_\ell$.

Because $k_\ell \geq \ell$ for $t \geq t_1$, $[k/k_\ell] \leq [k/\ell]$, and so

\[(6.4.46) \quad \prod_{\tau = t}^{t+k-1} \lambda_2(1+c_1\chi_1(\tau)+c_2\chi_2(\tau)) \leq \prod_{\tau = t}^{t+k-1} \lambda_2\sqrt{x_2(1+c_1\chi_1(\tau))}\prod_{\tau = t}^{t+k-1} \lambda_2\sqrt{x_2(1+c_2\chi_2(\tau))} \leq \lambda_2^{k/2}(1+c_1)^{1-k/\ell} \cdot \lambda_2^{k/2}(1+c_2)^{1-k/\ell} = (1+c_1)[\lambda_2^{\frac{k}{2k}}]^{\frac{2k}{2k}} \cdot (1+c_2)[\lambda_2^{\frac{k}{2k}}]^{\frac{2k}{2k}} \leq (1+c_1)\left[\frac{\lambda_3}{(1+c_1)^2}\right]^{\frac{k}{2k}} \cdot (1+c_2)\left[\frac{\lambda_3}{(1+c_2)^2}\right]^{\frac{k}{2k}},\]

where we make use of the definition of $\ell$. This shows (6.4.45).
ad 3. Let $\tilde{v}_t := c_0 (1+c_1)(1+c_2)v_t$, $\forall t \geq t_0$. Let

$$\mu := \max \left\{ 1, \left[ \frac{\lambda_2}{\lambda_1} \right]^{t_1-t_0+k-1} \right\}.$$ 

Then

$$(6.4-47) \quad \mu \tilde{v}_t \geq \tilde{v}_t \quad \text{for all } t \in [t_1, t_1 + k) \cap Z,$$

because

$$\tilde{v}_t \geq \sum_{j=0}^{t-t_0} \lambda_1^{t-t_0-j} \tilde{v}_{t-j};$$

and

$$\tilde{v}_t \leq \sum_{j=0}^{t-t_0} \lambda_2^{t-t_0-j} c_0 \tilde{v}_{t-j}.$$ 

ad 4. Let $m := \min \{ k \in \mathbb{N} | \frac{\lambda_1}{\lambda_2}^k \geq \mu \text{ and } k \geq 1 \};$ $m$ is well-defined because $\lambda_1 > \lambda_2$. It will now be shown that

$$(6.4-48) \quad \forall t \in [t_1 + m, t_1 + m + k) \cap Z: \quad \tilde{v}_t \geq \tilde{v}_t.$$ 

The proof of this inequality is an application of (6.4-45). For each $t \in [t_1, t_1 + k) \cap Z$, $\tilde{v}_t$ is compared with $\tilde{v}_t$ and $\tilde{v}_{t+m}$ with $\tilde{v}_t$; then (6.4-47) is applied:

$$\forall t \in [t_1, t_1 + k) \cap Z:$$

$$\tilde{v}_{t+m} = \sum_{t=t_1}^{t+t} \lambda_2 (1+c_1 x_1(\tau) + c_2 x_2(\tau)) \tilde{v}_t + \sum_{j=1}^{m} \lambda_3^{m-j-1} c_0 \tilde{v}_{t+j} \leq$$

$$(\text{apply (6.4-45)}) \leq \lambda_3^m \tilde{v}_t + \sum_{j=1}^{m} \lambda_3^{m-j-1} (1+c_1)(1+c_2) c_0 \tilde{v}_{t+j} \leq$$

$$(\text{apply definition of } m) \leq \lambda_1^m \tilde{v}_t + \sum_{j=1}^{m} \lambda_1^{m-j} (1+c_1)(1+c_2) c_0 \tilde{v}_{t+j} \leq$$

$$(\text{apply (6.4-47)}) \leq \lambda_1^m \tilde{v}_t + \sum_{j=1}^{m} \lambda_1^{m-j} (1+c_1)(1+c_2) c_0 \tilde{v}_{t+j} =$$
(by definition of $\bar{v}_{t+\delta} = \bar{v}_{t+m}$).

**ad 5.** It will now be shown that if for any $t$, one has $\bar{v}_t \geq \bar{v}_t$, then $\bar{v}_{t+\delta} \geq \bar{v}_{t+\delta}$ holds. The proof is again an application of (6.4-45), $\lambda_3 < \lambda_1$ is also used:

$$
\bar{v}_{t+\delta} = \sum_{\tau=t+1}^{t+\delta} \lambda_2 (1+c_1 \chi_1(\tau) + c_2 \chi_2(\tau)) \bar{v}_\tau +
$$

$$
+ \sum_{j=1}^{\delta} \sum_{\tau=t+j+1}^{t+\delta} \lambda_3 (1+c_1 \chi_1(\tau) + c_2 \chi_2(\tau)) \bar{v}_\tau
$$

$$
\leq \lambda_1 \bar{v}_t + \lambda_1 \sum_{j=1}^{\delta} (1+c_1 \chi_1(\tau) + c_2 \chi_2(\tau)) \bar{v}_{t+j} = \bar{v}_{t+\delta}.
$$

**ad 6.** Combining 4 and 5 gives

(6.4-49) $\forall t \geq t_1 + m : \bar{v}_t \geq \bar{v}_t$.

**ad 7.** Let $t_2 := t_1 + m$ and let

(6.4-50) $\bar{v} := \left\{ \max \left\{ 1, \frac{\lambda_2}{\lambda_1} (1+c_1+c_2) \right\} \right\} \bar{v}_{t_2}$.

Then, just as in (6.4-47), one finds

$$
\forall t \in [t_0, t_2] \cap \mathbb{Z} : \bar{v}_t \leq \bar{v}_t.
$$

Let $c_3 := \max \{ \mu, c_0 (1+c_1+c_2) \}$. Then it follows that

(6.4-51) $\forall t \geq t_0 : \bar{v}_t \leq c_3 \bar{v}_t$.

(b) This is a direct consequence of what was shown in step 2 above. Let $\delta > 0$, $c_1 \geq 0$ and $c_2 \geq 0$ be given. Take $\lambda_3 \in (\lambda_2, \lambda_1) \cap (\lambda_2, \lambda_2 (1+\delta))$. Then (6.4-45) implies:

$$
\forall t \geq t_1, \forall k \geq 1, \sum_{\tau=t}^{t+k-1} \lambda_2 (1+c_1 \chi_1(\tau) + c_2 \chi_2(\tau)) < \frac{\lambda_3 \lambda_2}{\lambda_2} (1+\delta)^k < (1+\delta)^k.
$$
Let $n_1 > k$ be such that

$$(1+\delta)^{\frac{1}{\lambda_2}} \frac{n_1}{\lambda_3} > (1+c_1+c_2)^{t_1-t_0}.$$  

Then

$$\forall t \text{ with } t_0 \leq t < t_1, \forall k \geq n_1 : \prod_{\tau=t}^{t+k-1} (1+c_1x_1(\tau)+c_2x_2(\tau)) < (1+c_1+c_2)^{t_1-t_0+k-1} \prod_{\tau=t_1}^{t+k-1} (1+c_1x_1(\tau)+c_2x_2(\tau)) < (1+c_1+c_2)^{t_1-t_0+k-1} < (1+\delta)^k,$$

and so

$$\forall t \geq t_1, \forall k \geq n_1 : \prod_{\tau=t}^{t+k-1} (1+c_1x_1(\tau)+c_2x_2(\tau)) < (1+\delta)^k,$$

this follows directly from the inequality above.

Q.E.D.

6.4-52. Theorem. There exists a constant $\gamma$, independent of time $t$ and independent of the data with the following property. If $\{\hat{\theta}_t\}$ is well-defined by the decoupled algorithm with parameter sequence $\{\hat{\theta}_t^{\hat{v}}\}$ which is such that it satisfies property 0 with $\alpha_t^1 > 0$ small enough or property 1, then the following inequality holds

$$\forall t \geq t_0 : d(\hat{\theta}_{t+1}, \hat{\theta}_t) \leq a_t \gamma.$$  

Proof. Suppose $\{\hat{\theta}_t\}$ is well-defined. Combining the previous two lemmas one obtains the inequality

$$\forall t \geq t_0 : d(\hat{\theta}_{t+1}, \hat{\theta}_t) \leq a_t \gamma^2.$$

If $\gamma > K'$ then $g_1(\gamma) = 0$ (cf. (6.2.8-4)) and therefore $\hat{\theta}_{t+1} = \hat{\theta}_t$ (cf. (6.2.9-11)). It follows that, if $\tilde{\gamma} = c_3^{2}(K')^2$,

$$\forall t \geq t_0 : d(\hat{\theta}_{t+1}, \hat{\theta}_t) \leq a_t \gamma.$$

Q.E.D.
6.4-53. **Corollary.** Consider the coupled algorithm. There exists a positive number $K^+$ such that

$$
(6.4-54) \quad \forall t \geq t_0: d(\theta_{t+1}, \theta_t) \leq a_t K^+.
$$

**Proof.** Because the coupled algorithm satisfies property 1 (cf. theorem (6.4-3)) the previous theorem applies. One has

$$
\begin{align*}
d(\theta_{t+1}, \theta_t) & \leq d(\theta_{t+1}, \delta_{t+1}) + d(\delta_{t+1}, \hat{\theta}_t) + d(\hat{\theta}_t, \theta_t) \\
& \leq \delta_{t+1} + a_{t+1} K + \delta_t = a_t (\delta_{t+1} + K + \delta_t).
\end{align*}
$$

Because $[\delta_t = 0]_{t=t_0}^\infty$ converges to zero for $t = \infty$ (by assumption, cf. section 6.2.9), the sequence $(\delta_{t+1} + K + \delta_t)_{t=t_0}^\infty$ converges to $K$. Therefore it has a maximum $K^+$ (say).

Q.E.D.

6.4-55. **Corollary.** Consider the decoupled algorithm and assume that property 0, with $a_m > 0$ small enough or property 1 is satisfied. There exists a $t_3 > t_0$, $t_3$ data-independent, such that

$$
(6.4-56) \quad \forall t \geq t_3: g^2(t) = 1.
$$

**Proof.** It suffices to show that $g_1(v_t) K^{-1} h_{t/2}$ is bounded by a data-independent constant (cf. (6.2.8-6) and recall $\lim_{t \to \infty} K_t = \infty$). Lemma (6.4-8) states

$$
g_1(v_t) K^{-1} h_{t/2} \leq \frac{\sigma^2}{v_t} \quad \text{for all } t \geq t_0.
$$

Analogously to the proof of theorem (6.4-52) it follows that

$$
g_1(v_t) K^{-1} h_{t/2} \leq K \quad \text{for all } t \geq t_0.
$$

Q.E.D.

6.4-57. **Theorem.** Consider the decoupled algorithm and assume property 0 is
satisfied with \( a_t' > 0 \) small enough or assume property 1. The following inequality holds:

\[ \| \xi_t \| \leq c \sqrt{v_t} \text{ for all } t \geq t_0. \]

Proof. This follows immediately by combining the inequality (6.4-12) of lemma (6.4-8), with lemma (6.4-41).

Q.E.D.

Using the results of this subsection a smaller class of parameter sequences \( \{a_t^+\} \) can be used. This will be the class of parameter sequences that satisfy property 2 below.

Let

\begin{equation}
(6.4-58) \quad \bar{a}_t := \max_{a \geq t} a, \quad \text{and} \quad \bar{a} = (\bar{a}_t)_{t=\infty}^c.
\end{equation}

Then \( \lim_{t \to \infty} a_t = 0 \), \( \bar{a} = \infty \), and \( \{a_t\} \) is monotonically non-increasing.

6.4-59. Property 2. There exists a data-independent constant \( K^+ > 0 \) such that property 1 holds with \( a_t' = \bar{a}_t K^+ \) for all \( t \).

6.4-60. Theorem. The sequence \( \{a_t^+\}_{t \geq t_0} \) that is produced by the (coupled) algorithm satisfies property 2 (with \( K^+ = K^+ \) as in corollary (6.4-53)).

Proof. From corollary (6.4-53) and the fact that \( a_t \leq \bar{a}_t \) for all \( t \), it follows that

\[ \forall t \geq t_0: d(\theta_{t+1}, \theta_t) \leq \bar{a}_t K^+. \]

So condition (ii) of property 1 with \( a_t' = \bar{a}_t K^+ \) is satisfied. It can easily be checked that also the other conditions are satisfied.

Q.E.D.

This theorem can be compared with theorem (6.4-3). Notice that property 2 implies property 1 and therefore the results presented under the assumption of
property 0 with $a^* > 0$ small enough or of property 1, also hold under the assumption of property 2.

6.5. Spaces of interpolation curves of parameter sequences and their topologies

In the following sections use will be made of interpolation curves of parameter sequences. In this section the topological and metrical structure of several spaces of interpolation curves and parameter sequences will be treated, and their relation to the algorithm.

To be able to handle coordinate chart changes in the analysis, consider the following parameter space

$$ (6.5.1) \quad \Theta^+ := \{(\theta, i, j) | j \in J, \ i \in I(j), \ \theta \in D^i_j \}. $$

The topology will be such that $\Theta^+$ consists of different components

$$ \overline{D}^i_j := \{(\theta, i, j) | \theta \in D^i_j \} $$

(for the definition of a component, cf. e.g. [Dug], p. 111). So each $\overline{D}^i_j$ is open and closed in this topology. Within each set $\overline{D}^i_j$, the topology of $\overline{D}^i_j$ is used: $\{(\theta, i, j) | \theta \in N \in \overline{D}^i_j \}$ is open iff $N$ is open in the relative topology of $\overline{D}^i_j$. This specifies the topology of $\Theta^+$ completely. The projection

$$ p: \Theta^+ \rightarrow M, \ (\theta, i, j) \mapsto \theta $$

is continuous.

Next, consider for a given constant $c > 0$, the topological space $L^*_C$ of curves in $M$ with global Lipschitz constant $c$:

$$ (6.5.2) \quad L^*_C := \{X: R \rightarrow M | \forall t, s \in R: d(X(t), X(s)) \leq c|t-s|\}, $$

with the compact-open topology (cf. e.g. [Dug], chapter XII).

Now consider curves $r \rightarrow (X(r), i(r), j(r)) \in \Theta^+$ which have the 'Lipschitz'-property $X \in L^*_C$. The set of all such curves provided with the compact-open topology, forms a topological space $\Theta^*_C$.

If the curves are defined on an interval $[a, b] \subseteq R$, the corresponding sets will be denoted by $L^*_C[a, b]$, resp. $L^*_C[a, \infty)$, and if they are defined on an interval $(-\infty, b]$ or $[a, \infty)$ the corresponding sets will be denoted by
If the constant \( c \) is clear from the context, the index \( c \) will be dropped. An important subset of \( \mathcal{L}_c^+ \) will be the set \( \mathcal{L}_{c_B}^+ \) of all curves in \( \mathcal{L}_c^+ \) that satisfy the prescription (6.3.2-21). (Similarly the notation \( \mathcal{L}_{c_B}^+(\pm \infty, 0] \), \( \mathcal{L}_{c_B}^+([0, \infty) \) etc. will be used).

Other important subsets of \( \mathcal{L}_c^+ \) are the sets \( \mathcal{L}_{lij}^+ \) of all curves in \( \mathcal{L}_c^+ \) that have a fixed coordinate chart index \((i,j)\). They are in fact also subsets of \( \mathcal{L}_{c_B}^+ \) (no coordinate change takes place and the curve remains within \( \mathcal{B}_{fj}^+ \), so the prescription (6.3.2-21) is satisfied). Furthermore, it is not difficult to see that \( \mathcal{L}_{lij}^+ \) is topologically equivalent to its natural projection \( \mathcal{L}_{lij} \) on \( \mathcal{L} \).

Similarly \( \mathcal{L}_{lij}[a,b] \) is topologically equivalent to its natural projection, \( \mathcal{L}_{lij}[a,b] \) on \( \mathcal{L}[a,b] \).

A basic idea of [Ku-C1] is to associate a parameter curve with each parameter sequence, in such a way that the parameter curve forms an interpolation of the parameter sequence on a shrinking time scale. This time scale is adapted to the stochastic approximation coefficients \( \{a_t\} \); instead of time instants \( t \),

\[
\frac{t-1}{t} \leq a \leq \frac{t}{t+1}
\]

in each local coordinate chart \( \varphi_j(\mathcal{B}_{fj}) \subseteq \mathcal{H}_t \), in accordance with section 6.2.9(a). The linear interpolation curve in \( \varphi^+ \) that is produced by the coupled algorithm will be denoted by \( (\theta(r), i(r), j(r)) \). Because of the shrinking time scale, one has

\[
(\theta(\mathcal{L}_t a), i(\mathcal{L}_t a), j(\mathcal{L}_t a)) = (\theta_t, i_t, j_t), \forall t \geq t_0.
\]

6.5-3. Theorem. The curve \( (\theta(r), i(r), j(r)) \), \( r \geq 0 \), that is produced by the (coupled) algorithm is an element of \( \mathcal{L}_{c_B}^+[0, \infty) \), with \( c = \mathcal{K}^+ \).

Proof. The curve satisfies the prescription (6.3.2-21) by construction of the algorithm (cf. also section 6.2.9 (a)). The Lipschitz property with \( c = \mathcal{K}^+ \) follows from corollary (6.4-53).

Q.E.D.

This theorem gives some indication of the relevance of the space \( \mathcal{L}_{c_B}^+[0, \infty) \) and similar spaces for our problem. Next we will present some properties of these
spaces.

6.5-4. Notation. Let \( X^+ \in \mathcal{L}^+ \) resp. \( \mathcal{L}^+[0,\infty) \). Then \( \forall t \geq t_0, X^+_t \) will denote the translated curve
\[
X^+_t(r) := X^+_t(\sum_{\tau = t_0}^{t-1} a_\tau + r), \quad \text{with } r \in \mathcal{R} \text{ resp. } r > 0.
\]
Clearly \( X^+_t \in \mathcal{L}^+ \) resp. \( \mathcal{L}^+[0,\infty) \).

6.5-5. Remark. We will also use 'backward interpolation curves' of the form
\[
Y^+(r) = X^+_t(\sum_{\tau = t_0}^{t-1} a_\tau - r) = X^+_t(-r).
\]
Then \( Y^+(0) = (\theta_{t-1}, i_{t-1}, j_{t-1}) \); \( Y^+(-a_{t-1}) = (\theta_{t-1}, i_{t-1}, j_{t-1}) \),
\[
Y^+(a_{t-1} - a_{t-2}) = (\theta_{t-2}, i_{t-2}, j_{t-2}) \text{ etc.}
\]
Clearly if \( X^+ \in \mathcal{L}^+ \), then \( Y^+ \in \mathcal{L}^+ \).
A fundamental fact for the analysis is the following application of the Arzela-Ascoli theorem.

6.5-6. Theorem. (a) \( \forall a, b \in \mathcal{R} \text{ with } a < b \), \( L[a,b] \) is compact.
(b) Any subset of \( L[a,b] \) is relatively compact (i.e. has compact closure).

Proof. The equicontinuity of \( L[a,b] \) and therefore of any subset follows easily from the Lipschitz property. Therefore, according to the theorem of Arzela-Ascoli, \( L[a,b] \) and any subset of \( L[a,b] \) is relatively compact in \( C^0([a,b], \mathcal{M}) \), the set of all continuous curves \( X: [a,b] \to \mathcal{M} \). It now suffices to show that \( L[a,b] \) is closed in \( C^0([a,b], \mathcal{M}) \). This is not difficult, and is left to the reader. Q.E.D.

6.5-7. Theorem (a) Any connected component of \( \mathcal{L}^+[a,b] \) is compact (and therefore each subset of such a component is relatively compact).
(b) For each index \( (i, j) \), \( \mathcal{L}^+_{ij}[a,b] \) is compact.

Proof. (a) With any curve \( X^+ \in L[a,b], X^+(r) = (X(r), i(r), j(r)) \) is associated the 'index curve' \((i(r), j(r)), r \in [a,b]\). If any two curves \( X^+, X^+ \in L^+[a,b] \)
have different index function then they lie in different components. This can be seen as follows. Suppose \( r_o \in [a, b] \) is such that \( I(r_o) = i_o \neq \bar{I}(r_o) \). The sets
\[
\{Z^+ = (Z(r),i(r),j(r)) \in L^+[a,b] | i(r_o) = i_o \}
\]
and
\[
\{Z^+ = (Z(r),i(r),j(r)) \in L^+[a,b] | i(r_o) \neq i_o \}
\]
are both open in \( L^+[a,b] \), and their union is \( L^+[a,b] \); \( X^+ \) is element of the first set and \( \bar{X}^+ \) is element of the second set.

Because of this, it suffices to show that each subset of all curves \( X^+(r) \) with a fixed index curve \( (i(r),j(r)) \) is compact. Such a subset is given by
\[
\{(X,i,j) \in L^+[a,b] | X \in L[a,b], \forall r \in [a,b]: X(r) \in \bar{D}_{i}(r)j(r) \}.
\]
This set is homeomorphic to
\[
\{X \in L[a,b] | \forall r \in [a,b]: X(r) \in \bar{D}_{i}(r)j(r) \}
\]
which is a closed subset of \( L[a,b] \). (It is easily seen that the complement of this set is open). Because \( L[a,b] \) is compact, the same holds for any closed subset.

(b) A special case of a fixed index curve is a constant index curve \( (i(r),j(r)) = (i,j) \). Using this, the result follows directly from the proof of (a).

Q.E.D.

Let \( t \geq t_o \), \( t \in \mathbb{Z} \). Then to each curve \( Y^+ \in L^+(-\infty,0] \) corresponds a parameter sequence \( \{(\theta^+_{t-t_o},^t_Y^+,^{t-t_o}Y_j)\}_{t=0}^{t-t_o} \) according to the ('backward') formula
\[
(6.5-8) \quad (\theta^+_{t-t_o},^t_Y^+,^{t-t_o}Y_j) = \begin{cases} 
Y^+(0) & \text{if } t = 0, \\
Y^+(t-1) & \text{if } t = 1,2,\ldots,t-t_o \\
Y_j^+(t-1) & \text{if } j = t-t_o
\end{cases}
\]
(cf. remark (6.5-5)). Let \( \eta_L: L^+(-\infty,0] \rightarrow (0)^{t-t_o+1} \) be the mapping which maps
\[ y^+ \in L^+(\rightarrow,0) \text{ to } \{(\theta_{t-\tau,t-\tau},j_{t-\tau})\}_{\tau=0}^{t-\tau} \]
given by (6.5-8).

Let us formally extend the definition of the sequence \(a_\tau\) to all \(t < t_0\) by taking \(a_t = a_{t_0}\) for all \(t < t_0\). Let \(a = (a_t)_{t=t_0}^\infty\) denote the resulting sequence. Then \(n^a\) can be defined as:

\[
 n^a_t : L^+(\rightarrow,0) \times (\Theta_t)^N
\]

(6.5-9)

\[
 y^+ \rightarrow \{(\theta_{t-\tau,t-\tau},j_{t-\tau})\}_{\tau=0}^\infty \text{ with }
\begin{cases}
  y^+(0) & \text{if } \tau = 0, \\
  y^+(-\sum_{j=t-\tau} a_j), & \text{if } \tau = 1, 2, \ldots
\end{cases}
\]

6.5-10. Remarks. (i) The sets \((\Theta_t)^{t=t_0+1}\) and \((\Theta)^N\) will be considered as topological spaces with the topology of pointwise convergence. This coincides with the compact-open topology, of course.

(ii) In \(n^a_t\), the sequence \(a = (a_t)\) can be replaced by any sequence \(\tilde{a} = (\tilde{a}_t)\) with \(\tilde{a}_t \geq 0\), \(\forall t\). Notation: \(n^\tilde{a}_t\).

It is not difficult to show:

6.5-11. Theorem. For each \(t\) and each \(\tilde{a} = (\tilde{a}_t)\) with \(\tilde{a}_t \geq 0\) for all \(t\), \(n^\tilde{a}\) is continuous.

The topological spaces \(L(\rightarrow,0), L^+(\rightarrow,0), (\Theta)^N\) and \((\Theta_t)^N\) can be provided with a metric in the following manner.

6.5-12. Definition. Let \(d\) denote the metric on \(\mathcal{H}\).

(a) Let \(d^+ : \Theta \times \Theta^+ \times [0,=] \) be the mapping given by

\[
d^+((\theta,i,j), (\tilde{\theta},\tilde{i},\tilde{j})) = \begin{cases}
 d(\theta,\tilde{\theta}) & \text{if } i = \tilde{i} \text{ and } j = \tilde{j}, \\
 = & \text{if } i \neq \tilde{i} \text{ or } j \neq \tilde{j}.
\end{cases}
\]
(b) Let $\rho_c^+: \mathcal{L}^+(\mathbb{R}, \mathbb{R}) \times \mathcal{L}^+(\mathbb{R}, \mathbb{R}) \to [0,1]$ be given by
\[
\rho_c^+((x^+, x^+), (\tilde{x}^+, \tilde{x}^+)) = \sup_{n \in \mathbb{N}} \frac{1}{n} \sup_{r \in \mathbb{R}} d^+(x^+(r), \tilde{x}^+(r)).
\]

(c) Let $\rho_c^+: \mathcal{L}^+(-\mathbb{R}, \mathbb{R}) \times \mathcal{L}^+(-\mathbb{R}, \mathbb{R}) \to [0,1]$ be given by
\[
\rho_c^+((x^+, x^+), (\tilde{x}^+, \tilde{x}^+)) = \sup_{n \in \mathbb{N}} \frac{1}{n} \sup_{r \in \mathbb{R}} d^+(x^+(r), \tilde{x}^+(r)).
\]

(d) Let $\rho: \mathcal{M} \times \mathcal{N} \to [0,1]$ be given by
\[
\rho(a, b) = \sup_{n \geq 0} \frac{1}{n+1} \sup_{0 \leq i \leq n} d(a_{i+1}, b_{i+1}).
\]

(e) Let $\rho^+: (\mathcal{N}^+) \times (\mathcal{N}^+) \to [0,1]$ be given by
\[
\rho^+(a^+, b^+) = \sup_{n \geq 0} \frac{1}{n+1} \sup_{0 \leq i \leq n} d^+(a_{i+1}, b_{i+1}).
\]

6.5-13. **Theorem.** $\rho_c^+, \rho_c^+, \rho, \rho^+$ are metrics of $\mathcal{L}^+(\mathbb{R}, \mathbb{R}), \mathcal{L}^+(-\mathbb{R}, \mathbb{R}), \mathcal{M}$ and $\mathcal{N}$ respectively, and these metrics are compatible with the given topologies of these spaces (i.e. the corresponding metric topology coincides with the original topology of each space).

Proof. For $\rho_c^+$, cf. [Dug], chapter XII, p. 272, 8.5. The other cases are straightforward modifications of the case with $\rho_c$. This is left to the reader. O.E.D.

Using these metrics, theorem (6.5-11) can be strengthened:

6.5-14. **Proposition.** Let $a \geq \max_l \{\max_t \tilde{a}_t\}$. Then for each $t \in \mathbb{Z}$:
\[
\forall x^+, \tilde{x}^+ \in \mathcal{L}^+(-\mathbb{R}, \mathbb{R}), \rho_c^+(\tilde{a}_t, \tilde{a}_t) \leq a \cdot \rho_c^+(x^+, \tilde{x}^+).
\]

Proof. It suffices to show the inequality for all $a \geq \max_l \{\max_t \tilde{a}_t\}$. One has
\[
\rho_c^+(x^+, \tilde{x}^+) = \sup_{n \in \mathbb{N}} \frac{1}{n} \sup_{-n \leq r \leq 0} d^+(x^+(r), \tilde{x}^+(r)).
\]
\[ \geq \sup \left[ \min_{n \in \mathbb{N}} \left( \sup_{\sum_{j=0}^{t} (x^t_j - \sum_{j=0}^{t} a_j) j \leq t} \right) \right] \]

To simplify the notation let

\[ \theta^+_{t-t} := x^t \left( - \sum_{j=0}^{t} a_j \right) \in \theta^+ \]

and

\[ \bar{\theta}^+_{t-t} := x^t \left( - \sum_{j=0}^{t} a_j \right) \in \theta^+ \]

It follows that

\[ \rho_c (x^c, \bar{x}^c) \geq \sup \left[ \min_{n \in \mathbb{N}} \left( \sup_{0 \leq a \leq n} \right) \right] \]

\[ \geq \frac{1}{a} \sup \left[ \min_{n \in \mathbb{N}} \left( \sup_{0 \leq a \leq n} \right) \right] \]

where as before \[ \frac{n}{a} \] denotes the entier of \[ \frac{n}{a} \]. This last expression is equal to

\[ \frac{1}{a} \sup \left[ \min_{n \in \mathbb{N}} \left( \sup_{0 \leq a \leq n} \right) \right] = \frac{1}{a} \rho_c (x^c, \bar{x}^c) \]

Q.E.D.

6.5-15. Remark. Because \( \forall t < t' : a'_{t - t}' = a'_{t - t}' \), by definition, and because \( \lim_{t \to \infty} a'_{t - t}' = a'_{t - t}' \), the \( \{ a'_{t - t}' \}_{t = -\infty}^{\infty} \) has a finite maximum. So the proposition can be applied to \( \{ a'_{t - t}' \} \).

The connection between some of the spaces of interpolation curves treated here and the classes of parameter sequences that were presented in the previous section will be treated next.

The mapping \( \bar{a}_{t} \), defined earlier on the space \( L^*_c (\rightarrow, 0) \), can also be defined on \( L^*_c \) as follows. Let \( \bar{a}_{t} \) be the mapping which maps \( \bar{x}^c \in L^*_c \) to \( \{ \theta^+_{t-t} \}_{t = -\infty}^{\infty} \) given by (compare (6.5-8))
\[ \theta^+_{t+t} = \begin{cases} 
  x^t_{j=t} & \text{if } t > 0, \\
  x^0 & \text{if } t = 0, \\
  x^{-(t-\tau)}_{j=t+\tau} & \text{if } t - t \leq \tau < 0, \text{ and} \\
  x^{t-1}_{j=t} & \text{if } \tau < t - t.
\end{cases} \]

Notice that, because of the shift-invariance of \( L^+_c \), the image \( \Pi_c(T_c^+) \) is the same for each \( t \in \mathbb{Z} \). A similar conclusion holds for any shift-invariant subspace of \( L^+_c \) like \( L^+_{CP} \).

6.5-17. Property 3. The parameter sequence \( \{\theta^+_t\}_{t=0}^\infty \) is an element of \( \Pi_c(T^+_c) \).


Proof. Let \( \theta^+ \) be as in (6.4-58). Let \( X^+ \in \text{L}^+_{CP} \) be such that \( \{\theta^+_t\}_{t=0}^\infty = \Pi_c(X^+) \).

Using (6.5-16) and the Lipschitz property of \( X \) one finds that for \( s > t \)

\[ d(\theta^+_t, \theta^+_s) = d(X(\Sigma a^+_j), X(\Sigma a^+_j)) \leq c a^+_s \leq c a^+_s. \]

A similar formula holds for \( s \leq t \). This shows that \( \{\theta^+_t\} \) satisfies condition (ii) of property 1 with \( a^+_s \). Condition (i) of property 1 follows directly from the definition of \( \Pi_c \). A coordinate change in the parameter sequence takes place from \( \theta^+_s \) to \( \theta^+_t \) iff a coordinate change in \( X^+ \) takes place on the

\[ \begin{align*}
  \text{interval from } & \Sigma a^+_j \text{ to } \Sigma a^+_j, \\
  j=1 & \text{ to } j=t.
\end{align*} \]

The interval has length \( a^+_s \) and because of the Lipschitz property of \( X \), the points \( \theta^+_s \) and \( \theta^+_{s+1} \) lie at a distance less than \( c a^+_s \) from the point on the curve \( X \) at which the coordinate change takes place. Because \( X^+ \in \text{L}^+_{CP} \) it satisfies the prescription (6.3.2-21). This implies that conditions (iii), (iv) and (v) of property 1 hold for \( \{\theta^+_t\} \) with \( a^+_s = c a^+_s, \forall s \).

Q.E.D.
The following theorem can be compared to theorem (6.4-60).

6.5-19. **Theorem.** The sequence \( \{ s^+_t \}_{t \geq t_0} \) that is produced by the coupled algorithm satisfies property 3 (with \( c = K^+ \) as in corollary (6.4-53)).

**Proof.** This follows from theorem (6.5-3), using \( a_{t+1} \leq a_t \).

O.E.D.

6.6. On a problem of P-a.s. convergence

6.6.1. **Introduction**

In this introduction we try to explain the main ideas concerning the relations between P-a.s. properties in the decoupled case and the coupled case by way of some illustrative propositions.

6.6.1-1. **Example.** Consider a probability space \((\Omega, \mathcal{H}, \mathbb{P})\) and a function

\[
F: \Gamma \times \Omega \rightarrow \Phi
\]

\[(\gamma, \omega) \rightarrow F(\gamma, \omega),\]

with \(\Gamma\) and \(\Phi\) topological spaces, \(\Gamma\) separable (i.e. \(\Gamma\) is a Hausdorff space which contains a countable dense set). Then

6.6.1-2. **Proposition.** Suppose that for each \(\omega \in \Omega\), \(F\) is continuous as a function of \(\gamma \in \Gamma\). Let \(S \subseteq \Phi\) be a closed set such that for each \(\gamma \in \Gamma\):

(6.6.1-3) \( F(\gamma, \omega) \in S, \ P\text{-a.s.} \)

Let \( \Omega \times \Gamma \) be arbitrary. Then

(6.6.1-4) \( F(\Omega, \omega) \in S, \ P\text{-a.s.} \)

**Proof.** Let \( \{ \gamma_1, \gamma_2, \ldots \} \) be a countable, dense subset of \(\Gamma\). Then for each \(i \in \mathbb{N}\), \(F(\gamma_i, \omega) \in S, \ P\text{-a.s.} \). Therefore, for each \(i \in \mathbb{N}\) there exists a set \(E_i \subseteq \Omega\) of 'exceptions', with \(P(E_i) = 0\) and

\[
\forall \omega \in \Omega \setminus E_i : F(\gamma_i, \omega) \in S.
\]
Let \( E := \bigcup_{i \in \mathbb{N}} E_i \). Then \( F(E) = 0 \) and 
\[
\forall i \in \mathbb{N}: \forall \omega \in \Omega \setminus E: F(\gamma_i, \omega) \in S.
\]

Now let \( \gamma \) be arbitrary. Because \( \{\gamma_i\} \) is dense in \( \Gamma \), there exists a subsequence \( \{\gamma_{i(k)}\}_{k \in \mathbb{N}} \) that converges to \( \gamma \). It follows that 
\[
\forall \omega \in \Omega \setminus E: F(\gamma, \omega) = \lim_{k \to \infty} F(\gamma_{i(k)}, \omega) \in S,
\]
because \( S \) is closed. So 
\[ (6.6.1-5) \quad \forall \gamma \in \Gamma: \forall \omega \in \Omega \setminus E: F(\gamma, \omega) \in S. \]

This implies clearly 
\[
\forall \omega \in \Omega \setminus E: F(G(\omega), \omega) \in S,
\]
and so 
\[
F(G(\omega), \omega) \in S, \text{ P-a.s.}
\]

\[ \text{Q.E.D.} \]

This theorem is meant to give an idea of the kind of reasoning that is involved; it will not be used. In our application of this kind of idea, \( \gamma \) will be a sequence or curve in the parameter space and \( F \) will be a sequence, while \( \omega \) represents the data \( \{\gamma_k\} \).

Now consider the following situation. Let \( \{F_k(\gamma, \omega)\}_{k=1}^{\infty} \) be a sequence of functions 
\[ (6.6.1-6) \quad F_k: \Gamma \times \Omega \to \Phi, \]
where \( \Gamma \) is now a compact topological space, \( \Phi \) is a normed linear space with norm \( 1,1 \), \( (\Omega, \mathcal{F}, \mathbb{P}) \) is a probability space, like in the example above. Consider the following set of functions of \( \gamma \).
(6.6.1-7) \( \{ F_k(\cdot;\omega) | k \in \mathbb{N}, \omega \in \Omega \} \).

Suppose it is equicontinuous on \( \Gamma \), i.e. for each \( \gamma_0 \in \Gamma \) and for each \( \varepsilon > 0 \) there exists a neighbourhood \( N \) of \( \gamma_0 \) such that

\[
\forall k \in \mathbb{N}, \forall \omega \in \Omega, \forall \gamma \in N: \| F_k(\gamma, \omega) - F_k(\gamma_0, \omega) \| < \varepsilon.
\]

(For the general definition of equicontinuity, cf. e.g. [Dug]). Then the following holds.

6.6.1-8. Proposition. Suppose that for each \( \gamma \in \Gamma \),

\[
\lim_{k \to \infty} F_k(\gamma; \omega) = 0, \text{ P-a.s.}
\]

(a) Then \( \exists N \subseteq \Omega \), with \( P(N) = 0 \) such that

\[
\forall \omega \in \Omega \setminus N, \forall \{ \gamma_k \} \subseteq \Gamma: \lim_{k \to \infty} F_k(\gamma_k, \omega) = 0.
\]

(b) Let \( \{ G_k \}_{k \in \mathbb{N}} \) be a sequence of (arbitrary) mappings \( G_k: \Omega \to \Phi \). Then

\[
\lim_{k \to \infty} F_k(G_k(\omega); \omega) = 0, \text{ P-a.s.}
\]

Proof. (b) follows directly from (a), so it suffices to prove (a).

It will be shown that

(6.6.1-9) \( \forall \varepsilon > 0: \exists N \subseteq \Omega \) such that \( \forall \omega \in \Omega \setminus N \),

for all sequences \( \{ \gamma_k \}_{k \in \mathbb{N}} \): \( \limsup_{k \to \infty} \| F_k(\gamma_k, \omega) \| < \varepsilon \).

Once this is established the result follows easily: Let \( E \subseteq \Omega \) denote the exceptions set for a given \( \varepsilon > 0 \), then \( P(E) = 0 \). Let

\[
E = \bigcup_{n \in \mathbb{N}} \bigcap_{n' \in \mathbb{N}} \{ 1/n' \}
\]

then \( P(E) = 0 \) and

\[
\forall \omega \in \Omega \setminus E, \forall \{ \gamma_k \}_{k \in \mathbb{N}} \subseteq \Gamma: \lim_{k \to \infty} F_k(\gamma_k, \omega) = 0.
\]
Now (6.6.1-9) will be shown. Let $\varepsilon > 0$ be fixed. Because the set of functions in (6.6.1-7) is equicontinuous, for each $\gamma_0 \in \Gamma$ an open neighbourhood $N_{\gamma_0}$ can be found such that

\[(6.6.1-10)\ \forall k \in N, \forall \omega \in \Omega, \forall \gamma \in N_{\gamma_0}, \|F_k(\gamma, \omega) - F_k(\gamma_0, \omega)\| < \frac{\varepsilon}{2}.
\]

Clearly $\{N_{\gamma} | \gamma_0 \in \Gamma\}$ forms an open cover of $\Gamma$. Because $\Gamma$ is compact there exists a finite subcover, say $N_{\gamma_1} \cup N_{\gamma_2} \cup \cdots \cup N_{\gamma_n}$. For each $i \in \{1, 2, \ldots, n\}$

\[(6.6.1-11)\ \lim_{k \to \infty} F_k(\gamma_i, \omega) = 0, \text{ P-a.s.}
\]

Let for each $i \in \{1, 2, \ldots, n\}$, $E_i$ be the exceptions set, $P(E_i) = 0$. Let

$$\overline{E} = \bigcup_{i=1}^{n} E_i,$$

then $P(\overline{E}) = 0$. For each $\omega \in \Omega \setminus \overline{E}$ and for each $i \in \{1, 2, \ldots, n\}$ there exists a $k_i(\omega)$ such that

\[(6.6.1-12)\ \forall k \geq k_i(\omega): \|F_k(\gamma_i, \omega)\| < \varepsilon/2.
\]

Let $k_0(\omega) := \max_{1 \leq i \leq n} k_i(\omega)$. Then one has for each $\omega \in \Omega \setminus \overline{E}$:

\[(6.6.1-13)\ \forall k \geq k_0(\omega), \forall i \in \{1, 2, \ldots, n\}: \|F_k(\gamma_i, \omega)\| < \varepsilon/2.
\]

Let $\{\gamma_i\}_{i=1}^{\infty}$ be an arbitrary sequence in $\Gamma$. Because $\cup N_{\gamma_1} = \Gamma$, for each $k \in N_{\gamma_1}$ there exists an $i(k) \in \{1, 2, \ldots, n\}$ such that $\gamma_k \in N_{\gamma_i(k)}$. Therefore applying (6.6.1-10) and (6.6.1-13),

$$\forall \omega \in \Omega \setminus \overline{E}, \forall k \geq k_0(\omega):$$

$$\|F_k(\gamma_k, \omega)\| \leq \|F_k(\gamma_i(k), \omega) - F_k(\gamma_i(k), \omega)\| + \|F_k(\gamma_i(k), \omega)\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So

$$P(\overline{E}) = 0 \text{ and } \forall \omega \in \Omega \setminus \overline{E}, \|\gamma_k \psi_k(\Gamma)\|_{k=1}^{\infty} \leq \limsup_{k \to \infty} \|F_k(\gamma_k, \omega)\| \leq \varepsilon.$$
i.e. (6.6.1-9) is shown.

6.6.2. On (equi-)continuity of some variables in the algorithm

To be able to apply the ideas of section 6.6.1 in the analysis of the algorithm, one needs to establish equicontinuity of several variables in the algorithm. More precisely each of these variables will be considered to be a family of functions. The family of functions is obtained by considering the decoupled algorithm, and considering the variable involved as a function of a parameter interpolation curve, for each time \( t \) and each data sequence \( \omega \). By letting \( t \) and \( \omega \) vary over all possibilities one obtains a family of functions.

6.6.2-1 Notation. Let \( \omega = \{y_t\}_{t=t_0}^m \) denote the data sequence, and let

\[ b_t = b_{t_0} = r_1(t)R_2(t)R(\theta_t,j_t)^{-1}h \]

(compare (6.2.8-5)).

So if \( \{\theta_t\} \) is well-defined and \( \lambda_{t,\theta_t} \neq 0 \),

\[ b_t = b_{t_0} = (\lambda_{t,\theta_t})^{-1}q(\theta_{t_1},j_{t_1})q(\theta_t,j_t) \]

(cf. also (6.2.9-2)).

For each \( t \) and \( \omega \), \( b_{t_0} \) can be considered in the decoupled algorithm as a function \( b_{t_0}(a) \) of the sequence of past parameter values

\[ a = \{\theta_{t-T} \in \theta_+ \}_{t=t_0}^m \]

Now if \( \{\theta_+\} \) satisfies property 3 then

\[ (6.6.2) \quad \{\theta_{t-T} \mid t_{t_0} \in \mathbb{R}_{\mathbb{R}}(\mathbb{R}_{\mathbb{R}}(-\omega,0)) \}. \]

So \( b_{t_0} \circ \mathbb{P}_t \) is a function of the elements of \( L_{cp}^{+}(-\omega,0) \) for each \( t \geq t_0 \) and each \( \omega \). To show equicontinuity of the \( b_{t_0} \circ \mathbb{P}_t \) the following technical lemma is needed.
6.6.2-3. **Lemma.** Let \( t_3 \) be as in corollary (6.4-55). For each \( \varepsilon' > 0 \) there exists a \( \delta > 0 \) such that for each \( \omega \in \Omega \), for all \( t \geq t_3 \) and for all \( \alpha, \beta \in \mathbb{P}_c^0(L^+_{\mathbb{C}P}(\sim,0)) \subseteq (\Theta^*)^0 \) the following implication holds:

\[
\rho^+(\alpha, \beta) < \delta \Rightarrow t_{\omega}^\beta(a) - b_{\omega}(\beta) \| < \varepsilon'.
\]

Proof. In fact the following will be shown.

\[
\forall \varepsilon' > 0, \exists \delta > 0 \text{ such that } \forall \omega \in \Omega, \forall t \geq t_3, \forall \alpha, \beta \in \mathbb{P}_c^0(L^+_{\mathbb{C}P}(\sim,0)): (\rho^+(\alpha, \beta) < \delta \Rightarrow t_{\omega}^\beta(a) - b_{\omega}(\beta) \| < \varepsilon')
\]

where \( \vec{a} = (\ldots, \vec{a}_{t_0}, \vec{a}_{t_0}, \vec{a}_{t_0}, \vec{a}_{t_0+1}, \vec{a}_{t_0+2}, \ldots, \vec{a}_{t_4-1}, \vec{a}_{t_4}, \vec{a}_{t_4}, \ldots) \) with \( t_4 \)

sufficiently large such that \( \vec{a}_{t_4} > 0 \) is sufficiently small to be able to apply the results derived before under the assumption of property 0 with \( a_\omega > 0 \) sufficiently small.

The proof consists of three parts. Part (i) shows that 'what happened long ago does not have much effect' and part (ii) shows the continuity with respect to the 'recent past'. Part (iii) combines (i) and (ii). Let \( \varepsilon' > 0 \) be fixed.

Choose \( \lambda \in (\lambda_0, \lambda_1) \) (compare (6.2.8-3)), and let \( \{(\tilde{D}_1, \tilde{D}_2, \tilde{D}_3)\} \) be as in theorem (6.3.4-21) with \( \varepsilon = \lambda - \lambda_0 \).

Define \( \delta' \) and \( \delta'' \) as follows

\[
(6.6.2-4) \quad \delta' := \varepsilon \cdot \frac{-4}{d} \cdot \frac{1}{K(H)} \cdot \frac{1}{K(1+c_3)^{-1}(K')^{-1}(\hat{K}(R^{-1}))^{-1}},
\]

\[
(6.6.2-5) \quad \delta'' = (\varepsilon')^2 \cdot \frac{1}{c_3} \cdot \frac{1}{2K(H)} \cdot \frac{1}{K(R^{-1})} \cdot \frac{1}{2}.
\]

According to lemma (6.4-41) there exists for \( \delta = (\lambda_1)^{-1} \cdot 1 \) (and \( c_1 \geq 0, c_2 \geq 0 \) as in the proof of lemma (6.4-8)) a number \( n_1 \in \mathbb{N} \) such that

\[
\forall t \geq t_0, \forall k \geq n_1, t^k - (1+c_1)(1+c_2)^k < (1+\delta)^k = \frac{\lambda_1 k}{\lambda},
\]

and so
\( (6.6.2-6) \quad \forall \tau \geq t_0, \forall k \geq n_1: \prod_{\tau=t}^{t+k-1} \left[ \left( \frac{\lambda}{\lambda_1} \right) \left( 1 + c_1 x_1(\tau) + c_2 x_2(\tau) \right) \right] < \left( \frac{\lambda}{\lambda_1} \right)^k / 2. \)

Now let \( n_0 \) with \( n_0 \geq n_1 \) and \( n_0 \geq (c a_{t_4})^{-1} \) be such that

\( (6.6.2-7) \quad \left( \frac{\lambda}{\lambda_1} \right)^{n_0/2} < \frac{\min(\delta^i, \delta^n)}{2 c_3 R}. \)

This is possible because \( 0 < \lambda < \lambda_1 \). Here \( c_3 \) is as in lemma (6.4-61).

(i) The following will be shown. Let \( t \geq t_3 \) be fixed. If the first \( n_0 + 1 \)
components of \( a, b \in H_0^1(\mathbb{R}_0) \) are equal, i.e., \( a_1 = b_1, i = 0, 1, 2, \ldots, n_0, \)

then \( \exists \beta \in H_0^1(\mathbb{R}_0) \), \( \| \beta \| \leq \epsilon / 4. \)

Of course, if for some \( \omega \), \( v_\omega > K' \) then \( \beta(v_\omega) = 0 \), hence \( b_\omega(a) = 0 \) and so either \( b_\omega(a) - b_\omega(b) \| = 0 \) in this case. Therefore, for this case the assertion is correct. Now suppose \( \omega \) is such that \( v_\omega < K' \). It then follows easily from the definition of \( v_\omega \) that \( \forall \tau \geq 0: v_{t-\tau} \leq K_1^1 \). Because \( a, b \in H_0^1(\mathbb{R}_0) \),

one can assume without loss of generality that property 0 is satisfied with \( s_{\omega, \tau} > 0 \) small enough or property 1 is satisfied, so theorem (6.4-57) is applicable. One obtains

\( (6.6.2-8) \quad \forall \tau \geq 0: \| \xi_{t-\tau} \| \leq c_3 K' \lambda_1^{-1}. \)

(\text{where by convention, one takes} \( \xi_t = 0 \) \text{if} \( t < t_0 \).)

Now consider the difference

\[ \Delta \xi_{t-\tau} = \xi_{t-\tau}(a) - \xi_{t-\tau}(b), \]

in an obvious notation. Now notice that if \( \tau \leq n_0 \), then, because \( a_\tau = b_\tau \), \( \Delta \xi_{t-\tau} \) depends only on \( \Delta \xi_{t-n_0} \) and on the \( a_\tau = b_\tau, \tau = 0, \ldots, n_0 \), and not on the data \( y_{t-\tau}, \tau = 0, \ldots, n_0 \). In a manner, completely analogous to the derivation of equation (6.4-26) in the proof of lemma (6.4-61), one can derive

\( (6.6.2-9) \quad \forall \tau \in \{1, 2, \ldots, n_0\}: \| \Delta \xi_{t-\tau+1} \| \leq \lambda (1 + c_1 x_1(t-\tau) + c_2 x_2(t-\tau)) \| \Delta \xi_{t-\tau} \| \)

and therefore
\[(6.6.2-10) \quad \| \Delta \xi_t \| \leq \sum_{t=1}^{n_0} \lambda (1+c_1 x_1(t-r)+c_2 x_2(t-r)) \| \Delta \xi_{t-n_0} \|.
\]

If \( t < n_0 \) then \( \xi_{t-n_0} = 0 \) and so \( \Delta \xi_{t-n_0} = 0 \), which implies that \( \Delta \xi_t = 0 \).

Consider the case \( t > n_0 \). Because

\[
\| \Delta \xi_{t-n_0} \| \leq \| \xi_{t-n_0}(a) \| + \| \xi_{t-n_0}(b) \|,
\]

application of (6.6.2-8) to (6.6.2-10) leads to the inequality:

\[(6.6.2-11) \quad \| \Delta \xi_t \| \leq \sum_{t=1}^{n_0} \lambda (1+c_1 x_1(t-r)+c_2 x_2(t-r)) \cdot 2 c_3 K^{\lambda^{-n_0}}.
\]

Applying (6.6.2-6) and (6.6.2-7) leads to

\[(6.6.2-12) \quad \| \Delta \xi_t \| \leq \min(\delta', \delta''). \]

Now \( \Delta \xi_t \) will be related to \( \Delta b_{t-w} := b_{t-w}(a)-b_{t-w}(b) \).

This can be done as follows: \( \Delta b_{t-w} = R x R^{-1} \Delta h_t \), so

\[(6.6.2-13) \quad \| \Delta b_{t-w} \| \leq \bar{R} R^{-1} \| \Delta h_t \|.
\]

From the definition of \( h \) it follows that

\[(6.6.2-14) \quad \| \Delta h_t \| \leq \sum_{i=1}^{d} \{ |\Delta \psi^T_{i,t} \xi_t | + |\psi^T_{i,t} \Delta \xi_t | + |\Delta \psi^T_{i,t} \xi_t | \}.
\]

From \( \xi_t = H(e^+_t) \xi_t \) it follows that \( \Delta \xi_t = H^+(e^+_t) \Delta \xi_t \).

So making use of theorem (6.4-57),

\[(6.6.2-15) \quad \| \xi_t \| \leq \bar{R}(H) \| \xi_t \| \leq \bar{R}(H)c_3 \xi_t \leq \bar{R}(H)c_3 K',
\]

and
This implies

$$\|\Delta t \|^2 \leq 2d\|\Delta z\|^2 + d\|\Delta z\|^2 \leq 2d\|R(N)^2c \delta \|^2 + d\|R(N)^2(\delta)\|^2 \leq$$

$$\leq \frac{\delta}{\delta} + \frac{\delta}{\delta}R(N^{-1})^{-1}.$$  

Combining this with (6.6.2-13) gives

$$\|\Delta t_{\tau} \|^2 < \epsilon/4.$$  

(ii) Fix all but the first $n_0+1$ components of the parameter sequence. I.e. Let

$$\{t_{\tau_{\tau}}, t=\tau_{\tau}+1\}_{\tau=0}^{n}$$

be fixed. Then $b_{t_{\tau}}$ can be considered as a function of

$$\{t_{\tau_{\tau}}, t=\tau_{\tau}+1\}_{\tau=0}^{n}$. The domain of this function is

$$B_{b_{t_{\tau}}}(\tau_{\tau_{\tau}}, \tau_{\tau_{\tau}}+1, \tau_{\tau_{\tau}}+1) \in B_{b_{t_{\tau}}}(\tau_{\tau_{\tau}}, \tau_{\tau_{\tau}}+1, \tau_{\tau_{\tau}}+1),$$

so $b_{t_{\tau}} \in (\tau_{\tau_{\tau}})^{n+1}$.

Note that $b_{t_{\tau}}$ is closed and therefore compact. The following property will be shown. For a given $\epsilon \|N_0 > 0$ a $\delta'' > 0$ can be found such that for all

$$a, b \in (\tau_{\tau_{\tau}})$$

with $(a_{\tau_{\tau_{\tau}}}, a_{\tau_{\tau_{\tau}}+1}, \ldots, a_{\tau_{\tau_{\tau}}}) \in D_{b_{t_{\tau}}}(\tau_{\tau_{\tau}}, \tau_{\tau_{\tau}}+1, \ldots, \tau_{\tau_{\tau}}) \in D_{b_{t_{\tau}}}$ and

$$\{a_{\tau_{\tau_{\tau}}}, i=1, \ldots, n_0+1\} \in (\tau_{\tau_{\tau}})^{n+1},$$

the following implication holds

$$\forall \epsilon \in (\tau_{\tau_{\tau}}, \tau_{\tau_{\tau}}+1), d^+((\tau_{\tau_{\tau}}, \tau_{\tau_{\tau}}), \tau_{\tau_{\tau}}') < \frac{1}{2} \epsilon \Rightarrow (\forall \tau \in [t_{\tau_{\tau}}, t_{\tau_{\tau}}+1), b_{t_{\tau}}(a_{\tau_{\tau}} - b_{t_{\tau}}(b_{\tau_{\tau}}) < \frac{1}{2} \epsilon),$$

with $d^+$ as in (6.5-12). Note that $\delta''$ may depend on the choice of the fixed part $\{t_{\tau_{\tau_{\tau}}}, t=\tau_{\tau_{\tau}}+1\}_{\tau=0}^{n+1}$ of the parameter sequence.

Because $\tau_{\tau_{\tau}}$ is a disjoint union $\tau_{\tau_{\tau}} = \cup_{i \in I} \tau_{i,j}^+ \cup_{j \in J}$ of compact subsets $\tau_{i,j}^+$, the Cartesian product $(\tau_{\tau_{\tau}})^{n+1}$ is a disjoint union of Cartesian products.
with

\[(i_0, j_0, i_1, j_1, \ldots, i_n, j_n) \in \{(i, j) | i \in I(j), j \in J \}^{n+1}\]

and this last set is finite. Each set \(D_b \cap (D_i^{1+} \times \ldots \times D_{i_n}^{1+} \times D_{j_0}^{1+} \times \ldots \times D_{j_n}^{1+})\) is compact and so continuity of a function on each such set implies uniform continuity. Let

\[((i_0, j_0), \ldots, (i_n, j_n))\]

be fixed. Just as in (i), if \(v_t > k'\), then \(b_{t\omega}(y) = 0\) for all \(y\), so \(b_{t\omega}(y) = b_{t\omega}(\mathbf{g}) = 0 < \frac{1}{k'}\) in this case.

Now consider the case \(v_t \leq k'\). Then the inequality (6.6.2-8) holds.

It implies

\[(6.6.2-21) \forall t \in \{0, 1, \ldots, n_0\} : \|y_{t^{-\tau}}\| \leq c_{j} k' \lambda_{1}^{-n_0}.\]

So the vectors \(\xi_{t}, \xi_{t-1}, \ldots, \xi_{t-n_0}\) are uniformly bounded by \(c_{j} k' \lambda_{1}^{-n_0}\), independently of \(t, \omega\) (provided \(v_t \leq k'\)) and independently of the choice of the parameter sequence. A similar argument can be set up for the observations \(y_t\).

Because \(v_t \leq k'\) one has by definition of \(v_t\), \(\|y_{t}\| \leq v_t \leq k'\), and more generally

\[(6.6.2-22) \|y_{t^{-\tau}}\| \leq v_{t^{-\tau}} \leq k' \lambda_{1}^{-t}, \tau = 0, 1, 2, \ldots\]

and so

\[(6.6.2-23) \forall t \in \{0, 1, 2, \ldots, n_0\} : \|y_{t^{-\tau}}\| \leq \frac{1}{k'} \lambda_{1}^{-n_0},\]

independently of \(t, \omega\) (provided \(v_t \leq k'\)). From this the property (6.6.2-20) can be proved in a straightforward manner, by writing out the formula for \(b_{t\omega}\) in terms of \(\beta_{t}, \beta_{t-1}, \ldots, \beta_{t-n_0} ; y_t, y_{t-1}, \ldots, y_{t-n_0}\) and

\((\xi_t, \xi_{t-1}, \ldots, \xi_{t-n_0 + 1}, \xi_{t-n_0}).\)

Using the fact that \(g_2(t) = 1, \forall t \geq t_3\) (so \(g_2\) cannot cause discontinuities here) and using the upperbounds for the \(\|y_{t^{-\tau}}\|\) and the \(\|y_{t^{-\tau}}\|, \tau = 0, 1, 2, \ldots, n_0\) and using compactness of the parameter space,
(6.6.2-20) follows. The details are left to the reader.

(iii) Combining (i) and (ii) can be done as follows:

Suppose \( p^+(a, \beta) < \min((4+n_0^{-1}, \delta) ) =: \delta \), with \( a, \beta \in \mathbb{N}_{\mathbb{CP}}(-\infty, 0] \).

Consider
\[
a = (a_0^+, a_1^+, a_2^+, \ldots, a_{n_0}^+, a_{n_0+1}^+, \ldots),
\]
\[
\tilde{a} := (a_0^+, a_1^+, a_2^+, \ldots, a_{n_0}^+, a_{n_0}^+, a_{n_0+1}^+, \ldots),
\]
\[
\gamma := (\beta_0^+, \beta_1^+, \ldots, \beta_{n_0}^+, \beta_{n_0}^+, \beta_{n_0+1}^+, \ldots),
\]
and
\[
\beta = (\beta_0^+, \beta_1^+, \ldots, \beta_{n_0}^+, \beta_{n_0}^+, \beta_{n_0+1}^+, \ldots).
\]

Because \( a \in \mathbb{N}_{\mathbb{CP}}^+(\mathbb{CP}) \) it follows easily that \( \tilde{a} \in \mathbb{N}_{\mathbb{CP}}^+(\mathbb{CP}) \).

Now recall that \( n_0 \) has been chosen, by definition, such that \( n_0^{-1} \leq \frac{c_{\mathbb{CP}}}{n_0} \), so
\[
\delta < n_0^{-1} \leq c_{\mathbb{CP}}.
\]

It follows that \( d^+(a_{n_0}^+, \beta_{n_0}^+ \gamma) < c_{\mathbb{CP}} \), and that \( a_{n_0}^+, \beta_{n_0}^+ \gamma \) lie in the same coordinate chart \( D_{i,j} \).

Using also that \( a, \beta \in \mathbb{N}_{\mathbb{CP}}^+(\mathbb{CP}) \) it follows that \( \gamma \in \mathbb{N}_{\mathbb{CP}}^+(\mathbb{CP}) \).

Therefore (i) and (ii) can be applied to the pairs \( a, \tilde{a}, \gamma, \) and \( \gamma, \beta \):

\[
\begin{align*}
&\| b_{t_0}(a) - b_{t_0}(\beta) \| \leq \| b_{t_0}(a) - b_{t_0}(\tilde{a}) \| + \| b_{t_0}(\tilde{a}) - b_{t_0}(\gamma) \| \\
&\| b_{t_0}(\gamma) - b_{t_0}(\beta) \| < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon.
\end{align*}
\]

Q.E.D.

6.6.2-24. Theorem. Let \( t_3 \) be as in corollary (6.6-55).

The set of functions
\[
\{ b_{t_0} \circ \mathbb{N}_{\mathbb{CP}}^+(\mathbb{CP}) \to \mathbb{R} \mid t \geq t_3, \omega \in \Omega \}
\]
is equicontinuous.

Proof. This follows immediately from lemma (6.6.2-3) and proposition (6.5-14).

Q.E.D.
Let \( S_{-r} \), \( r \geq 0 \) be the shift defined by

\[
S_{-r} : L^{p}_{c}(\mathbb{R}^{d}) \rightarrow L^{p}_{c}(\mathbb{R}^{d}),
\]

\[
(S_{-r}Y)(s) = Y(s-r).
\]

6.6.2-25. **Corollary.** Let \([b, c] \subseteq [0, \infty)\) be an arbitrary closed interval. The set of functions

\[
\{ h_t \circ \Pi_{t} \circ S_{-r} : L^{p}_{c}(\mathbb{R}^{d}) \rightarrow \mathbb{R} | t \geq t_3, \ r \in [b, c], \omega \in \Omega \}
\]

is equicontinuous.

Proof. This follows from lemma (6.6.2-3) and proposition (6.5-14), using the fact that for all \( s > 0 \) and all \( r \in (0, s) \) the following implication holds

\[
\rho_{c}(\nu^{+}, \nu^{+}) < \frac{1}{s} \Rightarrow \rho_{c}(\nu^{+} \circ S_{r}, \nu^{+} \circ S_{r}) < \frac{1}{s-r}.
\]

The details are left to the reader. Q.E.D.

Next the accumulated effects of the steps taken by the algorithm will be investigated. Consider sums of the form

\[
\sum_{t \in \mathbb{N}} \mathbb{A} \mathbb{B}, \quad \mathbb{A} \mathbb{B} \in \mathbb{N}
\]

where the \( \mathbb{N} \) are intervals such that the sums \( \sum \mathbb{A} \mathbb{B} \) are approximately equal to a constant independent of \( k \). Sums like (6.6.2-26) can and will be considered as integrals of piecewise constant functions, defined on the 'contracted time scale'. Let the partial sums of the series \( \sum \mathbb{A} \mathbb{B} \) be denoted

\[
\sum_{t=0}^{T} \mathbb{A} \mathbb{B}, \quad \mathbb{A} \mathbb{B} \in \mathbb{N}
\]

by \( s_t, t > t_0 \) and let \( s_{t_0} = 0 \) and \( s_t = -T \mathbb{A} \mathbb{B} \) if \( t < t_0 \). Let the piecewise
constant function \( b^0(r) \) be given by

\[
(6.6.2-27) \begin{cases}
    b^0(r) = b_{t_0} & \text{if } s_{t_0} = 0 < r < s_{t_0+1} = a_{t_0}, \\
    b^0(r) = b_t & \text{if } s_t \leq r < s_{t+1} \text{ for some } t > t_0.
\end{cases}
\]

6.6.2-28. **Proposition.** Let \( N = \{ t | s_t \in [x,y] \} \), for some \( x, y \in \mathbb{R}, x \leq y \). Let \( t' = \min N, t'' = \max N \). Then

\[
(6.6.2-29) \int_x^y b^0(r) dr = \sum_{t \in N} a_t b_t + (s_{t'-x})b_{t'-1} - (y-s_{t''+1})b_{t''},
\]

**Proof.**

\[
\int_x^y b^0(r) dr \quad \int_x^y b^0(r) dr \quad \int_x^y b^0(r) dr \quad \int_x^y b^0(r) dr = \sum_{t \in N} a_t b_t - (s_{t'+1} - y)b_{t''}.
\]

Q.E.D.

To study the asymptotic behaviour use will be made of sequences of intervals, to be more precise: sequences of disjoint intervals in \([0,\infty)\), with monotonically increasing sequence of right endpoints, of equal interval length \( q \) and such that the distance between any two intervals is greater than or equal to \( \delta \) for some predetermined \( \delta > 0 \). Each such sequence of intervals can be associated with a triple \( \mathcal{S} = (S, q, \delta) \) with \( 0 < \delta < q \) and \( S = \{ r_k \}_{k=1}^\infty \) a monotonically increasing sequence of positive numbers such that

\[
(6.6.2-30) \begin{cases}
    r_1 \geq q, \quad \text{and} \\
    \forall k \geq 2: r_k - q \geq r_{k-1} + \delta.
\end{cases}
\]

\[
\left[ r_1 - q, r_1 \right], \quad \left[ r_2 - q, r_2 \right], \quad \cdots
\]

6.6.2-31. **Notation.** The set of all such triples \( \mathcal{S} \) will be denoted by \( \mathcal{S} \).

6.6.2-32. **Definition.** Let \( \{ \delta_t^x \} = (s_t, l_t, j_t) \) \( T \) be a parameter sequence with
\( \theta_t^+ \in \Theta^+ \) for each \( t \), and \( T \in Z \) or \( T = - \). Let \( \tilde{S} \in \mathfrak{S}^* \). If \( k \in \mathbb{N} \) is such that \( s_{T+1} > r_k \), then the coordinate-chart-index set \( I_{(\tilde{S},(\theta_t^+)^T)} \) is defined by

\[
(6.6.2-33) \quad I_{(\tilde{S},(\theta_t^+)^T)} = \{(1,1) : s_{1,1} \in [r_k-q, r_k]\}.
\]

If \( T = - \) then the coordinate-chart-index set \( I(\tilde{S},(\theta_t^+)^-\infty) \) is defined as

\[
(6.6.2-34) \quad I(\tilde{S},(\theta_t^+)^-\infty) = \bigcup_{k \in \mathbb{N}} I_{(\tilde{S},(\theta_t^+)^-\infty)}.
\]

Let \( Y^+ \in \mathfrak{L}^+ \), then \( Y^+(r) = (Y(r),i(r),j(r)) \). Let \( \tilde{S} \in \mathfrak{S}^* ; I_{(\tilde{S},Y^+)} \) is defined by

\[
I_{(\tilde{S},Y^+)} = \{(i,j) : \exists r \in [r_k-q, r_k] \text{ such that: } (i,j) = (i(r),j(r))\}
\]

and

\[
I(\tilde{S},Y^+) = \bigcup_{k=1}^{\infty} I_{(\tilde{S},Y^+)}.
\]

6.6.2-35. Remarks. (i) If one deals with a parameter sequence \( \{\theta_t^+\}_{T_0}^{T} \), \( T_0 \in Z \) or \( T = - \) one can formally extend it to \( - \) by taking \( \theta_t^+ = \theta_t^+ \) for all \( t < T_0 \), and apply the previous definition to the resulting sequence.

(ii) \( I(\tilde{S},(\theta_t^+)^+)) \) and \( I(\tilde{S},Y^+) \) can be interpreted as the set of all the 'relevant' coordinate-chart indices for the combination \( (\tilde{S},(\theta_t^+)^+) \) resp. \( (\tilde{S},Y^+) \).

6.6.2-36. Definition. Let \( \tilde{S} \in \mathfrak{S}^* \) be fixed. For each \( k \in \mathbb{N} \), for each \( \omega \in \Omega \) and for each \( \{\theta_t^+\}_{T}^{T} \) with \( T \in Z \) or \( T = - \) such that \( s_{T+1} > r_k \), let \( f_k = f_{\omega_k}(\{\theta_t^+\}_{T}^{T}) \) be given by

\[
(6.6.2-37) \quad f_k = \begin{cases} 0 & \text{if } I_k > 1, \text{ and} \\ r_k & \text{if } I_k = 1, \\ \int_{r_k-q+\delta}^{r_k} b^\omega (r) dr, & \text{if } I_k = 1, \\ \end{cases}
\]

where \( I_k = I_{(\tilde{S},(\theta_t^+)^T)} \).

For fixed \( \tilde{S}, \omega \) and \( k \), \( f_k \) can be considered as a 'function' of the interpolation
curve $Y^+ \in L^+_{cP}(-\infty, 0]$ (just like the variables $b_k$). To state this precisely, let $\Pi^T_a$ (or $\Pi^T_a$ with $a = \{a_t\}$ if we want to stress the dependence on the sequence $a$) be defined by

$$\Pi^T_a: L^+_{cP}(-\infty, 0] \to (0^+)_{N^0}$$

(6.6.2-38)

$$\Pi^T_a(\theta^+) = \{(\theta^+_{t-T})_{t=0}^\infty\} \text{ with } \theta^+_{t-T} = Y^+(a_{t-T})$$

and $t$ such that $0 \leq r - s_t < a_t$ (compare (6.5-9)). Analogously to (6.5-11) one has

6.6.2-39. Theorem. For each $r$ and each $\tilde{a} = \{\tilde{a}_t\}$ with $\tilde{a}_t \geq 0$ for all $t$, $\Pi^T_{\tilde{a}}$ is continuous.

The proof is left to the reader.

Now $f_{k, \omega}^{T, k}$ is a function on $L^+_{cP}(-\infty, 0]$ for each $k$ and $\omega$ (for fixed $S$).

The following result is a basic one of this section

6.6.2-40. Theorem. Let $S^*$ be fixed. The set of functions

$$\{f_{k, \omega}^{T, k} \vert \Pi^T_a: L^+_{cP}(-\infty, 0] \to \mathbb{R}^d \vert k \in N, \omega \in \Omega\}$$

is equicontinuous.

Proof. For each $k \in N$, $f_{k, \omega}^{T, k}$ is a weighted sum of $d$-vectors $b_{k, \omega} \circ \Pi^T_a(\theta^+)$ of which the (positive) weights are adding up to $q-\delta$ ($= \text{the length of the interval of integration}$). Therefore it is sufficient to show that these $b_{k, \omega} \circ \Pi^T_a \circ S_{k, \omega}^{-r_k}$ are equicontinuous. Here $S_{k, \omega}^{-r_k}$ is the shift over a length $r_{k, \omega}^{-r_k}$. This length is bounded, because $0 \leq r_{k, \omega}^{-r_k} < q-\delta + \max a_k$ holds for all $k$ for which $b_{k, \omega}$ appears with a positive weight in the definition of $f_{k, \omega}$. Therefore corollary (6.6.2-25) applies and the result follows.

Q.E.D.
It will be important for our purposes to show that asymptotically, for \( k \to \infty \), the functions \( f_k \circ \Pi^+(\gamma^+) \) depend only on the behaviour of the interpolation function \( \gamma^+ \) on the interval \([-q,0]\).

6.6.2-41. Theorem.
Let \( S = (s,q,\delta) \in S \). Then \( \forall \varepsilon > 0 \), \( \exists k_2 \in \mathbb{N} \) such that \( \forall k \geq k_2, \forall \omega \in \Omega \) the following holds: if \( \gamma^+_1, \gamma^+_2 \in L^+_{cp}(-\infty,0] \) and \( \gamma^+_1|_{[q,0]} = \gamma^+_2|_{[-q,0]} \), then
\[
\| f_k \circ \Pi^+(\gamma^+_1) - f_k \circ \Pi^+(\gamma^+_2) \| < \varepsilon.
\]

Proof. Let \( \varepsilon > 0 \) be given. From lemma (6.6.2-3) it follows that there exists an \( n_0 \in \mathbb{N} \) such that for all \( t \geq t_\delta \) and for all \( \omega \in \Omega \),
\[ \forall a, b \in \Pi^+(t_{cp}(-\infty,0]) \text{ with } a_1 = b_1, i = 0, 1, 2, 3, \ldots, n_0, \text{ one has} \]
\[ (6.6.2-42) \| b_{tw}(a) - b_{tw}(b) \| < \varepsilon q^{-1}. \]

Now let \( k_2 \) be such that for all \( k \geq k_2 \) the following three conditions hold simultaneously. Let \( \tau_k := \min \{ t \in \mathbb{T} \mid t \geq r_k - q + \delta \} \).

(i) \( \tau_k > t_\delta \) (i.e., \( r_k - q + \delta > s_{t_\delta} \)).

(ii) \( \sup_{t \geq \tau_k} a_t < \min(\hat{s}_{t_\delta}, q - \delta) \)

(this implies \( r_k - q + \delta/2 < s_{\tau_k - 1} < r_k - q + \delta \) and \( s_{\tau_k} < r_k \)).

(iii) \( s_{\tau_k - 1} - s_{\tau_k - 1 - n_0} < \frac{\delta}{2} \) (this implies, using (ii),

that \( s_{\tau_k - 1 - n_0} > r_k - q \)).

It follows that for all \( k \geq k_2 \), if \( \gamma^+_1, \gamma^+_2 \in L^+_{cp}(-\infty,0] \) and \( \gamma^+_1|_{[-q,0]} = \gamma^+_2|_{[-q,0]} \), then for all \( t \) with \( s_t \in [r_k - q + \delta, r_k] \), the sequences \( \Pi_t \circ S_{-(r_k - s_t)}(\gamma^+) \) with \( t = 1, 2 \) have equal first \( n_0 \) terms:
\[
\Pi_t \circ S_{-(r_k - s_t)}(\gamma^+) = [Y_t^+(-\sum_{j=t-1}^{t} a_j + s_t - r_k)]_{i=0}^{m} = [Y_t^+(s_t - r_k)]_{i=0}^{m},
\]
according to (6.5–9) and the definition of $s_t$ (just before (6.6.2–27)).
Because $s_t \in [r_{k-\delta}^{-q}, r_k]$, it follows from (i), (ii) and (iii) that for
$i = 0,1,\ldots,n_0$, one has $s_{t-1} - r_k \in [-q,0]$, and so for all $i = 0,1,2,\ldots,n_0$,
$y_1^+(s_{t-1} - r_k) = y_2^+(s_{t-1} - r_k)$. Combining this with (6.6.2–42) one obtains for all
$t$ with $s_t \in [r_{k-\delta}, r_k]$,

$$\begin{align*}
(6.6.2–43) \quad \|b_{t_0} \circ \Pi_t \circ S_{-(r_k - r_{t_0})} (y_1^+) - b_{t_0} \circ \Pi_t \circ S_{-(r_k - r_{t_0})} (y_2^+)\| &< \epsilon q^{-1}.
\end{align*}$$

What does this imply for the expression $f_k \circ \Pi_k (y_1^+) - f_k \circ \Pi_k (y_2^+)$?

First notice that $I_k(\Pi_k(y_1^+)) = I_k(\Pi_k(y_2^+))$ because
$y_1^+[-q,0] = y_2^+[-q,0]$. Therefore there are only two possibilities, namely
(a) $|I_k| > 1$ in both cases (i.e. for $y_1^+$ and $y_2^+$) or
(b) $|I_k| = 1$ in both cases. In case (a), according to (6.6.2–37),

$$f_k \circ \Pi_k (y_1^+) - f_k \circ \Pi_k (y_2^+) = 0,$$

and in case (b), combining (6.6.2–37) with (6.6.2–43),

$$f_k \circ \Pi_k (y_1^+) - f_k \circ \Pi_k (y_2^+) =$$

$$= \|r_k \int_0^{r_k} (b^1(r) - b^2(r))dr\| \epsilon q^{-1} < \epsilon,$$

where $b^\ell(r)$, $\ell = 1,2$ is defined just as $b^\ell(r)$ in (6.6.2–27), with as parameter
sequence $\Pi_k (y_1^+)$, $\ell = 1,2$ respectively.

Q.E.D.

6.6.3. On the relation between the coupled and the decoupled algorithms.

Having derived the necessary equicontinuity properties of the relevant
variables in the previous subsections, we are now ready to apply the ideas
that were introduced in section 6.6.1.

6.6.3–1. Definition. The probability measure $\mathbb{P}$ on $(\Omega, \mathbb{H})$ (i.e. on the data)
induces a probability measure on the variables of the decoupled algorithm
(described at the beginning of section 6.4) and all variables derived from
those, like $\{f_k\}$ and $\{b^\ell\}$. This probability measure will be denoted by $\bar{\mathbb{P}}$. The
corresponding expectation operator will be denoted by $\bar{E}$.

6.6.3-2. Remarks. (1) Because the decoupled algorithm is only well-defined if
the parameter sequence $\{\theta_k^+|t^+| \in \mathcal{O}^+\}$ is specified the same holds for $\bar{P}$ and $\bar{E}$.
(2) In the coupled algorithm, $\bar{P}$ and $\bar{E}$ are still well-defined. Because the
result of taking expectations with respect to $\bar{P}$ depends on the sequence of
parameters and this sequence of parameters is now data-dependent, an expected
value with respect to $\bar{P}$ is also data-dependent (in general).

6.6.3-3. Theorem. Let $\bar{S} \in S$ be fixed. The set of functions

$$\{\bar{E} f_k \circ \Pi_k : L_{CP}^{*+}(-\infty,0] + \mathbb{R}^d | k \in N, \omega \in \Omega\}$$

is equicontinuous.

Proof. This follows directly from theorem (6.6.2-40). Indeed let $\varepsilon < 0$. There
exists a $\delta > 0$ such that if $d_C(Y_1^+, Y_2^+) < \delta$, then for all $k$ and for all $\omega \in \Omega$:

$$|\bar{E} f_k \circ \Pi_k(Y_1^+) - \bar{E} f_k \circ \Pi_k(Y_2^+) | < \varepsilon.$$

This implies that for all $k$ and for all $\omega \in \Omega$:

$$|\bar{E} f_k \circ \Pi_k(Y_1^+) - \bar{E} f_k \circ \Pi_k(Y_2^+) | =$$

$$= |\bar{E}(f_k \circ \Pi_k(Y_1^+) - f_k \circ \Pi_k(Y_2^+)) | \leq$$

$$\leq |\bar{E} f_k \circ \Pi_k(Y_1^+) - f_k \circ \Pi_k(Y_2^+) | \leq \bar{E} \varepsilon = \varepsilon.$$

Q.E.D.

6.6.3-4. Corollary. Let $\bar{S} \in S$ be fixed. The set of functions

$$\{f_{k\omega} \bar{E} f_k : L_{CP}^{*+}(-\infty,0] + \mathbb{R}^d | k \in N, \omega \in \Omega\}$$

is equicontinuous.

The proof follows immediately from theorems (6.6.2-40) and (6.6.3-3).

In this section the result will be derived that if for each parameter sequence
$\{\theta_k^+\}$ satisfying property 3 (cf. (6.5-17)) – with interpolation function
\[ Y^+ \in L_{cP}^* \text{ such that } \theta^+_t = Y^+(s^-) \text{ for all } t \in \mathbb{Z} \text{ and for each } \mathcal{S} \text{ for which } \]

\[ |I(\mathcal{S}, Y^+) = 1 \text{ (and hence } |I(\mathcal{S}, \{s^+_t\}) = 1), \text{ one has } \]

\[ \lim_{k \to \infty} f_k - \tilde{E}f_k = 0, \text{-a.s.}, \]

then for the coupled algorithm one has

\[ (\forall \mathcal{S} \in \mathbb{S} \text{ with } |I(\mathcal{S}, Y^+) = 1, \lim_{k \to \infty} f_k - \tilde{E}f_k = 0), \text{-a.s.} \]

First let us formulate precisely the hypothesis concerning \( \tilde{E} \)-a.s. convergence of \( f_k - \tilde{E}f_k \). (The hypothesis will be proven to be true in section 6.7.2. Here we are only dealing with its implications for the coupled algorithm).

6.6.3-5. Hypothesis. For all parameter sequences \( \{\theta^+_t\} \) satisfying property 3 with interpolation function \( Y^+ \in L_{cP}^* \) such that \( \theta^+_t = Y^+(s^-), \forall t \in \mathbb{Z} \) and for all \( \mathcal{S} \in \mathcal{S} \) such that \( |I(\mathcal{S}, Y^+) = 1 \), there exists a subset \( N \) of \( \Omega \) with \( P(N) = 0 \), such that for all \( \omega \in \Omega \setminus N \),

\[ \lim_{k \to \infty} f_{k\omega} - \tilde{E}f_k = 0. \]

6.6.3-6. Theorem. Let \( \mathcal{S} \in \mathbb{S} \) be fixed. Suppose the hypothesis (6.6.3-5) holds. Then there exists a subset \( N \) of \( \Omega \) with \( P(N) = 0 \) such that for each parameter sequence \( \{\theta^+_t\} \) satisfying property 3 with interpolation function \( Y^+ \in L_{cP}^* \) such that \( \theta^+_t = Y^+(s^-) \) for all \( t \in \mathbb{Z} \) and for each subsequence \( \mathcal{S} = \{r_k\}_{k=1}^\infty \) of \( \mathcal{S} = \{r_k\}_{k=1}^\infty \) for which

\[ |I((\mathcal{S}, \mathcal{Q}, \mathcal{S}), Y^+) = 1, \]

one has

\[ \forall \omega \in \Omega \setminus N: \lim_{j \to \infty} f_{k(j)} - \tilde{E}f_{k(j)} = 0. \]

Proof. The proof is rather long. Let \( \{\tilde{f}_k\}_{k=1}^\infty \) be defined by

\[ (6.6.3-7) \quad \tilde{f}_k = \begin{cases} 0 & \text{if } \forall j \in \mathbb{N}: k(j) \neq k, \\ f_{k(j)} & \text{if } \exists j \in \mathbb{N}: k(j) = k. \end{cases} \]
It is clear that

$$\lim_{k \to \infty} \tilde{f}_k - \tilde{f}_k = 0 \iff \lim_{j \to \infty} f_{k(j)} - f_{k(j)} = 0.$$ 

Because $|I((\bar{s}, q, \delta), Y^+)| = 1$ there is only a finite number of possibilities for $I((\bar{s}, q, \delta), Y^+)$. Therefore it is sufficient to prove the result for a fixed index $(i, j) = I((\bar{s}, q, \delta), Y^+)$, because a finite union of null sets (i.e. sets of measure zero) is a null set.

In section 6.5 we saw that the spaces $L_{ij}[a, b]$, with $[a, b] \subseteq \mathbb{R}$, are compact (Theorem 6.5-7(b)). The idea of the proof is now to relate the functions $\tilde{f}_k \circ \Pi^k : \mathbb{R}^d \to \mathbb{R}^d$ to functions $f_k, \tilde{f}_k : \mathbb{R}^d \to \mathbb{R}^d$ using Theorem 6.6.2-41, and then to use Theorem 6.6.1-8.

The functions $f_k, k \in \mathbb{N}$, are defined as follows. To each $Y \in \mathbb{L}_{ij}^+(\mathbb{R}, 0)$

associate $Y' \in \mathbb{L}_{ij}^+(\mathbb{R}, 0)$, defined by

$$Y'(s) = Y(-q) \text{ for all } s < -q.$$ 

Note that indeed $Y' \in \mathbb{L}_{ij}^+(\mathbb{R}, 0)$; i.e. it does satisfy the Lipschitz condition and it remains in the coordinate chart $D'_{L_i}$. So $Y' \in \mathbb{L}_{ij}^+(\mathbb{R}, 0) \subseteq \mathbb{L}_{ij}^+(\mathbb{R}, 0)$.

Now let

$$f_k(Y) := \tilde{f}_k \circ \Pi^k (Y'), \forall k \in \mathbb{N}.$$ 

The functions $f_k, k \in \mathbb{N}$, are defined as follows. With each element $Y \in \mathbb{L}_{ij}^+(\mathbb{R}, 0)$ associate an element $Y' \in \mathbb{L}_{ij}^+$ as follows.

Let $Y^{-1}$ denote the 'reverse' of $Y$

$$Y^{-1}(t) := Y(-q-t), \forall t \in [-q, 0].$$ 

Let
\[
(6.6.3-10) \begin{cases}
Y''(s) |_{r_k-q, r_k} = \begin{cases}
Y(s-r_k) & \text{if } k \text{ is odd}, \\
Y^{-1}(s-r_k) & \text{if } k \text{ is even},
\end{cases} \\
Y'' |_{r_k, r_{k+1}-q} = \text{constant} = \begin{cases}
Y(q) & \text{if } k \text{ is odd}, \\
Y(0) & \text{if } k \text{ is even and}
\end{cases} \\
Y'' |_{(-\infty, r_{1}-q]} = Y(0).
\end{cases}
\]

Note that indeed \( Y'' \in \mathbb{L}_{ij}^{*,+} \) (the Lipschitz condition holds and \( Y''(s) \in \mathcal{B}_{ij}^{*,+} \) for all \( s \in \mathbb{R} \)). Let

\[
(6.6.3-11) \quad f_k(Y) := f_k(Y(s_t)) \bigg|_{t=-\infty}^{t=0} = f_k \circ \mathcal{R}_k \circ (S^{-r_k}_k(Y)) \bigg|_{t=-\infty}^{t=0}, \forall k \in \mathbb{N}.
\]

Consider the following four assertions

(i) \( \forall Y \in \mathbb{L}_{ij}^{*,+}[-q,0]: \exists N, P(N) = 0 \) such that

\[
\forall \omega \in \mathbb{N}: \lim_{k \to \infty} f_k(Y) - \mathcal{F}_k(Y) = 0,
\]

(ii) \( \forall Y \in \mathbb{L}_{ij}^{*,+}[-q,0]: \exists N, P(N) = 0 \) such that

\[
\forall \omega \in \mathbb{N}: \lim_{k \to \infty} \frac{f_k(Y)}{k} - \frac{\mathcal{F}_k(Y)}{k} = 0,
\]

(iii) \( \exists N, P(N) = 0 \) such that \( \forall Y, Y_k \in \mathbb{L}_{ij}^{*,+}[-q,0] \}_{k=1}^{\infty}, \forall \omega \in \mathbb{N}: \)

\[
\lim_{k \to \infty} f_k(Y_k) - \mathcal{F}_k(Y_k) = 0,
\]

(iv) \( \exists N, P(N) = 0 \) such that for all \( \omega \in \mathbb{N} \), for all \( \{s_t^+\} \) satisfying property 3 and for all subsequences \( S \) of \( S \) such that

\[
\mathcal{I}(S, q, t, \{s_t^+\}) = \{(1,j)\}:
\]

\[
\lim_{k \to \infty} \frac{\mathcal{F}_k}{k} = 0.
\]

The following sequence of implications will be shown (making use of theorems derived before):

Hypothesis (a) (1) (b) (ii) (c) (iii) (d) (iv)

(a) For arbitrary \( Y \in \mathbb{L}_{ij}^{*,+}[-q,0] \), consider the parameter sequence \( \{s_t^+ = Y''(s_t) | t \in \mathbb{Z} \} \). From (6.6.3-11) it is clear that
\[ f''_k(Y) - \tilde{f}''_k(Y) = f''_k(\theta^+_t) - \tilde{f}''_k(\theta^+_t). \]

The hypothesis implies that there exists a set \( N \subseteq \Omega \) with \( P(N) = 0 \), such that

\[ \forall \omega \in \Omega \setminus N: \lim_{k \to \infty} f_k(\theta^+_t) - \tilde{f}_k(\theta^+_t) = 0. \]

(Note that the coordinate chart index \((i,j)\) is constant and equal to \((i,j)\) for all \( t \). So \( \forall Y \in L^*_{i,j}[-q,0], \exists N, P(N) = 0 \) such that \( \forall \omega \in \Omega \setminus N: f_k(Y) - \tilde{f}_k(Y) = 0 \), so the hypothesis implies \((i)\) indeed.

(b) Let \( Y \in L^*_{i,j}[-q,0] \) be arbitrary. Because

\[
\begin{cases}
Y(s) = Y'(s) = Y''(s+r^*_k) & \text{if } k \text{ is even, and} \\
Y(s) = Y'(s) = Y''(r^*_k - s) = (Y^{-1})''(s+r^*_k) & \text{if } k \text{ is odd.}
\end{cases}
\]

Application of theorem (6.6.2-41) gives us

\[
\forall \omega \in \Omega, \forall Y \in L^*_{i,j}[-q,0]:
\]

(6.6.3-12)

\[
\lim_{k \to \infty} f^1_{2k}(Y) - f''_{2k}(Y) = 0 \quad \text{and} \quad \lim_{k \to \infty} f^1_{2k-1}(Y) - f''_{2k-1}(Y^{-1}) = 0,
\]

and

\[
\forall Y \in L^*_{i,j}[-q,0]:
\]

(6.6.3-13)

\[
\lim_{k \to \infty} \tilde{f}^1_{2k}(Y) - \tilde{f}''_{2k}(Y) = 0 \quad \text{and} \quad \lim_{k \to \infty} \tilde{f}^1_{2k-1}(Y) - \tilde{f}''_{2k-1}(Y^{-1}) = 0.
\]

So \((i)\) implies (taking for each \( Y \in L^*_{i,j}[-q,0] \) the union of the null sets for \( Y \) and \( Y^{-1} \), which results in another null set) \( \forall Y \in L^*_{i,j}[-q,0], \exists N, P(N) = 0 \) such that

\[ \forall \omega \in \Omega \setminus N: \lim_{k \to \infty} f_k(Y) - \tilde{f}_k(Y) = 0. \]
So indeed (i) implies (ii).

(c) Corollary (6.6.3-4) states that the set of functions

$$\{ f_k \in L^+_{CP}(\omega, 0) \mid \mathcal{F}^k_k \in \mathbb{R}^d, k \in \mathbb{N}, \omega \in \Omega \}$$

is equicontinuous. The same holds a fortiori if the domain of the functions is restricted to a subset of $L^+_{CP}(\omega, 0)$. Let us restrict the domain to the set of all $Y$ with $Y \in L^+_{CP}(\omega, 0)$. From the equicontinuity of the resulting set of functions it follows that the set of functions

$$\{ f_k \in L^+_{CP}(\omega, 0) \mid \mathcal{F}^k_k \in \mathbb{R}^d, k \in \mathbb{N}, \omega \in \Omega \}$$

is equicontinuous as well. The space $L^+_{L^2}[-q, 0]$ is compact (cf. theorem (6.5-7)(b)). Therefore theorem (6.6.1-8)(a) is applicable; it tells us that (ii) implies (iii).

(d) Let $\{ \delta^+ \}$ be a sequence satisfying property 3, and let $\mathcal{F}^+ \in L^+_{CP}(\omega, 0)$ be an interpolation curve, i.e. $\delta^+ = Y^+(q, t), \forall t \in \mathbb{Z}$. Let $\mathcal{S} = \{ r_k(j) \}_{j=1}^m$ be a subsequence of $S$ such that $I(\mathcal{S}, q, \delta^+ Y^+((1, j)))$. Let for each $k \in \mathbb{N}, Y \in L^+_{CP}(\omega, 0)$ be defined by

$$Y_k(r) = Y^+(r_k, r), \forall r \in (-\infty, 0].$$

Then $Y_k(j) \in L^+_{L^2}[-q, 0]$ for each $j \in \mathbb{N}$.

By construction of the $Y_k$, one has

(6.6.3-14) \[ \tilde{\mathcal{F}}_k(\delta^+) - \mathcal{F}^k_k(\delta^+) = \tilde{\mathcal{F}}_k \circ \mathcal{F}^k_k(Y_k) - \mathcal{F}^k_k \circ \mathcal{F}^k_k(Y_k), \forall k \in \mathbb{N}. \]

Now (iii), together with theorem (6.6.2-41) implies

\[ \exists N, p(N) = 0, \text{ such that } \forall \omega \in \Omega \setminus N; \]

(6.6.3-15) \[ \lim_{k \to \infty} \tilde{\mathcal{F}}_k(\delta) - \mathcal{F}^k_k(\delta) = \lim_{k \to \infty} \tilde{\mathcal{F}}_k \circ \mathcal{F}^k_k(Y_k) - \mathcal{F}^k_k \circ \mathcal{F}^k_k(Y_k) = \]

\[ = \lim_{k \to \infty} \tilde{\mathcal{F}}_k(Y_k[-q, 0]) - \mathcal{F}^k_k(Y_k[-q, 0]) = 0. \]

Q.E.D.
The problem that is left is that the exceptions sets in the previous theorem depend on the choice of the sequence \( S' \in S \). And as the set \( S \) is uncountable, one can not conclude directly that there exists one exceptions set \( N \) with \( P(N) = 0 \), which contains all the exceptions for all possible choices of \( S' \). It will, however, be shown that such a set exists.

6.6.3-16. Theorem. The hypothesis (6.6.3-5) implies: \( \exists E \subseteq \Omega \) with \( P(E) = 0 \), such that for all \( (\theta^+_t)^+ \) satisfying property 3 - with an interpolation curve \( \gamma^+_{t^{+}} \in \mathcal{I}_{CP}^+ \) such that \( \theta^+_t = \gamma^+(s_t), \forall t \in \mathbb{Z} \) - and for all \( S' \in S \) such that

\[
|I(S', \gamma^+_t)| = 1,
\]

one has

\[
\forall \omega \in \Omega \setminus E: \lim_{k \rightarrow \infty} f_k - \widetilde{E} f_k = 0.
\]

Before giving the proof let us state a corollary.

6.6.3-17. Corollary. The hypothesis (6.6.3-5) implies the following. Suppose \( (\theta^+_t(\omega)) \) is an \( \omega \)-dependent parameter sequence, satisfying property 3 and with interpolation curve \( \gamma^+_{t^{+}} \) as in the previous theorem. Then

\[
\exists E, P(E) = 0, \forall \omega \notin E, \forall S' \in S \text{ with } |I(S', \gamma^+_t)| = 1, \lim_{k \rightarrow \infty} f_k - \tilde{E} f_k = 0.
\]

Proof of the theorem.

Let \( \overline{P} = \{(p_1, p_2) \mid p_1 \in \mathbb{Q}, p_2 \in \mathbb{Q}, p_1 > p_2 > 0\} \). Then \( \overline{P} \) is countable. For each \( p \in \overline{P} \), let us define

\[
(6.6.3-18) \quad S'_p = (S_{p_1}, p_1, p_2) := ((p_1 + p_2)^k)_{k=1}^{m}, (p_1, p_2).
\]

Then \( (S'_p) \mid p \in \overline{P} \) forms a countable subset of \( S \). Application of theorem (6.6.3-6) to \( S'_p \) leads to an exceptions set that will be denoted by \( E_p \) (in the theorem it is denoted by \( N \)), with the property \( P(E_p) = 0 \). Now let

\[
(6.6.3-19) \quad E = \bigcup_{p \in P} E_p.
\]

Then \( P(E) = 0 \) because \( \overline{P} \) is countable. Note that the intervals corresponding to \( S'_p \) are
(6.6.3-20) \[ ((p_1 + p_2)(t-1) + 2p_2, (p_1 + p_2)t), t \in \mathbb{N} \]

Let \( \{ \delta_k^+ \} \) be a parameter sequence satisfying property 3 and \( Y^+ \in L^+_{CP} \) a corresponding interpolation function as before. Let \( \tilde{S} = (S, q, \delta) = (\{ r_k \}, q, \delta) \) be some element of \( S \) such that \( |I(\tilde{S}, Y^+)| = 1 \).

Let \( \epsilon > 0 \). It will clearly be sufficient to show that

(6.6.3-21) \[ \forall \omega \in \mathbb{N}; E: \limsup_{k \to \infty} f_k - E f_k \leq \epsilon. \]

Fix \( p \in P \), in order to derive an inequality.

For each \( k \in \mathbb{N} \), let

(6.6.3-22) \[ L_k := \{ t \in \mathbb{N} | ((p_1 + p_2)(t-1) + 2p_2, (p_1 + p_2)t) \subseteq [r_k - q + \delta, r_k] \}; \]

notice that

(6.6.3-23) \[ (q - \delta - 2p_1)/(p_1 + p_2) \leq |L_k| \leq (q - \delta + 2p_2)/(p_1 + p_2). \]

Let for all \( t < L_k \)

(6.6.3-24) \[ H_{k+1} := ((p_1 + p_2)(t-1) + 2p_2, (p_1 + p_2)t), \]

and

(6.6.3-25) \[ H_k := \bigcup_{t \in L_k} H_{k+1}. \]

Let \( \lambda \) denote Lebesgue measure. One has

(6.6.3-26) \[ \lambda([r_k - q + \delta, r_k] \cap H_k) \leq (q - \delta) - |L_k| (p_1 + p_2) \leq \]

\[ \leq (q - \delta) - \left( \frac{q - \delta - 2p_1}{p_1 + p_2} \right) (p_1 - p_2) \leq \frac{(p_2 - p_1)^2}{p_1 (1 + p_2) (p_1 - p_2)} \leq \]

\[ \leq (q - \delta) \left( \frac{p_2}{p_1} \right) + p_1. \]
Note that this is small if $\frac{p_2}{p_1}$ and $p_1$ are small enough.

Let $\overline{S}_p = \{(p_1 + p_2)\ell(i)_i\}_{i=1}^\infty$ be the subsequence of $S_p$ with the property

$$\{\ell(1)|i \in \mathbb{N}\} = \bigcup_{k=1}^{k'} L_k.$$  

Let $\{f^P_i_{i=1}^m\}$ denote the corresponding $f$-sequence, i.e.

$$f^P_i = \int_{H_{k_i \ell(i)}} b^0(o)ds$$

with $k$ such that $\ell(i) \in L_k$.

For each $k \leq N$, one has

$$\|f_k^{\overline{E}_k} - f_k^P\| \leq \int_{r_k^{-q+\delta}} [b^0(o) - \overline{b}^0(o)]d\sigma \leq$$

$$\leq \|\int_{[r_k^{-q+\delta}, \overline{H}_{k_i \ell(i)}]} [b^0(o) - \overline{b}^0(o)]d\sigma\| + \sum_{\ell(k_i \ell(i)) \in H_k} \int_{H_k} [b^0(o) - \overline{b}^0(o)]d\sigma.$$  

Let $k_k = \max\{|k(i)| k_i \leq k\}$ for each $k$. From section 6.4 (see e.g. the proof of corollary (6.4-55)) it follows that $b^0(o)$ is bounded by a data-independent constant, because the parameter sequence satisfies property 3 and so a fortiori property 1. Let this constant be denoted by $K_b$. Then it follows that

$$\|f_k^{\overline{E}_k} - f_k^P\| \leq 2K_b \lambda([r_k^{-q+\delta}, \overline{H}_{k_i \ell(i)}]) + \sum_{k_k = k(-k-1)} \|f^P_i - f^P_i\|.$$  

Applying theorem (6.6.3-5) to $S_p$, and using $E_p \subseteq E$, $P(E) = 0$, we find that the hypothesis (6.6.3-5) implies:

$$\forall \omega \in E : \lim_{i \to \infty} \|f_k^{\overline{E}_k} - f_k^P\| = 0.$$  

Because $\{|L_k|\}_{k=1}^\infty$ is bounded (cf. (6.6.3-23)), and applying (6.6.3-26), one finds

$$\forall \omega \in \Omega \setminus E : \limsup_{k \to \infty} \|f_k^{\overline{E}_k} - f_k^P\| \leq 2K_2 \{q-\delta\} \frac{p_2}{p_1}.$$  

This is the inequality mentioned right after (6.6.3-21). It is now trivial to
see that \( p = (p_1, p_2) \in \mathbb{P} \) can be chosen such that the right-hand side of (6.6.3-31) is smaller than \( \varepsilon > 0 \).

Q.E.D.

6.7. Exponential decay properties of the algorithm and the implications for convergence

6.7.1. About some exponential decay properties

The main purpose of this section 6.7 is to prove that the hypothesis (6.6.3-5) holds. To do this, our main tools will be the 'exponential decay properties' of the algorithm. In this first subsection definitions of several decay properties will be given and some results will be derived. Applications to the algorithm will be treated in section 6.7.2.

Let us start with defining exponential decay. In the following let \( B \) denote a d-dimensional vector space \( (d \in \mathbb{N}) \) with inner product \( \langle , \rangle \) and norm \( \| , \| \) and let the vectors of \( B \) be represented with respect to some orthonormal basis.

6.7.1-1. Definition. (i) A sequence \( \{ b_k | b_k \in B \}_{k=1}^{\infty} \) converges exponentially to zero or, equivalently, decays exponentially if the following holds

\[ \exists c > 0, \forall \lambda \in (0,1): \forall k \in \mathbb{N}: \| b_k \| < c \lambda^k. \]

(ii) Let \( \tilde{N} = \{(k,\ell) | k \in \mathbb{N}, \ell \in \mathbb{N}, k \geq \ell \} \). A double sequence \( \{ b_{k,\ell} | b_{k,\ell} \in B \}_{(k,\ell) \in \tilde{N}} \) is said to be exponentially decaying if

\[ \exists c > 0, \forall (k,\ell) \in (0,1): \forall (k,\ell) \in \tilde{N}, \| b_{k,\ell} \| < c \lambda^{k-\ell}. \]

Remark. In this definition \( \lambda \in (0,1) \) can be replaced by \( e^{-\alpha} \) with \( \alpha > 0 \). This explains the word 'exponentially'.

6.7.1-4. Notation. If \( \{ b_{k,1} \}_{k=1}^{\infty} \) satisfies (6.7.1-2) (i) we will say that \( \{ b_{k,1} \}_{k=1}^{\infty} \) has e.d.' (exponential decay) or that \( \{ b_{k,1} \}_{k=1}^{\infty} \) is an e.d. sequence' (exponentially decaying sequence). Similarly, if \( \{ b_{k,\ell} | (k,\ell) \in \tilde{N} \} \) satisfies (6.7.1-2) (ii) we will say that \( \{ b_{k,\ell} | (k,\ell) \in \tilde{N} \} \) has e.d.' or that \( \{ b_{k,\ell} | (k,\ell) \in \tilde{N} \} \) is an e.d. double sequence'.

6.7.1-5. Theorem. (i) Let \( \{ b_k \}_{k=1}^{\infty}, \{ c_k \}_{k=1}^{\infty}, \{ a_j(k) \}_{k=1}^{\infty}, j \in \mathbb{N}, \) be e.d. sequences. Then
\( (a) \ \{b_k^k c_k \}_k \) is an e.d. sequence,

\( (b) \ \{b_k^T c_k \}_k \) is an e.d. sequence,

\( (c) \ \{k b_k^k \}_k \) is an e.d. sequence,

\( (d) \ \forall \lambda \in (0,1): \ \{b_k^* = \sum_{j=0}^{k} \lambda_j^j b_{k-j} \}_k \) is an e.d. sequence,

\( (e) \) if the \( b_k \) are nonnegative scalars and \( \nu > 0 \) is arbitrary, then \( \{b_k^\nu \}_k \) is an e.d. sequence, and

\( (f) \) if \( p(x^{(1)}, \ldots, x^{(n)}) \) is a polynomial in \( n \) variables without constant term, i.e., \( p(0, \ldots, 0) = 0 \), then the sequence \( \{p(d_k^{(1)}, \ldots, d_k^{(n)})\}_k \) has e.d.

\( (ii) \) Let \( \{b_k^k \} \in \mathbb{N}, \ {c_k^k \} \in \mathbb{N}, \ {d_k^{(1)} \} \in \mathbb{N}, \ j \in \mathbb{N}, \) be e.d. double sequences. Then

\( (a) \ \{b_k^k + c_k^k \} \in \mathbb{N} \) is an e.d. double sequence,

\( (b) \ \{b_k^T c_k^k \} \in \mathbb{N} \) is an e.d. double sequence,

\( (c) \ \{(k-t)b_k^k \} \in \mathbb{N} \) is an e.d. double sequence,

\( (d) \ \forall \lambda \in (0,1): \ \{b_k^* = \sum_{j=0}^{k} \lambda_j^j b_{k-j} \} \in \mathbb{N} \) is an e.d. double sequence,

\( (e) \) if the \( b_k^k \) are nonnegative scalars and \( \nu > 0 \) is arbitrary, then \( \{b_k^\nu \} \in \mathbb{N} \) is an e.d. sequence, and

\( (f) \) if \( p(x^{(1)}, \ldots, x^{(n)}) \) is a polynomial in \( n \) variables without constant term i.e., \( p(0, \ldots, 0) = 0 \), then the double sequence \( \{p(d_k^{(1)} d_k^{(2)} \ldots, d_k^{(n)}) \} \in \mathbb{N} \) has e.d.

Proof. We will give the proof of \( (i) \); the proof of \( (ii) \) is completely analogous.

\( (a) \) is trivial

\( (b) \) is trivial in the scalar case; the vector case follows easily by applying repeatedly \( (a) \) and the scalar case of \( (b) \).

\( (c) \) \( \{b_k \} \) has e.d., so \( \exists \epsilon > 0, \exists \lambda \in (0,1) \) such that \( \forall k \in \mathbb{N}: \ |b_k| < \epsilon \lambda^k \) which implies \( \forall k \in \mathbb{N}: \ |b_k| < \epsilon \lambda^k \).
Take $\lambda_2 \in (\lambda_1, 1)$. Then $\lim_{k \to \infty} \frac{k \lambda^k}{k} = 0$, so $\exists k_1 \in \mathbb{N}$ such that $\forall k \geq k_1: k \lambda^k \leq \lambda_2^k$.

Now let

$$(6.7.1-6) \quad c_2 := \max\{c, \max_{1 \leq k \leq k_1} \frac{\lambda}{\lambda_2^k}\},$$

then

$$(6.7.1-7) \quad \forall k \in \mathbb{N}: \|k \lambda \| < c_2 \lambda_2^k.$$

(d) Because $\{b_k\}$ has e.d., $\exists c_3 > 0, \exists \lambda_3 \in (0,1)$ such that $\|b_k\| < c_3 \lambda_3^k$ for all $k \in \mathbb{N}$. Consider

$$\|b_k\| = \| \sum_{j=0}^{k-1} \lambda_3^j b_{k-j} \| \leq \sum_{j=0}^{k-1} \lambda_3^j c_3 \lambda_3^k \leq k \max(\lambda_1, \lambda_3) c_3^k,$$

and $\{k \max(\lambda_1, \lambda_3)^k c_3^k\}_{k=1}^\infty$ is an e.d. sequence according to (b). So $\{b_k\}_{k=1}^\infty$ is an e.d. sequence.

(e) In this case one has

$$\exists \epsilon > 0, \forall \lambda \in (0,1), \forall k \in \mathbb{N}: 0 < b_k^k < c \lambda^k.$$

It follows that for arbitrary $\mu > 0$

$$\forall k \in \mathbb{N}: 0 < b_k^\mu < c^\mu (\lambda^\mu)^k,$$
and $c^\mu > 0, \lambda^\mu \in (0,1)$ of course, so $\{b_k^\mu\}$ is an e.d. sequence.

(f) This follows easily by repeated application of (a) and the scalar case of (b).

Q.E.D.

Let us now consider e.d. sequences of matrices. First let us show that for the definition it does not matter whether the spectral norm or the Frobenius norm is used.

6.7.1-8. Lemma. $\{A_k\}_{k=1}^\infty$ is an e.d. sequence of $d_1 \times d_2$ matrices with respect to the spectral norm iff it is an e.d. sequence with respect to the Frobenius norm.

The proof follows easily from the fact that for any matrix $A$,
\[ (6.7.1-9) \quad \|A^k\|_S \leq \|A^k\|_F \leq \sqrt{n} \cdot \|A^k\|_S, \]

where \( \| \cdot \|_S \) is the spectral norm and \( \| \cdot \|_F \) the Frobenius norm, as before. By convention, with an e.d. sequence of matrices \( \{A_k\}_{k=1}^\infty \) will be meant an e.d. sequence with respect to the Frobenius norm or equivalently with respect to the spectral norm (i.e. \( \|A_k\|_F \) or equivalently \( \|A_k\|_S \) is an e.d. sequence of numbers).

**Remark.** In the following the real case will be considered. The results for the complex case are analogous.

6.7.1-10. **Lemma.** \( \{A_k\}_{k=1}^\infty \) is an e.d. sequence of \( d_1 \times d_2 \) (real) matrices iff for each \( d_2 \)-vector, \( \{A_k x\}_{k=1}^\infty \) is an e.d. sequence of \( d_1 \)-vectors.

**Proof.** If \( \{A_k\}_{k=1}^\infty \) is an e.d. sequence then \( \exists c > 0, \forall \lambda \in (0,1); \|A_k\|_S < c \lambda^k \), and so if \( c_1 \geq c \|x\| > 0 \), then \( \|A_k x\|_S < \|A_k\|_S \|x\| < c_1 \lambda^k \), which shows that \( \{A_k x\}_{k=1}^\infty \) has e.d. On the other hand, if \( \{A_k x\}_{k=1}^\infty \) has e.d. for each \( d_2 \)-vector \( x \), then it follows easily that each component-sequence \( \{a_{k,j}^i\}_{k=1}^\infty \) where \( e_i \) denotes the \( i \)th unit vector, has e.d. and therefore \( \{A_k\}_{k=1}^\infty \) is an e.d. sequence. The details are left to the reader. Q.E.D.

6.7.1-11. **Lemma.** Let \( A \) be a square \( d \times d \) matrix. \( \{A^k\}_{k=1}^\infty \) is an e.d. sequence iff \( A \) is asymptotically stable.

**Proof.** If \( \{A^k\} \) is an e.d. sequence then clearly \( \lim_{k \to \infty} A^k = 0 \), and so all eigenvalues have modulus less than one, so \( A \) is asymptotically stable. On the other hand if \( A \) is asymptotically stable then its eigenvalues lie in the open unit disk. Let \( \lambda \in (0,1) \) be larger than the largest modulus of any eigenvalue of \( A \). Then

\[ (6.7.1-12) \quad \lim_{k \to \infty} \|A^k\|_S / \lambda^k = 0, \]

and so \( \|A^k\|_S / \lambda^k \) is bounded by some positive number \( c \) (say).

So \( \forall k \in \mathbb{N} \); \( \|A^k\|_S < c \lambda^k \), and so \( \{A^k\} \) is e.d. Q.E.D.
Remark. Notice that \( \{A_k^k\} \) is an e.d. sequence' is definitely a weaker statement than 'IA^k < 1'. As a simple example, consider \( A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \)
for some \( \lambda \in (0,1) \). Then \( \|A^k\|_S \geq 1+\lambda^2 > 1 \).

6.7.1-13. Lemma. Let \( A \) be a real square asymptotically stable matrix and \( \Gamma \) an arbitrary square matrix of the same size as \( A \). Then \( \{A^k \Gamma (A^k)^T \Gamma \} \) has e.d.

Proof. Consider \( \|A^k \Gamma (A^k)^T \Gamma \|_S \leq \|A^k \|_S^2 \|\Gamma\|_S \) and apply the previous lemma.
Q.E.D.

6.7.1-14. Lemma. Let \( \{A_k\}_{k=1}^\infty \) be a sequence of \( d \times d \) real positive semi-definite symmetric matrices. Then \( \{A_k\}_{k=1}^\infty \) is an e.d. sequence iff \( \{\text{tr } A_k\}_{k=1}^\infty \) is an e.d. sequence.

Proof. If \( A \) is (real) positive semi-definite symmetric, its largest eigenvalue is \( \|A\|_S \), and \( \|A\|_S \leq \text{tr } A \leq d \|A\|_S \). The lemma follows easily from this inequality.
Q.E.D.
6.7.1-15. Remarks. (i) Similar results hold for double matrix-sequences
\( \{ A_{k,l} \} \) a \( d_1 \times d_2 \) matrix, \((k,l) \in \mathbb{N} \).
(ii) Theorem (6.7.1-5) can be generalized to the matrix case in a
straightforward manner.

For stochastic vectors let us introduce the concept of exponential decay of
dependence.

6.7.1-16. Definition. Let \( \{ x_k \}^m_{k=1} \) be a sequence of random variables taking their
values in \( \mathbb{B} \). It will be said to have the property of exponential decay of
dependence if the following holds.

(i) For each \( p \in \mathbb{N} \) the sequence \( \{ E|x_k|^p \}^m_{k=1} \) is bounded and
(ii) for each pair \((k,l) \in \mathbb{N} \) there exist random variables \( x_{k,l} \) and \( \epsilon_{k,l} \) such
that
(a) \( x_k = x_{k,l} + \epsilon_{k,l} \),
(b) \( x_{k,l} \) is stochastically independent of \( x_{l}, x_{l-1}, x_{l-2}, \ldots \) and
(c) \( \forall p \in \mathbb{N}: \{ E|x_{k,l}|^p \}^m_{k=1} \) is an e.d. double sequence.

6.7.1-17. Notation. If \( \{ x_k \}^m_{k=1} \) has this property we will say that \( \{ x_k \}^m_{k=1} \) has
E.D.D.' or that \( \{ x_k \}^m_{k=1} \) is an E.D.D. sequence.

For those cases in which the \( \epsilon_{k,l} \) are Gaussian, condition (c) of definition
(6.7.1-16) simplifies considerably. To show this, the following lemma will be
used.

6.7.1-18. Lemma. There exists a sequence of polynomials \( \{ q_p(\mu,\sigma^2) \}^m_{p=1} \) in the
(scalar) variables \( \mu \) and \( \sigma^2 \), with the following property. If \( x \) is a scalar
Gaussian variable with mean \( \mu \in \mathbb{R} \) and variance \( \sigma^2 > 0 \), then

\( \{ 6.7.1-19 \} \quad E x^p = q_p(\mu,\sigma^2). \)

Furthermore for each \( p \in \mathbb{N} \), \( q_p(0,0) = 0. \)

Proof. This is a standard result from statistics. (It can easily be shown by
making use of the characteristic function of the Gaussian distribution with
mean $\mu$ and variance $\sigma^2 \geq 0$). Q.E.D.

6.7.1-20. Theorem. Let \( \{x_k\}_{k=1}^{\infty} \) be a sequence of random variables taking their values in \( B \). It has e.d.d. if

(i) for each \( p \in \mathbb{N} \) the sequence \( \{E|x_k|^p\}_{k=1}^{\infty} \) is bounded and

(ii) for each \( (k,l) \in \mathbb{N} \) there exist random variables \( x_{k,l} \) and \( \varepsilon_{k,l} \) such that

(a) \( x_k = x_{k,l} + \varepsilon_{k,l} \)

(b) \( x_{k,l} \) is stochastically independent of \( x_{k-1}, x_{k-2}, \ldots \)

(c) \( \varepsilon_{k,l} \) has a Gaussian distribution and

(d) \( \{E|x_{k,l}|^2|(k,l) \in \mathbb{N}\} \) is an e.d. double sequence.

Proof. Let the mean and covariance matrix of \( x_{k,l} \) be denoted by \( u_{k,l} \) and \( \Sigma_{k,l} \) respectively and let \( \sigma_{k,l}^2 := \operatorname{tr} \Sigma_{k,l} \). Because

\[
(6.7.1-21) \quad \|u_{k,l}\|^2 \leq E|x_{k,l}|^2 \leq (E|x_{k,l}|^2)^{\frac{1}{2}}
\]

and

\[
(6.7.1-22) \quad \sigma_{k,l}^2 = E|x_{k,l}|^2 - \|u_{k,l}\|^2 \leq E|x_{k,l}|^2
\]

and \( \{E|x_{k,l}|^2\} \) is an e.d. double sequence, it follows that \( \{u_{k,l}\}, \{\sigma_{k,l}^2\} \) are e.d. double sequences. Using lemma (6.7.1-13) it follows that \( \{x_{k,l}\} \) is an e.d. double matrix sequence.

Because the conditions (i), (ii)(a) and (ii)(b) of definition (6.7.1-16) are assumed to be fulfilled, it remains to show (ii)(c) of definition (6.7.1-16), i.e. that for all \( p \in \mathbb{N} \), \( \{E|x_{k,l}|^p|(k,l) \in \mathbb{N}\} \) is an e.d. double sequence. From the well-known inequality

\[
(6.7.1-23) \quad (E|x_{k,l}|^1)^2 \leq E|(E|x_{k,l}|^2)^{\frac{1}{2}}|^2
\]

it follows easily that it suffices to consider only even values of \( p \). So let \( p = 2j, j \in \mathbb{N} \). First consider the scalar case \( d = 1 \). Then \( \varepsilon_{k,l} = q_{2j}(u_{k,l}, \sigma_{k,l}^2) \), according to lemma (6.7.1-18). Because \( \{u_{k,l}\} \) and \( \{\sigma_{k,l}^2\} \) have e.d., the same holds for \( \{q_{2j}(u_{k,l}, \sigma_{k,l}^2)|(k,l) \in \mathbb{N}\} \), for each \( j \in \mathbb{N} \), according to theorem (6.7.1-5)(ii)(f).

Now consider the vector case. Use will be made of the following inequality. If
\(x^{(1)}, x^{(2)}, \ldots, x^{(d)}\) are arbitrary scalar random variables then

\[
(6.7.1-24) \quad E[x^{(1)} x^{(2)} \cdots x^{(d)}] \leq \prod_{i=1}^{d} \left[ E(x^{(i)})^2 \right]^{1-2^{-d}}.
\]

This inequality follows by induction from the well-known inequality for arbitrary scalar random variables \(a\) and \(b\):

\[
(6.7.1-25) \quad E|ab| \leq (Ea^2)^{\frac{1}{2}}(Eb^2)^{\frac{1}{2}}.
\]

Let \(e^{(i)}_{k\ell}\) denote the \(i\)-th component of \(e_{k\ell}\), \(i \in \{1, 2, \ldots, d\}\).
One has for each \(j \in \mathbb{N}\)

\[
(6.7.1-26) \quad E[e^{(i)}_{k\ell} e^{(j)}_{k\ell}] = E\left[ \sum_{i=1}^{d} (e^{(i)}_{k\ell})^2 \right]^{\frac{1}{2}} \leq \prod_{0 \leq j_1 < j \leq j_1 + j_2, \ldots, j_1 + j_2 + \cdots + j_d = 1}^{d} \left[ E(e^{(i)}_{k\ell})^2 \right]^{1-2^{-d}},
\]

where \((6.7.1-24)\) is used. From the scalar case it follows that for each \(i \in \{1, 2, \ldots, d\}\) the double sequence \(\{E(e^{(i)}_{k\ell})^2 | (k, \ell) \in \mathbb{N}\}\) has e.d. It follows from theorem (6.7.1-5)(ii)(f) that the double sequence that is obtained by taking the right-hand side of \((6.7.1-26)\) and letting \((k, \ell)\) take all possible values in \(\mathbb{N}\), has e.d. And therefore \(\{E[e^{(i)}_{k\ell} e^{(j)}_{k\ell}] | (k, \ell) \in \mathbb{N}\}\) has e.d.

Q.E.D.

It is perhaps not so surprising that the covariance matrix sequence of an e.d.d. sequence \(\{x_k\}\) has e.d. This is what will be shown next.

Let \(\text{cov}(x, x_k) := E(x - Ex_k)(x - Ex_k)^T\). (As stated before we restrict ourselves to the real case, for the complex case similar results hold).

6.7.1-27. Theorem. Let \(\{x_k\}_{k=1}^{\infty}\) be an e.d.d. sequence of vectors taking their values in \(\mathbb{B}\). The double sequence of covariance matrices \(\{\text{cov}(x, x_k) | (k, \ell) \in \mathbb{N}\}\) has e.d.

Proof. For all \((k, \ell) \in \mathbb{N}\), one has
\[ \text{cov}(x_k^*, x_k^*) = E((x_k^* - Ex_k^*)(x_k^* - Ex_k^*))^T = \]
\[ = E((x_k^* - Ex_k^*)(x_k^* - Ex_k^*))^T + E(\varepsilon_{kk}^* - E\varepsilon_{kk}^*)(x_k^* - Ex_k^*)(x_k^* - Ex_k^*)^T. \]

Because \( x_k \) and \( x_k^* \) are stochastically independent they have vanishing covariances. It follows that

\[ (6.7.1-28) \quad \text{cov}(x_k^*, x_k^*) = E(\varepsilon_{kk}^* - E\varepsilon_{kk}^*)(x_k^* - Ex_k^*)^T \]

Let us now take the Frobenius norm.

\[ \|\text{cov}(x_k^*, x_k^*)\|_F = \|E(\varepsilon_{kk}^* - E\varepsilon_{kk}^*)(x_k^* - Ex_k^*)\|_F \leq \]
\[ (6.7.1-30) \quad \|E(\varepsilon_{kk}^* - E\varepsilon_{kk}^*)(x_k^* - Ex_k^*)\|_F = E\|\varepsilon_{kk}^* - E\varepsilon_{kk}^*\|_F \cdot \|x_k^* - Ex_k^*\|_F \leq \]
\[ (E\|\varepsilon_{kk}^* - E\varepsilon_{kk}^*\|_F^2)^{1/2}(E\|x_k^* - Ex_k^*\|_F^2)^{1/2} \leq (E\|\varepsilon_{kk}^*\|_F^2)^{1/2}(E\|x_k^*\|_F^2)^{1/2}. \]

Now \( (E\|\varepsilon_{kk}^*\|_F^2)_{kk} \) is an e.d. double sequence and \( (E\|x_k^*\|_F^2)_{kk} \) is a bounded sequence, so \( ((E\|\varepsilon_{kk}^*\|_F^2)^{1/2}(E\|x_k^*\|_F^2)^{1/2})_{kk} \) is an e.d. double sequence, hence \( \{\|\text{cov}(x_k^*, x_k^*)\|_F \mid (k,k) \in \mathbb{N} \} \) is an e.d. double sequence and the theorem follows.

Q.E.D.

6.7.1-31. Corollary. If \( \{x_k^*\}_{kk} \) has e.d.d., then \( \{r \text{cov}(x_k^*, x_k^*)\}_{kk} \) has e.d.d.

In many important cases, taking a function of an e.d.d. sequence gives another e.d.d. sequence. This is treated next.

6.7.1-32. Theorem. Suppose \( \{x_k = (x_k^{(1)}, x_k^{(2)}, \ldots, x_k^{(d)})\}_{kk} \) is an e.d.d. sequence of random vectors taking their values in \( \mathbb{R} \).

(a) Suppose \( p = p(x_1^{(1)}, x_2^{(2)}, \ldots, x_d^{(d)}) \) is a polynomial in \( d \) variables, then the sequence \( \{p(x_k^{(1)}, x_k^{(2)}, \ldots, x_k^{(d)})\}_{kk} \) has e.d.d.

(b) Consider \( F : \mathbb{R} \to \mathbb{R} \) another finite dimensional vector space with inner product. Suppose \( F \) satisfies a global Lipschitz condition

\[ (6.7.1-33) \quad \exists c > 0, \forall x, y \in \mathbb{R} : \|F(x) - F(y)\| \leq c\|x - y\|. \]
Then \( \{F(x_k)\}_{k=1}^\infty \) has e.d.d.

(c) If \( F : B + \overline{B} \) is a \( C^1 \) mapping with compact support then \( \{F(x_k)\}_{k=1}^\infty \) has e.d.d.

(d) If \( \{F_k\}_{k=1}^\infty \) is a sequence of mappings satisfying a uniform Lipschitz condition

\[
\exists c > 0, \forall x, y \in B, \forall k \in \mathbb{N}: \|F_k(x) - F_k(y)\| \leq c\|x - y\|.
\]

Then \( \{F_k(x_k)\}_{k=1}^\infty \) has e.d.d.

(e) If \( F : \overline{B} \times B + \overline{B} \) is a continuous mapping with continuous partial derivatives with respect to the components of the \( x \)-vector, and \( F \) has compact support, then for each sequence \( \{x_k \in B\}_{k=1}^\infty \), the sequence \( \{F(x_k, x_k)\}_{k=1}^\infty \) has e.d.d.

(f) If \( \lambda \in (0, 1) \) then \( y_k = \sum_{j=0}^{k-1} \lambda^j x_{k-j} \) has e.d.d.

(g) If \( \{A_j\}_{j=1}^\infty \) is a sequence of linear mappings \( A_j : B + B \), and if \( \lambda \in (0, 1) \) exists such that \( \|A_j\| < \lambda \) for all \( j \), then \( y_k = \sum_{j=0}^{k-1} \lambda^j k k_{k-1} \ldots A_{k-j+1} x_{k-j} \) has e.d.d.

Proof. (a) First consider the case of two variables \( x_k^T = (u_k, v_k) \), with

\[
x_k = (u_k, v_k) \quad \text{and} \quad e_k^T = (e_k^u, e_k^v).
\]

The simplest cases are

(i) \( p(u, v) = u + v \) and (ii) \( p(u, v) = uv \).

ad(i) Let \( z_k = p(u_k, v_k) = u_k + v_k \). Let \( z_k = u_k + v_k \) and \( e_k = e_k^u + e_k^v \).

Then \( z_k \) is clearly stochastically independent of \( z_k, z_{k-1}, z_{k-2}, \ldots \), and so \( \{z_k\} \) is an e.d. double sequence \(((k, l) \in \mathbb{N})\).

This shows (i).

ad(ii) Let \( z_k = p(u_k, v_k) = u_k v_k \). Let \( z_k = u_k v_k \), then \( z_k \) is stochastically independent of \( z_k, z_{k-1}, z_{k-2}, \ldots \). Let \( e_k = u_k v_k - u_k v_k \).

Then one has

\[
e_k = u_k v_k - u_k v_k = (u_k - u_k) v_k + u_k (v_k - v_k) = e_k^u v_k + e_k^v u_k - e_k^v v_k
\]

and so for each \( p \in \mathbb{N} \)

\[
(6.7.1-35) \quad e_k = u_k v_k - (u_k - u_k) v_k = e_k^u v_k + e_k^v u_k - e_k^v v_k
\]
(6.7.1-36) \[ e_{k,l}^z \leq (|e_{k,l}^u||v_k| + |e_{k,l}^v||u_k|)P. \]

The right-hand side is a polynomial in the three variables \(|e_{k,l}^u||v_k|, |e_{k,l}^v||u_k|\) and \(|e_{k,l}^u||e_{k,l}^v|\), without constant term. Using (6.7.1-24) one can find an upper estimate for \(E(\epsilon_{k,l}^z)^P\) in terms of positive (but not necessarily integer) powers of moments of \(|e_{k,l}^u||v_k|, |e_{k,l}^v||u_k|\) and \(|e_{k,l}^u||e_{k,l}^v|\).

Similarly as is done in the proof of (6.7.1-20) one can show that for each p, \(\{E(\epsilon_{k,l}^z)^P\}(k,t)\) is an e.d. double sequence.

The arguments used for (1) and (ii) can be extended without problems to the case of d variables and to the case of polynomials p of any finite degree in those d variables (or one can use inductions). The details are left to the reader.

(b) Let \(z_k := F(x_k)\) and \(z_{k,l} := F(x_{k,l})\), then \(e_{k,l}^z = z_k - z_{k,l}\). Then \(z_{k,l}\) is independent of \(z_k, z_{k-1}, z_{k-2}, \ldots\) and \(h_{k,l}^z = |F(x_k) - F(x_{k,l})| \leq c |e_{k,l}^z| = c |e_{k,l}^u||e_{k,l}^v|\). This implies

\[ E(\epsilon_{k,l}^z)^P \leq c^P E(\epsilon_{k,l}^u)^P, \]

so \(\{E(\epsilon_{k,l}^z)^P\}(k,t) \in \tilde{N}\) is an e.d. double sequence for each \(p \in N\) and so \((z_k)\) has e.d.d.

(c) \(F\) satisfies a Lipschitz condition, so (b) is applicable.

(d) Let \(z_k = F(x_k)\) and \(z_{k,l} = F(x_{k,l})\). The proof is now completely similar to the proof of (b).

(e) \(F\) satisfies a Lipschitz condition with respect to \(x\) that is independent of \(\theta\).

Therefore this case reduces to case(d).

(f) Let \(y_{k,l} := \sum_{j=0}^{k-1} \lambda^j e_{k,j,l}\), then \(y_{k,l}\) is stochastically independent of \(x_k, x_{k-1}, \ldots\) and therefore of \(y_k, y_{k-1}, y_{k-2}, \ldots\). One has

\[ (6.7.1-37) \quad e_{k,l}^y = y_k - y_{k,l} = \sum_{j=0}^{k-1} \lambda^j e_{k,j,l} + \sum_{j=k-1}^{k-1} \lambda^j e_{k-j,l}. \]

It follows that
Because \((x_k)\) has e.d.d., there exists a \(c_1 > 0\) such that \(\forall k \in \mathbb{N}: \|x_k\| < c_1\)
and there exists a \(c_2 > 0\) and a \(\lambda_2 \in (\lambda_1, 1)\) such that \(E|x_k|^2 < c_2\lambda_2^{-k}\) for all
\((k, \ell) \in \mathbb{N}\). Substituting this one obtains

\[
(6.7.1-39) \quad E|x_k|^2 \leq \sum_{j=0}^{k-1} \lambda_2^j c_2 + \lambda_2^{k-1} \left( \lambda_2^{-1} \{ (k-1) c_2 + c_1 \} \right). \]

The right-hand side is the general term of a \((k, \ell)\)-double sequence, with
\((k, \ell) \in \mathbb{N}\), which has e.d.d., according to theorem (6.7.1-5) (1) (c) and
(a). It follows that \(\{E|x_k|^2\} (k, \ell) \in \mathbb{N}\) has e.d.d.
Similarly as in the proof of (a) and of (6.7.1-20) this argument can be
extended to show that for each \(p \in \mathbb{N}\), \(\{E|x_k|^p\} (k, \ell) \in \mathbb{N}\) has e.d.d., and so
\(\{y_k\}\) has e.d.d. The details are left to the reader.

(g) Let \(y_k := \sum_{j=0}^{k-1} A_k A_{k-1} \cdots A_{k-j+1} x_{k-j, \ell}\) then \(y_k\) is stochastically
independent of \(x_1, x_2, \ldots\) and therefore of \(y_1, y_2, \ldots\). One has

\[
(6.7.1-40) \quad y_k = y_k - y_{k-1} = \sum_{j=0}^{k-1} A_k A_{k-1} \cdots A_{k-j+1} x_{k-j, \ell} + \sum_{j=k-1}^{k-1} A_k A_{k-1} \cdots A_{k-j+1} x_{k-j}. \]

Because for all \(j\), \(\|A_j\| < \lambda\) it follows that

\[
(6.7.1-41) \quad E|y_k|^2 \leq \sum_{j=0}^{k-1} \lambda^j |x_{k-j, \ell}|^2 + \sum_{j=k-1}^{k-1} \lambda^j |x_{k-j}|^2. \]

This is equal the inequality (6.7.1-37) with \(\lambda_2\) instead of \(\lambda\). From here the
proof is identical to the proof of (f).

Q.E.D.

The previous theorem shows that the concept of an e.d.d. sequence of random
variables is a rather flexible one. This will be useful in the analysis of the
algorithm. However, a somewhat more general concept will in fact be needed,
which is introduced next.

6.7.1-42. Definition. Let \(N = \{N_i\}_{i=1}^\infty\) be a sequence of disjoint intervals
of \(\mathbb{N}\) with \(N_i \subseteq N_{i+1}\) (i.e. \(x \in N_i, \forall y \in N_{i+1}, x \leq y\)) and with interval lengths
going to infinity for \(i \rightarrow \infty\) and let
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\[ N_0 := \bigcup_{i=1}^{\infty} N_i \text{ and } N_i := \{(k,\ell) | k \geq \ell, k \in N_i, \ell \in N_i\}. \]

(a) Let \( \{b_{k\ell} \mid (k,\ell) \in \bar{N}_i\} \) be a deterministic double sequence of elements of \( B \). Then \( \{b_{k\ell} \mid (k,\ell) \in \bar{N}_i\} \) is said to have the property of exponential decay with respect to the interval sequence \( N \) (abbreviated as 'e.d.i-N', or 'e.d.i' if \( N \) is clear from the context), if the following holds:

\[ \exists c > 0, \exists \lambda \in (0,1) \text{ such that } \forall (k,\ell) \in \bar{N}_i: \|b_{k\ell}\| < c\lambda^{k-\ell}. \]

(b) Let \( \{x_k \mid k \in N_0\} \) be a sequence of random variables taking their values in \( B \). Then \( \{x_k \mid k \in N_0\} \) is said to have the property of exponential decay of dependence with respect to the interval sequence \( N \) (abbreviated as 'e.d.d.i-N' or 'e.d.d.i' if \( N \) is clear from the context), if the following holds:

\[ \forall (k,\ell) \in \bar{N}_i, \exists x_{k\ell}, c_{k\ell} \text{ such that } \]

(i) \[ x_k = x_{k\ell} + c_{k\ell}, \quad \forall (k,\ell) \in \bar{N}_i, \forall i \in N, \]

(ii) \[ x_{k\ell} \text{ is stochastically independent of } \{x_j \mid j \in N_i, j \leq \ell\}, \forall (k,\ell) \in \bar{N}_i, \forall i \in N, \text{ and } \]

(iii) \[ \forall p \in N, \exists c > 0, \exists \lambda_p \in (0,1) \text{ such that } \forall i \in N, \forall (k,\ell) \in \bar{N}_i: \text{E}e^{-c_{k\ell} \lambda_i p} < c\lambda^{k-\ell}. \]

6.7.1-43. Theorem. Let \( \{x_k \mid k \in N_0\} \) be an e.d.d.i.-N sequence. The properties (a) - (g) of theorem (6.7.1-32) hold if

(i) 'e.d.d.' is replaced by 'e.d.d.i-N' and

(ii) in (f) and (g) the summation is replaced by summations that are going back only to the beginning of the relevant interval in \( N \). I.e. in the analogon of (f)

\[ \{y_k = \sum_{j=0}^{k-m N} \lambda^j x_{k-j} \mid i \text{ such that } k \in N_i \}_k \in N_0 \]

is the sequence that has e.d.d.i. and in the analogon of (g),

\[ \{y_k = \sum_{j=0}^{k-m N} A_k \cdots A_{k-j+1} x_{k-j} \mid i \text{ such that } k \in N_i \}_k \in N_0 \]

is the sequence that has e.d.d.i.

The proof is completely analogous to the proof of theorem (6.7.1-32).
6.7.1-44. Lemma. If \( x_k \) is an e.d. sequence, then \( x_k \) has e.d.d.1-N.

This is an obvious but useful result.

6.7.1-45. Lemma. If \( x_k \) is a sequence of random vectors such that

(i) \( x_k = 0 \) if, for some \( i \in N, k \in N \) and \( k \neq \min(N) \),
(ii) for each \( p \in N \) the sequence \( \{E|x_k|^p\}_{k \in N} \) is bounded,

then \( x_k \) has e.d.d.1-N.

This can be shown simply by taking \( x_{k,i} = 0 \) for all \( (k,i) \in N_i \). The details are left to the reader.

6.7.2. Applications to the algorithm

First it will be shown that several quantities that appear in the algorithm have e.d.d. or e.d.d.1, especially \( \{b_k\} \). This result will then be used to show that \( \{\hat{e}_k - \bar{e}_k\} \) converges to zero for \( k \to \infty \), \( \hat{e} \)-a.s. This implies that the hypothesis (6.6.3-5) is true.

6.7.2-1. Theorem. The sequence of outputs \( \{y_t\} \) of the system to-be-identified is an e.d.d. sequence.

Proof. By assumption (cf. section 6.2.1) the 'true' model has an innovations representation (cf. section 2.4 and (4.8-1)) with \( D = \mathbb{I} \) (without loss of generality this may be assumed) and, for notational reasons, \( \omega_t \) instead of \( \eta_t \):

\[
\begin{align*}
\text{(6.7.2-2)} & \quad x_{t+1} = Ax_t + B\eta_t, \quad t \in \mathbb{Z}, \quad x_t \in \mathbb{R}^n, \\
& \quad y_t = Cx_t + \omega_t, \quad y_t \in \mathbb{R}^p, \quad \omega_t \in \mathbb{R}^p
\end{align*}
\]

with \( A \) asymptotically stable etc. (cf. sections 2.4 and 4.8). For simplicity of notation assume in this proof that \( t_0 = 1 \), and consider \( \{y_t\}_{t=1}^{\infty} \). It is clear that

\[
\text{(6.7.2-3)} \quad y_t = CA^{-1}x_1 + \sum_{r=1}^{t-1} CA^{-1}B\eta_{t-r} + \omega_t.
\]
Now for \((t,s) \in \mathbb{N}\), let

\[
(6.7.2-6) \quad y_{ts} = \begin{cases} 
  t-s-1 \sum \limits_{r=1}^{\min(t,s)} CA^{r-1} B_{t-r} \omega_{t-r} + \omega_t \text{ if } t > s, \\
  0 \quad \text{otherwise.}
\end{cases}
\]

Then \(y_{ts}\) is stochastically independent of \(\omega_s, \omega_{s-1}, \omega_{s-2}, \ldots\) and therefore of 
\(Y_s, Y_{s-1}, Y_{s-2}, \ldots\).

Let

\[
(6.7.2-5) \quad c^y_{ts} = y_t - y_{ts} = CA^{t-s-1} x_1 + \sum \limits_{r=t-s}^{t-1} CA^{r-1} B_{t-r}. \quad \text{for } r > t-s.
\]

It follows that

\[
(6.7.2-6) \quad E_{t-s} c^y_{ts} = CA^{t-1} \text{Ex}_{t-1} (A^{t-1})^T C^T + \sum \limits_{r=t-s}^{t-1} CA^{r-1} B \sum_{r=1}^{\min(t,s)} (A^{r-1})^T C^T.
\]

From lemma (6.7.1-13) it follows that \(\{CA^{r-1} B \sum_{r=1}^{\min(t,s)} (A^{r-1})^T C^T\}_{r=1}^\infty\) is e.d. Let the sequence \(\{c_r\}_{r=1}^\infty\) be defined by

\[
(6.7.2-7) \quad c_r = \text{tr} CA^{r-1} B \sum_{r=1}^{\min(t,s)} (A^{r-1})^T C^T, \quad r = 1, 2, 3, \ldots.
\]

Then \(\{c_r\}\) has e.d.; this follows from lemma (6.7.1-13) and theorem (6.7.1-5) (1) (a), making use of the fact that \(i_r\) is a linear combination of the entries of \(A^{r-1} B \sum_{r=1}^{\min(t,s)} (A^{r-1})^T C^T, r = 1, 2, 3, \ldots\).

It follows that

\[
\begin{cases} 
  t-1 \sum \limits_{r=t-s}^{t-1} C_{(t,s)\in\mathbb{N}} \\
  \sum \limits_{r=t-s}^{t-1} C_{(t,s)\in\mathbb{N}}
\end{cases}
\]

is an e.d. double sequence, as can easily be shown. The sequence

\(\{\text{tr} CA^{t-1} \text{Ex}_{t-1} (A^{t-1})^T C^T\}_{t=1}^\infty\) is e.d. (same proof as for the fact that \(\{c_r\}\) has e.d.). It follows easily that \(\{\text{Ex}_{t-s} s^2\} = \{\text{tr} E_{t-s} s^2\}\) is an e.d. double sequence. Because the \(\epsilon_{ts}\) are all Gaussian, theorem (6.7.1-20) tells us that \(\{y_t\}\) has e.d.

\(\blacksquare\)

6.7.2-8. Corollary. \(\{ty_t\}\) has e.d.d.

Proof. \(F(y) = ty_t\) is mapping with Lipschitz constant 1:
\begin{equation}
|F(y) - F(x)| = |y - x| \leq \|y - x\|.
\end{equation}

Therefore theorem (6.7.1-32) (b) is applicable and the corollary follows.

Q.E.D.

6.7.2-10. **Theorem.** \{v_t\} has e.d.d.

Proof. Let \( t_0 = 1 \) to simplify the notation (without loss of generality). From the definition (6.2.8-3) of \( v_t \) it follows that

\begin{equation}
(6.7.2-11) \quad v_t = \sum_{r=0}^{t-1} \lambda_r^t \|y_{t-r}\|.
\end{equation}

Corollary (6.7.2-8) and theorem (6.7.1-32) can now be applied to conclude that \( \{v_t\} \) has e.d.d.

Q.E.D.

Next we want to show that \( \{g_1(v_t)\} \) has e.c.d. To be able to do this we need the following technical lemma.

6.7.2-12. **Lemma.** Let for all \((t,s) \in \mathbb{N}\)

\begin{equation}
(6.7.2-12) \quad v_{t,s} = \begin{cases} 
\sum_{r=0}^{t-s-1} \lambda_r^t \|y_{t-r,s}\| & \text{if } t > s, \\
0 & \text{if } t = s.
\end{cases}
\end{equation}

The following holds: let \( v > 0 \), then

\begin{equation}
(6.7.2-13) \quad \exists \epsilon > 0, \exists \epsilon_0 > 0, \forall (s,t) \in \mathbb{N}, \forall \epsilon \in (0, \epsilon_0): P(v - \epsilon < v_{s,t} \leq v) < c \cdot \epsilon^\frac{1}{4}.
\end{equation}

The proof will be given in appendix 6B.

6.7.2-14. **Theorem.** \( \{g_1(v_t)\} \) has e.d.d.

Proof. Let
(6.7.2-15) \( c_{ts}^g := g_1(v_t) - g_1(v_{ts}) \),

where \( v_{ts} \) is as defined in lemma (6.7.2-12). Clearly \( v_{ts} \) and therefore \( g_1(v_{ts}) \) is stochastically independent of \( y_s, y_{s-1}, \ldots \) and therefore of \( v_s, v_{s-1}, \ldots \) and of \( g_1(v_s), g_1(v_{s-1}), \ldots \). It is clear that \( c_{ts}^g \in \{-1, 0, 1\} \). One has (cf. (6.2.8-4) for the definition of \( g_1 \))

\[
\begin{align*}
\forall p \in \mathbb{N}: E|c_{ts}^g|^p &= E|c_{ts}^g| = P[g_1(v_t) \neq g_1(v_{ts})] = \\
&= P[v_t > K' \text{ and } v_{ts} \leq K'] + P[v_t \leq K' \text{ and } v_{ts} > K'].
\end{align*}
\]

Let \( c_{ts}^v = v_t - v_{ts} \). Then

\[
(6.7.2-17) \ P[v_t > K' \text{ and } v_{ts} \leq K'] = P[c_{ts}^v > K' - v_{ts} \text{ and } v_{ts} \leq K'].
\]

Let \( F(v_{ts}) \) denote the distribution function of \( v_{ts} \), then

\[
(6.7.2-18) \ P[c_{ts}^v > K' - v_{ts} \text{ and } v_{ts} \leq K'] = \int_{v_{ts}=0}^{K'} P[c_{ts}^v > K' - v_{ts} | v_{ts}] dF(v_{ts}) = \\
\int_{v_{ts}=0}^{K'} (E[c_{ts}^v|v_{ts}] \frac{v_{ts}}{E|c_{ts}^v|}) + \int_{v_{ts}=K'}^{K'} (E[c_{ts}^v|v_{ts}] \frac{v_{ts}}{E|c_{ts}^v|}) dF(v_{ts}) \leq \\
\int_{0}^{K'} (E[c_{ts}^v])^\frac{1}{2} \frac{v_{ts}}{E|c_{ts}^v|} dF(v_{ts}) + P[K'-(E|c_{ts}^v|)]^\frac{1}{2} v_{ts} \leq K'] \leq (E|c_{ts}^v|)^\frac{1}{2} + P[K'-(E|c_{ts}^v|)]^\frac{1}{2} v_{ts} \leq K'].
\]

According to theorem (6.7.2-10) \( v_t \) has e.d.d. and therefore \( (E|c_{ts}^v|)_t \in \mathbb{N} \) is an e.d. double sequence. It follows that

\[
(E|c_{ts}^v|)_t \in \mathbb{N} \text{ is an e.d. double sequence too. And making use of lemma (6.7.2-12) it follows that } P[K'-(E|c_{ts}^v|)]^\frac{1}{2} v_{ts} \leq K'] \in \mathbb{N} \text{ is an e.d. double sequence. Combining (6.7.2-17) and (6.7.2-18), it follows that } P[v_t > K' \text{ and } v_{ts} \leq K'] \in \mathbb{N} \text{ is an e.d. double sequence. In a completely similar fashion, by interchanging the roles of } v_t \text{ and } v_{ts} \text{ and}
\]

replacing $\varepsilon_{ts}^t$ by $-\varepsilon_{ts}^t$ one can show that \( \{P[v_t \leq K' \text{ and } v_{ts} > K'] | (t,s) \in \mathbb{N}\} \) is an e.d. double sequence. Equation (6.7.2-16) now implies that for each \( p \in \mathbb{N} \), \( \{E[\varepsilon_{ts}^t]^P \} \) is an e.d. double sequence, and therefore \( \{e_t(v_t)\} \) is e.d.d.

Q.E.D.

Now consider the decoupled algorithm, and let \( \{e_t\} \) satisfy property 3 with

interpolation curve \( y^+ \in I^+ \) such that \( \theta_t = y^+(s_t), \forall t \in \mathbb{Z} \). Let

\( \tilde{s} \in \tilde{S}, \tilde{\mathcal{S}} = \{(r_k)_{k=1}^m, q_0, 0\}, \) be such that \( |I(\tilde{s}, y^+)| = 1 \), say \( I(\tilde{s}, y^+) = \{(i, j)\} \) and let \( \tilde{N} = (N_k)_{k=1}^m \) be the corresponding sequence of intervals defined by

\[
N_k = \{t | s_t \in [r_{k-1}, r_k] \}.
\]

6.7.2-19. Theorem. \( \{\varepsilon_t\} \) has e.d.d.-N (with respect to \( \tilde{P} \)).

Proof. Combining the inequality (6.4-12) of lemma (6.4-8) with lemma (6.4-41) (a) one finds there exists a \( c_3 > 0 \), \( c_3 \) data-independent, such that

\[
\forall t \geq 0 : \|\xi_t\| \leq c_3 v^t.
\]

Because \( \{v_t\} \) has e.d.d. (theorem (6.7.2-10)), it follows that \( \{E[\xi_t^t]^P\}_{t=0}^m \) is bounded for each \( p \in \mathbb{N} \).

Now, according to theorem (6.3.4-21) one can choose (and we assume that this has been taken care of) the refinement of the coordinate-charts-cover of the manifold such that within each chart \( D_{ij} \) the spectral norm of \( F(\phi_j(\theta_t); i, j) \) is smaller than \( \lambda + \varepsilon < 1 \). Now consider equation (6.3.3-15). If \( t+1, t \in N_k \), then \( \xi(t+1; i, j) \) can be decomposed as the sum of a vector due to \( \xi(\min N_k; i, j) \) and the rest, which depends only on \( y_t, y_{t-1}, \ldots, y_{\min(N_k)} \). The first part has e.d.d.-N according to the combination of lemma (6.7.1-45) (with \( x_t = \xi(t; i, j) \) if \( t = \min N_k \) for some \( k \) and \( x_t = 0 \) otherwise) with theorem (6.7.1-43) (g) (with \( F(\phi_j(\theta_t); i, j) \) instead of \( A_t \)). Because \( \{y_t\} \) has e.d.d., \( \{G(\phi_j(\theta_t); i, j)\} \) has e.d.d.-N and therefore the second part has e.d.d.-N according to theorem (6.7.1-43) (g) (again with \( F(\phi_j(\theta_t); i, j) \) instead of \( A_t \)). Applying (6.7.1-43) (a) one finds that \( \{\xi_t\} \) has
6.7.2-20. **Theorem.** \( \{b_t\} \) has **e.d.i.-N** (w.r.t. \( \tilde{F} \)).

**Proof.** Consider the definition of \( b_t \) in (6.6.2-1). Substituting for \( h(\varepsilon_t,j) \) according to (6.2.8-5), one finds

\[
(6.7.2-21) \quad b_t = g_1(\nu_t)g_2(t)\mathbb{R}(\theta_t,j)^{-1}\Psi(t,j)^T e(t,j).
\]

Consider (6.3.3-16); the matrix \( H(\hat{\varepsilon}(\theta);1,j) \) is bounded on \( \tilde{E}_j^* \); \( \{\varepsilon(t;1,j)\} \) has e.d.i.-N (theorem (6.7.2-19)) and \( \{\nu_t\} \) has e.d.d. (theorem (6.7.2-1)). Therefore \( z(t,j) \) has e.d.i.-N, where

\[
z(t,j)^T = (e(t,j)^T, \nu_1(t,j)^T, \ldots, \nu_d(t,j)^T),
\]

as before. Applying theorem (6.7.1-43) (a) (or in fact a slight generalization of this, namely to the case of a vector of polynomials instead of one polynomial) one finds that \( \{\Psi(t,j)^T e(t,j)\} \) has e.d.i.-N. From corollary (6.4-55) it follows easily that \( \{g_1(\nu_t)\} \) has e.d.d. Theorem (6.7.2-14) states that \( \{g_1(\nu_t)\} \) has e.d.d. And \( \{\mathbb{R}(\theta_t,j)^{-1}\} \) is a sequence of bounded nonstochastic matrices. It follows from theorem (6.7.1-43) (a) that \( \{b_t\} \) has e.d.i.-N.

**Q.E.D.**

6.7.2-22. **Corollary.** The double sequence

\[
(\tilde{E}(b \tilde{E}_s)^T(b \tilde{E}_s)^T) \text{ for } (t,s) \in \tilde{N}_1 \text{ has e.d.i.-N,}
\]

i.e. \( \exists c > 0, \exists \lambda \in (0,1) \) such that \( \forall (t,s) \in \tilde{N}_1 : |(\tilde{E}(b \tilde{E}_s)^T(b \tilde{E}_s)^T)| < c \lambda^{t-s} \).

The proof is analogous to that of corollary (6.7.1-31), the only difference is that here it concerns 'e.d.i.-N' instead of instead of 'e.d.d.', and 'e.d.i.' instead of 'e.d.'. The details are left to the reader. From this corollary the following important lemma can be derived. (Let \( \theta_t, \tilde{N}, N \) etc. be again as
described just before theorem (6.7.2-19)).

6.7.2-23. Lemma. 

\[ \sum_{k=1}^{w} \| \tilde{E}_{k} \|_2^2 < \infty. \]

Proof. Consider the definition (6.6.2-37) of \( f_k \). Because \( |I(\tilde{S}, \tilde{y}^k)| = 1 \) it follows that \( |I_k| = 1 \) for each \( k \in \mathbb{N} \); in fact \( I_k = \{(i,j)\} \) for each \( k \). So for each \( k \in \mathbb{N} \):

\[ f_k = \int_{r_k-q+\delta}^{r_k} b^0(r)dr. \]

It follows that

(6.7.2-24) \[ \| f_k - \tilde{E}_k \|_2^2 = \int_{r_k-q+\delta}^{r_k} (b^0_\sigma - \tilde{b}^0_\sigma)d\sigma \]

and

(6.7.2-25) \[ \sum_{k=1}^{w} (f_k - \tilde{E}_k)^T (f_k - \tilde{E}_k) = \sum_{k=1}^{w} \int_{r_k-q+\delta}^{r_k} (b^0_\sigma - \tilde{b}^0_\sigma)(b^0_\sigma - \tilde{b}^0_\sigma)d\sigma. \]

Consider the definition (6.6.2-27) of \( b^0(\cdot) \). Using the previous corollary one can derive

(6.7.2-26) \[ \int_{r_k-q+\delta}^{r_k} (b^0_\sigma - \tilde{b}^0_\sigma)(b^0_\sigma - \tilde{b}^0_\sigma)d\sigma \leq \sum_{(t,s) \in K} a_t a_s c_{\lambda} |t-s|, \]

with \( c > 0, \lambda \in (0,1) \) as in corollary (6.7.2-22).

A fortiori, it follows that

(6.7.2-27) \[ \| f_k - \tilde{E}_k \|_2^2 \leq c \sum_{t \geq t_0, s \geq s_0} a_t a_s c_{\lambda} |t-s|, \]

For simplicity of notation, take \( t_0 = 1 \). Now consider
The lemma follows. Q.E.D.

To show that from this it follows that the hypothesis (6.6.3-5) is true the following lemma is needed.

6.7.2-29. **Lemma.** Let \( \{x_k\}_{k=1}^\infty \) be an arbitrary sequence of random vectors \( x_k \in B \) with mean zero and covariance matrix \( \Gamma_k \). If \( \sum_{k=1}^\infty \text{tr} \Gamma_k < \infty \), then \( \lim_{k \to \infty} x_k = 0 \) with probability one.

**Proof.** Let \( \varepsilon > 0 \) and apply Chebyshev's inequality

\[
(6.7.2-30) \quad P(\|x_k\| > \varepsilon) \leq \frac{\text{Ex}_k x_k}{\varepsilon^2} = \frac{\text{tr} \Gamma_k}{\varepsilon^2}.
\]

Therefore

\[
(6.7.2-31) \quad \forall \varepsilon > 0: \sum_{k=1}^\infty \text{P}(\|x_k\| > \varepsilon) \leq \sum_{k=1}^\infty \text{tr} \Gamma_k / \varepsilon^2 < \infty.
\]

According to the lemma of Borel-Cantelli it follows that \( \lim_{k \to \infty} x_k = 0 \) with probability one.

Q.E.D.

Combining lemmas (6.7.2-23) and (6.7.2-29) one finds the main result of this section.

6.7.2-32. **Theorem.** The hypothesis (6.6.3-5) is true.

6.7.2-33. **Remark.** It follows of course that those results in section 6.6 (cf. especially theorem (6.6.3-16) and (6.6.3-17)) that are derived under the condition that the hypothesis is true, are all true!
6.8. The associated differential equation

6.8.1. An integral formula for the decoupled algorithm

In the second part of this section (section 6.8.2) the ordinary differential equation (o.d.e) that is associated with the algorithm will be derived. The construction of the function that satisfies the o.d.e. was an idea of [Ku-C].

However, in section 6.9 it will turn out that this function is constant, and its value is in fact an equilibrium point of the o.d.e. Therefore any claim that the algorithm will eventually follow or approximate a nonconstant solution curve of the differential equation seems unjustified, or is at least not justified by the approach followed here.

In this section (6.8.1) an integral formula will be derived for the decoupled algorithm, which is needed for the derivation of the o.d.e. in section 6.8.2.

To start with, two sorts of variables that depend on a parameter point (instead of a parameter curve or sequence) will be defined.

6.8.1-1. Notation. Let $\delta^+ = (\theta, i, j) \in \Theta^+$ and consider the decoupled algorithm with constant parameter sequence $\{\theta_t^+ = \theta^+\}_{t=0}^\infty$. Then in the notation of the variables occurring in the decoupled algorithm, $\{\theta_t^+\}_{t=0}^\infty$ will be replaced by $\delta^+$. For example, $b_t(\theta^+) = b_t(\{\theta^+\}_{t=0}^\infty)$.

Now consider the vector of all random variables that occur in the decoupled algorithm at time $t$, or are derived from it. It is

\[(6.8.1-2) \quad \langle y_t, x_t, z_t, \xi(t+1;i,j), s(t,j), \psi(t,j), h_t, b_t \rangle.\]

If the parameter sequence has constant value $\delta^+$, then the probability distribution of this vector converges to a steady state distribution for $t \to \infty$. (From corollary (6.4-55) it follows that $g_2(t) = 1$ for $t$ large enough and it can be considered as nonrandom for $t \to \infty$. Therefore it can be left out of our considerations concerning the asymptotic behaviour of the algorithm).

6.8.1-3. Definition. Let $\langle y, v, g_1(v), \xi(\theta), s(\theta), \psi(\theta,j), h(\theta,j), b(\theta,j) \rangle$ be a random vector with as its probability distribution the steady state distribution of (6.8.1-2). The components of the vector will be called steady state random variables for the decoupled algorithm at the parameter point $\delta^+$.

Note that $s(\theta)$ does not depend on $j$. That this is correct follows from the
construction of the prediction error algorithm (cf. section 6.2, esp. section 6.2.2); \( e(\theta) \) is the steady state random variable corresponding the prediction error \( \varepsilon(\theta) \) defined in (6.2.2-3). To be able to work with local coordinates, instead of \( e(\theta) \) also the notation \( e(\phi; j) \) will be used, where \( \phi = \phi_j(\theta) \) stands for the local coordinates of \( \theta \) in \( C_j \) (as before). Because the random variables in the definition correspond to the decoupled algorithm their expectation will be denoted by \( \mathbb{E}e(\theta^+), \mathbb{E}e(\phi; j) \) etc.

6.8.1-4. Remarks. (i) One way to construct formally a steady state random vector for the decoupled algorithm is by taking a constant parameter sequence in the algorithm and starting formally at \( t = -\infty \), which makes it into an asymptotically stable time-invariant filter. Because the true system is asymptotically stable, the result is mathematically well-defined and the resulting variables have the steady-state probability distribution at each time \( t \).

(ii) If \( \theta^+ = (\theta, i, j) \) then \( b(\theta^+) = b(\theta, i, j) \) will denote the same as \( b(\theta, j) \), this also holds for \( h, v, r \) etc. The algorithm is constructed such that the following equality holds (cf. section 6.2 especially section 6.2.2).

6.8.1-5. Theorem. Let \( \theta^+ = (\theta, i, j) \in \theta^+ \). One has

\[
(6.8.1-6) \quad \mathbb{E}b(\theta, j) = -R(\theta, j)^{-1} \frac{\partial}{\partial \theta} \mathbb{E}g_1(v)\varepsilon(\phi; j)^T\varepsilon(\phi; j)/2, 
\]

i.e. \( \mathbb{E}b(\theta, j) \) equals minus the Riemannian gradient of the function

\[
(6.8.1-7) \quad V(\theta) := \frac{1}{2} \mathbb{E}g_1(v)\varepsilon(\theta)^T\varepsilon(\theta),
\]

in terms of the local coordinates of the chart \( (C_j, \phi_j) \).

If \( K' \) in the definition of \( g_1 \) is taken large enough, the probability that \( g_1(v) \) equals zero will be very small, and \( V(\theta) \) will be close to \( V(\theta) \). There is an asymptotic relation between the value of \( b_k \) for a varying parameter sequence and its value for a related constant parameter sequence. The same holds for \( \mathbb{E}b_k \). But asymptotically \( b_k \) with constant parameter sequence has the same distribution as \( b \) and thus one obtains an asymptotic relation between \( \mathbb{E}b_k \) with a (specific type of) varying parameter sequence and \( \mathbb{E}b \). The precise formulation, in terms of interpolation curves, is as follows.
6.8.1-8. Theorem. Let the coordinate-chart index \((i,j)\) be fixed and let \((q, \delta)\) with \(q > \delta > 0\), be fixed. (a) For each \(\varepsilon > 0\) there exists a \(t_2\) such that if \(Y^+ \in \mathbf{L}^+_{CP}(-\infty, 0]\) and \(Y^+|_{[-q, 0]} \in \mathbf{L}^+_{ij}[-q, 0]\) then

\[
\forall t \geq t_2, \forall \omega \in [0, q-\delta], \exists b_{\omega} \text{ s.t. } \| \mathbf{S}_{\omega}(Y^+) - b_{\omega}(Y^+(\omega)) \| < \varepsilon,
\]

where by abuse of notation \(Y^+(\omega)\) denotes the parameter sequence with constant value \(Y^+(\omega)\) (cf. (6.8.1-1)).

(b) For each \(\varepsilon > 0\) there exists a \(t_5\) such that if \(Y^+ \in \mathbf{L}^+_{CP}(-\infty, 0]\) and \(Y^+|_{[-q, 0]} \in \mathbf{L}^+_{ij}[-q, 0]\), then

\[
\forall t \geq t_5, \forall \omega \in [0, q-\delta], \exists b_{\omega} \text{ s.t. } \| \mathbf{S}_{\omega}(Y^+) - b_{\omega}(Y^+(\omega)) \| < \varepsilon.
\]

(c) Let \(\{Y^+_t\}_{t=t_0}^\infty\) be a sequence such that

\[
(1) \quad \forall t \geq t_0: Y^+_t \in \mathbf{L}^+_{CP}(-\infty, 0],
\]

\[
(ii) \quad \forall t \geq t_0: Y^+_t|_{[-q, 0]} \in \mathbf{L}^+_{ij}[-q, 0] \text{ and}
\]

\[
(iii) \quad \lim_{t \to \infty}(Y^+_t|_{[-q, 0]}) = Y^+|_{[-q, 0]} \text{ (convergence in the topology of } \mathbf{L}^+_{ij}[-q, 0], \text{ cf. section 6.5}).
\]

Then for each \(\varepsilon > 0\) there exists a \(t_7\) such that

\[
6.8.1-9 \quad \forall t \geq t_7, \forall \omega \in [0, q-\delta]: \| \mathbf{S}_t \circ \mathbf{S}_{\omega}(Y^+) - \mathbf{S}_{\omega}(Y^+(\omega)) \| < \varepsilon.
\]

Let \(\{\tau(t)\}_{t=t_0}^\infty\) be a sequence of nonnegative numbers with limit zero. Then for each \(\varepsilon > 0\) there exists a \(t_8\) such that

\[
6.8.1-10 \quad \forall t \geq t_8, \forall \omega \in [0, q-\delta]: \| \mathbf{S}_t \circ \mathbf{S}_{\omega}(\tau(t)) (Y^+) - \mathbf{S}_{\omega}(Y^+(\omega)) \| < \varepsilon.
\]

Proof. (a) Because of lemma (6.6.2-3) it is sufficient to show that for each \(\delta' > 0\) there exists a \(t_2\) such that if \(Y^+ \in \mathbf{L}^+_{CP}(-\infty, 0]\) and \(Y^+|_{[-q, 0]} \in \mathbf{L}^+_{ij}[-q, 0]\), then...
\[ \forall t \geq t_2, \forall \sigma \in [0, q-\delta]: \rho^+ (\pi^+_t \circ S_{-\sigma} (Y^+), (Y(-\sigma))) < \delta. \]

This property can be shown as follows. Let \( n_0 \in \mathbb{N} \) be such that \( n_0^{-1} < \min(\delta, 1) \). Let \( t_2 \) be such that \( \forall t \geq t_2: \sum_{s=t-n_0}^{t-1} a_s < \min(\delta, \frac{\delta}{c}) \) (where \( c \) is the Lipschitz constant!). From the definition of \( \rho^+ \) (cf. (6.5-12)) it follows that

\[
\rho (\pi^+_t \circ S_{-\sigma} (Y^+), (Y^+(-\sigma))) \leq \\
(6.8.1-11) \\
\leq \max \left\{ \sup_{0 \leq n \leq n_0^{-1}} \left[ \sup_{1 \leq j \leq n+1} d^+ (Y^+(-\sigma))_j \right] ; \sup_{n \geq n_0} \left( \frac{1}{n+1} \right) \right\} \leq \\
\leq \max \left\{ c \sum_{s=t-n_0}^{t-1} a_s \frac{1}{n+1} \right\} < \max(\delta, \delta) = \delta.
\]

(b) Because

\[
\hat{E}(b_t \circ \pi^+_t \circ S_{-\sigma} (Y^+) - b_t (Y^+(-\sigma))) \leq \hat{E}(b_t \circ \pi^+_t \circ S_{-\sigma} (Y^+) - b_t (Y^+(-\sigma))) ,
\]

it follows from (a) that for each \( \varepsilon > 0 \) there exists a \( t_2 \) such that if \( Y^+ \in T^+_{c^\varepsilon} (-\infty, 0] \) and \( Y^+ \in [-q, 0] \),

(6.8.1-12) \( \forall t \geq t_2, \forall \sigma \in [0, q-\delta]: \hat{E}(b_t \circ \pi^+_t \circ S_{-\sigma} (Y^+) - \hat{E}(Y^+(-\sigma))) < \varepsilon/2. \)

Therefore it suffices to show that for given \( \varepsilon > 0 \) there exists a \( t_6 \) such that

(6.8.1-13) \( \forall t \geq t_6, \forall \sigma \in \pi^+_t \circ S_{-\sigma} (Y^+) \circ \hat{E}(b_t (Y^+)) < \varepsilon/2. \)

Once this has been shown then taking \( \theta^+ = Y^+(0), t_5 = \max(t_2, t_6) \) and combining (6.8.1-10) and (6.8.1-11) the result follows. Now consider (6.8.1-11). According to remark (6.8.1-4), \( b(\theta^+) \) may be taken to be \( b^\ast \) if \( b^\ast \) is defined as the \( b_t \) that results if the algorithm is started (formally) at \( t = -\infty \) (instead of \( t = t_0 \)) and a constant parameter sequence \( \{\theta^+\} \) is employed. The difference of \( b^\ast \) with the \( b_t \) that results if the algorithm is started at \( t = t_0 \) and the same constant parameter sequence \( \{\theta\} \) is used, is caused (only) by the fact
that $\xi_t^*$ has a nonzero value while $\xi_{t_0} = 0$. (The notation is obvious: $\xi_t^*$ is the vector $\xi(t; 1, 1)$ that results if in the decoupled algorithm starting at $t = \infty$ the constant parameter sequence $\{\theta^+_t\}$ is applied, $\xi_t$ is the vector $\xi(t; 1, 1)$ that results if in the decoupled algorithm starting at $t = t_0$ the constant parameter sequence $\{\theta^*_t\}$ is applied). The difference

$$\Delta \xi_t = \xi_t^* - \xi_t$$

is equal to $F^{t-t_0} \Delta \xi_{t_0} = F^{t-t_0} \xi_{t_0}^*$. Using the fact that $F_{t_0}$ is bounded by a number smaller than one, uniformly for all $\theta^+_t \in D^{+}_{L \omega}$ (cf. theorem (6.3.4-21)) it follows that $\bar{F}(b_{t_0}^* - b_t^*)$ has exponential decay, uniformly for all $\theta^+_t \in D^{+}_{L \omega}$, and the result follows.

(c) First (6.8.1-9) will be shown. Consider the following inequality

$$1 \bar{F}_{t_0} \circ \pi_t^a \circ S_{-\omega} (y_t^*) - \bar{F}_t (y_t^* - \omega) || \leq$$

$$\leq 1 \bar{F}_{t_0} \circ \pi_t^a \circ S_{-\omega} (y_t^*) - \bar{F}_t \circ \pi_t^a \circ S_{-\omega} (y_t^* || +$$

$$+ 1 \bar{F}_{t_0} \circ \pi_t^a \circ S_{-\omega} (y_t^* - \bar{F}_t (y_t^* - \omega) ||.$$
Because \( \lim_{t \to \infty} (Y^t_t | [-q,0]) = Y^+ | [-q,0] \), there exists a \( t' \geq t_0 \) such that for all \( t \geq t' \),

\[
\sup_{-q \leq r \leq 0} d^+ (Y^+_t(r), Y^+(r)) < \delta', \quad \forall \sigma \in [-q+\delta,0]: \sup_{-\delta \leq r \leq 0} d^+ (Y^+_t(r-\sigma), Y^+(r-\sigma)) < \delta'.
\]

Now consider proposition (6.5-14) (b) and the \( n_0(t) \) defined there. Because \( \lim_{t \to \infty} a_t = 0 \), one has \( \lim_{t \to \infty} n_0(t) = \infty \), and so there exists a \( t' \geq t_0 \) such that

\[
\forall t \geq t': \frac{1}{n_0(t)+1} < \delta'. \quad \text{Application of proposition (6.5-14) (b) gives:}
\]

\[
(6.8.1-16) \quad \forall t \geq \max(t', \, t''): \rho^+ (H^a_t (S^{-}_\sigma (Y^+_t)), H^a_t (S^{-}_\sigma (Y^+))) < \delta'.
\]

Combination of (6.8.1-15) with (6.8.1-16) gives:

\[
(6.8.1-17)
\]

\[
\forall \tau \geq \max(t', \, t'', \, t_3), \forall \sigma \in [-q+\delta,0]: \exists \widetilde{E}_t \circ \|_{\tau} \circ S^{-}_\sigma (Y^+_t) - \widetilde{E}_t \circ \|_{\tau} \circ S^{-}_\sigma (Y^+) \leq \frac{\varepsilon}{2}.
\]

According to (b) there exists a \( t_5 \) such that

\[
(6.8.1-18) \quad \forall t \geq t_5, \forall \sigma \in [-q+\delta,0]: \exists \widetilde{E}_t \circ \|_{\tau} \circ S^{-}_\sigma (Y^+) - \widetilde{E}_t \circ \|_{\tau} \circ S^{-}_\sigma (Y^+)) \leq \frac{\varepsilon}{2}.
\]

Let \( t_7 = \max(t_6, t', t'', t_3) \). Then substitution of (6.8.1-17) and (6.8.1-18) into (6.8.1-14) leads to the desired result.

It remains to show (6.8.1-10). Let \( t'' \) be such that for all \( t \geq t'' \), \( \tau(t) < \delta/2 \). For all \( t \geq t'' \) let \( Y^+_t := S^{-}_\tau(t) (Y^+_t) \). Then, using (i) and (ii) and the Lipschitz condition:

\[
\forall \tau \in [-q+\delta,0]: d^+ (Y^+_t(r), Y^+_t(r)) = d^+ (Y^+_t(r-\tau(t)), Y^+_t(r)) \leq c.\tau(t).
\]

Therefore \( \lim_{t \to \infty} Y^+_t | [-q,0] = Y^+ | [-q,0] \). Replacing in (6.8.1-9) \( q \) by \( q-\frac{\delta}{2} \), \( \delta \) by \( \frac{\delta}{2} \), and \( Y^+_t \) by \( Y^+_t \), and \( t_7 \) by \( (t_6 := \max(t_7, t'') \), one obtains (6.8.1-10).

Q.E.D.
6.8.1-19. Theorem. Let \( \{s_t^+\} \) be a parameter sequence satisfying property 3 with interpolation function \( Y^+ \in L^1_{cp} \) such that \( q^+_t = Y^+(s_t^+) \), for all \( t \in \mathbb{Z} \). There exists a set \( E \subseteq \Omega \), \( P(E) = 0 \) with the following property. Let \( \mathbf{S} \in \mathbf{S} \) be such that

1. \( |I(\mathbf{S}, Y^+)| = 1 \); let \( \{(i,j)\} \) denote \( I(\mathbf{S}, Y^+) \), and

(11) \( \{S_{r_k^+}^{(Y^+) \left[-q,0\right)}\}_{k=1}^{\infty} \) converges in \( L^1_{ij}[-q,0] \); let \( X^+ \in L^1_{ij}[-q,0] \) denote the limit.

Then the following holds

\[
\lim_{k\to\infty} f_k = \lim_{k\to\infty} \mathbf{f}_k = \lim_{k\to\infty} \int_0^{r_k^+} \mathbf{B}^0(r)dr = \\
\int_0^{r_k^+} \mathbf{B}^0(X(-\sigma))d\sigma, \forall \omega \in \Omega \setminus E.
\]

(6.8.1-20)

Proof. Equation (6.8.1-20) contains three equalities that have to be verified. The last one will be treated first. From the definition of \( b^0 \) (cf. (6.6.2-27)) it follows that

\[
(6.8.1-21) \quad b^0(r_k^- - \sigma) = b_t \circ \Pi_t \circ s_t \circ (Y^+)
\]

if \( t = t(k) \) is such that \( s_t^+ \leq r_k^- - \sigma < s_{t+1} \) and \( t \geq t_0 \).

Let \( Y_k := S_{r_k^+}^{(Y^+)} \) and \( \tau(k) = r_k^- s_t(k) - \sigma \) for all \( k \). Then \( \lim_{k} Y_k \left[-q,0\right) = X \)

and

\[
(6.8.1-22) \quad b^0(r_k^- - \sigma) = b_t \circ \Pi_t \circ s_{-\sigma - \tau(k)}(t_k^+), \text{ with } t = t(k).
\]

It is not difficult to conclude from theorem (6.8.1-18) (c) that the result presented there holds equally well for a sequence \( \{t(k)\}_{k=1}^\infty \) instead of all \( t \geq t_0 \). Applying this one finds that \( \{\mathbf{B}^0(r_k^- - \sigma)\} \) converges uniformly for all \( \sigma \in [0,q-\delta] \) to \( \mathbf{B}^0(X^+(-\sigma)) \). Therefore

\[
(6.8.1-23) \quad \lim_{k\to\infty} \int_0^{r_k^- - \sigma} \mathbf{B}^0(r_k^- - \sigma)d\sigma = \int_0^{q-\delta} \mathbf{B}^0(X^+(-\sigma))d\sigma.
\]

(The existence of the integral on the right-hand side follows from the continuity of \( \mathbf{B}^0(X^+(-\sigma)) \) as a function of \( \sigma \)). Thus the last equality of
(6.8.1-20) has been shown to hold.

The second equality of (6.8.1-20) follows directly from the definition of \( f_k \) and condition (1) on \( \tilde{S} \) (cf. (6.6.2-37)). The first equality holds for all \( \omega \notin E \) and follows from theorem (6.6.3-16) in combination with theorem (6.7.2-32).

Q.E.D.

From this theorem together with theorem (6.8.1-5) one finds

\[
\forall \omega \notin E: \lim_{k \to \infty} f_k = -\frac{1}{2} \int_{0}^{\infty} \left( \mathbb{R}(X^\omega(-\sigma))^{-1} \frac{\partial}{\partial \sigma} \mathbb{E}_1(v) \right) \mathcal{L} \{ \phi_j(X(-\sigma)); j \} ds.
\]

6.8.2. The o.d.e. for the (coupled) algorithm

Consider the 'coupled' algorithm. This means that apart from the equations of the decoupled algorithm (cf. section 6.4) one has equations (6.2.9-11), (6.2.9-12) and the rules for coordinate change that follow (6.2.9-12). For the 'coupled' algorithm one has the following equality.

6.8.2-1. Theorem. Let \( \omega \in \Omega \) be fixed. Let \( \{ \theta^+_t \}_{t=t_0}^\infty \) and \( \{ \theta^+_t \}_{t=t_0}^\infty \) be the two parameter sequences that are produced by the algorithm (as described in section 6.2.9). Let \( Y^+ \in L^\infty_{cp} \) be an interpolation curve such that \( Y^+(s) = \theta^+_t \) for all \( t \geq t \) (such a curve exists according to theorem (6.5-19)). Let \( \tilde{S} = ((r_k)_{1}^{\infty}, q, \delta) \in \tilde{S} \) be such that \( I(\tilde{S}, Y^+) = 1 \), say \( I(\tilde{S}, Y^+) = ((1, j)) \).

Then for all \( k \in \mathbb{N} \):

\[
(6.8.2-2) \quad f_k = \phi_j(Y(r_k)) - \phi_j(Y(r_k-q+\delta)) + \tilde{\tau}(k),
\]

and

\[
(6.8.2-3) \quad \lim_{k \to \infty} \tilde{\tau}(k) = 0.
\]

Proof. The proof is given in four steps. In each step a sequence is defined which converges to zero. The sum of those four sequences is \( (\tilde{\tau}(k))_{k=1}^{\infty} \) and it then follows that \( \lim_{k \to \infty} \tilde{\tau}(k) = 0. \)
(1) Let
\[ N_k := \{ t \mid \sigma_t \in [r_k-q+\delta, r_k] \}, \quad t^*(k) := \min N_k \text{ and } t(k) := \max N_k. \]

Applying proposition (6.6.2-28) to the definition of \( f_k \) (6.6.2-37) one finds
\[ f_k = \int_{r_k-q+\delta}^{r_k} b^0(\sigma) d\sigma = \sum_{t \in N_k} a_{t^* t} + \tilde{\tau}_1(k), \]
with
\[ \tilde{\tau}_1(k) = [a_{t^* t}(r_k-q+\delta)]b_{t^* t}(r_k) - [r_k-s_{t^* t}(k)+1]b_{t^* t}(r_k). \]

Now \( 0 < a_{t^* t}(r_k-q+\delta) < a_{t^* t}(r_k) \) and \( 0 < r_k-s_{t^* t}(k)+1 < a_{t^* t}(k)+1 \).
Because \( a_k \to 0 \) for \( k \to \infty \) and because \( \{b_t\} \) is bounded by a data-independent constant (this follows from corollary (6.4.55)), \( \lim_{k \to \infty} \tilde{\tau}_1(k) = 0. \)

(2) Because in the 'coupled' algorithm, \( \phi_t \) is well-defined one has (compare (6.6.2-1)) for all \( t \in \cup N_k \) \( a_{t^* t} b_t = \phi(\theta_{t^* t}^{-1}, j) - \phi(\theta_t, j) \) if \( \lambda_t = 1. \)

Now \( \lambda_t \neq 1 \) occurs only if in the next step a coordinate change takes place (cf. (6.2.9-1)ff). Therefore \( \lambda_t = 1 \) if \( t \in N_k \setminus \{ t(k) \}. \)

So
\[ \sum_{t \in N_k} a_{t^* t} b_t = \sum_{t \in N_k} [\phi(\theta_{t^* t}^{-1}, j) - \phi(\theta_t, j)] + a_{t^* t} b_{t^* t} = \]
\[ = \phi(\theta_{t^* t}(r_k), j) - \phi(\theta_t, j) + \tilde{\tau}_2(k), \]
where \( \tilde{\tau}_2(k) = a_{t^* t} b_{t^* t}. \) Clearly \( \lim_{k \to \infty} \tilde{\tau}_2(k) = 0, \) according to similar arguments as given in (1) above.

(iii) Let (for \( k \) sufficiently large such that \( \theta_{t^* t}(r_k) \in C_j, \theta_t(r_k) \in C_j \))
\[ \tilde{\tau}_3(k) = \{ \phi(\theta_{t^* t}(r_k), j) - \phi(\theta_{t^* t}(r_k), j) \} - \{ \phi(\theta_t(r_k), j) - \phi(\theta_t(r_k), j) \}. \]

From the coupling equation (6.2.9-12) (if also (6.2.9-4) ff), the fact that \( \lim_{t \to \infty} \delta_t = 0 \) and the equivalence of the inner metric with the local coordinate metric (cf. (6.3.4-23)), it then follows that \( \lim_{k \to \infty} \tilde{\tau}_3(k) = 0. \)
(iv) Let

$$\tilde{\tau}_4(k) = \{t(k) - \phi(Y(r_k)) - \phi(\theta_{t''}(k)) - \phi(\theta_{t''}(k))\}.$$ 

Because $Y(r)$ is an interpolation curve,

$$Y(s_{t''}(k)) = \theta_{t''}(k)$$ and $Y(s_{t''}(k)) = \theta_{t''}(k).$

Because $r_{-s_{t''}(k)} = 0$, $s_{t''}(k) = r_{-q+\delta} = 0$, and because $Y$ satisfies a Lipschitz condition (and again using the equivalence of inner metric and local-coordinates-metric), one has $\lim_{k \to \infty} \tilde{\tau}_4(k) = 0$. Now take

$$\tilde{\tau}(k) = \tilde{\tau}_1(k) + \tilde{\tau}_2(k) + \tilde{\tau}_3(k) + \tilde{\tau}_4(k)$$ and the theorem follows.

Q.E.D.

The combination of this theorem with corollary (6.8.1-24) leads to the differential equation associated with the algorithm.

6.8.2-4. Theorem. There exists a set $E$, $P(E) = 0$, with the following property.

Let $\omega \in \Omega E$. Let $\{t^+\}_{t_0}$ and $\{t^-\}_{t_0}$ be the two parameter sequences that are produced by the algorithm (for this $\omega$). Let $Y^+ \in L^+\mathcal{C}$ be an interpolation curve such that $Y^+(s_{t^+}) = \theta_{t^+}$ for all $t \geq t_0$. Let for all $\delta \in (0, q)$, $S = \{(r_k), q, \delta\} \in S$ be such that

(i) $|I(S, Y^+)| = 1$; let $((i, j))$ denote $I(S, Y^+)$ and

(ii) $(S_k^+, Y^+_{k})_{k=1}^\infty$ converges in $L^+_{-q, 0}]$; let $X^+ \in L^+_{-q, 0}$ denote the limit.

Then the following holds

$$\forall \delta \in (0, q): \phi_j(X(0)) - \phi_j(X(-q+\delta)) =$$

(a) $-\frac{1}{2} \int_{-q+\delta}^{0} R(X(t))^{-1} \frac{3}{2} \varepsilon_{t^+} \phi_j(X(t); j) \, dt$.

(b) For all $r \in (-q, 0): \frac{3}{2} \theta_{t^+} \phi_j(X(r)) = \frac{1}{2} R(X(t))^{-1} \frac{3}{2} \varepsilon_{t^+} \phi_j(X(t); j) \, dt$.
(c) For all \( r \in (-q,0) \)

\[
(6.8.2-5) \quad \dot{X}(r) = -\nabla_{R^G} V(X(r)),
\]

where \( \nabla_{R^G} \) denotes the Riemannian gradient of the function \( V \) (cf. section 6.2.6 and (6.8.1-7)).

Proof. First remark that if \( \delta' \) is such that \( (r_k, q, \delta') \) satisfies (i) and (ii) then the same holds for all \( (r_k, q, \delta) \) with \( \delta \in (0, q) \).

(a) Let \( \delta \in (0, q) \) be arbitrary. Application of theorem (6.8.2-1) gives

\[
\lim_{k \to \infty} f_k = \lim_{k \to \infty} \phi_j(Y(r_k)) - \phi_j(Y(r_k - q + \delta)) = \phi_j(X(0)) - \phi_j(X(-q + \delta)).
\]

Now corollary (6.8.1-24) gives the result.

(b) Let \( r = -q + \delta \) with \( q \) fixed and \( \delta \) varying over \( (0, q) \). Then \( r \in (-q, 0) \).

Substitution of \( \delta = q + r \) in (a) and differentiation with respect to \( r \) gives

\[
-\frac{3}{\delta r} \phi_j(X(r)) = R(X(r))^{-1} - \frac{3}{\delta \phi} \phi_j(X(r)); j} 1^2 2.
\]

(c) The vector \( \frac{3}{\delta r} \phi_j(X(r)) \) is the expression in local coordinates of the tangent vector \( \dot{X}(r) \). As explained in section 6.2.6, the vector

\[
K(X(r))^{-1} \frac{3}{\delta \phi} \phi_j(X(r)); j} 1^2 2
\]

is the expression in local coordinates of the Riemannian gradient vector of the function \( V \) (defined in (6.8.1-7)). In a coordinate free notation, (b) can be expressed as

\[
\forall r \in (-q, 0): -\dot{X}(r) = \nabla_{R^G} V(X(r)),
\]

from which (6.8.2-5) follows by multiplication of both sides with \(-1\).

Q.E.D.
6.8.2-6. **Remark.** Part (c) of this theorem is a basic result about the asymptotic behaviour of the algorithm from which all (other) results about the asymptotic behaviour will be derived.

Part (c) of this theorem will now be generalized in the sense that condition (i) will be dropped, and (ii) will be changed accordingly.

6.8.2-7. **Theorem.** There exists a set $E$, $P(E) = 0$, with the following property. Let $\omega \in \Omega \setminus \varepsilon$. Let $\{\theta_t\}$ and $\{\hat{\theta}_t\}$ be the two parameter sequences that are produced by the algorithm (for this $\omega$). Let $Y^+ \in L^{+}_{cP}$ be an interpolation curve such that $Y^+ (s_t) = \theta^+_t$ for all $t \geq t_o$.

(As usual let $Y$ denote the projection of $Y^+$ in $L^*_c$). Let $\bar{S} = \{(r_k, q, \delta) \in S\}$ be such that $(S_{r_k} (Y)|_{[-q, 0]})_{k=1}^{\infty}$ converges in $L^*_c [-q, 0]$ and let $X \in L^*_c [-q, 0]$ denote the limit. Then $\forall \varepsilon \in (-q, 0)$:

\[
(6.8.2-8) \quad \dot{x}(r) = -V \times (x(r)).
\]

**Proof.** Let $X^+_k (r) = Y^+ (r_k - r)$ for all $r \in [-q, 0]$, then $X^+_k \in L^{+}_{cP} [-q, 0]$ and $X_k \in L^*_c [-q, 0]$. Then $\lim_{k=\infty} X_k = X \in L^*_c [-q, 0]$. The following three remarks, labeled (A), (B) and (C) will be useful.

(A) For each subsequence $\{k(l)\}_{l=1}^{\infty}$ of $\{k\}_{k=1}^{\infty}$ one has $\lim_{k=\infty} x_k (l) = X$.

A subsequence of $\{k(l)\}_{l=1}^{\infty}$ is of course also a subsequence of $\{k\}_{k=1}^{\infty}$. In order not to complicate the notation further, all such subsequences will also be denoted by $\{k(l)\}_{l=1}^{\infty}$.

(B) Because $X \in L^*_c [-q, 0]$, $X$ is continuous. Also $V$ is continuous (it is even differentiable). Therefore it suffices to show that the differential equation (6.8.2-8) holds on $(-q, 0) \setminus F$, $F$ some finite set. This can be shown by considering the corresponding integral equation in local coordinates. To make this clear consider an o.d.e. in $\mathbb{R}^n$: $\dot{x}(t) = f(x(t))$, $\forall t \in (0, t_1) \cup (t_1, t_2)$. 


If \( x \) is continuous on \([0,t_2]\) and \( f \) is continuous then it follows that

\[
\forall t \in [0,t_1]: x(t) - x(0) = \int_0^t f(x(\tau))d\tau, \text{ and}
\]

\[
\forall t \in [t_1,t_2]: x(t) - x(t_1) = \int_{t_1}^t f(x(\tau))d\tau.
\]

But then clearly

\[
\forall t \in [0,t_2]: x(t) - x(0) = \int_0^t f(x(\tau))d\tau.
\]

The right-hand side is differentiable, so the left-hand side is differentiable as well and

\[
\forall t \in (0,t_2): x(t) = f(x(t)).
\]

(C) For \( \varepsilon > 0 \) let \( N_\varepsilon(F) \) denote the \( \varepsilon \)-neighbourhood of \( F \), i.e.

\[
N_\varepsilon(F) = \{ x \mid d(x,F) < \varepsilon \}.
\]

Clearly, if the differential equation holds on \((-q,0) \setminus N(F)\) for each \( \varepsilon > 0 \), then it holds on \((-q,0) \setminus F\). According to (B) this is sufficient.

The proof will now be given in three steps.

(1) As noted before, \( X \in L_{[-q,0]} \) so \( X \) is continuous. Consider the finite cover \( \{U_p\}_p \) of \( M \) from proposition (6.3.2-22). Using the compactness of \([-q,0]\) it is not difficult to show that there exists a finite partition of \([-q,0]\), namely \(-q = -q(0) < -q(1) < \ldots < -q(N) = 0\), such that for each \( n = 1,2,\ldots,N \), there is a \( p \), such that \( X([-q(n-1),-q(n)]) \subseteq U_p \). According to (B) it is sufficient to show that the differential equation (6.8.2-8) holds on each interval \((q(n-1),q(n))\), \( n = 1,2,\ldots,N \). For each \( n \) the proof will be the same, therefore the proof will only be given for \((-q(N-1),0)\). To simplify the notation, \((-q,0)\) will be used instead of \((-q(N-1),0)\). One has now (without loss of generality) \( X([-q,0]) \subseteq U_p \).

(ii) Now consider \( \{X_k([-q,0])\}_{k=1}^\infty \). Because \( \lim_{k \to \infty} X_k = X \) in \( L_{[-q,0]} \) and \( X([-q,0]) \subseteq U_p \), with \( X([-q,0]) \) compact and \( U_p \) open, there exists a \( k_1 \) such
that for all \( k \geq k_1 \), \( X_k([-q,0]) \subseteq \cup_p \). According to proposition (6.3.2-22),

\[ X^+_k \] (which is an element of \( L^+_{cp}[-q,0] \)) can have at most two coordinate changes occurring on \([-q,0] \). So three cases can be distinguished, namely: (a) no coordinate changes occur, (b) one coordinate change occurs and (c) two coordinate changes occur. In case (a) one has a finite number of possibilities for the coordinate chart index \((i,j)\); because

\[ (i,j) \in \tilde{I} := \{(i,j) | i \in I, j \in I(j)\}, \] the number of possibilities is in fact \( |\tilde{I}| \).

In case (b) one can distinguish again a finite number of possibilities, namely a first coordinate chart index, \((i_1,j_1)\) on an interval \([-q,-q_1(k)]\) (for some appropriately chosen \(-q_1(k)\)) and a second one \((i_2,j_2)\) on the interval \([-q_1(k),0]\). So the number of possibilities is in fact smaller than or equal to \( |\tilde{I}|^2 \). Similarly in case (c), the number of possibilities is smaller than or equal to \( |\tilde{I}|^3 \). The total number of possibilities is therefore smaller than or equal to \( |\tilde{I}| + |\tilde{I}|^2 + |\tilde{I}|^3 < \infty \). Because the number of possibilities is finite, at least one of them must occur infinitely often. Choose a subsequence \((k(\ell))_{\ell=1}^\infty\) such that \( X^+_k \) has the same possibility for its coordinate chart indices for all \( \ell \in \mathbb{N} \).

(iii) If (a) is the case, then there is an \((i,j) \in \tilde{I}\) such that \( X^+_k \) is \( X(\ell,k(\ell),i,j) \) for all \( \ell \in \mathbb{N} \). Then the previous theorem can be applied, with \((r_k^\ell)_{k=1}^\infty\) instead of \((r_k)_{k=1}^\infty\) and it follows that \( X \) satisfies the differential equation. If (b) is the case, \(-q_1(k(\ell)) \in [-q,0] \) is the number at which a coordinate change takes place in \( X^+_k \). Because \([-q,0] \) is compact, \((-q_1(k(\ell)))_{\ell=1}^\infty\) has a convergent subsequence.

As explained in (A) this subsequence will again be denoted by \((-q_1(k(\ell)))_{\ell=1}^\infty\). Let its limit be \(-q_1\). For each \( \epsilon > 0 \) there exists an \( \ell_1 \) such that for all \( \ell \geq \ell_1 \), \(-q_1(k(\ell)) \geq -q_1 - \epsilon\) and therefore for all \( \ell \geq \ell_1 \), \( X^+_k \) has only one coordinate chart index. Therefore the previous theorem can be applied to \( X[-q,-q_1-\epsilon] \) and so \( X[-q,-q_1-\epsilon] \) satisfies the differential equation in this case. A similar argument shows that \( X[-q_1+\epsilon,0] \) satisfies the differential equation in this case (b). Because \( \epsilon > 0 \) was chosen arbitrarily, according to (C) the proof is complete for this case (b). If (c) is the case one can use a similar argument as in case (b).

Q.E.D.
From this theorem the following main result of this section can now easily be derived.

6.8.2-9. Theorem. There exists a set \( E, P(\xi) = 0 \), with the following property. Let \( \omega \in \Omega \setminus E \). Let \( \{ \theta_t \} \) and \( \{ \hat{\theta}_t \} \) be the parameter sequences that are produced by the algorithm (for this \( \omega \)). Let \( Y \in L_c([-\infty, \infty)] \) be an interpolation curve such that \( Y(t) = \theta_t \) for all \( t \geq t_0 \). Let \( \{ r_k \} \) be a monotonically increasing divergent sequence of positive real numbers.

(a) Let \( q_1, q_2 \in \mathbb{R}, q_1 < q_2 \), and let \( k_1 \) be such that \( r_{k_1} > q_1 \). Let

\[
X_k := S_{r_k} (Y)|_{[q_1, q_2]} \quad \text{for all } k \geq k_1.
\]

If \( \{ X_k \} \) converges to \( X \) in \( L_c([-\infty, \infty)] \), then

\[
\dot{X}(r) = -\nabla \cdot (X(r)) \quad \text{for all } r \in (q_1, q_2).
\]

(b) Let \( X_k = S_{r_k} (Y)|_{[0, \infty)} \) for each \( k \in \mathbb{N} \). If \( \{ X_k \} \) converges to \( X \) in \( L_c([0, \infty)) \), then

\[
\dot{X}(r) = -\nabla \cdot (X(r)) \quad \text{for all } r \in (0, \infty).
\]

Proof. (a) This follows directly from the previous theorem by taking \( q = q_2 - q_1 \), by taking a subsequence \( \{ r_k(\xi) \}_{\xi=1}^\infty \) such that \( r_k(\xi+1) - r_k(\xi) > q + \delta \) for each \( \xi \) (such that \( S = \{ (r_k(\xi))_{\xi=1}^\infty, q, \delta \} \in \mathcal{S} \)) and by shifting the curves over a length \( q_2 \).

(b) Take \( q_1 = 0 \). For each \( q_2 > 0 \), it follows from (a), by restricting the \( X_k \) and \( X \) to the interval \([0, q] \), that

\[
\dot{X}(r) = -\nabla \cdot (X(r)) \quad \text{for all } r \in (0, q_2).
\]

Because this holds for each \( q_2 > 0 \), one can conclude that

\[
\dot{X}(r) = -\nabla \cdot (X(r)) \quad \text{for all } r > 0.
\]

Q.E.D.
6.9. The asymptotic behaviour of the algorithm

In this section the results of the previous section will be used to derive convergence results for the algorithm. To start with consider the following definition.

6.9.1. Definition. Let \( M_0 \) be a metric space with metric \( d \).
(a) Let \( \{x_k\}_{k=0}^{\infty} \) be a sequence of points in \( M_0 \). A point \( x \in M_0 \) will be called a limit point if for each \( \epsilon > 0 \) and for each \( k_1 \geq k_0 \) there exists a \( k \geq k_1 \) such that \( d(x_k, x) < \epsilon \).
(b) Let \( X : [0, \infty) \to M_0 \) be a function. A point \( x \in M_0 \) will be called a limit point at infinity if for each \( \epsilon > 0 \) and for each \( r \geq 0 \) there exists an \( r_1 \) such that \( d(X(r), x) < \epsilon \).

6.9.2. Remarks. (i) In case (a), \( x \) is a limit point of \( \{x_k\} \) if and only if there exists a subsequence of \( \{x_k\} \) with limit \( x \).
(ii) In case (b), \( x \) is a limit point at infinity if and only if there exists a sequence \( \{r_k\} \) with \( \lim_{k \to \infty} r_k = \infty \) such that \( \lim_{k \to \infty} X(r_k) = x \).

Now let \( \{\theta_{t_0}^\omega\}_{t_0}^{\infty} \) and \( \{\theta_{t_0}^\omega\}_{t_0}^{\infty} \) be the parameter sequences produced by the algorithm (for given \( \omega \)) and let \( Y \in C_0^\omega [0, t_0] \) be an interpolation function with \( Y(s_t) = \theta_t \) for all \( t \geq t_0 \) (such an interpolation function exists, cf. (6.5-19)).

6.9.3. Lemma. (a) The set \( \mathcal{X} \) of limit points of \( \{\theta_{t_0}^\omega\}_{t_0}^{\infty} \) is equal to the set of limit points of \( \{\tilde{\theta}_{t_0}^\omega\}_{t_0}^{\infty} \) and equal to the set of limit points at infinity of \( Y \).
(b) The set \( \mathcal{V} \) of limit points of \( \{V(\tilde{\theta}_{t_0})\}_{t_0}^{\infty} \) is equal to the set of limit points of \( \{V(\tilde{\theta}_{t_0})\}_{t_0}^{\infty} \) and equal to the set of limit points at infinity of \( Y \).

Proof. (a) Because of the coupling equation, \( d(\theta_{t_0}^\omega, \tilde{\theta}_{t_0}^\omega) \to 0 \) for \( t \to \infty \) and therefore any limit point of \( \{\theta_{t_0}^\omega\} \) is a limit point of \( \{\tilde{\theta}_{t_0}^\omega\} \) and vice versa. Because \( Y(s_t) = \theta_t \) for all \( t \geq t_0 \), any limit point of \( \{\theta_{t_0}^\omega\} \) is a limit point at infinity of \( Y \). On the other hand if \( x \) is a limit point at infinity of \( Y \) then there exists a sequence \( \{r_k\} \), with \( r_k \to \infty \) for \( k \to \infty \), such that
\( Y(r_k + x) \) for \( k = \infty \). For each \( k \) let \( t(k) \) be such that \( |s_{t(k)} - \tau_k| < a_{t(k)} \).

Because of the Lipschitz condition, one has \( d(Y(r_k), Y(s_{t(k)})) < c \cdot a_{t(k)} \).

Because \( a_{t(k)} \to 0 \) for \( k \to \infty \), one has \( Y(s_{t(k)}) \to x \) for \( k \to \infty \).

But \( Y(s_{t(k)}) = \theta_{t(k)} \) and it follows that \( x \) is a limit point of \( \{ \theta_t \} \) (cf. remark (6.9-2)).

(b) Because \( V_\theta \) is continuous on \( M \) and \( M \) is compact, \( V_\theta \) is uniformly continuous. One can now proceed analogously to the proof of (a). The details are left to the reader.

Q.E.D.

Because \( M \), the manifold of parameter points, is compact and \( V_\theta \) continuous, one has in fact

6.9-4. Lemma. (a) \( \dot{\mathbf{v}} = V_\theta (\dot{x}) \), (b) \( \mathbf{v} \neq \emptyset \) and (c) \( \mathbf{v} \) is compact.

Proof. (a) Because \( V_\theta \) is continuous it is clear that \( V_\theta (x) \subseteq V \). On the other hand if \( \mathbf{v} \in V \) then there exists a sequence \( \{ t(k) \}_{k=1}^\infty \) such that \( V_\theta (\theta_{t(k)}) + v \).

Because \( \theta_{t(k)} \in M \) and \( M \) is compact, \( \{ \theta_{t(k)} \} \) has a convergent subsequence, with some limit \( \mathbf{s} \in X \). Clearly \( V_\theta (\mathbf{s}) = v \) and so \( v \in V_\theta (X) \). This shows that \( V \subseteq V_\theta (X) \) and so \( \mathbf{v} = V_\theta (X) \).

(b) Because \( M \) is compact, \( \mathbf{X} \neq \emptyset \), so \( \mathbf{v} \neq \emptyset \).

(c) \( X \) is closed and \( X \subseteq M \), so \( X \) is compact. Therefore \( \dot{\mathbf{v}} = V_\theta (x) \) is compact.

Q.E.D.

Because \( \mathbf{v} \) is also the set of limit points at infinity of the continuous function \( V_\theta \circ Y \) one can show the following.

6.9-5. Lemma. \( \mathbf{v} \) is convex.

Proof. It has to be shown that if \( v_1, v_2, v_1 \in \mathbf{v}, v_1 < v_2 \) and \( v \in \mathbf{R} \) such that \( v_1 < v < v_2 \), then \( v \in \mathbf{v} \). So assume \( v_1, v_2 \in \mathbf{v}, v_1 < v_2 \) and \( v_1 < v < v_2 \). Then there is a monotonically increasing divergent sequence \( \{ r_k \}_{k=1}^\infty, r_k \in [0, \infty) \) for each \( k \in \mathbf{N} \), such that \( V \circ Y(r_{2k+1}) = v_1 \) and \( V \circ Y(r_{2k}) = v_2 \) for \( k \to \infty \). Because \( v_1 < v < v_2 \) it follows that there exists a \( k_1 \) such that for all \( k \geq k_1 \), \( V \circ Y(r_{2k+1}) < v \) and \( V \circ Y(r_{2k}) > v \). Because \( V \circ Y \) is continuous it follows that for each \( k \geq k_1 \) there exists a number \( s_k \in (r_{2k}, r_{2k+1}) \) with...
\( V \circ Y(s_k) = v \). Therefore \( v \in V \).

Q.E.D.

6.9.6. Corollary. \( V \) is a nonempty, compact interval \([v_1, v_2] \subset \mathbb{R}\), with \( v_1 \leq v_2 \).

Similar results can be obtained for \( X \).

6.9-7. Theorem. The set \( X \) is nonempty, compact and connected.

Proof. Because \( M \) is compact, the sequence \( \{t_n\} \) must have a limit point, so \( x \neq \emptyset \). The set \( X \subset M \) of limit points of \( \{t_n\} \) is closed and therefore compact.

Suppose \( X \) is not connected, to obtain a contradiction. Then there are two disjoint open sets \( U_1 \) and \( U_2 \) that cover \( X \) in a nontrivial way, i.e.

\[ x \subset U_1 \cup U_2 \text{ and } x \cap U_1 \neq \emptyset \text{ and } x \cap U_2 \neq \emptyset. \]

Let \( W = M \setminus (U_1 \cup U_2) \), then \( W \) is compact. Because \( x \cap U_1 \) and \( x \cap U_2 \) are nonempty, for each \( \tilde{r} > 0 \), the interpolation curve \( Y(\tilde{r}+r), r \geq 0 \), must switch back and forth between \( U_1 \) and \( U_2 \) (infinitely often). Therefore for each \( \tilde{r} > 0 \) the curve \( Y(\tilde{r}+r), r \geq 0 \), enters \( W \) (infinitely often). Because \( W \) is compact it follows that \( Y \) must have a limit point at infinity in \( W \), i.e. \( x \cap W \neq \emptyset \). This is in contradiction with \( x \subset U_1 \cup U_2 \). Therefore \( X \) is connected.

Q.E.D.

Notice that the differential equation has not been used yet. This will be done next. Let \( \{r_k\} \) be a divergent monotonically increasing sequence of positive real numbers. Let

\[ X_k := S_{r_k} (Y)|_{[0, \omega)} \text{, i.e. } X_k(r) = Y(r+r_k), \forall r, r_k \geq 0, \forall k \in \mathbb{N}. \]

\( \{X_k\} \) will be called a sequence of translations of the interpolation curve \( Y \).

6.9-8. Lemma. Let \( E \subset \Omega \) with \( P(E) = 0 \) be as in theorem (6.8.2-9) and let \( \omega \notin E \). Let \( \lambda \in L^1_{c}(0, \omega) \) be the limit of a sequence of translations of the interpolation curve, as just described. Then \( V \circ X : [0, \omega) \to \mathbb{R} \) is monotonically nonincreasing.

Proof. According to theorem (6.8.2-9) \( b ) \( X \) is a solution of the differential equation \( X = -\frac{\partial}{\partial \lambda} Y \).
Therefore one has

\[\frac{3}{2r} V_R g(X(r)) = \langle V_R g(X(r)), \dot{X}(r) \rangle_R = -\langle \dot{X}(r), \dot{X}(r) \rangle_R \leq 0,\]

where \(\langle \cdot, \cdot \rangle_R\) denotes the inner product corresponding to the Riemannian metric in the tangent space at a given point of the manifold.

Q.E.D.

A critical point of \(V_g\) is (of course) by definition any parameter point \(\theta\) such that \(V_R g(\theta) = 0\). A critical value of \(V_R g\) is a number \(v \in \mathbb{R}\) for which a critical point \(\theta\) exists such that \(v = V_R g(\theta)\). (Of course if \(v\) is a critical value and \(v = V_R g(\theta)\) for some \(\theta\), one cannot conclude that \(\theta\) is a critical point!). It will now be shown that if \(X\) is a limit curve as before, then \(V_R g \circ X(r)\) is a critical value of \(V\) for each \(r \geq 0\).

6.9-10. Theorem. Let \(E\), with \(P(E) = 0\), be as in theorem (6.8.2-9) and let \(\omega \in E\). Let \(X \in L^1_c[0,\omega]\) be the limit of a sequence of translations of the interpolation curve, as described before. Then for each \(r \geq 0\), \(V_R g \circ X(r)\) is a critical value of \(V_g\).

Proof. Distinguish two cases.

(i) \(V_R g \circ X(r), r \in [0,\omega),\) is constant,
(ii) \(V_R g \circ X(r), r \in [0,\omega)\) is not constant.

\(\text{(i) Because } V_R g \circ X(r) \text{ is constant, one has}
\]

\[0 = \frac{d}{dr} V_R g \circ X(r) = \langle V_R g(X(r)), \dot{X}(r) \rangle_R =
\]

\[= -\langle \dot{X}(r), \dot{X}(r) \rangle_R, \text{ for all } r > 0\]

and therefore \(\dot{X}(r) = 0\) for all \(r > 0\), and so \(V_R g(X(r)) = -\dot{X}(r) = 0\) for all \(r > 0\), so \(X(r)\) is a critical point for all \(r \geq 0\). and by continuity, \(X(r)\) is also a critical point for \(r = 0\). Therefore \(V_R g \circ X(r)\) is a critical value of \(V_g\) for each \(r \geq 0\). This proves the theorem in case (i).
ad(ii). Now suppose \( V_g \circ \sigma_X, r \in [0, \infty) \) is not constant. From lemma (6.9-8) it is known that \( V_g \circ \sigma_X \) is monotonically nonincreasing. It follows that there must be numbers \( r_1, r_2 \) with \( 0 < r_1 < r_2 \), such that \( V_g \circ \sigma_X(r_1) > V_g \circ \sigma_X(r_2) \). Let \( \epsilon_1 > 0, \epsilon_2 > 0 \) be such that

\[
V_g \circ \sigma_X(r_1) - \epsilon_1 > V_g \circ \sigma_X(r_2) + \epsilon_2.
\]

Consider the open sets (relatively to \( X \))

\[
N_1 := \{ \theta \in X | V_g(\theta) > V_g \circ \sigma_X(r_1) - \epsilon_1 \},
\]

and

\[
N_2 := \{ \theta \in X | V_g(\theta) < V_g \circ \sigma_X(r_2) + \epsilon_2 \}.
\]

Then

\[
\sigma_X(r_1) \in N_1 \text{ and } \sigma_X(r_2) \in N_2 \text{ and } N_1 \cap N_2 = \emptyset.
\]

Now consider the interpolation curve \( Y \) and the corresponding real function \( V_g \circ Y \). Because \( \sigma_X(r_1) \) and \( \sigma_X(r_2) \) are limit points at infinity of \( Y \), \( Y(r) \) enters and leaves \( N_1 \) and \( N_2 \) infinitely often. Therefore there exists a monotonically increasing divergent sequence of real numbers \( \{q_k\}_{k=1}^{\infty} \) such that

(a) \( Y(q_{2k+1}) \in \partial N_1 \) and \( Y(q_{2k}) \in \partial N_2 \) for all \( k \in \mathbb{N} \), and so

\[
V_g \circ Y(q_{2k+1}) = V_g \circ \sigma_X(r_1) - \epsilon_1 \text{ and } V_g \circ Y(q_{2k}) = V_g \circ \sigma_X(r_2) + \epsilon_2 \text{ for all } k \in \mathbb{N}
\]

(b) the image of the open interval \( (q_{2k}, q_{2k+1}) \) is outside \( N_1 \) and \( N_2 \), i.e.

\[
Y((q_{2k}, q_{2k+1})) \subseteq N_1^c \cap N_2^c.
\]

So at each \( q_{2k} \), \( Y \) leaves \( N_2 \) and at each \( q_{2k+1} \), \( Y \) enters \( N_1 \).

Now define

(6.9-11) \( T_k := q_{2k+1} - q_{2k}, \forall k \in \mathbb{N} \).
Two cases will be distinguished:

(1) \( \exists T \in \mathbb{R} \): \( \liminf_{k \to \infty} T_k = T \), and

(2) \( \lim_{k \to \infty} T_k = \infty \).

ad(1) There exists a subsequence of \( (T_k) \) that converges to \( T \). It is not difficult to see that without loss of generality it may by assumed that

\[ \lim_{k \to \infty} T_k = T. \]

Consider the sequence of translation curves \( \{ Z_{k} \}_{k=1}^{\infty} \) with

\[ \forall k \in \mathbb{N}, \quad Z_{k} = \gamma(q_{2k} + r)|_{[-\frac{1}{2}, T+\frac{1}{2}]} \in L_{c}[{-\frac{1}{2}, T+\frac{1}{2}}]. \]

From the compactness of \( L_{c}[{-\frac{1}{2}, T+\frac{1}{2}}] \) (cf. theorem (6.5-6)) it follows that the sequence has a convergent subsequence, with limit \( Z \), say. According to theorem (6.8.2-9)(a), \( Z(r) \), with \(-\frac{1}{2} < r < T+\frac{1}{2}\), is a solution of the differential equation

\[ Z(r) = -V \frac{\partial}{\partial r} (Z(r)), \quad -\frac{1}{2} < r < T+\frac{1}{2}. \]

Therefore, according to the analog of lemma (6.9-8), \( Z \) is monotonically nonincreasing, so

\[ Z(0) \geq Z(T). \]

On the other hand, by construction,

\[ Z_{k}(0) = V_{g} \circ X(r_{2}) + \epsilon_{2} \text{ for each } k \in \mathbb{N} \]

and

\[ Z_{k}(T_{k}) = V_{g} \circ X(r_{1}) - \epsilon_{1} \text{ for each } k \in \mathbb{N}, \]

from which it follows easily (using the Lipschitz condition) that

\[ Z(0) = V_{g} \circ X(r_{2}) + \epsilon_{2} \]

and
Therefore $Z(0) < Z(T)$, which contradicts (6.9-12)! The conclusion is that this case cannot occur!

Ad(2) In this case consider the sequence of translation curves $\{Z_k\}_{k=1}^{\infty}$ with

\begin{equation}
Z_k = Y(q_{2k} \cdot r)|_{[0, \infty)} \in L^1_c([0, \infty)).
\end{equation}

According to the Arzela-Ascoli theorem $\{Z_k\}$ has a convergent subsequence, with limit $Z$, say. $Z$ satisfies the differential equation and according to lemma (6.9-8), $V_g \circ Z$ is monotonically nonincreasing. On the other hand, for each $k \in \mathbb{N}$,

$$V_g \circ Z_k(r) \geq V_g \circ X(r_2) + \varepsilon_2 \quad \forall r \in [0, T_k]$$

and

$$V_g \circ Z_k(0) = V_g \circ X(r_2) + \varepsilon_2.$$ 

From this it follows easily that

$$V_g \circ Z(0) = V_g \circ X(r_2) + \varepsilon_2$$

and

$$V_g \circ Z(r) \geq V_g \circ X(r_2) + \varepsilon_2, \quad \forall r \geq 0.$$ 

But $V_g \circ Z$ is monotonically nonincreasing and therefore $V_g \circ Z(r)$ is constant for all $r \geq 0$. Similarly as in (1) it follows that $Z(0)$ is a critical point of $V_g$ and $V_g \circ Z(0) = V_g \circ X(r_2) + \varepsilon_2$ is a critical value of $V_g$. Because $\varepsilon_2$ can be chosen arbitrarily from the open interval $(0, V_g \circ X(r_2) - V_g \circ X(r_1))$ it follows that each number in the open interval $(V_g \circ X(r_2), V_g \circ X(r_1))$ is a critical value of $V_g$. Now the set of critical values of $V_g$ is compact (it is the image of the set of critical points, which is a closed set in $\mathcal{N}$, because $V_g \in C^\infty \subseteq C^1$, and therefore compact; it follows
that the set of critical values is compact. It follows that the closed interval

\[ [v \circ X(r_2), v \circ X(r_1)] \]

consists of critical values of \( V_g \).

Now \( (r_1, r_2) \) is an arbitrary pair for which \( r_1 < r_2 \) and

\[ v \circ X(r_2) < v \circ X(r_1). \]

It follows that for each \( r \geq 0 \), \( v \circ X(r) \) is a critical value of \( V_g \).

Q.E.D.

6.9-15. Corollary. Let \( E \) be as before and \( \omega \notin E \). The set \( V \) consists of critical values of \( V_g \).

Proof. Let \( v \in V \). Then there exists a monotonically increasing divergent sequence \( \{r_k\}_{k=1}^{\infty} \) such that \( v \circ Y(r_k) \to v \) for \( k \to \infty \). Let \( X_k(r) = Y(r_k + r) \left|_{[0, \infty)} \right. \) for each \( k \in \mathbb{N} \). Then \( \{X_k\}_{k=1}^{\infty} \) has a convergent subsequence with limit \( X \in L^1_c(0, \infty) \) and \( v \circ X(0) = \lim_{k \to \infty} v \circ X_k(0) = v \). From the previous theorem it follows that \( v \) is a critical value.

Q.E.D.

This corollary can be combined with corollary (6.9-6) to show that \( V \) consists of one point only, by using the well-known theorem of Sard.

6.9-16. Theorem. Let \( \omega \notin E \). The set \( V \) consists of one critical value of \( V_g \).

Proof. Because \( V_g: M \to \mathbb{R} \) is \( C^\infty \), it follows from the theorem of Sard (cf. [Ch-Ha], p. 34ff) that the complement of the set of critical values of \( V_g \) is dense in \( \mathbb{R} \). Because \( V \) is a closed interval \([v_1, v_2]\), \( v_1 \leq v_2 \) (according to corollary (6.9-6)) and \( V \) consists of critical values of \( V_g \), it follows that \( v_1 = v_2 \) must hold and \( V \) is a one-point-set. So \( V \) consists of one critical value of \( V_g \).

Q.E.D.

6.9-17. Corollary. Let \( \omega \notin E \). Let \( v \in \mathbb{R} \) be such that \( V = \{v\} \). The sequences
\[ \{V_t(\hat{\theta}(x))\}_{t \to t_0}^\infty, \{V_t(\hat{\theta}(y))\}_{t \to t_0}^\infty \text{ converge to } v \text{ and } \lim_{r \to \infty} V_g \circ Y(r) = v. \]

Proof. Because \( V_g \) is continuous and \( M \) is compact, \( V_g(M) \) is compact. For each \( t \in g_t \), \( V_g(\hat{\theta}(x)) \in V_g(M) \). Because \( \{V_t(\hat{\theta}(x))\} \) has only one limit point \( v \), it follows that \( \{V_t(\hat{\theta}(x))\} \) converges to \( v \). Similarly to the proof of lemma (6.9-3) it can be shown that from \( V_g(\hat{\theta}(x)) + v \) for \( t \to +\) it follows that \( V_g(\hat{\theta}(x)) + v \) for \( t \to +\) and \( V_g \circ Y(r) + v \) for \( r \to +\).

Q.E.D.

This corollary constitutes already one important result about the asymptotic behaviour of the algorithm. Next, the set \( X \) will be investigated further.

6.9-18. Theorem. Let \( \omega \notin E \), \( E \) as before. The set \( X \) consists of critical points of \( V_g \).

Proof. Let \( \theta \in X \). There exists a monotonically increasing divergent sequence \( \{r_k\} \) such that \( \lim_{k \to \infty} V(r_k) = \theta \). Let \( X_k(r) = Y(r_k)^+\theta \), then \( \{X_k\} \) contains a convergent subsequence with limit \( X \), and \( X(0) = \theta \). It follows from the previous corollary that for each \( r > 0 \), \( V_g \circ X(r) = v \). Therefore, for all \( r > 0 \),

\[
0 = \frac{d}{dr} V_g \circ X(r) = \left< \nabla_{R_g}(X(r)), X^\prime(r) \right>_R = \\
= -\left< \nabla_{R_g}(X(r)), \nabla_{R_g}(X(r)) \right>_R.
\]

So \( 0 = \dot{X}(r) = V_g V(X(r)) \) for each \( r > 0 \). It follows that \( X(r) \) is constant, equal to \( X(0) = \theta \) and \( V_g \circ \theta \) = 0, i.e. \( \theta \) is a critical value of \( V_g \).

Q.E.D.

6.9-19. Corollary. Let \( \omega \notin E \). Let \( X \in L_c(0,\infty) \) be the limit of a sequence of translations of the interpolation curve. Then \( X \) is a solution of the equations \( 0 = X(r) = V_g V(X(r)), r \in (0,\infty) \). This implies that \( X \) is constant and equal to a critical point \( \theta \) of \( V_g \) such that \( V_g(\theta) = v \).

So the differential equation has now been replaced by a static(critical-point-) equation. The main conclusion can now be stated

6.9-20. Main theorem. There exists a subset \( E \subset O \), \( P(E) = 0 \), with the
following property. Let \( \omega \in \Omega \). Let \( \{ \theta_t \}_{t=t_0}^{\infty} \) and \( \{ \hat{\theta}_t \}_{t=t_0}^{\infty} \) be the sequences produced by the algorithm.

(a) The sequences \( \{ V_g(\theta_t) \}_{t=t_0}^{\infty} \) and \( \{ V_g(\hat{\theta}_t) \}_{t=t_0}^{\infty} \) converge to a critical value \( \nu \) of \( V_g \).

(b) The sequence \( \{ \theta_t \}_{t=t_0}^{\infty} \) (and \( \{ \hat{\theta}_t \}_{t=t_0}^{\infty} \) as well) converges to its set of limit points \( X \). \( X \) is a nonempty, compact, connected subset of the set of critical points corresponding to the critical value \( \nu \) of \( V_g \).

6.9-21. Corollary. Let \( \omega \not\in \Omega \). If all the critical points of \( V_g \) are isolated then \( \{ \theta_t \}_{t=t_0}^{\infty} \) (as well as \( \{ \hat{\theta}_t \}_{t=t_0}^{\infty} \)) converges to a critical point \( \theta \) of \( V_g \) and

\[ \{ V_g(\theta_t) \}_{t=t_0}^{\infty} \] converges to the critical value \( \nu = V_g(\theta) \).

6.9-22. Remarks. (i) Whether or not all critical points of \( V_g \) are isolated depends on the choice of \( M \). For a given \( M \) it could in principle be investigated whether or not \( V_g \) has this property. From Morse theory it is known that generically functions have isolated critical points (cf. [MI])

(ii) From the properties of the steady state Kalman filter it is known that it is the unique filter that leads to the minimal prediction error covariance matrix. Because it has been assumed here that the true system is an element of \( M \), it follows that the criterion function \( V \) has a unique global minimum at the true system. Because for large values of \( K' \), \( V_g \) is only a slight perturbation of \( V \), one may expect \( V_g \) to have the same property for large \( K' \). In that case the point \( \theta_0 \) at which \( V_g \) has its global minimum is an isolated critical point of \( V_g \) and \( V_g(\theta_0) \) is of course a critical value. It follows that if \( \{ V_g(\theta_t) \}_{t=t_0}^{\infty} \) converges to \( V_g(\theta_0) \) then \( \{ \theta_t \}_{t=t_0}^{\infty} \) converges to \( \theta_0 \).

Further remarks and comments will be given in the next section.
6.10. Some final remarks

6.10.1. On the question of global convergence to the true parameter point
In the previous section it was found that the parameter sequence \( \{ \theta_t \} \) (as well
as \( \{ \hat{\theta}_t \} \)) converges to some connected set of critical points of \( V_g \), while one
would like to have convergence to the true parameter point i.e. 'consistency'
of the algorithm. With respect to this two remarks can be made.
(i) The true parameter point \( \hat{\theta} \) is the unique global minimum of \( V \) (this follows
from the properties of the steady state Kalman filter). \( V_g \) can be considered
as a perturbation of \( V \). The size of this perturbation depends on the choice of
\( K' \) in the definition of \( \delta_1 \) (cf. (6.2.8-4)), especially in relation to the true
innovations covariance matrix \( \tilde{\Sigma} \). By choosing \( K' \) large enough in relation
to \( \tilde{\Sigma} \), the point(s) \( \theta_0 \) at which \( V_g \) take(s) its global minimum can be put
arbitrarily close to \( \hat{\theta} \). This is shown in Appendix 6C. In the further
discussion it will be assumed that \( \theta_0 \) is close to \( \hat{\theta} \) and - for linguistic
simplicity - that \( \theta_0 \) is unique. (If \( \theta_0 \) is not unique the points where \( V_g \) takes
its global minimum will all be close to \( \hat{\theta} \) for \( K' \) large enough and therefore
they will be close together. The discussion remains essentially the same if
one excludes this possibility).
(ii) The set of critical points of \( V_g \) can be split up in (a) the global
minimum point \( \theta_0 \), (b) other local minima and (c) all other critical points,
like saddle points and local maxima.

ad(b). It is not clear whether there will be local minima other than the
global minimum. The choice of the manifold \( \mathcal{M} \) of course plays a role in this
question. It is in principle possible to check whether one has (almost)
arrived at the true parameter value, by comparing the sample covariances
(which can be computed on-line) with the theoretical covariance matrices that
correspond with the estimated parameter point. (If it becomes more or less
clear that the algorithm converges to a non-global, local minimum - or to any
other critical point(s) - one might consider the possibility of restarting the
algorithm at another parameter point. To this end a variety of random restart
procedures familiar from global optimization theory could be employed).
Investigation of \( V(\theta) = V(\theta; \hat{\theta}, \tilde{\Sigma}) \) as a function on \( \mathcal{M} \times \mathcal{M} \times \text{Pos} \) could perhaps
reveal more about the occurrence of non-global local minima.

ad(c) In this case the situation is in some sense somewhat more hopeful than
for the non-global, local minima. Each critical point \( \theta_c \) of this kind has the
following property. In each open neighbourhood of the critical point there are
points with lower value of the criterion function \( V_g \) than \( V_g (\hat{\theta}) \).
Especially if the Hessian of $V_\theta$ at such a point is nonsingular, it might very well be possible to show, that the parameter sequence \( \{ \theta_k \} \) produced by the algorithm will not converge to \( \theta^*_C \). Results in this direction have been treated by Ljung (cf. [Lj 75]). The underlying idea is that because of the noise in the system the parameter sequence will not 'get stuck' on such a critical point. However, there are technical difficulties in the proof and I have so far not succeeded in resolving them. But even if one could show such a result, then still in practice one should expect the algorithm to behave rather poorly in the neighbourhood of such a critical point, because the gradient is close to zero. Perhaps a solution to this problem can be found by taking second order information into account. Compare e.g. [Lich] for a comparable situation in the theory of optimization over a manifold. This requires future research.

The point of avoiding convergence to critical points of this kind is especially important in the light of the results of [Del 82], which state that 'in most cases', a smooth function on a manifold of fixed McMillan degree must have more than one critical point. This follows from an investigation of the topological structure of such manifolds. The topological theory does not necessarily imply that there will be non-global local minima. But there are results on the minimal number of saddle points and local maxima. These results should warn us that saddle points and local maxima will indeed play a role in the problem of identification within such manifolds. On the other hand, this discussion shows that the claims made by [De - By] that algorithms like the one treated here cannot be globally convergent are (at least) stated in a confusing manner. The problem of global convergence is still open and not settled by their approach.

6.10.2. Some remarks on applications and possible extensions of the algorithm

(i) It should perhaps be stressed again that the algorithm is well-suited for many constrained identification problems. The manifold \( M \) is the set of all parameter points that satisfy the constraints. An advantage of this algorithm over other methods like those described in [Ku-Cl] is that one does not need any projection facility, the constraints are satisfied automatically, by construction. Of course, the requirement that \( M \) is compact and is a differentiable manifold without boundaries is rather restrictive. Cf. also (v) below, however.

(ii) Due to the rather tolerant formulation of the coupling 'equation'

(6.2.9-6) many algorithms, which look rather different at first sight, fall...
into the presented class of algorithms (or only small modifications are needed). Examples are the coordinate-free prediction error algorithm presented in [Hnz 85b], and the (generalized) Gauss-Newton algorithm discussed in [Hnz 85a]. (We hope to give more details about these algorithms and their analysis in the near future).

(iii) If \( M \) could be allowed to have a nonempty boundary, this would be an important extension of the algorithm. Two kinds of boundaries have to be distinguished. (a) A boundary in the sense of differentiable manifolds, as treated e.g. in [Boo]. The boundary is then itself a manifold (without boundary!) of dimension \( d-1 \) (where \( d \) is the dimension of \( M \), as usual) and the coordinate charts are homeomorphic to a relatively open subset of the closed half-space \( x = (x_1, \ldots, x_d) | x \in \mathbb{R}^d, x_1 \geq 0 \). In this case the extension of the algorithm appears to be rather straightforward. If the parameter point \( \hat{\theta} \) is at the boundary then, in terms of the local coordinates, one computes the antigradient (i.e. minus the Riemannian gradient) and if it points into the closed half-space it can be used right away, if it points out of the closed half-space then it has to be projected first on the closed half-space, or, equivalently, on the space \( \{x = (x_1, \ldots, x_d) | x_1 = 0 \} \), in terms of local coordinates. The practical problem, however, is to find such coordinates. This is in many cases quite difficult, and future research is needed.

(b) A boundary in the topological sense, if \( M \) is embedded in some larger topological space. (Case (a) can be regarded as a special case of this one). In this case one allows boundaries of varying dimension (< \( d \)). This type of problem will appear for certain constrained identification problems. It also appears if one considers a manifold \( M \subset \mathbb{R}^{n, \hat{m}} \) which has lower McMillan-degree systems in its (topological) boundary. As a simple example one can think of the case \( \mathbb{R}^{n, 1, 1} \), which is a double, infinite sheeted Riemann surface (cf. chapter 5) in which the origin (i.e. the zero system) forms the boundary. This is clearly not an example of case (a), because the dimension of the manifold is two and the dimension of the boundary is zero. How to handle such cases requires more research.

(iv) A rather straightforward extension is the one to systems with exogenous inputs. One will have to formulate some kind of persistency-of-excitation condition to obtain identifiability of the relevant parameters. There is apparently no conceptual problem to work out this case along the same lines as is done here for the case without exogenous inputs.

(v) Perhaps it will turn out to be possible to generalize the algorithm of
(iv) To the so-called 'external variable' representation of linear systems. In
that case the algorithm has to identify the inputs and the outputs from the
given external variables. A change in the set of input variables and the set
of output variables would then correspond to some kind of change of
coordinates. Whether this can be worked out has to be investigated.
(vi) It will also be interesting to consider other probability distributions
for the innovations. In many cases the algorithm and the analysis are expected
to work the same. A special case is a 'chopped-off' Gaussian distribution. In
that case, there seems to be no need to use \( g_1 \) and \( g_2 \), which simplifies the
algorithm. In practice one should in fact use 'chopped-off' Gaussian
distributions in many cases, to remain consistent with the available knowledge
about the variables under consideration. This is because in many (all?) cases
one knows (perhaps rough) upper- and lower bounds for the variables. As a
simple example, take the length of a human. This is often presented as a
Gaussian variable. But it is clear that zero is a lower bound, and 10m (or the
length of the equator, or the diameter of the known universe) is an upper bound
that is clearly sufficient for all practical purposes.
(vii) Another possible way to extend the algorithm (which is also related to
(iii)(a) and (b)) is to a more general geometry than the Riemannian geometry
used here. Especially one may define a so-called Finsler metric on the space of
systems. This means that on each tangent space one has a norm, but not
(necessarily) an inner product. In that case it still seems possible to define
a gradient direction, and to formulate an analogous algorithm. Also the
extension of the analysis may be tractable.
(viii) Recursive identification is closely related to 'tracking', i.e. on-line
identification of (slowly) varying systems. To make the algorithm presented
here into a tracking algorithm, one just has to change the requirement
\[
\lim_{t \to \infty} a_t = 0 \quad \text{into} \quad \lim_{t \to \infty} a_t = a_\infty > 0 \quad (\text{or perhaps } \limsup_{t \to \infty} a_t > 0).
\]
It will be
interesting, but probably quite a bit more difficult, to find the asymptotic
properties of such an algorithm.
(ix) An advantage of the algorithm is that it can also be considered as an
adaptive filtering algorithm. If the parameter sequence converges to the true
parameter point, then the algorithm contains the steady state Kalman filter
for the system under consideration.
(x) An extension to the continuous time case appears to be quite feasible. In
fact, the differential geometric set-up that is used here, is very well suited
for the continuous time case.

(xii) Finally, of course, the proof of the pudding is in the eating and algorithms like this should prove their value in practical applications. Much remains to be done, to implement this kind of algorithm and to compare it with others. This will also be, I hope, the subject of future work.
Appendix 6A. Computation of $T(\theta)$, $\frac{\partial T(\theta)}{\partial \phi_{i}}$ and $\frac{\partial \phi_{i}}{\partial T_{j}}$.

Suppose one does not know (or does not want to compute) the transformation $\phi_{i}^{\top} \phi_{j}^{-1}$ explicitly, nor $T(\theta)$ as a function of $\theta$. Then for the computation of $T(\theta)$, $\frac{\partial T(\theta)}{\partial \phi_{i}}$, $k = 1, 2, \ldots, d$, and $\frac{\partial \phi_{i}}{\partial T_{j}}$, one can apply the following method, which makes use only of the knowledge of

$$(A(\phi_{i}), B(\phi_{i}), C(\phi_{i})) \in \mathbb{R}^{m \times n \times m}, \text{ at } \phi = \phi_{j}(\theta),$$

its partial derivatives with respect to the components $\phi_{k}^{i}$, $k = 1, 2, \ldots, d$, at $\phi = \phi_{j}(\theta)$,

$$(A(\phi_{i}), B(\phi_{i}), C(\phi_{i})) \in \mathbb{R}^{m \times n \times m}, \text{ at } \phi = \phi_{i}(\theta),$$

its partial derivatives with respect to the components $\phi_{k}^{i}$, $k = 1, 2, \ldots, d$, at $\phi = \phi_{i}(\theta)$. There is no claim that the following method is computationally very efficient. The main reason to include it is to show explicitly that knowledge of the quantities just described suffice for the computation at hand.

The method is as follows:

1. The Jacobian $\frac{\partial \phi_{i}}{\partial T_{j}}$ can be computed as follows. Consider $\theta, \theta_{1}, \theta_{2} \in C_{1} \cap C_{2}$, $\theta$ being the parameter point at which the change of coordinates has to take place. Consider the difference system

$$(6A-1) \quad A(\theta_{1}, \theta_{2}) = 0, \quad B(\theta_{1}, \theta_{2}) = 0, \quad C(\theta_{1}, \theta_{2}) = 0$$

(compare chapter 5, (5.2-6)).

Compute the Riemannian metric tensor $Q$ of this set of models at $\theta_{1} = \theta_{2} = \theta$, with respect to the tangent vector $(\phi_{i}, \phi_{j})^{T}$ (see chapter 5, (5.2-38) (a)); one obtains $Q$ from the formula

$$(6A-2) \quad \langle (\phi_{i}^{T}, \phi_{j}^{T}) \rangle^{2} = (\phi_{i}^{T}, \phi_{j}^{T})_{0} \begin{bmatrix} v_{i} \\ v_{j} \end{bmatrix}.$$

Partition $Q$ as $Q = [Q_{1}, Q_{2}]$, $Q_{1}$: $2n \times n$, $i = 1, 2$. Then the Jacobian is given by the formula
This can be shown by using the fact that the kernel of $Q$ is the set
$$\{ (y, y) \mid y \in \mathbb{R}^q \}.$$ 

One can also partition $Q$ as

$$(6A-4) \quad Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}, \quad Q_{ij} : n \times n, \quad i, j \in \{ 1, 2 \},$$

then

$$(6A-5) \quad J = -Q_{11}^{-1}Q_{12}.$$ 

(11) For $\theta \in \mathbb{R}^{m \times a}$, $T(\theta)$ can be calculated as follows: Recall the notation for the reachability matrix from section 4.4. From (6.2.7-4) it follows that (with local coordinates $\phi$ for $\theta$)

$$(6A-6) \quad R[A(\phi, i), B(\phi, i)] = T(\theta)R[A(\phi_j, j), B(\phi_j, j)].$$

From this it follows that

$$(6A-7) \quad T(\theta) = R[A(\phi, i), B(\phi, i)]R[A(\phi_j, j), B(\phi_j, j)]^T = L_{ij}^{-1}L_{jj}^{-1},$$

where $L_{ij}$ denotes the unique solution (for given $\phi, \phi_j$) of the Lyapunov equation

$$(6A-8) \quad L_{ij} = A(\phi, i)L_{ij}A(\phi_j, j)^T = B(\phi, i)B(\phi_j, j)^T,$$

for all $i$ and $j$. (So this also defines $L_{ij}$).

From the formula for $T(\theta)$ one can compute the derivative of $T(\theta)$ with respect to the local coordinate $\phi_1$: 
\[ (6A-9) \quad \frac{dT(k)}{\phi_k^z} = \frac{3L_{\phi_{i,j}}}{\phi_k^z} \left( I + \sum_{\phi_j^z} \frac{3L_{\phi_{j,j}}}{\phi_j^z} \right) \]

\[ = \frac{3L_{\phi_{1,j}}}{\phi_{1,j}} + \frac{3L_{\phi_{2,j}}}{\phi_{2,j}} + \frac{3L_{\phi_{3,j}}}{\phi_{3,j}} + \frac{3L_{\phi_{4,j}}}{\phi_{4,j}} = \frac{3L_{\phi_{i,j}}}{\phi_{i,j}} + \frac{3L_{\phi_{j,j}}}{\phi_{j,j}} \]

\[ \frac{3L_{\phi_{1,j}}}{\phi_{1,j}} \quad \text{and} \quad \frac{3L_{\phi_{2,j}}}{\phi_{2,j}} \quad \text{are the unique solutions of} \quad \frac{3L_{\phi_{1,j}}}{\phi_{1,j}} \quad \text{respectively, the Lyapunov equations} \]

\[ (6A-10) \quad \frac{3L_{\phi_{1,j}}}{\phi_{1,j}} - A(\phi_{i,j}) - \frac{3L_{\phi_{j,j}}}{\phi_{j,j}} A(\phi_{j,j}) = \frac{3B(\phi_{i,j})}{\phi_{i,j}} - B(\phi_{j,j}) + \frac{3A(\phi_{j,j})}{\phi_{j,j}} \]

\[ (6A-11) \quad \frac{3L_{\phi_{1,j}}}{\phi_{1,j}} - A(\phi_{i,j}) - \frac{3L_{\phi_{j,j}}}{\phi_{j,j}} A(\phi_{j,j}) = B(\phi_{i,j}) - \frac{3B(\phi_{j,j})}{\phi_{j,j}} + A(\phi_{i,j}) - \frac{3A(\phi_{j,j})}{\phi_{j,j}} \]

\[ (6A-12) \quad \frac{3L_{\phi_{1,j}}}{\phi_{1,j}} - A(\phi_{i,j}) - \frac{3L_{\phi_{j,j}}}{\phi_{j,j}} A(\phi_{j,j}) = \frac{3B(\phi_{i,j})}{\phi_{i,j}} - B(\phi_{j,j}) + B(\phi_{j,j}) \]

\[ = \frac{3A(\phi_{i,j})}{\phi_{i,j}} - L_{\phi_{j,j}} A(\phi_{j,j}) + A(\phi_{i,j}) A_{\phi_{j,j}} \]
Appendix 6B. Proof of lemma (6.7.2-12)

The idea of the proof is to fix all the components of all the innovations, except one. It will then be shown that a $c > 0$ and an $\varepsilon_0 > 0$ can be found as required in the lemma, independent of the values of the fixed innovation-components. From this the result follows directly.

Let $w_t^{(1)}$ denote the $t$-th component of the innovations vector $w_t$.

Let $F$, with $F \subset H$, be the $\sigma$-algebra generated by $w_t^{(2)}, w_t^{(3)}, \ldots, w_t^{(p)}, w_{t-1}^{(1)}, w_{t-1}^{(2)}, \ldots, w_{t-1}^{(p)}, w_{t-2}^{(1)}, \ldots$, i.e. all components of $w_t, w_{t-1}, w_{t-2}, \ldots$, except the component $w_t^{(1)}$. It will be shown that (with $v > 0$ fixed)

(6B-1) $\exists c > 0, \exists \varepsilon_0 > 0, \forall \varepsilon \in (0, \varepsilon_0), \forall (s,t) \in \bar{N}, \forall w \in \Omega: P[v - \varepsilon \leq w_t^{(1)} \leq v + \varepsilon] < c \varepsilon^4$.

From this it follows, by taking the expectation, that

(6B-2) $\forall c > 0, \exists \varepsilon_0 > 0, \forall \varepsilon \in (0, \varepsilon_0): P(v - \varepsilon \leq w_t^{(1)} \leq v + \varepsilon) = EP[v - \varepsilon \leq w_t^{(1)} \leq v + \varepsilon] < c \varepsilon^4$,

and the lemma follows. So it remains to show (6B-1). Let for any real number $x$,

(6B-3) $(x)_+ := \max(x, 0)$.

If $t = s$ one has $v_{ts} = 0$, so if $\varepsilon_0$ is taken smaller than $v$, then

(6B-4) $P[v - \varepsilon \leq w_t^{(1)} \leq v + \varepsilon] = 0 < c \varepsilon^4$.

So it remains to prove (6B-1) for $s > t$ (instead of $s > t$).

For $s > t$, consider

(6B-5) $P[v - \varepsilon \leq w_t^{(1)} \leq v + \varepsilon] = P[v - \varepsilon - \lambda_1 v_{t-1, s} \leq w_t^{(1)}] \leq v - \lambda_1 v_{t-1, s}^{(1)}$.

$= P[(v - \varepsilon - \lambda_1 v_{t-1, s})^+ \leq w_t^{(1)}] \leq v - \lambda_1 v_{t-1, s}^{(1)}$.

$\leq P[(v - \varepsilon - \lambda_1 v_{t-1, s})_+^2 \leq v - \lambda_1 v_{t-1, s}^{(1)}] \leq (v - \lambda_1 v_{t-1, s})^2$. 


From (6.7.2-4) it follows that for \( t > s \), \( y_{ts} \) can be decomposed as follows

\[
(6B-6) \quad y_{ts} = y_s + \begin{bmatrix} u^{(1)}_t \\ t \\ \vdots \\ 0 \end{bmatrix},
\]

where \( y_s \) is \( F \)-measurable, and \( u^{(1)}_t \sim n(0, \sigma_{11}^2) \), where \( \sigma_{11} \) is the \((1,1)\)-element of \( \Sigma \) (cf. (6.7.2-2)). It follows that, conditional on \( F \),

\[
(6B-7) \quad \| y_{ts} \|^2 = f^2 + x^2,
\]

where \( f \) is \( F \)-measurable and chosen to be nonnegative and \( x \sim n(x_0, \sigma_{11}^2) \), with \( x_0 \) \( F \)-measurable (\( x = u^{(1)}_t + x_0 \) in fact). Substituting this one obtains

\[
(6B-8) \quad P[(\nu - \lambda_1 y_{t-1}, s)^2_+ \leq \| y_{ts} \|^2 \leq (\nu - \lambda_1 y_{t-1}, s)^2 | F] = \\
= P[(\nu - \lambda_1 y_{t-1}, s)^2_+ - f^2 \leq x^2 \leq (\nu - \lambda_1 y_{t-1}, s)^2 - f^2 | F] = \\
= P[(\nu - \lambda_1 y_{t-1}, s)^2 - f^2)^2_+ \leq |x| \leq ((\nu - \lambda_1 y_{t-1}, s)^2 - f^2)^2_+ | F].
\]

Because \( x \sim n(x_0, \sigma_{11}^2) \), the probability density function of \( x \) exists and this function is smaller than or equal to \( 2(2\pi \sigma_{11}^2)^{-\frac{1}{2}} \) for all \( x \). Hence the probability density function of \( |x| \) exists and is smaller than or equal to

\[
(6B-9) \quad \frac{1}{2(2\pi \sigma_{11}^2)^{-\frac{1}{2}}} [((\nu - \lambda_1 y_{t-1}, s)^2 - f^2)^2_+] =: \Delta(\epsilon).
\]

Three disjoint possibilities can be distinguished.

(a) \( (\nu - \lambda_1 y_{t-1}, s)^2_+ > f (\geq 0) \),
(b) \( (\nu - \lambda_1 y_{t-1}, s)^2_+ \leq f \) and \( (\nu - \lambda_1 y_{t-1}, s) > f \),
or
\[(\nu + \lambda_1^+ v_{t-1, s}^+ ) \leq f.\]

ad(a) \((\nu - \nu^\epsilon_1^+ v_{t-1, s}^+) > f\) implies \((\nu - \nu^\epsilon_1^- v_{t-1, s}^- ) > f\), and \(\nu > f\). In this case

\[
\Delta(\epsilon) = 2(2\pi \sigma_{11}^2)^{-\frac{1}{4}} \int F \kappa(\kappa^2 - f^2)^{-\frac{1}{4}} dk \leq 2(2\pi \sigma_{11}^2)^{-\frac{1}{4}} \int f(\kappa^2 - f^2)^{-\frac{1}{4}} dk = \]

\[
\leq 2(2\pi \sigma_{11}^2)^{-\frac{1}{4}} \epsilon^\frac{1}{2} (2f + \epsilon)^{-\frac{1}{4}} \epsilon^\frac{1}{2} \leq 2(2\pi \sigma_{11}^2)^{-\frac{1}{4}} \epsilon^\frac{1}{2} (2f + \epsilon)^{-\frac{1}{4}} \epsilon^\frac{1}{2}.
\]

So in this case:

\[
\forall \epsilon \in (0, \epsilon_o): \Delta(\epsilon) \leq 2(2\pi \sigma_{11}^2)^{-\frac{1}{4}} (2f + \epsilon)^{-\frac{1}{2}} \epsilon^\frac{1}{2}.
\]

ad(b) \((\nu - \nu^\epsilon_1^- v_{t-1, s}^- ) > f\) implies \(\nu > f\).

Furthermore, in this case \((\nu - \nu^\epsilon_1^- v_{t-1, s}^- )^2 - f^2 \) = 0, so

\[
\Delta(\epsilon) = 2(2\pi \sigma_{11}^2)^{-\frac{1}{4}} \int f(\kappa^2 - f^2)^{-\frac{1}{4}} dk = \]

\[
= 2(2\pi \sigma_{11}^2)^{-\frac{1}{4}} \int f(\kappa^2 - f^2)^{-\frac{1}{4}} dk \leq 2(2\pi \sigma_{11}^2)^{-\frac{1}{4}} \int f(\kappa^2 - f^2)^{-\frac{1}{4}} dk,
\]

and one can proceed as in (a) and obtain (6B-11) for this case too.

ad(c) \((\nu - \nu^\epsilon_1^- v_{t-1, s}^- ) \leq f\) implies \((\nu - \nu^\epsilon_1^- v_{t-1, s}^- ) \leq f\) and so \(\Delta(\epsilon) = 0\) in this case, and (6B-11) follows trivially in this case. So (6B-11) holds in all cases. Combining (6B-11) with (6B-9) and (6B-5) one finds

\[
(6B-13) \quad \text{Pr} \leq \nu^* \leq \nu \leq |F| \leq \Delta(\epsilon) \leq 2(2\pi \sigma_{11}^2)^{-\frac{1}{4}} (2f + \epsilon)^{-\frac{1}{4}} \epsilon^\frac{1}{2}.
\]

Taking \(\epsilon \in (0, \nu)\) arbitrary and \(c = 2(2\pi \sigma_{11}^2)^{-\frac{1}{4}} (2f + \epsilon)^{-\frac{1}{4}} \epsilon^\frac{1}{2}\), (6B-1) follows.

Q.E.D.
Appendix 6C. On the relationship between $V$ and $V_g$

The function $V_g$ depends on $K'$ because $g_1$ does. For fixed $\tilde{g}$ and $\tilde{Z}$, the function $V_g$ converges to $V$ in supnorm over $M$, as $K' \to \infty$. This will first be shown.

6C-1. Lemma. \( \forall \delta > 0, \exists K' \) such that \( \forall K' \geq K'': \sup_{c} \| V(\theta) - V(\theta') \| < \delta. \)

Proof. The proof will consist of three steps.

(i) First it will be shown that the set of functions \( \{ V_{g} | K' > 0 \} \cup \{ V \} \) is equicontinuous. Let \( \theta_1 \in M \) be fixed. Consider

\[
 f(\theta_1, \theta_2) = \tilde{Z}(i\epsilon(\theta_1) t^2 - i\epsilon(\theta_2) t^2). 
\]

This is a continuous function of $\theta_2 \in M$ and it is zero if $\theta_2$ is equal to $\theta_1$. Let $N(K') = \{ \omega \in C | v > K' \}$, then, by definition of $g_1$ one has $\omega \in N(K')$ iff $g_1(\omega) = 0$. Let $N(\omega) = \emptyset$. Then one has

\[
 V_g(\theta) = \int_{N(K')} \| \epsilon(\theta) t^2 d\tilde{P}(\omega) \text{ and } V(\theta) = \int_{N(K')} \| \epsilon(\theta) t^2 d\tilde{P}(\omega). 
\]

It follows that for arbitrary $\theta_1, \theta_2 \in M$ and for each $K' > 0$,

\[
 |V_g(\theta_1) - V_g(\theta_2)| = \int_{N(K')} \| (\epsilon(\theta_1) t^2 - \epsilon(\theta_2) t^2) d\tilde{P}(\omega) | \leq \int_{N(K')} \| \epsilon(\theta_1) t^2 - \epsilon(\theta_2) t^2 | d\tilde{P}(\omega) \leq f(\theta_1, \theta_2). 
\]

And similarly

\[
 |V(\theta_1) - V(\theta_2)| \leq f(\theta_1, \theta_2). 
\]

Because $f(\theta_1, \theta_2)$ is continuous as a function of $\theta_2$ and $f(\theta_1, \theta_2) = 0$, it follows that \( \{ V_g | K' > 0 \} \cup \{ V \} \) is an equicontinuous set of functions.

(ii) Next it will be shown that for each fixed $\theta$, $V_g(\theta) \to V(\theta)$ as $K' \to \infty$.

Consider

\[
 V(\theta) - V_g(\theta) = \int_{N(K')} \| \epsilon(\theta) t^2 d\tilde{P}(\omega). 
\]

Because $f(\theta_1, \theta_2)$ is continuous as a function of $\theta_2$, the function $V_g(\theta) \to V(\theta)$ as $K' \to \infty$. This will now be shown.

(iii) Finally, it will be shown that $V_g(\theta)$ is bounded above and below by $V(\theta)$.
Because $K'_2 > K'_1$ implies $N(K'_2) \subseteq N(K'_1)$, because $\mathbb{E}(N \cap N(K')) = 0$ and because the integral $\int_{\mathbb{N}} |c(\theta)|^2 d\mathbb{P}(\omega)$ is convergent (it is equal to $V(\theta)$), it follows that

$$\lim_{K' \to \infty} \frac{1}{n} \int_{\mathbb{N}} |c(\theta)|^2 d\mathbb{P}(\omega) = 0,$$

so

$$\lim_{K' \to \infty} V_{g}(\theta) = V(\theta).$$

(iii) From (i) and (ii) and the compactness of $M$ it follows that $V_{g} \to V$ in supnorm. (Suppose this is not so. Then there is a $\delta > 0$ and there are sequences $\{x_{k}'\}^{\infty}_{k=1}$ and $\{x_{k}\}^{\infty}_{k=1}$ such that $\forall k \in \mathbb{N}: |V(x_{k}') - V_{g}(x_{k})| \geq \delta$.

Because $M$ is compact, $\{x_{k}\}$ has a convergent subsequence with limit $\tilde{x} \in M$. Using the equicontinuity it can then be shown that $\limsup_{K' \to \infty} |V(\tilde{x}) - V_{g}(\tilde{x})| \geq \delta$, which contradicts the pointwise convergence shown in (ii)).

Q.E.D.

It can now rather easily be shown that, (due to the fact that $V$ has its unique global minimum at the true parameter point $\tilde{\theta}$) $V_{g}$ has its global minimum at a point (or at points) arbitrarily close to $\tilde{\theta}$ if $K'$ is large enough (for fixed $\tilde{\theta}$ and $\bar{\tilde{\theta}}$).

6C-2. Theorem. (Let $\tilde{\theta}$ and $\bar{\tilde{\theta}}$ be fixed). Let $\epsilon > 0$ be given. Then there exists a $K'_0 > 0$ with the following property. If $K' > K'_0$ and $V_{g}$ has its global minimum at $\theta_{o} \in M$, then $d(\theta_{o}, \bar{\tilde{\theta}}) < \epsilon$.

Proof. $V$ takes on its global minimum value at only one point, namely $\tilde{\theta}$. Let $B_{\epsilon}(\tilde{\theta}) = \{\theta \in M | d(\theta, \tilde{\theta}) < \epsilon\}$. Let $V(\tilde{\theta}) + \delta$ be the global minimum of $V$ on the (compact) set $M \setminus B_{\epsilon}(\bar{\tilde{\theta}})$. Then $\delta > 0$, (because $\bar{\tilde{\theta}} \notin M \setminus B_{\epsilon}(\bar{\tilde{\theta}})$). Take $K'_0 > 0$ such that for all $K' > K'_0$, $\sup_{\theta \in M \setminus B_{\epsilon}(\bar{\tilde{\theta}})} |V-V_{g}| < \delta/2$. This is possible according to the previous lemma. Then for all $\theta \in M \setminus B_{\epsilon}(\bar{\tilde{\theta}})$, $V_{g}(\theta) > (V(\tilde{\theta}) + \delta) - \delta/2 = V(\tilde{\theta}) + \delta/2$, while $V(\bar{\tilde{\theta}}) < V(\tilde{\theta}) + \frac{\delta}{2}$.

Therefore $V_{g}$ takes on its global minimum in the set $B_{\epsilon}(\tilde{\theta})$. Q.E.D.
60-3. Remark. In choosing $K'$ one has to reckon with the scale of the process. If all outputs are multiplied by a constant scaling factor $\lambda$, then $\tilde{y}$ is multiplied with $\lambda^2$ and $\tilde{y}$ remains the same (!). To obtain the same asymptotic results, $K'$ should be multiplied by $\lambda$. 
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<tr>
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<td>Mathematical Centre Tracts 147, Amsterdam.</td>
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List of mathematical symbols

Ia. Sets-general notation

\[ \subseteq \]
inclusion

\[ |V| \]
cardinality of the set \( V \)

\[ \in \]
element of

\[ \notin \]
not an element of

\[ \equiv \]
equivalence class

\[ V \setminus W = \{ v \in V \mid v \notin W \} \]
the symmetric difference

\[ V^c \]
complement of the set \( V \)

\[ \partial V \]
topological boundary of the set \( V \)

\[ B(\theta, \delta) \]
ball with centre \( \theta \) and radius \( \delta \)

\[ \emptyset \]
the empty set

\[ \mathbb{N} = \{ 1, 2, 3, \ldots \} \]
the set of natural numbers

\[ \mathbb{N}_0 = \{ 0, 1, 2, \ldots \} = \mathbb{N} \cup \{ 0 \} \]

\[ \mathbb{Z} \]
the set of integers

\[ \mathbb{Q} \]
the set of rational numbers

\[ \mathbb{R} \]
the set of real numbers

\[ \mathbb{R}_+ = \{ x \in \mathbb{R} \mid x > 0 \} \]
the set of complex numbers

\[ \mathbb{C} \]
the set of complex numbers

\[ D(0,1) = \{ z \in \mathbb{C} \mid |z| < 1 \} \]

\[ \bar{D}(0,1) = \{ z \in \mathbb{C} \mid |z| \leq 1 \} \]

\[ D(\mathbf{z}_0, \mathbf{r}) = \{ z \in \mathbb{C} \mid |z-\mathbf{z}_0| < \mathbf{r} \} \]
(Ia. Sets - general notation)

\( G(0,1) = \{ z \in \mathbb{C} \mid |z| = 1 \} \)

\( G_n^k(\mathbb{R}) \)
the group of nonsingular nxn real matrices

\( C^k \)
the set of real functions which are \( k \) times continuously differentiable

\( C^0([a,b],\mathbb{M}) \)
the set of all continuous curves \( X: [a,b] \to \mathbb{M} \)

\( C^\infty \)

\( C^\omega \)

\( S^2 \)
the unit sphere in \( \mathbb{R}^3 \)

118
118
159

(Ib. Special sets used in the text)

\( T \)
time axis

\( U \)
input alphabet

\( U^* \)
set of input functions

\( Y \)
output alphabet

\( Y^* \)
set of output functions

\( F \)
set of nonanticipative input-output mappings

\( B \)
set of initial conditions

\( U_{\text{infT}} \)

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12
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(1b, Special sets used in the text)

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(Ib. *Special sets used in the text*)

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<tr>
<td>0</td>
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<td>((U,\delta))</td>
<td>coordinate neighbourhood</td>
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<td>(U)</td>
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<td>B</td>
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<td>group</td>
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<td>(\Gamma_{m',n,m}^m)</td>
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<td>(\mathcal{M}_{1,m',n,m}^m)</td>
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<td>(\nu_{\alpha}^m)</td>
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<td>(\nu_{m',\alpha}^m)</td>
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<td>(\nu_{m',\alpha}^m)</td>
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<tr>
<td>Pos(m)</td>
<td>manifold of all (m \times m) symmetric</td>
</tr>
<tr>
<td></td>
<td>positive definite matrices</td>
</tr>
</tbody>
</table>
(I.b. Special sets used in the text)

\[ M^m, n, m \]
\[ M^m, \Delta \]
\[ M^m, n, m \]
\[ M^m, \Lambda \]
\[ M^m, \Delta \]
\[ M^m, n, m \]
\[ \mathbb{R} \]
\[ M^+; M^- \]
\[ L^a_{m, n, m} \]
\[ X_{\mathbb{R}} \]
\[ L^2(\lambda) \]
\[ \Omega \]
\[ F \]
\[ L^2(\lambda)^+ \]
\[ B(\lambda) \]
\[ E \]
\[ L \]
\[ C_j \]
\[ J \]
\[ C_j'; C_j'' \]
\[ J''(\theta); J'(\theta) \]
\[ E_k; E_{ij}; E_{ij}' \]
\[ I(j) \]
\[ E_{ij} \]
\[ D''; D'; D_{ij} \]
\[ M_{ij}; M_{ij}' \]
\[ N_{ij}; N_{ij}' \]
\[ U_k \]
(1b. Special sets used in the text)

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<tr>
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<td>$i_c; L_{c}^+; L_c[a,b];$</td>
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<td>$i_j$</td>
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<tr>
<td>$X; V$</td>
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</table>
Iia. Mappings - general notation

- $\mathcal{W}$: set of all mappings $T \to W$
- $f: A \to B, a \mapsto b = f(a)$
- $f|_C$: the mapping $f$ restricted to $C \subseteq A$
- $f|_{(-\infty,t]}$ etc.
- $\text{Dom}(f)$: the domain of the mapping $f$
- $f \circ g$

Iib. Special mappings used in the text

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<td>$\beta$</td>
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<td>$S$</td>
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<td>$L = S^{-1}$</td>
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<td>$\tau$</td>
<td>a backward shift</td>
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<td>$\sigma$</td>
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<td>$T_M$</td>
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(IIb. Special mappings used in the text)

1+1
\( F_0 \)

\( p: B + X \)
projection

\( \beta: I_{m,n,p} + I_{m,n,p} \)

\( \Omega \)
norm of the system \( \Omega \)

\( \phi_{\Sigma} \)

\( \Sigma \)

\( \delta \)

\( \psi(x) \)

\( \rho(z); \phi(x) \)

\( \rho_\alpha(w) \)

\( \rho: \phi \)

\( \alpha: \phi \)

\( \|A\|_F \)

\( \chi; y \)

\( \rho \Delta \)

\( \psi \Delta \)

\( \varphi \Delta \)

\( \phi \alpha \)

\( F; \beta; F; \beta \)

\( \delta \)

\( \varphi \beta \)

\( \varphi \beta \)

\( G_\beta; G_\beta \)

\( F_\phi \)

\( d_M \)

Q

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(IIb. Special mappings used in the text)

\[
K, K^- \\
\phi_1, \phi_2, \phi_3 \\
d(\mu_1, \mu_2) \\
\langle , \rangle \qquad \text{Hellinger distance} \\
L^2(\lambda) \\
\varphi \\
d_B \\
F_T \\
\varphi \\
V \\
\phi_j \\
\tilde{\phi}_j \\
\phi_j = (\phi_1^j, \phi_2^j, \ldots, \phi_d^j)^T \\
x(\theta); x(\phi, j) \qquad \text{smooth local section} \\
\varrho_1; \varrho_2 \\
d_j \\
\chi_1(t); \chi_2(t) \\
X_t^+(r); Y_t^+(r) \\
\Pi_t \\
\Pi_t^a; \Pi_t^\alpha \\
\rho_c^+; \rho_c^-; \rho^+ \\
F_k \\
\phi \\
S_{-r}
\]
(IIb. Special mappings used in the text)

\[ f_k = f_{kw} \left( \left[ d^{+} \right] \right) \]
\[ \pi^r; \pi^r_a \]
\[ l, h_s \]
\[ \nu_R(\theta) \]
\[ X_k; X \]
\[ \langle \cdot, \cdot \rangle_R \]

IIIa. Matrices - general notation

\[ \text{rk}(A) = \text{rank}(A), \]
\[ \sigma(A) = \text{spec}(A), \]
\[ \text{det}(A) = |A|, \]
\[ \text{ker}(A) = \]
\[ \hat{\text{ker}}(H), \]
\[ \text{im}(A) \]

the rank of matrix A
the spectrum of the (square) matrix A, \[ \sigma(A) = \{ \lambda \in \mathbb{C} \mid \lambda I - A = 0 \} \]
the determinant of A
the left kernel of an \( n \times n \) matrix A,
\[ \hat{\text{ker}}(A) = \{ x \in \mathbb{R}^n \mid x^T A = 0 \} \]
H a Hankel matrix
image of \( A = \{ x \mid \exists y \text{ such that } x = Ay \} \)

IIIb. Special matrices used in the text

\[ R(A, B) \]
\[ O(A, C) \]
reachability matrix
observability matrix
### Special matrices used in the text

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<tr>
<td>$M(\tilde{\eta})$</td>
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<tr>
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<tr>
<td>$M_A(A(r)(\tilde{\eta})$, etc.</td>
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<tr>
<td>$J = J(A, B, C, D, \tilde{\eta})$</td>
<td></td>
<td>206</td>
</tr>
<tr>
<td>$\Psi_t$</td>
<td></td>
<td>232</td>
</tr>
<tr>
<td>$F(\theta); G(\theta); H(\theta); K$</td>
<td></td>
<td>232</td>
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### IIIB. Special matrices used in the text

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<tr>
<td>$A(\phi, j)$, $B(\phi, j)$, $C(\phi, j)$</td>
<td>240</td>
</tr>
<tr>
<td>$F(\phi, j)$, $G(\phi, j)$, $H(\phi, j)$</td>
<td>240</td>
</tr>
<tr>
<td>$R(\phi, j)$</td>
<td>241</td>
</tr>
<tr>
<td>$S(\theta; i, j)$</td>
<td>244</td>
</tr>
<tr>
<td>$T_{ij}$</td>
<td>260</td>
</tr>
<tr>
<td>$F(j; i, j)$, $G(j; i, j)$, $H(j; i, j)$</td>
<td>261</td>
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<tr>
<td>$E$</td>
<td>expectation operator</td>
</tr>
<tr>
<td>$\chi_G$</td>
<td>indicator function of the set $G$</td>
</tr>
<tr>
<td>$1, 1^2$</td>
<td>the Euclidean (resp. Hermitean norm)</td>
</tr>
<tr>
<td>$x^T$ with $x \in \mathbb{R}^n$</td>
<td>the transposed of $x$</td>
</tr>
<tr>
<td>$x^*$ with $x \in \mathbb{C}^n$</td>
<td>the transposed conjugate of $x$</td>
</tr>
<tr>
<td>$E(y_{t+1}</td>
<td>y_t)$</td>
</tr>
<tr>
<td>$L$</td>
<td>the lag operator</td>
</tr>
<tr>
<td>$\deg p(z)$, with $p(z)$ a polynomial (matrix)</td>
<td>82</td>
</tr>
<tr>
<td>$\min(a, b)$, $a, b \in \mathbb{R}$</td>
<td>87</td>
</tr>
<tr>
<td>$\max(a, b)$, $a, b \in \mathbb{R}$</td>
<td>87</td>
</tr>
<tr>
<td>$N(\mu, \sigma^2)$</td>
<td>Gaussian (or normal) distribution with mean $\mu$ and variance $\sigma^2$</td>
</tr>
<tr>
<td>$\langle, \rangle$</td>
<td>inner product</td>
</tr>
</tbody>
</table>

### IVb. Other special symbols used in the text

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<td>$\Sigma$</td>
<td>system</td>
</tr>
<tr>
<td>$\Delta^+$</td>
<td>14</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>14</td>
</tr>
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<td>$\frac{1}{n} v_n^\delta$</td>
<td>relative frequency</td>
<td>25</td>
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<tr>
<td>$y_{t_0+j \mid t_0-1}$</td>
<td></td>
<td>28</td>
</tr>
<tr>
<td>$t_i$</td>
<td></td>
<td>34</td>
</tr>
<tr>
<td>$s_i$</td>
<td></td>
<td>34</td>
</tr>
<tr>
<td>${a(1)}$</td>
<td></td>
<td>34</td>
</tr>
<tr>
<td>${a(k) \mid b(k)}$</td>
<td></td>
<td>37</td>
</tr>
<tr>
<td>$r_i$</td>
<td>row degrees</td>
<td>46</td>
</tr>
<tr>
<td>$p_i$</td>
<td>row degrees</td>
<td>46</td>
</tr>
<tr>
<td>$p_k$</td>
<td></td>
<td>49</td>
</tr>
<tr>
<td>$J$</td>
<td></td>
<td>91</td>
</tr>
<tr>
<td>$I$</td>
<td></td>
<td>91</td>
</tr>
<tr>
<td>$d_k$</td>
<td></td>
<td>101</td>
</tr>
<tr>
<td>$N(I+1)$</td>
<td></td>
<td>111</td>
</tr>
<tr>
<td>$\alpha$</td>
<td></td>
<td>163</td>
</tr>
<tr>
<td>$\lambda$</td>
<td></td>
<td>165</td>
</tr>
<tr>
<td>$\mu$</td>
<td></td>
<td>165</td>
</tr>
<tr>
<td>$\beta$</td>
<td></td>
<td>166</td>
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<tr>
<td>$r$</td>
<td></td>
<td>186</td>
</tr>
<tr>
<td>$\theta$</td>
<td></td>
<td>186</td>
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<tr>
<td>$\Im(A,B,C;x)$</td>
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<td>$v_t$</td>
<td>standardized innovations</td>
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</tr>
<tr>
<td>$w_t$</td>
<td>innovations</td>
<td></td>
</tr>
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<td>$\theta$</td>
<td>parameter</td>
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<td>$\varepsilon_t$</td>
<td>prediction error</td>
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</tr>
<tr>
<td>$a_k$</td>
<td>weighting constants</td>
<td>231</td>
</tr>
<tr>
<td>$v_t = \delta \varepsilon_t / \delta \theta$</td>
<td>231</td>
<td></td>
</tr>
<tr>
<td>$j = j(r)$</td>
<td></td>
<td>238</td>
</tr>
<tr>
<td>$\xi(t,j); \psi(t,j); \epsilon(t,j)$ etc.</td>
<td>240</td>
<td></td>
</tr>
<tr>
<td>$\sigma(\theta); \lambda^M(\theta); \lambda_0; \lambda_1$</td>
<td>245</td>
<td></td>
</tr>
<tr>
<td>$v_t$</td>
<td></td>
<td>245</td>
</tr>
<tr>
<td>$K'$</td>
<td></td>
<td>245</td>
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<td>247</td>
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<td>$\epsilon$</td>
<td>260</td>
</tr>
<tr>
<td>$\zeta(t; i, j)$</td>
<td>261</td>
</tr>
<tr>
<td>$K_e$</td>
<td>262</td>
</tr>
<tr>
<td>$I(\gamma)$</td>
<td>262</td>
</tr>
<tr>
<td>$a'_t$</td>
<td>266</td>
</tr>
<tr>
<td>$\alpha'_t$</td>
<td>266</td>
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<tr>
<td>$\mathcal{K}(C), \mathcal{K}(H)$ etc.</td>
<td>267</td>
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<td>269</td>
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<td>279</td>
</tr>
<tr>
<td>$K^+$</td>
<td>280</td>
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<td>$t_3$</td>
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<tr>
<td>$\bar{a}_t; \bar{a}$</td>
<td>281</td>
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<td>$\mathfrak{k}^+; \mathfrak{k}^+$</td>
<td>281</td>
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<tr>
<td>$r$</td>
<td>283</td>
</tr>
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<td>$d^+$</td>
<td>286</td>
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<td>$\omega$</td>
<td>294</td>
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<tr>
<td>$b_t = b_{tw} = b_{tw}(a)$</td>
<td>294</td>
</tr>
<tr>
<td>$s_t$</td>
<td>301</td>
</tr>
<tr>
<td>$S = (S, q, \delta)$</td>
<td>302</td>
</tr>
<tr>
<td>$\tilde{\varphi}$</td>
<td>306</td>
</tr>
<tr>
<td>$\tilde{E}$</td>
<td>307</td>
</tr>
<tr>
<td>$\nu_{ts}$</td>
<td>331</td>
</tr>
<tr>
<td>$(y, v, q_1(v), \xi(\theta^+), \epsilon(\theta), \psi(\theta, j), h(\theta, j), b(\theta, j))$</td>
<td>337</td>
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<td>$\tilde{\tau}(k)$</td>
<td>344</td>
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Page 14, definition 2.1-8: If \( t - \delta \notin T \), then define formally \( u(t - \delta) := u_0 \), where \( u_0 \) is some fixed value in \( U \) (e.g. \( u_0 = 0 \) if \( 0 \in U \)).

Page 34: A concise definition of \( p_k \), \( k = 1, 2, \ldots, m \), is as follows: Let \( \text{Dep}(i; H) \) be the property that row \( i \) of the Hankel matrix \( H \) depends on the previous rows \( 1, 2, \ldots, i-1 \). Then

\[
p_k := -1 + \min \{ p \in \mathbb{N}_0 \mid \text{Dep}(pm + k; H) \}.
\]

Alternatively, let \( \text{Ind}(i, H) \) denote the property that row \( i \) of the Hankel matrix is independent of the previous rows \( 1, 2, \ldots, i-1 \). Then

\[
p_k = \max \{ p \in \mathbb{N}_0 \mid \text{Ind}(pm + k; H) \}.
\]

Page 66/67, theorem 2.4.2-17: The unicity is (of course) within the class of square, real, proper rational (transfer) matrices.

Page 127: Lemma 4.4-5 follows in fact directly from the existence of the controllability (Kronecker) indices. They can be analysed in a completely dual fashion to the analysis of the observability indices, as given in 2.3.3.

Page 145: The metric (5.2-25) is a metric on the set of stable systems (as opposed to the (sub)set of asymptotically stable systems) and not more than that:

(i) If \( A \) is stable then \( \{ A^k \}_{k=0}^\infty \) is bounded and therefore

\[
\{ C A^{k-1} B B^* (A^*)^{k-1} C^* \}_{k=1}^\infty
\]

is bounded and so

\[
\frac{\text{tr}}{k^2} C A^{k-1} B B^* (A^*)^{k-1} C^* \quad \text{is bounded and well-defined.}
\]

(ii) If \( A \) is unstable (and \( (A, B, C) \) minimal of course) then

(a) \( \exists \lambda \in \sigma(A) \) with \( |\lambda| > 1 \) and there is an eigenvector \( x \in \mathbb{C}^n \setminus \{0\} \) with \( Ax = \lambda x \),
and/or

(b) \exists \lambda \in \sigma(A) \text{ with } |\lambda| = 1, \text{ there is a corresponding generalized

eigenvector } x \in C^N \setminus \{0\} \text{ and there is an eigenvector } y \in C^N \setminus \{0\}

such that

$$Ax = \lambda x + y$$

In case (a) $$\frac{A^kx}{k} = \frac{\lambda^k}{k} x \text{ and } \left|\frac{\lambda^k}{k}\right| \to \infty \text{ if } k \to \infty.$$ 

In case (b) $$\frac{A^kx}{k} = \frac{\lambda^k}{k} x + \frac{\lambda^{k-1}y}{k} = \frac{\lambda^k}{k} x + \lambda^{k-1}y.$$ 

Because \(|\lambda| = 1, \frac{\lambda^k}{k} x \text{ converges to zero if } k \to \infty, \text{ but } \lambda^{k-1}y \text{ does not}

converge to zero if } k \to \infty, \text{ its norm } \|\lambda^{k-1}y\| = \|y\| \text{ remains constant}

(\# 0). With an argument that is similar to the proof of theorem (2.3.4-3)

in Appendix 2A, it can be shown that in both cases (a) and (b),

$$\sum_{k=1}^{\infty} \frac{A^{k-1}B}{k} \text{ does not converge to zero and}$$

$$\text{tr}_{k=1}^{\infty} \frac{A^{k-1}B (A^{k-1}B)^*}{k^2} = -.$$ 

Page 148, remark (ii): Compare theorem 4.5.2.

Page 155: Add after remark (iii): remark (iv):

Another (and in fact somewhat simpler) representation of

(5.2-39)/(5.2-40) is given in [Hnn 88a,b].

Page 157, section 5.3.2: \(A\) is the time interval length. This concept

corresponds in fact with inf \(A^+\) in definition 2.1.-11.

Page 197 and further: Instead of speaking of a degenerate Riemannian metric,

it would probably be better to speak of a positive semi-definite

Riemannian metric. If there exists a point \(E\) and a nonzero tangent vector

\(\nu\), such that at \(E\), \(g(\nu,\nu) = 0\), then one could speak of a singular

positive semi-definite Riemannian metric.
Page 201: The last full sentence should read: "One can expect this to hold for all locally continuous canonical forms in which $n^2$ components of $(A, B, C)$ are fixed to be 0 or 1, and all other components are free to vary in an $n(m+m')$-dimensional open set, such that its closure contains systems with smaller McMillan degree."

Page 226: Formula (6.1-7) follows from the fact that

$$\frac{d\Phi}{df} - \frac{d\Phi}{df} = 0.$$ 

Page 352: The concept of a limit point at infinity corresponds with that of an $\omega$-limit point of $[H_i - S]$. 